The Quantum Hall Effect

Seminar talk by Tobias Witt

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Abstract When exposed to a magnetic field B electrons will move in a direction perpendicular to the drop of voltage. This phenomenon is known as the Hall effect and associated with a resistance $R \sim B$. For low temperatures and strong magnetic fields, however, $R = \frac{R_0}{n}$ is quantized as an integer fraction of the universal von-Klitzing-constant while following the step-shaped curve of figure 8. The aim of these notes is to explain why the QHE can already be understood from ordinary Quantum mechanics and Quantum statistics. For economic reasons this write-up does not dwell upon the subtleties of QM on covering spaces such as the relation between ground-state degeneracy and Dirac's quantisation condition.

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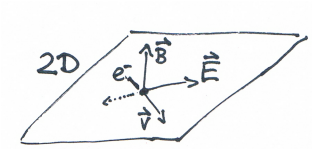

Figure 1: Electrons in the equilibrium of magnetic and electric forces

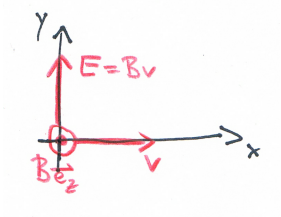

Figure 2: A convenient choice of frame

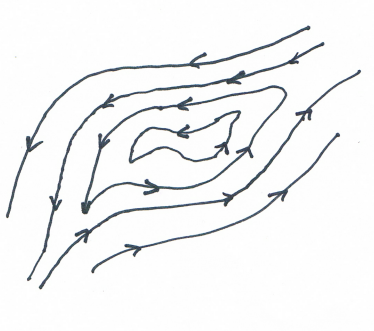

Figure 3: In the Drude model electrons flow along equipotential lines

1 Prerequisites

1.1 Classical HE in the Drude model

Consider non-interacting particles of charge Q and mass m moving in a
2-dimensional material. The motion is driven by an electric field $\vec{E}$ within in the plane while getting deflected by a magnetic field $\vec{B}$ perpendicular to the plane (see figure 1). A stable velocity $\vec{v}$ is obtained under the constraint

$$\begin{aligned} 0 = m\dot{\vec{v}} &= Q\,\vec{E} + Q\,\vec{v}\times\vec{B} \\ \Rightarrow\quad \vec{E} = \vec{B}\times v &= B\,\underbrace{\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}}_{=-\boldsymbol{\varepsilon}}\,\vec{v} \end{aligned} \tag{1}$$

also known as the *Drude equation*. In the Drude approximation we use eq. (1) to determine a velocity field $\vec{v}(\vec{x})$ in response to the electric field configuration $\vec{E}(\vec{x})$ while taking $\vec{B} = B\vec{e}_z$ to be constant for most applications.[1]
The matrix ε represents a rotation by 90° and since 2d-rotations commute, we see that eq. (1) exhibits rotational symmetry in the sense that if $(\vec{E},\vec{v})$ is a solution, so is $(R\vec{E},R\vec{v})$ for any $R\in SO(2)$.[2] By choice of frame we may always assume that $E\sim\vec{e}_y$ and $\vec{v}\sim\vec{e}_x$ like in figure 2. The proportionality $|\vec{E}| = B|\vec{v}|$ is reminiscent of Ohm's Law and so we will define a **Hall resistance**

$$\boxed{R = \frac{U}{I} \sim \frac{E_y}{v_x} = B} \tag{2}$$

that *increases linearly with the magnetic field*. As opposed to what we expect from standard electrics the current I flows perpendicular to the voltage drop meaning that **particle trajectories follow lines of constant potential**. In particular, distinct trajectories do not intersect, so the flow of particles is laminar.

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[1] Within a travelling time T our particle may witness a variation $\Delta\vec{E}$ and by eq.(1) this gives rise to an acceleration $\Delta\vec{v}$:

$$B\,|\dot{\vec{v}}| = \frac{B|\Delta\vec{v}|}{T} = \frac{|\Delta\vec{E}|}{T} = |v^i\partial_i\vec{E}\,| \le |\vec{v}|\cdot||\partial_i\partial_j V||$$

