Holography and large-N Dualities

Is String Theory Holographic?

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1 Introduction

The holographic principle [1] is based on the idea that there is a limit on information content of spacetime regions. For a given volume V bounded by an area A, the state of maximal entropy corresponds to the largest black hole that can fit inside V. This entropy bound is specified by the Bekenstein-Hawking entropy

$$S \leq S_{\mathrm{BH}} = \frac{A}{4G} \tag{1.1}$$

and the goings-on in the relevant spacetime region are encoded on "holographic screens".

The aim of these notes is to discuss one of the many aspects of the question in the title, namely: "Is this feature of the holographic principle realized in string theory (and if so, how)?". In order to adress this question we start with an heuristic account of how string like objects are related to black holes and how to compare their entropies. This second section is exclusively based on [2] and will lead to a key insight, the need to consider BPS states, which allows for a more precise treatment. The most fully understood example is a bound state of D-branes that appeared in the original article on the topic [3]. The third section is an attempt to review this construction from a point of view that highlights the role of AdS/CFT [4, 5]. We will focus on a version of the story which touches part of the subsequent mathematical physics literature related to the geometry of K3 surfaces [6, 7, 8, 9, 10].

2 Classical Strings and Black Holes

We consider a heuristic model for a (classical bosonic) string made out of "string bits" 2.1. It can be thought of as arising from a random walk process in d-dimensional

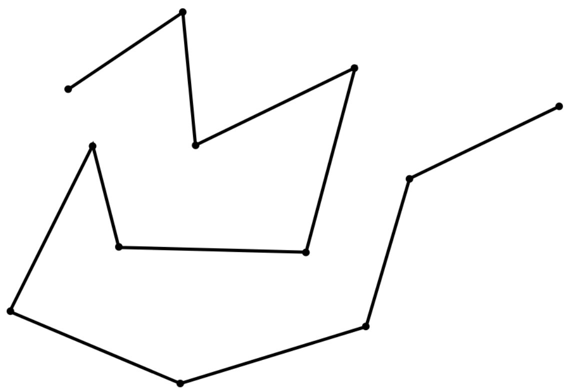

Figure 2.1: We assume that the bits are of length $l = \sqrt{\alpha'}$ and the whole string has mass M, length L and number of bits $N = L/l$.

spacetime. For such processes the average end-to-end distance can be calculated and is given by

$$\sqrt{\langle D^2 \rangle} = \sqrt{lL} \sim \sqrt{M}, \tag{2.1}$$

while its Schwarzschild radius is

$$R_{\mathrm{S}} = \frac{2GM}{c^2} \sim M. \tag{2.2}$$

By this analysis of the dependence on mass we deduce that sufficiently massive strings are indeed expected to form black holes. This can be rephrased as follows: E.g. in the case of open strings with zero momentum we have

$$M^2 = E^2 = \frac{1}{\alpha'}(N-1) \simeq \frac{N}{\alpha'} \qquad \text{if } N \gg 1. \tag{2.3}$$

Therefore a string at zero momentum and sufficiently high degree of excitation will look like a stationary Schwarzschild black hole for an outside observer.

The Bekenstein-Hawking entropy of this black hole is given by ($c = \hbar = 1$)

$$S_{\mathrm{BH}} = \frac{A}{4l_p^2} = 4\pi G M^2, \tag{2.4}$$

with the Planck-length l_p and proportional to the mass^2. What is the entropy of the random-walk string? We can estimate the number of microstates as

$$\Omega \sim d^{L/\sqrt{\alpha'}} \sim d^{M\sqrt{\alpha'}} \sim e^{M\sqrt{\alpha'}\log(d)}, \tag{2.5}$$

by using $M \sim TL \sim 1/\alpha' \, L$, where T is the tension. An approximation for the entropy is therefore

$$S_{\mathrm{str}} = \log \Omega \sim \sqrt{\alpha'} M. \tag{2.6}$$

The result of this heuristic approach is not too far from the correct one [2]

$$S_{\mathrm{str}} = 4\pi \sqrt{\alpha'} M \tag{2.7}$$

and in any case the entropy is proportional to the mass^1.

