Introduction to conformal field theory

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July 9, 2018

Abstract

This summary contains two talks held in the seminar on "Holography and large-N dualities" in summer 2018. The first two sections summarise the talk held by Torben Skrzypek who explained the mathematical prerequisites to conformal quantum field theory, as they are needed in the following. He worked out the specific form of conformal transformations, identified the conformal group and showed, how in the quantisation process the Lie algebra has to be extended to find a unitary represention. The first part mainly follows Schottenloher [1] but has also taken inspiration from Blumenhagen and Plauschinns book [2].

The last two sections outline the talk given by Jannik Fehre on the key techniques and features of two dimensional conformal field theory. As an example, he applied these insights to the free boson on the cylinder. The second part focusses on a discussion close to Blumenhagen and Plauschinn [2] but also uses some details from the lecture notes by Qualls [3].

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1.1 Local conformal transformations

Definition 1.1. *Let $(M, g), (N, g')$ be smooth Pseudo-Riemannian manifolds, $U \subset M, V \subset N$ open subsets of M and N and $\phi : U \to V$ a smooth, non singular map. ϕ is called* ***conformal*** *if there exists a smooth map $\Omega : U \to \mathbb{R}_+$ (the* ***conformal factor****) such that*

$$\phi^* g' = \Omega^2 g.$$

In local coordinates

$$g'_{\rho\sigma}(x') \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} = \Omega^2(x) g_{\mu\nu}(x).$$

For simplicity, we will work in flat space and with transformations from M to M and use $\eta_{\mu\nu}$ to denote the metric $\mathrm{diag}(-1, -1, ...1)$ with signature (p, q). For an infinitesimal transformation $\phi : x \mapsto x' = x + \epsilon(x) + \mathcal{O}(\epsilon^2)$ we get

$$\eta_{\rho\sigma}(\delta^\rho_\mu + \partial_\mu \epsilon^\rho)(\delta^\sigma_\nu + \partial_\nu \epsilon^\sigma) + \mathcal{O}(\epsilon^2) = \Omega^2(x) \eta_{\mu\nu}.$$

Computing the left hand side yields

$$\eta_{\mu\nu} + (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) + \mathcal{O}(\epsilon^2) = \Omega(x)^2 \eta_{\mu\nu} = \eta_{\mu\nu} + \kappa(x) \eta_{\mu\nu}$$

where we defined $\kappa : U \to \mathbb{R}$ by $\kappa(x) = \Omega^2(x) - 1$. This leaves us with a restricting equation for the infinitesimal transformations up to first order of the form

$$(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) = \kappa(x) \eta_{\mu\nu}.$$

Tracing with $\eta^{\mu\nu}$ yields

$$2\partial^\mu \epsilon_\mu = \kappa(x) d \quad \Rightarrow \quad \kappa(x) = \frac{2}{d}(\partial \cdot \epsilon)$$

with $d = p + q$ the dimension of the manifold. Thus we get the **conformal Killing equation**

$$(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) = \frac{2}{d}(\partial \cdot \epsilon) \eta_{\mu\nu}. \tag{1}$$

For $d = 2$ we can explicitly solve this system of two differential equations. For $d \geq 3$ we can perform further contractions to get

$$[-g_{\mu\nu} \Box + (d - 2) \partial_\mu \partial_\nu](\partial \cdot \epsilon) = 0 \tag{2}$$

$$2\partial_\mu \partial_\nu \epsilon_\rho = \frac{2}{d}(-\eta_{\mu\nu} \partial_\rho + \eta_{\rho\mu} \partial_\nu + \eta_{\nu\rho} \partial_\mu)(\partial \cdot \epsilon). \tag{3}$$

Equation (2) implies a linear structure of $(\partial \cdot \epsilon)$, thus we can make the widest possible ansatz

$$(\partial \cdot \epsilon) = A + B_\mu x^\mu$$
$$\epsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho$$

with $a_\mu, b_{\mu\nu}, c_{\mu\nu\rho}$ constant and $c_{\mu\nu\rho} = c_{\mu\rho\nu}$. We can further decompose $b_{\mu\nu}$ in a symmetric and an antisymmetric part. Another application of (2) requires the symmetric part to be proportional to the metric, which yields $b_{\mu\nu} = \alpha \cdot \eta_{\mu\nu} + m_{\mu\nu}$, where $m_{\mu\nu}$ is the antisymmetric part. As for $c_{\mu\nu\rho}$, we can apply equation (3) to express it by the vector $b_\mu = d^{-1} \cdot c^\rho_{\rho\mu}$, which gives the general form for an infinitesimal conformal transformation

$$x'^\mu = x^\mu + a^\mu + \alpha \cdot x^\mu + m^\mu_\nu x^\nu + 2(x \cdot b) x^\mu - (x \cdot x) b^\mu.$$

We can consider each parameter as the action of a generator on x, which correspond to the following transformations

Parameter	Transformation		Generator	
a_μ	$x'^\mu = x^\mu + a^\mu$	Translation	$P_\mu \ = \mathrm{i}\partial_\mu$	(Momentum)
α	$x'^\mu = (1+\alpha)x^\mu$	Dilation	$D \ = -\mathrm{i}x^\mu\partial_\mu$	
$m_{\mu\nu}$	$x'^\mu = x^\mu + m^\mu_\nu x^\nu$	Rotation	$L_{\mu\nu} = \mathrm{i}(x_\mu\partial_\nu - x_\nu\partial_\mu)$	(Ang. Momentum)
b_μ	$x'^\mu = x^\mu + 2(x\cdot b)x^\mu$ $-(x\cdot x)b^\mu$	SCT	$K_\mu \ = -\mathrm{i}(2x_\mu x^\nu\partial_\nu$ $-(x\cdot x)\partial_\mu)$	

.

We see that except for the usual Poincaré-group transformations of translation and rotation (which include Lorentz-boost), we also get a **dilation**, which scales the whole space and a **special conformal transformation** (SCT), which can be written in its finite form as

$$x'^\mu = \frac{x^\mu - (x\cdot x)b^\mu}{1 - 2(b\cdot x) + (b\cdot b)(x\cdot x)}$$

corresponding geometrically to an inversion at the unit sphere, a subsequent translation by $-b^\mu$ and yet another inversion. However, it is possible that the translation hits the coordinate origin and x' is sent to infinity by the last inversion. This is precisely the case when the denominator vanishes. Therefore we have to restrict our attention to regions, where the transformation is not singular, which breaks any arising global group structure. To find a conformal group, we have to conformally compactify the flat space we were working on thus far.