For eq. (1) to remain a valid approximation we require that $\boldsymbol{m|\dot{\vec{v}}| \ll Q|\vec{E}|}$ which is satisfied if

$$\frac{m}{B}\,|\vec{v}|\cdot||\partial_i\partial_j V|| \ll Q|\vec{E}|$$

With $B\,|\vec{v}| = |\vec{E}|$ this becomes

$$\boxed{\frac{||\partial_i\partial_j V||}{B^2} \ll \frac{Q}{m}}$$

so the Drude approximation can be applied for **magnetic fields** that are **strong** compared to the second derivatives of the electric potential V. This argument might seem circular. However, given an electric field configuration $\vec{E}$ we can solve for $\vec{v}$ to satisfy the Drude equation, then calculate $\dot{\vec{v}}$ and modify $\vec{E}$ such that $m\dot{\vec{v}} = Q\vec{E}' + Q\vec{v}\times\vec{B}$ is fullfilled. $\vec{E}'$ might no longer be the gradient of a potential but we should expect that small irritations in the field of force do not destroy our particle trajectories.

[2] Alternatively one can just argue that ε is an invariant tensor of $SO(2)$, i.e. $R^T\varepsilon R = \det(R)\varepsilon = \varepsilon$.
In fact, since ε is also a generator of $SO(2)$, we see that $R\in O(2)$ commutes with $SO(2)$ if and only if $\det R = 1$:

$$R^T\,\exp(\theta\,\boldsymbol{\varepsilon})R = \exp\Big(\theta\,R^T\boldsymbol{\varepsilon}R\Big) = \exp(\pm\theta\,\boldsymbol{\varepsilon})$$

Every matrix can be written in a basis of orthogonal matrices, for 2d:
$\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ and $\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$ with $\det = 1$, $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ and $\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ with $\det = -1$
Thus, our argument can be reversed: Rotational symmetry of the physical system requires that $\vec{E}$ and $\vec{v}$ are related by a matrix of the form $\eta\cdot\mathbb{1} + B\cdot\varepsilon$ with $\eta, B\in\mathbb{R}$

1.2 Quantizing single electrons in a magnetic background field

Given that $\vec{\nabla}\cdot\vec{B}\sim \mathrm{d}(*B)=0$, we can express our magnetic field as the curl of a vector potential: $\vec{B}=\vec{\nabla}\times\vec{A}\sim *(\mathrm{d}\,A)$
This is needed to formulate the Lagrangian

$$L=\frac{1}{2}m\dot{\vec{x}}^{\,2}+Q\,\dot{\vec{x}}\cdot\vec{A}(\vec{x})+Q\,V(\vec{x})$$

whose variation under $\vec{x}\longrightarrow\vec{x}+\delta\vec{x}$ reproduces the Lorentz force $Q\vec{v}\times\vec{B}$ and electric force $Q\,\vec{E}=Q\,\nabla V$ as encountered in eq.(1) [3]
From the Drude approximation we expect a motion that is effectively
insensitive to the charge and mass of particles, so for convenience we will set $m=1$, $Q=-1$ and consider the Lagrangian

$$L=\frac{1}{2}\dot{x}^{\,2}-\dot{x}\cdot A(x)-V(x) \tag{3}$$

With vector indices kept implicit the *canonical momentum* reads

$$p=\frac{\partial L}{\partial\dot{x}}=\dot{x}-A \tag{4}$$

and by a Legendre transform we obtain the *Hamiltonian*

$$H=p\dot{x}-L=\frac{1}{2}\dot{x}^2+V=\frac{1}{2}(p+A)^2+V \tag{5}$$

This is to be understood as a function of x and p only.
For quantum particles we think of position and momentum as operators $\hat{x}$ and $\hat{p}$ subject to the commutation relation

$$\big[\hat{x}_k,\hat{p}_l\big]=i\hbar\,\delta_{k\,l} \tag{6}$$

which is of course inspired by

$$\left[x_k\,,\,-i\hbar\,\frac{\partial}{\partial x^l}\right]=i\hbar\,\delta_{k\,l}\quad\text{and}\quad\left[i\hbar\,\frac{\partial}{\partial p^k}\,,\,p_l\right]=i\hbar\,\delta_{k\,l}$$