The disagreement of both calculations for the entropy is of course expected. In the purely statistical derivation of S_{str} the entropy will certainly be an extensive quantity and adding string bits of a given chunk of the mass M is expected to increase the entropy linearly. That this is different from the gravitational situation and the expression for S_{BH} is a well-known feature of gravitational physics. There is however an obvious reason why this comparison is questionable: A non-vanishing black hole entropy requires interactions because

$$G \sim g^2 \alpha', \tag{2.8}$$

where g is the (closed) string coupling, while we treated the random-walk "string" as free. Although there is further heuristic evidence that a Schwarzschild black hole is the strong coupling version of a highly excited string [2], an exact computation is only feasible if it remains valid when changing from $g = 0$ to $g \neq 0$. Unfortunately the strong coupling regime in string theory is in general very hard to control. This means we have to consider a state which is BPS, where a computation at weak coupling allows for making statements about the strong coupling version.

3 The Strominger-Vafa Construction

3.1 AdS/CFT for the D1/D5 System

The first and by now best understood example of such a BPS state is due to Strominger and Vafa and constructed from a bound state of D1- and D5-branes in type IIB string theory [3], see also [11]. For a correct number of branes the system is BPS and we can try to deduce some information of the strong coupling situation from the weak coupling computation. Although the number of microstates will not be invariant there is a certain topological index which acts as a bound on the degeneracy of BPS microstates.

Let X be a complex, projective[1] and compact K3 surface and consider type IIB string theory compactified to five dimensions on $\mathbb{R}^{1,4} \times X \times S^1$ with stacks of Q_1 D1-branes and Q_5 D5-branes wrapping cycles according to 3.1. Let $(2\pi)^4 V$ be the volume of X and R be the radius of the S^1. Using the superposition principle for supergravity solutions of

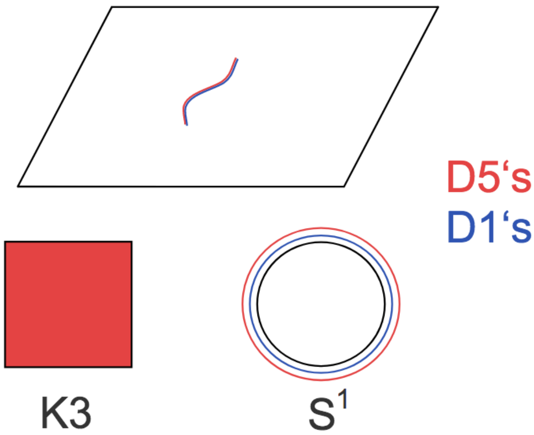

Figure 3.1: The D5-branes wrap all of the compact space and the D1-branes only wrap the S^1. A five dimensional physicist sees a worldline in $\mathbb{R}^{1,4}$.

intersecting branes [12] the metric of the non-compact part of this spacetime looks like

$$\mathrm{d}s^2 = -\left(H_1 H_5 \left(1+K\right)\right)^{-2/3} \mathrm{d}t^2 + \left(H_1 H_5 \left(1+K\right)\right)^{1/3} \left(\mathrm{d}r^2 + r^2 \mathrm{d}\Omega_3^2\right) \tag{3.1}$$

with

$$H_1(r) = 1 + \frac{r_1^2}{r^2} \qquad H_5(r) = 1 + \frac{r_5^2}{r^2} \qquad K(r) = \frac{r_m^2}{r^2} \tag{3.2}$$

and

$$r_1^2 = \frac{gQ_1 l_s^6}{V} \qquad r_5^2 = gQ_5 l_s^2 \qquad r_m^2 = \frac{g^2 N l_s^8}{R^2 V}, \tag{3.3}$$

where N is the momentum quantum number corresponding to motion along the S^1. This supergravity solution is valid if $gQ_1, gQ_5, gN \gg 1$ which we will enforce by assuming a very high number of D1- and D5-branes as well as a very high momentum quantum number.