1.2 Conformal compactification and the conformal group

The goal is to find a manifold $N^{p,q}$, where all conformal transformations are defined as smooth non-singular maps and enjoy a group structure. More explicitly we want to find a conformal embedding $\tau : \mathbb{R}^{p,q} \hookrightarrow N^{p,q}$ such that for every conformal $\phi : U \mapsto V$ with $U, V \subset \mathbb{R}^{p,q}$ a diffeomorphism $\phi^* : N^{p,q} \mapsto N^{p,q}$ with commuting diagram

$$\begin{tikzcd} U \arrow[r, "\phi"] \arrow[d, hook, "\tau"] & V \arrow[d, hook, "\tau"] \\ N^{p,q} \arrow[r, "\phi^*"] & N^{p,q} \end{tikzcd}$$

exists. A Manifold $N^{p,q}$ with those caracteristics is a **conformal compactification** of $\mathbb{R}^{p,q}$.

To that end, we embed $\mathbb{R}^{p,q}$ into the projective space $\mathbb{R}\mathrm{P}^{d+1}$ by

$$\begin{aligned} \tau\colon \mathbb{R}^{p,q} &\to \mathbb{R}\mathrm{P}^{d+1} \\ x^\mu &\mapsto \left(\frac{1 - x^\mu x_\mu}{2} : x^1 : x^2 : \ldots : x^d : \frac{1 + x^\mu x_\mu}{2}\right) \end{aligned}$$

where homogeneous coordinates have been used for $\mathbb{R}\mathrm{P}^{d+1}$. Now the closure $\overline{\tau(\mathbb{R}^{p,q})}$ is precisely given by the quartic

$$N^{p,q} := \{(\xi^0 : \ldots : \xi^{n+1}) \in \mathbb{R}\mathrm{P}^{d+1} | \langle \xi, \xi \rangle_{\mathbb{R}^{p+1,q+1}} = 0\}$$

which again can be double-covered by the manifold $\mathbb{S}^p \times \mathbb{S}^q \subset \mathbb{R}^{p+1,q+1}$ and thus has an induced metric. It can be shown by explicit calculation that τ maps conformally on $N^{p,q}$. Now we have to check, if we can find diffeomorphisms ϕ^*, that correspond to all possible conformal transformations according to the upper diagram. We use that the group of orthonormal transformations $O(p+1, q+1)$ on $\mathbb{R}^{p+1,q+1}$ acts diffeomorphic on $\mathbb{S}^p \times \mathbb{S}^q$ and because $N^{p,q}$ is double covered, we can restrict our attention to the subgroup $SO(p+1, q+1)$. Indeed, an explicit calculation shows that there is a one to one correspondence between the matrix-representation of $SO(p+1, q+1)$ and conformal transformations on $\mathbb{R}^{p,q}$, e.g.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \Lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SO(p+1,q+1) \leftrightarrow \text{Rotation } \Lambda \in SO(p,q).$$

Such we have found the **conformal group** $Conf(\mathbb{R}^{p,q}) \cong SO(p+1,q+1)$ acting diffeomorphic on the conformal compactification $N^{p,q}$ of $\mathbb{R}^{p,q}$. If we only want the connected part containing the identity, we get $SO(p+1,q+1)^+$. The conformal group is a Lie group, so it should be possible to find a Lie algebra of dimension $\dim SO(p+1,q+1) = \frac{(d+1)(d+2)}{2}$. Going back to the generators defined in the table of conformal transformations, the specific combinations

$$\begin{aligned} J_{0,\nu} &= \frac{1}{2}(P_\nu - K_\nu) & J_{0,d+1} &= D \\ J_{\mu,\nu} &= L_{\mu\nu} & J_{\mu,d+1} &= \frac{1}{2}(P_\mu + K_\mu) \end{aligned}$$

with $J_{ij} = -J_{ji}$ have the the familiar commutator structure for $\mathfrak{so}(p+1,q+1)$

$$[J_{ab}, J_{cd}] = i(\eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}).$$

This reassures us that

Theorem 1.1. *The conformal group $Conf(\mathbb{R}^{p,q})$ of flat spacetime with signature (p,q) is isomorphic to $SO(p+1,q+1)$.*

1.3 Special case of $d = 2$

In the derivation of the conformal transformations, the conformal Killing equation(1) was said to be directly solvable for the case of dimension $d = 2$. Especially in equation (2) we see that the second term will vanish in two dimensions, so our discussion so far didn't account for this less restrictive case. The global results will turn out quite similar, but locally we will face a different situation. In the following, we will work in Euclidean space with signature $(0,2)$. For a Lorentzian situation our results can be applied by Wick rotation of one coordinate. Going back to the conformal Killing equation (1)

$$(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu},$$

we only have two equations to solve due to the symmetry:

$$\begin{aligned} \partial_0\epsilon_0 + \partial_0\epsilon_0 &= \partial_0\epsilon_0 + \partial_1\epsilon_1 & &\Rightarrow \quad \partial_0\epsilon_0 = \partial_1\epsilon_1 \\ \partial_0\epsilon_1 + \partial_1\epsilon_0 &= 0 & &\Rightarrow \quad \partial_0\epsilon_1 = -\partial_1\epsilon_0. \end{aligned}$$

which are precisely the Cauchy-Riemann equations of complex analysis. We will therefore go to a complex description of the Euclidean plane by setting

$$\begin{aligned} z &= x^0 + \mathrm{i}x^1 & \epsilon &= \epsilon^0 + \mathrm{i}\epsilon^1 & \partial_z &= \frac{1}{2}(\partial_0 - \mathrm{i}\partial_1) \\ \bar{z} &= x^0 - \mathrm{i}x^1 & \bar{\epsilon} &= \epsilon^0 - \mathrm{i}\epsilon^1 & \partial_{\bar{z}} &= \frac{1}{2}(\partial_0 + \mathrm{i}\partial_1). \end{aligned}$$

The Cauchy-Riemann equations now simply state that

$$\partial_{\bar{z}}\epsilon = 0 \qquad\qquad \partial_z\bar{\epsilon} = 0$$

or in words $\epsilon(z)$ is a holomorphic function and $\bar{\epsilon}(\bar{z})$ is an antiholomorphic function. We see now that every holomorphic or antiholomorphic function gives a conformal transformation $z' = z + \epsilon(z) = f(z)$. We could even use meromorphic functions f as conformal transformation, if we restrict our attention to a region U with no singular points of f. This is convenient, because we can expand meromorphic functions as Laurent series and write a general conformal transformation as

$$z' = z + \sum_{n \in \mathbb{Z}} \epsilon_n(-z^{n+1})$$

where the minus sign and the counting is conventional. This is nothing else then expanding the transformation in a basis of generators, which are given by

$$l_n = -z^{n+1}\partial_z \qquad\qquad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}}.$$

We get a countably infinite number of local conformal generators, which is a special characteristic of 2-d conformal theory. We can compute the commutators of those generators and find two independent copies of the so-called **Witt-algebra** $\mathcal{W}$

$$\begin{aligned}
&[l_m, l_n] = z^{m+1}\partial_z(z^{n+1}\partial_z) - z^{n+1}\partial_z(z^{m+1}\partial_z) = (m-n)l_{m+n} \\
&[\bar{l}_m, \bar{l}_n] = (m-n)\bar{l}_{m+n} \\
&[l_m, \bar{l}_n] = 0.
\end{aligned}$$