Note that we will quantize only the test particle (i.e. the electron) and treat our vector potential $\vec{A}$ as a classical background field whose coupling to the wave function ψ is local when seen in position space. When $A(x)=\sum a_n x^n$ is expressed as a Taylor series, we just keep the coefficients and define an operator $A(\hat{x})=\sum a_n\hat{x}^n$ acting on the Hilbert space of wave functions.[4]
By $\big[\,\hat{x}^n\,,\,\hat{p}\,\big]=i\hbar\,n\,\hat{x}^{n-1}$ we see that $\big[\,A(\hat{x})\,,\,\hat{p}\,\big]=i\hbar\;\partial_x A(\hat{x})$ acts as a derivative. Similarly we have $\big[f(\hat{p})\,,\,\hat{x}\big]=-i\hbar\,\partial_p f(\hat{p})$ for any function of momentum. However, the same argument applies to our Hamiltonian $H(\hat{x},\hat{p})$ which is a mixed function of $\hat{x}$ and $\hat{p}$: From now on we take $\hbar=1$ and define the Heisenberg picture of an operator by $\mathcal{O}(t):=\mathrm{e}^{iHt}\,\mathcal{O}\,\mathrm{e}^{-iHt}$. Then the classical Hamilton equation implies [5]

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\hat{x}(t)=i\big[H,\hat{x}(t)\big]=\boldsymbol{\partial_p H}(\hat{x}(t),\hat{p}(t))=\boldsymbol{\dot{x}}(\hat{x}(t),\hat{p}(t))$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\,\hat{p}(t)=i\big[H,\hat{p}(t)\big]=\boldsymbol{-\partial_q H}(\hat{x}(t),\hat{p}(t))=\boldsymbol{\dot{p}}(\hat{x}(t),\hat{p}(t))$$

[3] Note that a gauge transformation $\vec{A}\longrightarrow\vec{A}+\vec{\nabla}\phi$ enters by a total time derivative $L\longrightarrow L+Q\frac{\mathrm{d}}{\mathrm{d}t}\phi$ and leaves the action $S=\int\mathrm{d}t\,L$ invariant

[4] In momentum space one has for example

$$\widetilde{A\,\psi}\,(p)=\int\mathrm{d}x\;\psi(x)\,A(x)\,\mathrm{e}^{-ip\cdot x}=\int\mathrm{d}x\;\psi(x)\,A\Big(i\frac{\partial}{\partial p}\Big)\,\mathrm{e}^{-ip\cdot x}=A\Big(i\frac{\partial}{\partial p}\Big)\,\widetilde{\psi}(p)$$

so the operator $A(\hat{x})$ commutes with Fourier transformation.

[5] One may use that

$$H(\hat{x},\hat{p})=\mathrm{e}^{iHt}\,H(\hat{x},\hat{p})\,\mathrm{e}^{-iHt}=H\left(\mathrm{e}^{iHt}\,\hat{x}\,\mathrm{e}^{-iHt}\,,\;\mathrm{e}^{iHt}\,\hat{p}\,\mathrm{e}^{-iHt}\right)=H\left(\hat{x}(t)\,,\,\hat{p}(t)\right)$$

and that $\hat{x}(t)\,,\,\hat{p}(t)$ obey the same commutation relations as $\hat{x}\,,\,\hat{p}$.

This guarantees that the expectation value of

$$\boldsymbol{\pi} := \dot{\boldsymbol{x}} = \hat{\boldsymbol{p}} + \boldsymbol{A}(\hat{\boldsymbol{x}}) \tag{7}$$

has an interpretation as the velocity of a localized state $|\Psi\rangle$, i.e.

$$\begin{aligned} \langle\Psi(t)|\,\boldsymbol{\pi}\,|\Psi(t)\rangle = \langle\Psi|\,\boldsymbol{\pi}(t)\,|\Psi\rangle = \langle\Psi|\,\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}(t)\,|\Psi\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\,\langle\Psi|\,\hat{x}(t)\,|\Psi\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\,\langle\Psi(t)|\hat{x}|\Psi(t)\rangle \\ \Rightarrow \quad \langle\boldsymbol{\pi}\rangle(t) = \frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{\boldsymbol{x}}\rangle \end{aligned} \tag{8}$$

given that $|\Psi(t)\rangle = \mathrm{e}^{-iHt}\,|\Psi\rangle$ satisfies the Schroedinger equation.
In what comes next we turn off the electric field $\vec{E}$ by setting $V = 0$. Thus, our Hamiltonian reads

$$H = \frac{1}{2}\,\boldsymbol{\pi^2} \tag{9}$$

For the components we have

$$\Big[\pi_x, \pi_y\Big] = \Big[p_x + A_x, p_y + A_y\Big] = -i(\partial_x A_y - \partial_y A_x) = -iB \tag{10}$$

where B was constant before quantization and thus remains a scalar factor after quantization. Going one step further one gets

$$\Big[\frac{1}{2}\pi^2\,,\,\pi_y\Big] = \Big[\frac{1}{2}\pi_x^2\,,\,\pi_y\Big] = -iB\,\pi_x$$

or more systematically

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{\pi} = i\Big[H\,,\,\vec{\pi}\Big] = B\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\vec{\pi} = \vec{B}\times\vec{\pi} \tag{11}$$

which is an operator analogue of the Lorentz force appearing in eq.(1).