[1]This assumption is only necessary for the viewpoint we want to take and will become clear later on.

If the system is BPS this solution describes an extremal, Reissner-Nordström black hole with Hawking temperature $T_{\mathrm{H}} = 0$. Its "macroscopic" entropy can be calculated by using the Bekenstein-Hawking formula (the horizon is at $r = 0$)

$$S_{\mathrm{BH}} = \frac{A}{4G_5} = \frac{1}{4G_5}\pi^2 r^3 \left(H_1 H_5 \left(1+K\right)\right)^{3/6}\Big|_{r=0} = 2\pi\sqrt{Q_1 Q_5 N}. \tag{3.4}$$

The basic motivation behind the construction of Strominger and Vafa is to determine a microscopic origin of this entropy at weak coupling. As it turns out, this is an application of AdS/CFT.

To explain how the AdS/CFT correspondence enters into the story we consider a slightly modified version of the geometry in which we unwrap the S^1 (or, equivalently, set $N = 0$)[2]. In this case the supergravity solution for the five dimensional non-compact part of the spacetime becomes

$$\mathrm{d}s^2 = H_1^{-1/2} H_5^{-1/2}\left(-\mathrm{d}t^2 + \mathrm{d}x^2\right) + H_1^{1/2} H_5^{1/2}\left(\mathrm{d}r^2 + r^2 \mathrm{d}\Omega_3^2\right). \tag{3.5}$$

If we vary the coupling the system has two different descriptions 3.2, see [4, 5]. AdS/CFT

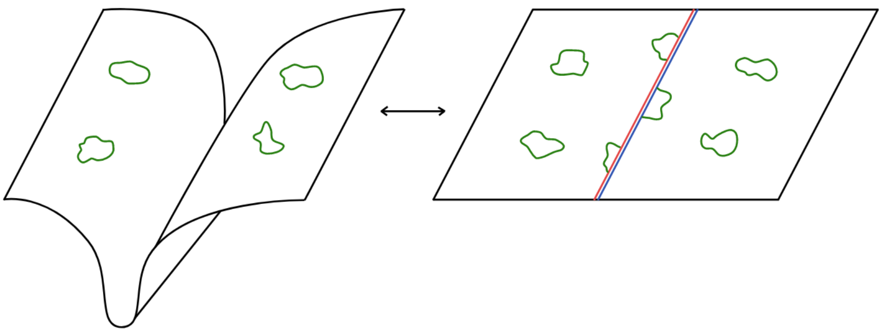

Figure 3.2: The black string case (left) is valid for $gQ_1, gQ_5 \gg 1$ and is related to a system of (proper) D-branes whose worldvolume theory is a gauge theory determined by open strings (right), valid for $gQ_1, gQ_5 \ll 1$, when we lower the coupling and keep fixed the number of branes. The system of D-branes has $\mathrm{SO}(1,1)\times\mathrm{SO}(4)$ as symmetry group and preserves $\mathcal{N} = (4,4)$ supersymmetry. The gauge theory lives on the *intersection* of all branes. We omitted the K3 surface on both sides.

will be valid in a further low-energy limit ($\alpha' \to 0$), in which we keep all dimensionless parameters fixed, in this case

$$\frac{r}{\alpha'} \qquad v := \frac{V}{(2\pi)^4(\alpha')^2} \qquad g_6 := \frac{g}{\sqrt{v}}. \tag{3.6}$$

There are two interpretations for this low energy limit, depending on the value of the coupling.