This is again only the local structure of the conformal transformations and fails to compose a group due to possible singular points. To get to the conformal group, we again have to conformally compactify $\mathbb{C}$. We guess that this will be diffeomorphic to the Riemann sphere, which is the one point compactification of $\mathbb{C}$ to $\mathbb{C} \cup \{\infty\}$. Now we have to find all conformal transformations, that are non-singular on this space. If we take a look at the generators, we find, that for $l_n = -z^{n+1}\partial_z$ to be non-singular at $z = 0$ we have to have $n \geq -1$ and for the point at infinity, we can instead transform to the coordinate $z = -w^{-1}$ and we find that $l_n = -(-w)^{-n+1}\partial_w$ is only finite for $n \leq 1$. So only the generators $\{l_{-1}, l_0, l_1\}$ are globally well defined. When we decompose l_0 in radial coordinates, we get

$$l_0 = -z\partial_z = -\frac{1}{2}r\partial_r + \mathrm{i}\frac{1}{2}\partial_\phi.$$

Now we can again write all globally defined conformal transformations and their geometrical interpretation as

Transformation	Generator	
Translation	l_{-1}	$= -\partial_z$
Dilation	$l_0 + \bar{l}_0$	$= -r\partial_r$
Rotation	$\mathrm{i}(l_0 - \bar{l}_0)$	$= -\partial_\phi$
SCT	l_1	$= -\partial_w$

.

If we put all of those transformations together in a general transformation, we get exactly the Möbius transformations

$$z \mapsto z' = \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{C}$ and as a further requirement $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$. We can indeed set this determinant to 1 because an overall factor does not change the transformation. This constitutes the group $SL(2, \mathbb{C})$. Note furthermore, that $(a, b, c, d) \sim (-a, -b, -c, -d)$ give the same transformation, so we actually only have $SL(2, \mathbb{C})/\mathbb{Z}_2$. We have thus found the global conformal group $Conf(\mathbb{R}^{2,0}) \cong SL(2, \mathbb{C})/\mathbb{Z}_2$.
Of course we could also simply use the formalism from the last section, embed $\mathbb{R}^{2,0}$ in the projective space and

find a correspondence between the $SO(3,1)^+$ and the conformal transformations. So to sum up all results for the 2-d case:

Theorem 1.2. *The local conformal transformations are generated by two copies of the infinite dimensional Witt-algebra $\mathcal{W}$.*
The global ***conformal group*** *is given by $Conf(\mathbb{R}^{2,0}) \cong {SL(2,\mathbb{C})}/{\mathbb{Z}_2} \cong SO(3,1)^+$.*

2 Central extensions and the Virasoro algebra

In the process of quantising a classical theory, we move from a desription in the classical phase space to a description in a complex Hilbert space $\mathbb{H}$. A physical **state** is now given by an equivalence class $[\psi] = \{\mathrm{e}^{\mathrm{i}a} |\psi\rangle \, | a \in [0, 2\pi)\}$ so it lives on a projective space $\mathbb{P}$. On this projective space, we can construct a **transition probability** $\delta : \mathbb{P} \times \mathbb{P} \to \mathbb{R}$ by taking arbitrary representatives $|\phi\rangle, |\psi\rangle$ of the equivalence classes $[\phi], [\psi]$ and defining

$$\delta([\phi], [\psi]) := \frac{|\langle \phi \mid \psi \rangle|^2}{\langle \phi | \phi \rangle^2 \langle \psi | \psi \rangle^2}.$$

This is not a metric, but induces a topology on $\mathbb{P}$. Now the group of transformations $T : \mathbb{P} \to \mathbb{P}$, which leave δ invariant is called $\mathrm{Aut}(\mathbb{P})$. The very important subgroup $U(\mathbb{P})$ thereof is induced by the group $U(\mathbb{H})$ of unitary operators of the Hilbert space. It is part of the exact sequence

$$1 \hookrightarrow U(1) \xhookrightarrow{t} U(\mathbb{H}) \xrightarrow{\pi} U(\mathbb{P}) \longrightarrow 1$$

which shall be our motivation and first example of a central extension.

2.1 Central extensions in quantising symmetries

Definition 2.1. *An* ***extension*** *of a group G by the group A is given by an exact sequence of group homomorphisms*

$$1 \hookrightarrow A \xrightarrow{t} E \xrightarrow{\pi} G \longrightarrow 1.$$

It is ***central*** *if A is abelian and $\mathrm{im}(t)$ is in the* ***center*** *of E, that is $\forall a \in A, \forall b \in E : t(a) \cdot b = b \cdot t(a)$.*

The same definition can be made for Lie algebras with the exact sequence

$$0 \hookrightarrow \mathfrak{a} \xrightarrow{t} \mathfrak{e} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0.$$

and $[\mathfrak{a}, \mathfrak{e}] = 0$ in the central case. To give a few examples:

- The trivial central extension is just

$$1 \hookrightarrow A \xrightarrow{t} A \times G \xrightarrow{\pi} G \longrightarrow 1.$$

- For every extension of a Lie group, also the Lie algebras are extended as

$$0 \hookrightarrow \mathrm{Lie}(A) \xrightarrow{t} \mathrm{Lie}(E) \xrightarrow{\pi} \mathrm{Lie}(G) \longrightarrow 0$$

  but in general not the other way around.

- The Lorentz group $SO(1,3)^+$ is extended to

$$1 \hookrightarrow \{1, -1\} \xrightarrow{t} SL(2,\mathbb{C}) \xrightarrow{\pi} SO(1,3)^+ \longrightarrow 1.$$

- Another beautiful example are the Euclidean or Poincaré groups

$$1 \hookrightarrow \mathbb{R}^{p,q} \xrightarrow{t} O(p,q) \ltimes \mathbb{R}^{p,q} \xrightarrow{\pi} O(p,q) \longrightarrow 1$$

  with the semidirect product defined by the group operation $(g,x)\cdot(g',x') := (gg', x\tau(g)x')$ on $O(p,q)\times\mathbb{R}^{p,q}$, where $\tau : O(p,q) \to \mathrm{Aut}(\mathbb{R}^{p,g})$ is a representation. Thus the Euclidean or Poincaré groups are central extensions of the orthonormal or Lorentz groups.