The matrix $\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$ has eigenvectors $(1,-i)$ and $(1,i)$
so we find **raising and lowering operators**

$$\begin{aligned} a &:= \frac{1}{\sqrt{2B}}\big(\pi_x - i\pi_y\big) \qquad \big[H, a\,\big] = -B\,a \\ a^\dagger &= \frac{1}{\sqrt{2B}}\big(\pi_x + i\pi_y\big) \qquad \big[H, a^\dagger\,\big] = +B\,a^\dagger \end{aligned}$$

where the normalization is such that $\big[a, a^\dagger\,\big] = 1$
Our Hamiltonian is now brought into the form

$$H = \frac{1}{2}\pi^2 = B\left(a^\dagger a + \frac{1}{2}\right) \tag{12}$$

reminiscent of an harmonic oscillator.
Given the positive-definiteness of our Hilbert space $\langle a^\dagger a\rangle \geq 0$ will never be negative and so we are left with a discrete energy spectrum:

$$E_n = B\left(n + \frac{1}{2}\right) \qquad n \in \mathbb{N} \tag{13}$$

The E_n are referred to as **Landau levels** .
Unlike what we are used to from a harmonic oscillator, these energy levels are degenerate. The degeneracy is best seen in **symmetric gauge**

$$\vec{A} = \frac{1}{2}\vec{B}\times\vec{x}$$

We can always define the modified operators

$$\widetilde{\pi}_x := p_x - A_x \qquad \widetilde{\pi}_y := p_y - A_y$$

such that $\big[\widetilde{\pi}_x, \widetilde{\pi}_y\big] = +iB$, but symmetric gauge is (up to a translation of the origin) the unique gauge in which they commute with the original ones:

$$\big[\pi, \widetilde{\pi}\big] \sim \partial_x A_y + \partial_y A_x = 0$$

Again we can define a pair of raising and lowering operators

$$b = \frac{1}{\sqrt{2B}}\big(\widetilde{\pi}_x + i\,\widetilde{\pi}_y\big) \qquad b^\dagger = \frac{1}{\sqrt{2B}}\big(\widetilde{\pi}_x - i\,\widetilde{\pi}_y\big) \qquad \text{with} \quad \big[b, b^\dagger\big] = 1$$

The Landau levels are invariant subspaces of b, $b^\dagger$ and since not only $a^\dagger a$ but also $b^\dagger b$ has a discrete spectrum, our Hilbert space will be spanned by states of the form

$$\boxed{|n,\, m\rangle \sim \big(a^\dagger\big)^n \big(b^\dagger\big)^m |0,0\rangle} \tag{14}$$

where $a\,|0,0\rangle = 0 = b\,|0,0\rangle$ is the ground state. Indeed, the vector potential $\vec{A}(\vec{x})$ is linear in $\vec{x}$ so we can express $\hat{x}$ and $\hat{p}$ as linear combinations of the π, $\widetilde{\pi}$ which themselves are linear combinations of a, $a^\dagger$, b, $b^\dagger$. This ensures that the space spanned by the $|n,m\rangle$ is closed under the action of observables $\mathcal{O}(\hat{x},\hat{p})$ and in particular under time evolution e^{-iHt}.

Our operators have an explicit representation that is schematically given by[6]

$$a \sim \pi_x - i\pi_y = \underbrace{p_x}_{-i\partial_x} - i\,\underbrace{p_y}_{-i\partial_y} + \underbrace{A_x}_{-y} - i\,\underbrace{A_y}_{x} = -i\big(\underbrace{\partial_x - i\partial_y}_{\partial_{\overline{z}}} + \underbrace{x - iy}_{z}\big)$$

$$\text{i.e.} \qquad \begin{aligned} a &\sim -i(\,\overline{\partial} + z) \\ b &\sim -i(\partial + \overline{z}) \end{aligned}$$