[2]In this case the area of the horizon and the entropy shrinks to zero, but this will not interfere with the point that is made.

a) For weak coupling $gQ \ll 1$, we have a decoupling of supergravity in the bulk and gauge theory on the branes. The gauge theory has a Coulomb branch, which corresponds to varying expectation values of gauge fields, and (unlike for example the $\mathcal{N} = 4$ SYM on a D3) also a Higgs branch, corresponding to varying expectation values of matter fields that arise from strings that stretch between the two types of branes. There is an argument [5] how the Coulomb branch can be removed by symmetries such that we are interested in the **Higgs branch of the (1+1) dimensional gauge theory with $\mathcal{N} = (4, 4)$ supersymmetry on the worldvolume**. This theory is known to be a SCFT.

b) For strong coupling $gQ \gg 1$ the supergravity solution is valid but also involves a certain decoupling. As g_{tt} is not constant, the energy measured by an observer at infinity E_∞ is different from the one measured at a fixed position p, E_p. The difference is determined by the redshift factor

$$E_\infty = H_1^{-1/4} H_5^{-1/4} E_p. \tag{3.7}$$

This implies that the low energy limit above contains two kinds of particles: Massless particles in the bulk, i.e. supergravity in the bulk, and any particle near the horizon which is heavily redshifted when observed from a great distance. In the limit these two systems decouple as on the one hand the wavelength of the bulk particles grows indefinitely due to redshifting and interaction with the geometry of fixed characteristic length scale becomes impossible, and on the other hand because near horizon particles cannot escape from the gravitational potential of the branes. Applying the limit as described above to the five dimensional solution (3.5) yields

$$\mathrm{d}s^2 = \frac{r^2}{\alpha' g_l \sqrt{Q_1 Q_5}} \left(-\mathrm{d}t^2 + \mathrm{d}x^2\right) + \alpha' g_6 \sqrt{Q_1 Q_5} \left(\frac{\mathrm{d}r^2}{r^2} + \mathrm{d}\Omega_3^2\right). \tag{3.8}$$

This **near horizon geometry** can be identified as $\mathrm{AdS}_3 \times S^3 \times X$ (after including the K3 again).

In both interpretations the system decouples from bulk supergravity and identifying what remains on both sides is the essence of the $\mathrm{AdS}_3/\mathrm{CFT}_2$-correspondence. The two dimensional SCFT lives on the conformal boundary of the relevant AdS-space. This identification at low energy remains valid when compactifying the x coordinate on a S^1 and our hope is to gain insight into the microscopic origin of the Bekenstein-Hawking formula by analysing the SCFT that is dual to type IIB string theory near the horizon.

3.2 The Instanton Moduli Space

One possibility [5, 11, 13] to fully characterize the low energy physics of the SCFT is to start with the $\mathrm{U}(Q_5)$ gauge theory on the D5-branes and think of the D1-branes a Q_1 instantons in this gauge theory. These instantons live on $X \times S^1$ and are translationally invariant with respect to time. We consider a limit in which $\mathrm{Vol}(X) \ll \mathrm{Vol}(S^1)$ such that compactification naively yields an effective field theory on $\mathbb{R}^{1,0} \times S^1$. The effective field theory has an instanton configuration which is determined by a choice of classical vacuum and these various effective versions are then related by moduli which paramatrize families of classical solutions in the effective description. After compactification, i.e. in the limit where the K3 is small compared to the S^1, the instanton moduli will depend on the geometric moduli of the K3 surface [3]. The dynamics of the effective theory therefore depends on a background of moduli fields and our system is actually a sigma model with worldvolume $\mathbb{R}^{1,0} \times S^1$ and the *instanton moduli space* as target space. In the current situation there is an assumption for which there is an interesting geometric characterization of the instanton moduli space [6, 9, 13, 14].

Suppose the Poincaré dual $PD([D5])$ is a non-primitive class in the cohomology of X, i.e. in the image of the Lefschetz operation

$$\begin{aligned} L : H^2(X, \mathbb{R}) &\longrightarrow H^4(X, \mathbb{R}) \\ [\eta] &\longmapsto [\eta \wedge \omega], \end{aligned} \tag{3.9}$$

where ω is the symplectic form on the K3 surface. The fields that determine the instanton moduli space depend on modular parameters of X. The assumption above ensures that, instead of all moduli of X, only those of a holomorphic[3] two cycle $C \subset X$ determine the instanton moduli space. For the sake of simplicity, we further assume $Q_1 = Q_5 = 1$. A generalization to a higher stack of branes is known [14].