Going back to the quantisation process, we are especially interested in whether or to which extend symmetries of the classical theory (e.g. conformal symmetry) are carried over to the quantum theory. We propose that at least the physical states in $\mathbb{P}$ should still obey the symmetry. This makes intuitive sense, but has to be postulated. Let the classical theory be symmetric under the group action of a Lie group G. Then we propose the existence of a **projective representation** $s : G \to U(\mathbb{P})$ such that we get the diagram

$$\begin{tikzcd}
1 \arrow[r, hook] & U(1) \arrow[r, "t'"] \arrow[d, "\mathrm{id}"] & E \arrow[r, "\pi'"] \arrow[d, "s'"] & G \arrow[r] \arrow[d, "s"] \arrow[dl, dashed, "?"] & 1 \\
1 \arrow[r, hook] & U(1) \arrow[r, hook, "t"] & U(\mathbb{H}) \arrow[r, "\pi"] & U(\mathbb{P}) \arrow[r] & 1.
\end{tikzcd}$$

The existence of a representation $s' : E \to U(\mathbb{H})$ is automatically implied, but is it also possible to find a **unitary representation** of G in $U(\mathbb{H})$? This would be called a **lift** of G ("?" in the diagram). The answer to this question lies in the cohomology theory of the corresponding Lie algebra. The necessary definitions are:

- Alternating group $\mathrm{Alt}^2(\mathfrak{g},\mathfrak{a}) := \{\Theta : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{a} | \text{bilinear and alternating}\}$
- Cocycles $Z^2(\mathfrak{g},\mathfrak{a}) := \{\Theta \in \mathrm{Alt}^2 | \Theta(x,[y,z]) + \Theta(y,[z,x]) + \Theta(z,[x,y]) = 0\}$
- Coboundaries $B^2(\mathfrak{g},\mathfrak{a}) := \{\Theta : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{a} | \exists\mu \in \mathrm{Hom}_{\mathbb{K}}(\mathfrak{g},\mathfrak{a}) : \theta(x,y) = \mu([x,y])\}$
- Cohomology $H^2(\mathfrak{g},\mathfrak{a}) := Z^2(\mathfrak{g},\mathfrak{a})/B^2(\mathfrak{g},\mathfrak{a})$

With this setup we can quote a theorem by Bargmann, which states that:

Theorem 2.1. *(**Bargmann**) Let G be a simply connected finite-dimensional Lie group with second cohomology*

$$H^2(\mathrm{Lie}(G), A\} = 0,$$

then every projective representation has a lift as a unitary representation.

Thus the second cohomology of the Lie algebra can be seen as obstructing the lift. If it doesn't vanish, we cannot use the classical symmetry group, but have to use its central extension in the quantum theory instead. We can further use the second cohomology to classify all possible central extensions. In fact, if we were given a central extension

$$0 \hookrightarrow \mathfrak{a} \xrightarrow{t} \mathfrak{e} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0,$$

the commutation relations in $\mathfrak{e}$ are given by a cocycle Θ as

$$[\tilde{a},\tilde{b}] = \widetilde{[a,b]} + \Theta(a,b)$$

Where the tilde over some $a \in \mathfrak{g}$ denotes some element of the respective pre-image of π. Now two different central extensions are group theoretically equivalent, if the cycycles in the commutation relations only differ by some coboundary. If, moreover, the cocycle is itself a coboundary, it is equivalent to the 0 of the cohomology and therefore the central extension splits and is equivalent to the trivial extension

$$\begin{tikzcd}
0 \arrow[r, hook] & \mathfrak{a} \arrow[r, "t'"] \arrow[d, "\mathrm{id}"] & \mathfrak{e} \arrow[r, shift left, "\pi'"] \arrow[d] & \mathfrak{g} \arrow[l, shift left, "s"] \arrow[r] \arrow[d, "\mathrm{id}"] \arrow[dl] & 0 \\
0 \arrow[r, hook] & \mathfrak{a} \arrow[r, "t"] & \mathfrak{a} \times \mathfrak{g} \arrow[r, "\pi"] & \mathfrak{g} \arrow[r] & 0.
\end{tikzcd}$$

Thus the cohomology measures the non-triviality of the central extension and serves as classification.
The theory of central extensions has been applied to describe anomalies in quantum theory and quantum field theory. For example, a number of chiral anomalies is discussed in this form in the book by Fujikawa [4] . There have also been attempts to explain the Higgs-mechanism as a cohomological artefact [5].

2.2 Quantising conformal symmetry - the Virasoro algebra

We will restrict our discussion to the two-dimensional case, since it is the most important one for the discussions following this talk. In the spirit of conformal symmetry, we will now project from $\mathbb{H}$ by also dropping the magnitude of the elements. Thus, we now have a central extension by $\mathbb{C}$ instead of $U(1)$. It has become clear in the discussion above that to quantise a theory with conformal symmetry, we will have to compute the second cohomology of it's Lie algebra, which in the two dimensional case is nothing else then the Witt algebra $\mathcal{W}$.

Theorem 2.2. *The second cohomology of the Witt algebra is $H^2(\mathcal{W}, \mathbb{C}) \cong \mathbb{C}$ and generated by*

$$\omega(l_n, l_m) := \delta_{n+m} \frac{n}{12}(n^2 - 1).$$

Proof. The proof consist of three steps, which we will only sketch

- $\omega \in Z^2(\mathcal{W}, \mathbb{C})$
- $\omega \notin B^2(\mathcal{W}, \mathbb{C})$
- $\Theta \in Z^2(\mathcal{W}, \mathbb{C}) \Rightarrow \exists \lambda \in \mathbb{C} : \Theta \sim \lambda\omega.$

ω is clearly bilinear and alternating. A direct computation of the definition of Z^2 shows the first step.
For the second step we assume the existence of $\mu \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{W}, \mathbb{C})$ with $\omega(x, y) = \mu([x, y])$. Then

$$\begin{aligned}
\omega(l_n, l_{-n}) &= \mu([l_n, l_{-n}]) \\
\frac{n}{12}(n^2 - 1) &= 2n\mu(l_0) \\
\mu(l_0) &= \frac{1}{24}(n^2 - 1) \;\; \forall n \in \mathbb{N},
\end{aligned}$$

which is a contradiction. For the last part, we take a general Θ and insert it into the cocycle equation with the elements l_0, l_m, l_m to get restrictions on its structure. Then by adding coboundaries, we can always get to the form

$$\Theta'(l_n, l_m) = \delta_{n+m} h(n)$$

with $h(0) = h(1) = 0$ and $h(k) = h(-k)$. Upon another use of the cocycle equation, we can prove the proportionality to ω. □