For the ground state $a\,|0,0\rangle = 0 = b\,|0,0\rangle$ one obtains the unique solution

$$|0,0\rangle \sim \mathrm{e}^{-|z|^2}$$

and the lowest Landau level (LLL) is built from states

$$|0,m\rangle \sim (b^\dagger)^m\,|0,0\rangle \sim z^m \mathrm{e}^{-|z|^2} = r^m e^{-r^2} \mathrm{e}^{-im\theta} \tag{15}$$

whose propability density is radially symmetric and sharply peaked at $r \sim \sqrt{m}$.
Indeed, by rewriting $\Psi(z,\overline{z}) = f(z,\overline{z})\,\mathrm{e}^{-|z|^2}$ the LLL-condition $a\,|\Psi\rangle = 0$ is equivalent to $f(z,\overline{z}) = f(z)$ being holomorphic and any such state can be expressed as a Taylor series of the states $|0,m\rangle$.

[6]Schematically means we are not interested in numerical factors; of course normally one would have $z = x - iy \Rightarrow \partial_z = \frac{1}{2}\big(\partial_x + i\partial_y\big)$ but the $\frac{1}{2}$ is not relevant for our discussion

1.3 The propagation of wave packets

Let us have a brief look at how the Drude model is recovered for quantum mechanical wave packets of the form

$$\Psi(x,y) = \int \mathrm{d}k\, \Psi_k(y)\, \mathrm{e}^{ikx} \tag{16}$$

In presence of a constant electric field $\vec{E} = E\,\vec{e}_y$ and with the vector potential taken to be in Landau gauge $\vec{A} = (-By, 0)$ our Hamiltonian reads

$$H = \frac{1}{2}\,\hat{p}_y^2 \,+ \frac{1}{2}\big(\,\hat{p}_x - B\,y\big)^2 + E\,y$$

By acting on wavefunctions of definite x-momentum[7] and with the notation $\overline{k} := k - B\,y$ this becomes

$$\begin{aligned} H_k &= \frac{1}{2}\,\hat{p}_y^2 \,+ \frac{1}{2}\big(\,k - B\,y\big)^2 + E\,y \\ &= \frac{1}{2}\,\hat{p}_y^2 \,+ \frac{1}{2}\,\overline{k}^2 - \frac{E}{B}\,\overline{k} + \frac{E}{B}\,k \\ &= \frac{1}{2}\,\hat{p}_y^2 \,+ \frac{B^2}{2}\,\Big(y - \frac{k}{B} + \frac{E}{B^2}\Big)^2 \;\; + \frac{\boldsymbol{E}}{\boldsymbol{B}}\,\boldsymbol{k} + const. \end{aligned}$$

The first part is just a displaced 1d harmonic oscillator whose eigenfunctions $|u_n\rangle$ form an orthonormal basis of $L^2(\mathbb{R})$, [8] i.e. every smooth parameter family $|f_k\rangle$ of $L^2(\mathbb{R})$-functions can be written as a sum $|f_k\rangle = \sum_n c_n(k)\,|u_n\rangle$ with smooth functions $c_n(k) = \langle u_n|f_k\rangle$. In particular, this applies to $|f_k\rangle\,(y) = \Psi_k(y)$.

Now assume that our wave packet is built only from the lowest oscillator level:

$$\Psi_k(y) = c_0(k)\; u_0\Big(y - \frac{k}{B} + \frac{E}{B^2}\Big) \;+\; 0$$

The dynamics is then given by

$$\Psi(x,y,t) = \int \mathrm{d}k\, \mathrm{e}^{-iE_0 t}\, \Psi_k(y)\;\, \mathrm{e}^{\,ik\left(x - \frac{E}{B}t\right)} = e^{-iE_0 t}\; \Psi\Big(x - \frac{E}{B}t,\; y,\; 0\Big)$$

With $\vec{E}$ pointing in y-direction the wavepacket propagates in positive x-direction as expected. Moreover,

$$\boxed{v = \frac{E}{B}}$$

is exactly the *Drude velocity* from section 1.1.[9]

[7] such as $\Psi_k(y)\mathrm{e}^{ikx}$

[8] see [3], Satz 1.4.1 for a proof

[9] A more general wavepacket might also oscillate in y-direction due to the mixing of oscillator levels.