Recall that there are 6 scalar fields

$$\phi_i \in \Gamma(C, N_{C|\mathrm{Target}}) \otimes \mathrm{ad}(\mathrm{U}(1)) \qquad i = 1, ..., 6, \tag{3.10}$$

where $N_{C|\mathrm{Target}}$ is the normal bundle of C inside the target space of the *string*, $\mathbb{R}^{1,4} \times X \times S^1$. These come from string excitations that are orthogonal to the branes and can be interpreted as Goldstone bosons for the spontaneous breakdown of Poincaré symmetry due to the branes 3.3. Four of them correspond to the embedding into $\mathbb{R}^4$ while the remaining two correspond to the embedding into X. As C is embedded holomorphically into X the latter can be combined into one complex scalar field which we will denote as Φ. There is a short exact sequence

$$0 \longrightarrow T_C = K_C^{-1} \longrightarrow T_X = K_C^{-1} \oplus N_C \longrightarrow N_C \longrightarrow 0 \tag{3.11}$$

of vector bundles on C from which we can calculate

$$\begin{aligned} \det(T_X) &= \det(K_C^{-1}) \otimes \det(N_C) \\ &= K_C^{-1} \otimes N_C \\ &= \mathcal{O}_X, \end{aligned} \tag{3.12}$$

[3]This condition is needed in order to respect supersymmetry [14].

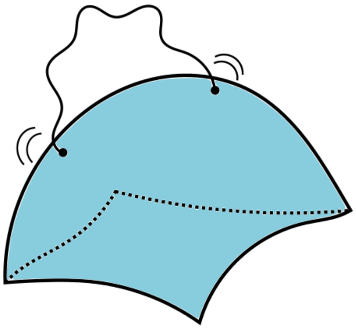

Figure 3.3: The modes of the string orthogonal to the brane allow for a description of its deformation into the normal directions.

where the last equality follows from X being Calabi-Yau. This implies $N_C \simeq K_C$ and in particular

$$\Phi \in \Omega^1(C) \otimes \mathrm{ad}(\mathrm{U}(1)), \tag{3.13}$$

i.e. we can think of the complex scalar field as a complex differential form on C. As we started with a gauge theory on $\mathbb{R}^{1,0} \times S^1 \times X$, taking the limit where the K3 is very small is an example of dimensional reduction as studied in [13]. There is a set of equations, the Hitchin equations, that need to be satisfied in order for the reduction to be well defined and as the relevant moduli are those of $C \subset X$ which are controlled by the field Φ, these become

$$\begin{aligned} F_{z\bar{z}} &= [\Phi_z, \Phi_{\bar{z}}] \\ D_z \Phi_{\bar{z}} &= 0 = D_{\bar{z}} \Phi_z, \end{aligned} \tag{3.14}$$

after we specified local complex coordinates on X (and therefore on C). The Hitchin equations simplify in this case as the gauge theory is abelian $F_{z\bar{z}} = 0$. The first equation therefore forces the U(1)-bundle to be flat while the second implies that Φ is holomorphic. This holomorphic form $\Phi \in H^{0,1}(C)$ can be further characterized by

$$H^{0,1}(C) \simeq H^1(C, \Omega^0_C) = H^1(C, \mathcal{O}_C) \tag{3.15}$$

by Dolbeault‘s theorem. The canonical bundle of C is related to the canonical bundle of X by the adjunction formula

$$K_C \simeq i^* \left(K_X \otimes \mathcal{O}(C)\right) \simeq i^*(\mathcal{O}(C)) \tag{3.16}$$

using the fact that X is Calabi-Yau. So Serre duality implies (we slightly abuse notation)

$$\Phi \in H^1(C, \mathcal{O}_C) \simeq H^0(C, \mathcal{O}(C)). \tag{3.17}$$

Note that in such a setting $\mathcal{O}(C)$ is the normal bundle of C in X and there is an isomorphism

$$H^0(C, N_{C|X}) \simeq T_{[C]}\mathcal{H}, \tag{3.18}$$

where $\mathcal{H}$ is the Hilbert scheme of X, i.e. the parameter space of closed subspaces of X. In our projective situation these subspaces are determined up to scaling and therefore the relevant parameter space is

$$\mathbb{P}H^0(C, \mathcal{O}(C)) \simeq \mathbb{P}^g, \tag{3.19}$$

for a curve of genus g.