The denominator 12 has been chosen for convenience in conformal field theory and is simply conventional. Now the only possible central extension of the Witt-algebra is

$$0 \hookrightarrow \mathbb{C}c \xrightarrow{t} \mathcal{V}\mathrm{ir} \xrightarrow{\pi} \mathcal{W} \longrightarrow 0$$

with the **Virasoro algebra**

$$[l_n, l_m] = (n-m)l_{n+m} + \delta_{n+m}\frac{n}{12}(n^2-1)c.$$

Having found the central extension, we are free to quantise conformal symmetry according to

$$\begin{tikzcd}
0 \arrow[r, hook] & \mathbb{C}c \arrow[r, "t'"] \arrow[d, "\mathrm{id}"] & \mathcal{V}\mathrm{ir} \arrow[r, "\pi'"] \arrow[d] & \mathcal{W} \arrow[r] \arrow[d, "s"] \arrow[dl, "\mathrm{X}"] & 0 \\
1 \arrow[r, hook] & \mathbb{C} \arrow[r, "t"] & U(\mathbb{H}) \arrow[r, "\pi"] & U(\mathbb{P}) \arrow[r] & 1.
\end{tikzcd}$$

To round up our discussion of the conformal group and the Virasoro algebra, we can take a look at the only elements of the Witt-algebra, that are defined globally and constitute the conformal group. We notice that $\bar{\mathcal{W}} = \{l_{-1}, l_0, l_1\}$ is closed and the second cohomology vanishes on this subgroup. So for the actual conformal group, there exists a lift

$$\begin{tikzcd}
0 \arrow[r, hook] & \mathbb{C} \arrow[r, "t''"] \arrow[d, "\mathrm{id}"] & \mathbb{C}\times\bar{\mathcal{W}} \arrow[r, "\pi''"] \arrow[d] & \bar{\mathcal{W}} \arrow[r] \arrow[d, "s'"] \arrow[dl] & 0 \\
1 \arrow[r, hook] & \mathbb{C} \arrow[r, "t'"] \arrow[d, "\mathrm{id}"] & \mathbb{C}\times Conf \arrow[r, "\pi'"] \arrow[d] & Conf \arrow[r] \arrow[d, "s"] \arrow[dl] & 1 \\
1 \arrow[r, hook] & \mathbb{C} \arrow[r, "t"] & U(\mathbb{H}) \arrow[r, "\pi"] & U(\mathbb{P}) \arrow[r] & 1.
\end{tikzcd}$$

Thus, global conformal symmetry is carried over to the quantum theory, while local conformal transformations have to be modified by the use of the Virasoro algebra.

3 Conformal field theory in 2 dimensions

3.1 Radial quantisation

We consider $2-d$ Euclidean conformal field theory with coordinates $(x^0 = \mathrm{i}t, x^1)$ which we combine into one complex variable $w = x^0 + \mathrm{i}x^1$. We will deal with a compactified space which means we identify $w \sim w + 2\pi\mathrm{i}$. This lets us naturally define a mapping back to the complex plane via $z = \exp w$. As usually done in QFT, we treat z and $\bar{z}$ as independent variables rather than x^0 and x^1.
Of central importance are the following definitions for fields $\phi(z,\bar{z})$:

- $\phi(z,\bar{z})$ is called <u>chiral / anti-chiral</u> if $\partial_{\bar{z}}\phi = 0$ / $\partial_z\phi = 0$.
- $\phi(z,\bar{z})$ has <u>conformal dimension</u> $(h,\bar{h})$ if under $z \mapsto \lambda z$ $(\lambda \in \mathbb{C})$ it transforms as

$$\phi(z,\bar{z}) \mapsto \lambda^h \bar{\lambda}^{\bar{h}} \phi(\lambda z, \bar{\lambda}\bar{z}).$$

- $\phi(z,\bar{z})$ is called a <u>primary field</u> of conformal dimension $(h,\bar{h})$ if under conformal transformations $z \mapsto f(z)$ it transforms as

$$\phi(z,\bar{z}) \mapsto \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \phi(f(z), \bar{f}(\bar{z})). \tag{4}$$

  ϕ is called <u>quasi-primary</u> if this only holds for global transformations $f \in {}^{\mathrm{SL}(2,\mathbb{C})}\!/\!_{\mathbb{Z}_2}$.

In order to quantise a field ϕ of conformal dimension $(h,\bar{h})$ we first Laurent expand it,

$$\phi(z,\bar{z}) = \sum_{n,\bar{n}\in\mathbb{Z}} z^{-n-h}\bar{z}^{-\bar{n}-\bar{h}}\phi_{n,\bar{n}},$$

such that $\phi_{n,\bar{n}}$ has scaling dimension $(n, \bar{n})$, and then promote the $\phi_{n,\bar{n}}$ to operators.
For later reference we state the general commutation relation for the Laurent modes of two chiral quasi-primary fields

$$\phi_i(z) = \sum_m \phi_{i,m} z^{-m-h_i}, \quad \phi_j(z) = \sum_n \phi_{j,n} z^{-n-h_j},$$

$$[\phi_{i,m}, \phi_{j,n}] = \sum_k C_{ij}^k p_{ijk}(m,n)\phi_{k,m+n} + d_{ij}\delta_{m,-n} \binom{m+h_i-1}{2h_i-1} \tag{5}$$

with

$$p_{ijk}(m,n) = \sum_{\substack{r,s\in\mathbb{Z}_0^+ \\ r+s=h_i+h_j-h_k-1}} A_{r,s}^{ijk} \binom{-m+h_i-1}{r} \binom{-n+h_j-1}{s},$$

$$A_{r,s}^{ijk} = (-1)^r \frac{(2h_k-1)!}{(h_i+h_j+h_k-2)!} \prod_{t=0}^{s-1}(2h_i-2-r-t) \prod_{u=0}^{r-1}(2h_j-2-r-u).$$

The constants C_{ij}^k and d_{ij} will appear later.
We will also use the commutator of the generators for conformal transformations L_m and the Laurent modes of a chiral primary field ϕ_n

$$[L_m, \phi_n] = ((h-1)m - n)\phi_{m+n}. \tag{6}$$

Under Hermitian conjugation the Euclidean coordinates transform as $(\mathrm{i}x^0, x^1) \mapsto (-\mathrm{i}x^0, x^1)$ which means that $z \mapsto \bar{z}^{-1}$. This motivates the definition

$$\phi^\dagger(z,\bar{z}) := \bar{z}^{-2h} z^{-2\bar{h}} \phi\left(\frac{1}{\bar{z}}, \frac{1}{z}\right) = \bar{z}^{-2h} z^{-2\bar{h}} \sum_{n,\bar{n}\in\mathbb{Z}} \bar{z}^{n+h} z^{\bar{n}+\bar{h}} \phi_{n,\bar{n}} = \sum_{n,\bar{n}\in\mathbb{Z}} \bar{z}^{n-h} z^{\bar{n}-\bar{h}} \phi_{n,\bar{n}}$$

from which follows that $(\phi_{n,\bar{n}})^\dagger = \phi_{-n,-\bar{n}}$.

We define the vacuum to be the state with the highest number of symmetries, that means $L_n|0\rangle = 0$ for as many n as possible. Due to the Virasoro algebra (with $c \neq 0$) we have to restrict ourselves to $n \geq -1$.