2 Quantization of the Hall resistance

2.1 An explicit calculation of the Laughlin charge pump

In section 1.3 we have shown that when exposed constant electro-magnetic fields $\vec{E} \sim \vec{e}_y$ and $\vec{B} \sim \vec{e}_z$ quantum particles propagate with the Drude velocity

$$\vec{v} = \frac{E}{B}\,\vec{e}_x$$

just as in classical mechanics. However, what is quantized with the Quantum Hall Effect is the electric current I [10]. To calculate such a current we need information not only about velocity but also about density, i.e. about the gap between neighbouring particles.

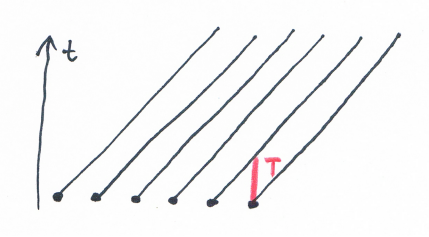


Figure 4: The electric current I relates to the time lapse T needed for an electron to take the position of its predecessor.

This information is provided by the spectrum of radially symmetric wavefunctions as derived in section 1.2. Adjusting to the shape of these wavefunctions we will calculate the Hall current I and resistance $R \sim \frac{1}{I}$ in the following set-up:

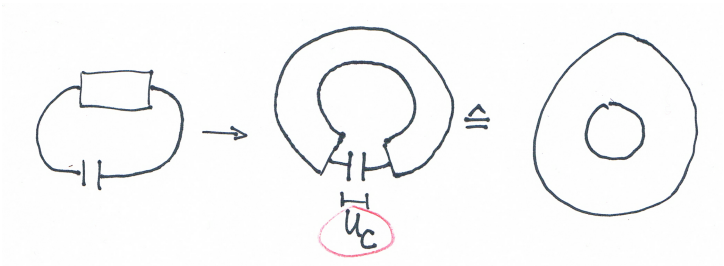


Figure 5: Experimental set-up

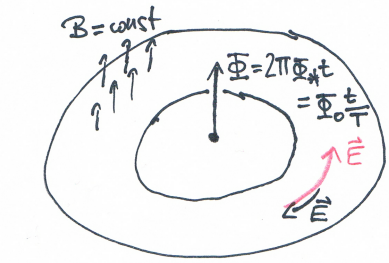


Figure 6: Turning on a magnetic flux Φ

We start by deforming a standard Hall probe until it can be thought of as an annulus. As before, the material will be penetrated by a constant magnetic field $\vec{B} = B\,\vec{e}_z = \nabla \times \vec{A}^B$ with $\vec{A}^B = \frac{1}{2}\vec{B} \times \vec{x}$. In the centre of our annulus, however, we place an increasing magnetic flux $\Phi = 2\pi\Phi_\star t = \Phi_0 \frac{t}{T}$.
By $U_{ind} = \oint \mathrm{d}\vec{x} \cdot \vec{E} = -\partial_t \Phi = -\frac{\Phi_0}{T}$ this induces a circular electric field

$$\vec{E} = -\frac{\Phi_\star}{r}\,\vec{e}_\theta \tag{17}$$

along the annulus. In polar coordinates we have $\nabla = \vec{e}_r\,\partial_r + \vec{e}_\theta\,\frac{1}{r}\partial_\theta$ and thus $\vec{E} = \nabla V$ can be expressed as gradient of the potential [11]

$$V = -\Phi_\star\,\theta \tag{18}$$

[10] At least if we keep the voltage U constant.
[11] Polar coordinates $\mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}^2 - \{0\}$, $(r,\,\theta) \longmapsto (r\cos\theta,\, r\sin\theta)$ provide a nice covering space on which to define such an otherwise "multivalued" potential