The space of solutions to the Hitchin equations (3.14) is called the *Hitchin moduli space* and denoted by $\mathcal{M}_g^H$. What we really need, however, is its compactification [13, 14] (see also [15]) for which we will use the same symbol. By what we have said the points of $\mathcal{M}_g^H$ are pairs $(C \subset X\ ,\ p \in \mathrm{Jac}(X))$ because a choice of flat U(1)-bundle corresponds to the choice of a point on the Jacobian of the curve. There is a conjectural [3, 9, 14] birational equivalence between the Hitchin moduli space of the D1/D5-system and the symmetric product of the K3 surface

$$\mathcal{M}_g^H \overset{\sim}{\dashrightarrow} \mathrm{Sym}^k(X) = {X^{\times k}}\big/{S_k}. \tag{3.20}$$

Here, $k = Q_1 Q_5$ is supposed to be the product of the number of branes. There is strong (but quite subtle) field theoretic evidence [11] for this conjecture to be true but in the special case at hand their is also geometric motiviation that the moduli space might have something to do with the symmetric product [14].

Idea: We assume that X is an elliptic surface [16], i.e. there is a surjection $X \to \mathbb{P}^1$ such that the generic fiber is an elliptic curve. Not all fibers can be smooth: If that was the case the multiplicativity of the topological Euler number $e(X) = e(\mathbb{P}^1)e(X_t) = 0$ would be a contradiction to $e(X) = 24$. A generic K3 surfaces has exactly 24 singular fibers where in this case the singularities are ordinary double points. Although a generic K3 surface is not elliptic, elliptic ones are rather frequent and we assume that X is a generic one of those.
The Hitchin moduli space is a fibration over the space of deformations of C in X

$$\begin{array}{ccc} \mathrm{Jac}(X_t) & \longrightarrow & \mathcal{M}_g^H \\ & & \big\downarrow \\ & & \mathbb{P}^g. \end{array} \tag{3.21}$$

The projection is simply forgetting the U(1)-bundle and the fiber over a $t \in \mathbb{P}^g$ is the Jacobian of the corresponding curve.
In the case of $g = 0$ we have $C \simeq \mathbb{P}^1$ with normal bundle in X given by $\mathcal{O}(-2)$. It follows that

$$\dim \mathcal{H} = \dim H^0(\mathbb{P}^1, \mathcal{O}(-2)) = 0, \tag{3.22}$$

which means that genus 0 curves are "rigid", i.e. they do not admit deformations into nearby holomorphic cycles. Moreover, $\mathrm{Jac}(\mathbb{P}^1) = 0$ and therefore $\mathcal{M}_g^H = \{\mathrm{pt.}\}$.
For the case of elliptic curves E, $g = 1$, the parameter space is $\mathbb{P}^1$ and $\mathrm{Jac}(E) = E$. The

Hitchin moduli space is the fibration over $\mathbb{P}^1$ with elliptic curves as generic fibers. By the argument above such a fibration generically includes 24 singular fibers and indeed reproduces the fibration of the (elliptic) K3 surface X with which we started.
The case of $g \geq 2$ is more involved and we have to make another assumption: All line bundles $\mathcal{O}(D)$ corresponding to any divisors D on C are of degree g. In such a setting there is a map

$$\begin{aligned} \mathrm{Sym}^g(C) &\longrightarrow \mathrm{Pic}^g(C) \\ D = x_1 + \cdots + x_g &\longmapsto \mathcal{O}_C(D), \end{aligned} \tag{3.23}$$

sending g arbitrary points on C, which naturally define an effective divisor on $\mathrm{Sym}^g(C)$, to its associated line bundle of degree g. Let P be any rational point on C, then we further have

$$\begin{aligned} \mathrm{Pic}^g(C) &\longrightarrow \mathrm{Jac}(C) \\ \mathcal{O}_C(D) &\longmapsto \mathcal{O}_C(D) \otimes \mathcal{O}_C(-g.P). \end{aligned} \tag{3.24}$$