Now considering the asymptotic in-state, defined as

$$|\phi\rangle := \lim_{z,\bar{z}\to 0} \phi(z,\bar{z})|0\rangle,$$

in order for $|\phi\rangle$ to be regular at $z = 0$, we have to require that

$$\phi_{n,\bar{n}}|0\rangle = 0 \quad \text{for} \quad n > -h, \bar{n} > -\bar{h}.$$

Those are the annihilation operators, analogously for the asymptotic out-state we see that $\phi_{n,\bar{n}}$ with $n \leq -h, \bar{n} \leq -\bar{h}$ are creation operators.

3.2 2- and 3-point function

Let ϕ_i be chiral quasi-primary fields labelled by the index i. Then, by exploiting the required symmetry of correlation functions and the transformation properties of quasi-primary fields, we are able to constrain the structure of the 2- and 3-point function up to a constant.
We consider first the 2-point function $\langle\phi_i(z)\phi_j(w)\rangle =: g(z,w)$.

- The invariance under L_{-1} (translations) implies $g(z,w) = \tilde{g}(z-w)$.
- The invariance under L_0 (dilations) implies $\tilde{g}(z-w) = \lambda^{h_i+h_j}\tilde{g}(\lambda(z-w)) \Rightarrow \tilde{g}(z-w) = \frac{d_{ij}}{(z-w)^{h_i+h_j}}$.
- The invariance under L_1 (special conformal transformations which especially contain $z \mapsto -z^{-1}$) implies $\tilde{g}(z-w) = \frac{\tilde{g}(-z^{-1}+w^{-1})}{z^{2h_i}w^{2h_j}} \Rightarrow h_i = h_j$.

Summarising we have

$$\langle \phi_i(z)\phi_j(w)\rangle = \frac{d_{ij}\delta_{h_i h_j}}{(z-w)^{2h_i}}, \tag{7}$$

where d_{ij} are constants of the theory.
The same reasoning can be applied to the 3-point function yielding

$$\langle \phi_i(z_1)\phi_j(z_2)\phi_k(z_3)\rangle = \frac{C_{ijk}}{(z_1-z_2)^{h_i+h_j-h_k}(z_2-z_3)^{h_j+h_k-h_i}(z_1-z_3)^{h_k+h_i-h_j}}, \tag{8}$$

where again C_{ijk} are constants.
d_{ij} and C_{ijk} are called structure constants.
Since we only employed the global symmetries which also are present in higher dimensions, similar results hold for CFTs in $d > 2$.
In order the 2-point function to be invariant under the rotation $z \mapsto \exp(2\pi \mathrm{i})z$ we have that for quasi-primary fields $h \in \frac{1}{2}\mathbb{Z}$.

3.3 The energy-momentum tensor

In a classical field theory of arbitrary dimension defined by the Lagrangian $\mathcal{L}$, there is the Noether theorem stating that if $\mathcal{L}$ is invariant under the simultaneous infinitesimal transformations $x_\mu \mapsto x_\mu + \delta x_\mu, \phi_r \mapsto \Delta\phi_r$ (here r labels the fields of the theory), then the current

$$\begin{aligned} j^\mu &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)}\Delta\phi_r - T^{\mu\nu}\delta x_\nu \\ \text{with} \quad T^{\mu\nu} &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)}\partial^\nu \phi_r - \mathcal{L}g^{\mu\nu} \end{aligned}$$

is conserved ($\partial_\mu j^\mu = 0$). The energy-momentum tensor $T^{\mu\nu}$ is in general not symmetric, but may be symmetrised via a gauge transformation $\mathcal{L} \mapsto \mathcal{L} + \partial_\mu f^\mu$.

In the case of a CFT (of arbitrary dimension) a Lagrangian not always exists, so we define the energy-momentum tensor instead as the linear map from infinitesimal conformal transformations $z \mapsto z + \epsilon(z)$ to the corresponding conserved current j,

$$j^\mu = T^{\mu\nu}\epsilon_\nu. \tag{9}$$

This agrees with the Noether theorem for translations $\epsilon(z) = \text{const}$. We also require T to be symmetric. Note that the normalisation of T remains open here, we will address this later.
For $\epsilon = \text{const.}$ we observe that T is conserved:

$$0 = \partial_\mu j^\mu = \partial_\mu(T^{\mu\nu}\epsilon_\nu) = (\partial_\mu T^{\mu\nu})\epsilon_\nu \quad \Rightarrow \quad \partial_\mu T^{\mu\nu} = 0.$$

Using this, the symmetry of T and the conformal Killing equation (1) we see, by considering arbitrary ϵ now, that T is traceless:

$$\begin{aligned}0 = \partial_\mu j^\mu &= (\partial_\mu T^{\mu\nu})\epsilon_\nu + T^{\mu\nu}(\partial_\mu \epsilon_\nu) = \frac{1}{2}T^{\mu\nu}(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu) \\ &= \frac{1}{2}T^{\mu\nu}\eta_{\mu\nu}(\partial\cdot\epsilon)\frac{2}{d} = \frac{1}{d}(\partial\cdot\epsilon)T^\mu_\mu \quad \Rightarrow \quad T^\mu_\mu = 0.\end{aligned}$$

Coming back to 2 dimensions, we get for our coordinates

$$T_{z\bar{z}} = \begin{pmatrix} T(z) & 0 \\ 0 & \bar{T}(\bar{z}) \end{pmatrix},$$

so T is diagonal and consists of a chiral and an anti-chiral part.

3.4 Operator product expansion

An operator product expansion is an expansion (OPE) of the (radial ordered) product of two operators at different spacetime points in terms of operators at just one of those points.

As a first example we consider a primary field ϕ and the energy-momentum tensor. To that end, we derive two expressions for the transformation behaviour of ϕ under an infinitesimal conformal transformation $\epsilon(z)$.
One of them is obtained by using (4), a Taylor expansion and the residual theorem:

$$\begin{aligned}\delta_{\epsilon,\bar{\epsilon}}\phi(z,\bar{z}) &= (h\partial_z\epsilon + \epsilon\partial_z + \bar{h}\partial_{\bar{z}}\bar{\epsilon} + \bar{\epsilon}\partial_{\bar{z}})\phi(z,\bar{z}) \\ &= \oint_{\mathcal{C}(z)} \frac{\mathrm{d}w}{2\pi\mathrm{i}}\epsilon(w)\left[\frac{h}{(w-z)^2}\phi(z,\bar{z}) + \frac{1}{w-z}\partial_z\phi(z,\bar{z})\right] + \text{anti-chiral}.\end{aligned}$$

Here $\mathcal{C}(w)$ is some adequate contour around w.
For the second expression we recall that in QFT a conserved current j^μ induces a conserved charge

$$Q = \int \mathrm{d}x^1 j_0 \quad \text{at} \quad x^0 = \text{const.}$$

(here in 2 dimensions) which generates the corresponding symmetry transformation for an operator A as $\delta A = [Q, A]$. In radial quantisation $x^0 = \text{const.}$ translates to $|w| = \text{const.}$, therefore:

$$\begin{aligned}Q &= \oint_{\mathcal{C}(0)} \frac{\mathrm{d}w}{2\pi\mathrm{i}} T(w)\epsilon(w) + \text{anti-chiral} \\ \Rightarrow \delta_{\epsilon,\bar{\epsilon}}\phi(z,\bar{z}) &= \oint_{\mathcal{C}(0)} \frac{\mathrm{d}w}{2\pi\mathrm{i}}[T(w)\epsilon(w), \phi(z,\bar{z})] + \text{anti-chiral}.\end{aligned}$$

Here arises the ambiguity of $|z|$ lying inside or outside of $\mathcal{C}(0)$. We address this issue with time ordering known from QFT in mind, which here becomes radial ordering:

$$\mathcal{R}(A(w)B(z)) := \begin{cases} A(w)B(z), & |w| > |z| \\ B(z)A(w), & |z| > |w| \end{cases}.$$

We define:

$$\begin{aligned}\oint_{\mathcal{C}(0)} \mathrm{d}w[A(w), B(z)] &:= \oint_{|w|>|z|} \mathrm{d}w A(w)B(z) - \oint_{|w|<|z|} \mathrm{d}w B(z)A(w) \\ &= \oint_{\mathcal{C}(z)} \mathrm{d}w \mathcal{R}(A(w)B(z))\end{aligned}$$

From now on, $\mathcal{R}$ will be implicit. Using this expression for the contour integral over the commutator, $\delta_{\epsilon,\bar{\epsilon}}\phi(z,\bar{z})$ becomes

$$\delta_{\epsilon,\bar{\epsilon}}\phi(z,\bar{z}) = \oint_{\mathcal{C}(z)} \frac{\mathrm{d}w}{2\pi\mathsf{i}}\epsilon(w)T(w)\phi(z,\bar{z}) + \text{anti-chiral}.$$

Comparing both expressions yields

$$T(w)\phi(z,\bar{z}) = \frac{h}{(w-z)^2}\phi(z,\bar{z}) + \frac{1}{w-z}\partial_z\phi(z,\bar{z}) + \text{reg.}, \tag{10}$$

where reg. denotes contributions regular at $w = z$. This is the OPE we were looking for.

As a second example we take a look at the OPE of T with itself. It will be useful to know the fact that the Laurent modes of the energy-momentum tensor are the generators of the conformal transformations. In order to see this, we Laurent expand T (with conformal dimension $h = 2$):

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n,$$
$$\text{where}\quad L_n = \oint_{\mathcal{C}(0)} \frac{\mathrm{d}z}{2\pi\mathsf{i}} z^{n+1}T(z).$$

Choosing the particular conformal transformation $\epsilon(z) = -\epsilon_n z^{n+1}$ the corresponding conserved charge gets

$$Q_n = -\epsilon_n \sum_{n\in\mathbb{Z}} \oint_{\mathcal{C}(0)} \frac{\mathrm{d}z}{2\pi\mathsf{i}} L_m z^{n-m-1} = -\epsilon_n L_n.$$

The identification of the Laurent modes of T with the conformal generators fixes the normalisation of T. One can now check that these modes satisfy the Virasoro algebra only if the OPE is given by

$$T(w)T(z) = \frac{c/2}{(w-z)^4} + \frac{2T(z)}{(w-z)^2} + \frac{\partial_z T(z)}{w-z} + \text{reg.}$$

This shows that T is not a primary field, but it is indeed quasi-primary.

It can be shown that the OPE of two chiral quasi-primary fields can be expressed in terms of only the chiral quasi-primary fields of the theory and their derivatives, so we can make the ansatz

$$\phi_i(w)\phi_j(z) = \sum_{k,n\geq 0} C_{ij}^k \frac{a_{ijk}^n}{n!} \frac{1}{(w-z)^{h_i+h_j-h_k-n}} (\partial_z)^n\phi_k(z), \tag{11}$$

where the constants are split such that a_{ijk}^n only depends on h_i, h_j, h_k and n. The coefficients can be determined by looking at the 3-point function

$$\langle\phi_i(z)\phi_j(1)\phi_k(0)\rangle.$$

One can use directly the formula for the 3-point function (8), or first expand $\phi_j(z)\phi_j(1)$ with (11) and then use the formula for 2-point functions (7). Comparing both expressions yields

$$a_{ijk}^n = \binom{2h_k+n-1}{n}^{-1}\binom{h_k+h_i-h_j+n-1}{n},$$
$$C_{ij}^k d_{kl} = C_{ijl},$$

where $\binom{\cdot}{\cdot}$ are the binomial coefficients and d_{kl}, C_{ijl} the structure constants.

The regular part of an OPE gives rise to a notion of normal ordering where creation operators are put to the left and annihilation operators to the right. For the OPE of two chiral fields $\phi(w)$ and $\chi(z)$ we can write

$$\phi(w)\chi(z) = \text{sing.} + \sum_{n=0}^{\infty} \frac{(w-z)^n}{n!} \mathcal{N}(\chi\partial^n\phi)(z).$$

The first term

$$\psi(z) := \oint_{\mathcal{C}(z)} \frac{\mathrm{d}w}{2\pi\mathrm{i}} \frac{\phi(w)\chi(z)}{w-z} = \sum_{n\in\mathbb{Z}} z^{-n-h_\phi-h_\chi}\psi_n$$

can be shown to be of the form

$$\psi_n = \sum_{k>-h_\phi} \chi_{n-k}\phi_k + \sum_{k\le -h_\phi} \phi_k\chi_{n-k}.$$

Higher orders work analogously. This defines the normal ordering operator $\mathcal{N}$.

3.5 Conformal Ward identity

Ward identities are manifestations of classical symmetries at quantum level. Here we derive an expression for the primary fields $\phi_1, \ldots, \phi_N$ using the OPE (10) (ϵ is some infinitesimal conformal transformation):

$$\begin{aligned}
&\oint_{\mathcal{C}(0)} \frac{\mathrm{d}w}{2\pi\mathrm{i}} \epsilon(w) \langle T(w)\phi_1(z_1,\bar z_1)\cdots\phi_N(z_N,\bar z_N)\rangle \\
&= \sum_{i=1}^{N} \langle \phi_1(z_1,\bar z_1)\cdots \left(\oint_{\mathcal{C}(z_i)} \frac{\mathrm{d}w}{2\pi\mathrm{i}} \epsilon(w) T(w)\phi_i(z_i,\bar z_i) \right) \cdots \phi_N(z_N,\bar z_N)\rangle \\
&= \sum_{i=1}^{N} \langle \phi_1(z_1,\bar z_1)\cdots \left(\oint_{\mathcal{C}(z_i)} \frac{\mathrm{d}w}{2\pi\mathrm{i}} \epsilon(w) \left(\frac{h_i}{(w-z_i)^2} + \frac{1}{w-z_i}\partial_{z_i} \right) \phi_i(z_i,\bar z_i) \right) \cdots \phi_N(z_N,\bar z_N)\rangle \\
&= \oint_{\mathcal{C}(0)} \frac{\mathrm{d}w}{2\pi\mathrm{i}} \epsilon(w) \sum_{i=1}^{N} \left(\frac{h_i}{(w-z_i)^2} + \frac{1}{w-z_i}\partial_{z_i} \right) \langle \phi_1(z_1,\bar z_1)\cdots\phi_N(z_N,\bar z_N)\rangle,
\end{aligned}$$