The composition $AJ : \mathrm{Sym}^g(C) \xrightarrow{\sim} \mathrm{Jac}(C)$ can be thought of as a degree g variant of the Abel-Jacobi map. It is automatically surjective and in this case also injective because we only care about effective divisors. We can therefore think of a point in $\mathcal{M}_g^H$ as a curve $C \subset X$ with g unordered points specified. By forgetting the curve, this data corresponds to the K3 surface X with g unordered points specified or, put differently, a point in the symmetric product. Conversely, fixing g points on X, all curves that pass through these points are parametrized by hyperplanes in $\mathbb{P}^g$. There are g of them and they generically intersect at exactly one point. So for any g points there is generically one curve which passes through all of them. In particular it has g marked points.

These arguments may provide motivation for believing the conjecture, but a proof would involve an analysis of singularities [14]. Additionally it might be interesting to try to relate the genus g as an exponent in the symmetric product with Q_1Q_5 which arises in a more field theoretic description (3.20).

Remark: The object $\mathcal{M}_g^H \simeq \mathrm{Sym}^g(X)$ is an orbifold with orbifold singularities. These can be resolved in a certain way [9, 11] and the result is hyperkähler and Calabi-Yau. Without further specification, we will treat this resolution as the correct instanton moduli space and denote it by $\mathfrak{M}$. The important point is that their orbifold cohomologies coincide [17], which somewhat justifies an often stated equivalence in the literature. It also follows that the (complex) dimension of $\mathfrak{M}$ is not $2Q_1Q_5$, but $2(Q_1Q_5+1)$ [5, 9, 11].

3.3 The Elliptic Genus

Now that we identified the target space of the effective field theory that is dual to type IIB string theory near the horizon of the D1/D5-system, we can finally make a statement about the black hole microstates. We want to count the BPS ground states of this sigma model on $\mathfrak{M}$ in order to find an estimate of their degeneracy which remains valid even when we vary the coupling. For such a task there is a very helpful tool in guise of a topological index [8, 18].

Definition. Let M be a Kähler manifold of complex dimension d and consider an elliptic curve $E \subset M$ with modulus τ and coordinate z. We define

$$q = \exp(2\pi i\tau) \qquad y = \exp(2\pi i z). \tag{3.25}$$

Then the quantity

$$\chi(M; q, y) = \mathrm{Tr}_{\mathscr{H}(M)}(-1)^F y^{F_L} q^{L_0 - \frac{d}{8}} \overline{q}^{\overline{L}_0 - \frac{d}{8}} \tag{3.26}$$

is called *elliptic genus* of the sigma model on M. Here, $\mathscr{H}$ denotes the Hilbert space of states, $F = F_L + F_R$ denotes the Fermion numbers and L_0 is the level 0 Virasoro operator.

It is instructive to recall Witten‘s index

$$\mathrm{Tr}_{\mathscr{H}(M)}(-1)^F = \chi(M; 1, 1) \tag{3.27}$$

which is an index for ground states of the associated sigma model and computes the Euler characteristic of the target space. The basic idea behind Witten‘s index is that supersymmetry leads to a cancellation of all non-ground states due to the factor $(-1)^F$ [19]. This index counts the number of zero energy bosonic ground states minus the number of zero energy fermionic ground states, which is a geometric invariant of the target space.