where $|w| > |z_i|\ \forall i$. Since this holds for any $\epsilon(z) = -\epsilon_n z^{n+1}$, we can cancel the integral of both sides:

$$\langle T(w)\phi_1(z_1,\bar z_1)\cdots\phi_N(z_N,\bar z_N)\rangle = \sum_{i=1}^{N} \left(\frac{h_i}{(w-z_i)^2} + \frac{1}{w-z_i}\partial_{z_i} \right) \langle \phi_1(z_1,\bar z_1)\cdots\phi_N(z_N,\bar z_N)\rangle.$$

This is the conformal Ward identity for primary fields.

4 The free boson

As an example we look at the free boson living on a cylinder. We start with the action for a real scalar field $X(x^0, x^1)$

$$S = \frac{1}{4\pi\kappa} \int \mathrm{d}x^0\, \mathrm{d}x^1 \sqrt{|h|} h^{\alpha\beta} \partial_\alpha X \partial_\beta X$$

with the metric $h_{\alpha\beta}$ and $h = \det h_{\alpha\beta}$. In Euclidean space the metric becomes after our coordinate transformation:

$$\begin{pmatrix} x^0 \\ x^1 \end{pmatrix} \mapsto \begin{pmatrix} z \\ \bar z \end{pmatrix},$$

$$h_{\alpha\beta} \mapsto g_{ab} = \frac{\partial x^\alpha}{\partial x^a} \frac{\partial x^\beta}{\partial x^b}.$$

Therefore the action now reads

$$S = \frac{1}{4\pi\kappa}\int \mathrm{d}z\,\mathrm{d}\bar{z}\partial_z X\partial_{\bar{z}}X.$$

The classical equation of motion is obtained by varying the action with respect to X:

$$\frac{\delta S}{\delta X} = -\frac{\delta}{\delta X}\frac{1}{4\pi\kappa}\int \mathrm{d}z\,\mathrm{d}\bar{z}(\partial_z\partial_{\bar{z}}X)\delta X = -\frac{1}{4\pi\kappa}\partial_z\partial_{\bar{z}}X \stackrel{!}{=} 0 \quad \Rightarrow \quad \partial_z\partial_{\bar{z}}X = 0.$$

This motivates the definition of

$$\begin{aligned} j(z) &:= \mathrm{i}\partial_z X,\\ \bar{j}(\bar{z}) &:= \mathrm{i}\partial_{\bar{z}}X. \end{aligned}$$

Clearly $j(z)$ is a chiral and $\bar{j}(\bar{z})$ an anti-chiral field.
To have the action S be conformally invariant, X must have the conformal dimension $(0,0)$ and $j(z)$ and $\bar{j}(\bar{z})$ are primary fields of dimensions $(1,0)$ and $(0,1)$, respectively.

From QFT we know that the 2-point function

$$K(z,\bar{z},w,\bar{w}) := \langle X(z,\bar{z})X(w,\bar{w})\rangle$$

is the Green's function for the equation of motion,

$$\partial_z\partial_{\bar{z}}K = -2\pi\kappa\delta(z-w),$$

from which follows that

$$K = -\kappa\log|z-w|^2.$$

Comparing this to (7) shows us that X cannot be a quasi-primary field. On the other hand we have

$$\langle j(z)j(w)\rangle = -\partial_z\partial_{\bar{z}}K = \frac{\kappa}{(z-w)^2}$$

which indeed is consistent with (7) for $d_{jj} = \kappa$. Similarly we se that $d_{\bar{j}\bar{j}} = \kappa$ and $d_{j\bar{j}} = 0$.

The energy-momentum tensor can be computed as the functional derivative of the action with respect to the metric:

$$T_{ab} = 4\pi\kappa\gamma\frac{1}{\sqrt{|g|}}\frac{\delta S}{\delta g^{ab}} = \gamma\begin{pmatrix} jj & 0\\ 0 & \bar{j}\bar{j}\end{pmatrix}$$

with the normalisation γ. Strictly speaking, we would have to verify that this definition of T coincides with our earlier definition (9). However, we see that the formal requirements on the structure are satisfied.
We want the expectation value of T as an operator to vanish, therefore we write

$$\begin{aligned} &T(z) = \gamma\mathcal{N}(jj)(z)\\ &\Rightarrow T_n = L_n = \gamma\mathcal{N}(jj)_n = \gamma\sum_{k>-1} j_{n-k}j_k + \gamma\sum_{k\leq -1} j_k j_{n-k}. \end{aligned}$$

The normalisation can be fixed by considering

$$[L_m, j_n] = -2\gamma\kappa n j_{m+n} \stackrel{!}{=} ((h_j-1)m-n)j_{m+n} = -nj_{m+n} \quad \Rightarrow \quad \gamma = \frac{1}{2\kappa},$$

where we employed (6).

As the final result here we will derive the central charge c for this theory. To this end we compute

$$\langle 0|L_{+2}L_{-2}|0\rangle = \langle 0|\left[L_{+2}L_{-2}\right]|0\rangle = \frac{c}{2}.$$

On the other hand we have

$$\begin{aligned}
&L_{-2}|0\rangle = \frac{1}{2\kappa}j_{-1}j_{-1}|0\rangle, \\
&\langle 0|L_{+2} = \frac{1}{2\kappa}\langle 0|(j_2 j_0 + j_1 j_1) = \frac{1}{2\kappa}\langle 0|(-j_0 j_2 + j_1 j_1) = \frac{1}{2\kappa}\langle 0|j_1 j_1 \\
&\quad \Rightarrow \quad \langle 0|L_{-2}L_{+2}|0\rangle = \frac{1}{4\kappa^2}\langle 0|j_1 j_1 j_{-1} j_{-1}|0\rangle = \frac{1}{2}
\end{aligned}$$

The last equality follows by commuting the js through with the general formula (5) for commutators of chiral quasi-primary fields which simplifies to

$$[j_m, j_n] = \kappa m \delta_{m,-n}.$$

Combining both expressions yields

$$c = 1.$$

This is no coincidence since the factor of $\frac{1}{12}$ in the Virasoro algebra was chosen such that here $c = 1$.

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