The elliptic genus is much more closely related to the full partition function. Essentially it is the specialization $\overline{y} = 1$ of the full partition function in the Ramond-Ramond sector, after choosing the boundary conditions for the fermions correctly. As a consequence of the missing sign, as for Witten‘s index, there are no contributions of left moving states with $\overline{L}_0 - d/8 > 0$. This means that only the right moving Ramond ground states contribute to the expression such that the elliptic genus is holomorphic in q (or τ). An equivalent definition would therefore be

$$\chi(M; q, y) = \mathrm{Tr}_{\mathscr{H}(M)}(-1)^F y^{F_L} q^{L_0 - \frac{d}{8}}. \tag{3.28}$$

χ being holomorphic requires the exponent $L_0 - \frac{d}{8}$ to be an integer, such that the elliptic genus loses its dependence on modular parameters of M. A physical interpretation of the elliptic genus as a "counting function" of perturbative BPS states is therefore not quite correct: Such a function would certainly depend on the moduli. The topological index $\chi(M; q, y)$ will therefore only be useful to compute a *bound* on the degeneracy of BPS states and therefore the microstates of the black hole.

The power of the elliptic genus in the current situation is based on a mathematical fact: If the target space is Calabi-Yau, $\chi(M; q, y)$ is a *weak Jacobi form* [18, 20] of weight $w = 0$ and index $r = d/2$, a function

$$F : \mathfrak{H} \times \mathbb{C} \longrightarrow \mathbb{C}, \tag{3.29}$$

where $\mathfrak{H}$ is the upper half plane, which behaves somewhat like a modular form. Its transformation behavior with respect to the action of $\mathrm{SL}(2,\mathbb{C})$ is

$$\begin{aligned} F\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right) &= (c\tau+d)^w \exp\left(\pi i\frac{rcz^2}{c\tau+d}\right)F(\tau,z) \\ F(\tau,z+m\tau+n) &= \exp\left(-\pi i r(m^2\tau+2mz)\right)F(\tau,z), \end{aligned} \tag{3.30}$$

with integers m,n,w and (possibly) half-integer r (there are subtleties for odd d [8]).

At this point we will cheat: The instanton moduli space $\mathfrak{M}$ is an orbifold, rather than a manifold. Strictly speaking we would have to compute the orbifold version of the elliptic genus which, fortunately, is strongly related to the elliptic genus of X in our case. We will ignore this subtlety and refer the reader to [7]. When we assume $\mathfrak{M}$ to be a Calabi-Yau manifold we have an expansion [7, 20]

$$\chi(\mathfrak{M};q,y)=\sum_{n\geq 0,F_L} c(n,F_L)\,q^n y^{F_L}. \tag{3.31}$$

The index n is determined by the level-matching condition[4]

$$L_0-\overline{L}_0=L_0=N \tag{3.32}$$

where N is the quantized momentum along the S^1 and because the operator $\overline{L}_0$ does not affect the elliptic genus. This implies $n=N$. The coefficients $c(N,F_L)$ correspond to the degeneracy of the BPS state for the given parameters. For very large N, a condition we assumed throughout, their leading asymptotics can be obtained from a Hardy-Ramanujan formula[5] (see also [10])

$$c(N,F_L)\sim\exp\left(4\pi\sqrt{\frac{d}{8}N}\right). \tag{3.33}$$

Using the dimension of $\mathfrak{M}$ as explained at the end of the last section, the leading degeneracy of BPS states for $N\gg 1$ is given by

$$c(N,F_L)\sim\exp\left(2\pi\sqrt{(Q_1Q_5+1)N}\right). \tag{3.34}$$

The statistical or "microscopic" entropy is then bounded by

$$S=\log c(N,F_L)\sim 2\pi\sqrt{(Q_1Q_5+1)N}, \tag{3.35}$$

which for $Q_1,Q_5\gg 1$ agrees to leading order with the Bekenstein-Hawking formula.

[4]This really is an expression for the eigenvalues of the operator L_0. They are fixed to be integers.
[5]In the literature such an approximation is sometimes also attributed to Cardy [21].

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