As a side effect, Φ sources an Aharonov-Bohm-type vector potential

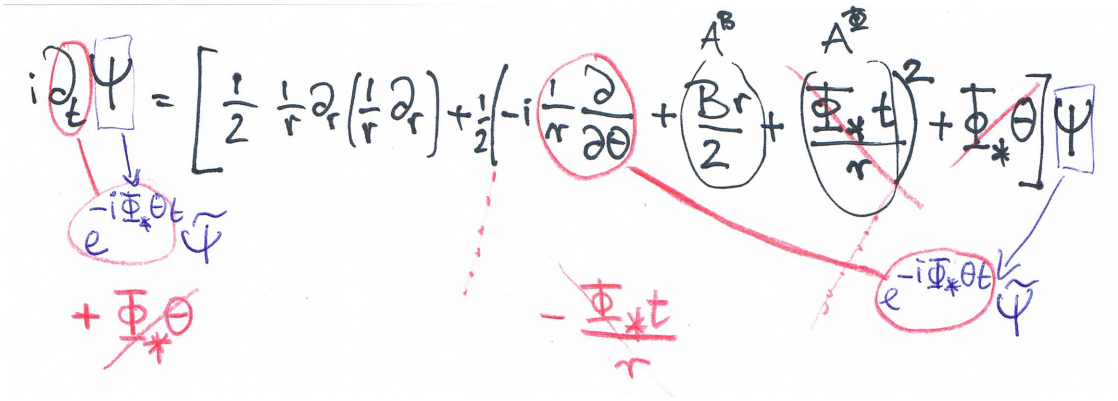
$$\vec{A}^{\Phi} = \frac{\Phi}{2\pi r}\,\vec{e}_\theta = \frac{\Phi_\star\, t}{r}\,\vec{e}_\theta$$

Both $\vec{A}^{\Phi}$ and V can be simultaneously gauged away if only

$$V = +\Phi_\star\,\theta$$

Thus, we will simply turn on an additional voltage U_C in order to reverse the direction of $\vec{E}$.

Putting together all pieces of the Hamiltonian we obtain the time-dependent Schroedinger equation:



Given that two states $\Psi = \mathrm{e}^{-i\Phi_\star\,\theta\,t}\,\widetilde{\Psi}$ are related by a gauge transformation, Ψ satisfying the original Schroedinger equation can be restated as $\widetilde{\Psi}$ satisfying the Schroedinger equation without flux:

$$i\,\partial_t\Psi = H_{\Phi_\star}\Psi \quad\Longleftrightarrow\quad i\,\partial_t\widetilde{\Psi} = H_{\Phi_\star=0}\,\widetilde{\Psi} \tag{19}$$

At $t = 0$, however, both pictures coincide and we take our initial conditions to be the symmetric gauge eigenfunctions[12] from Section 1.2

$$\text{i.e.} \qquad \Psi(0) = \widetilde{\Psi}(0) = f_{m,m}$$

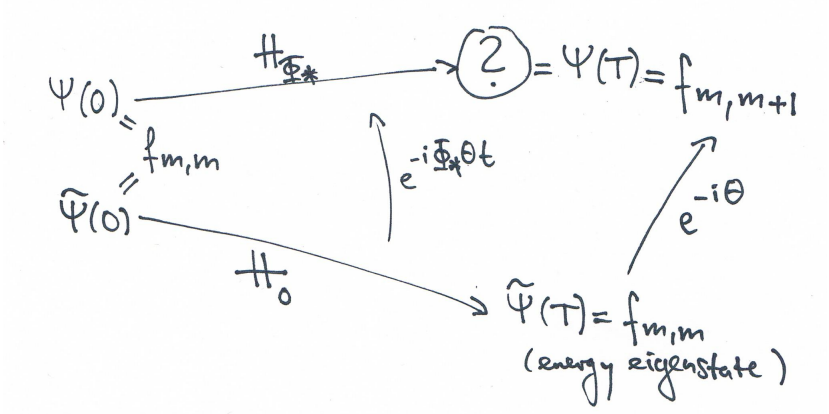
with $\boldsymbol{f_{n,m} :\sim r^n\,\mathrm{e}^{-r^2}\,\mathrm{e}^{-im\theta}}$ being a larger class of normalized functions.

$\left|\widetilde{\Psi}\right\rangle$ is an eigenstate of $H_{\Phi_\star=0}$ and will remain invariant under time evolution while $\Psi(r,\theta)$ gets modified by a factor $\mathrm{e}^{-i\Phi_\star t\,\theta}$

Let $t = T$ be the time at which we reach

$$\boxed{\Phi_\star T = 1} \tag{20}$$

Then, the winding of our wavefunction will have increased by $m \longrightarrow m+1$:



[12] In this calculation we only look at the lowest Landau level.

The radial part, however, didn't follow. To resolve this, remark that whether two functions of the type $f_{n,m}$ overlap or not is determined only by their angular parts:

$$\langle f_{\tilde{n},\tilde{m}} | f_{n,m} \rangle \sim \int_0^{2\pi} \mathrm{d}\theta \, \mathrm{e}^{i\,(\tilde{m}-m)\theta} = 2\pi\, \delta_{\tilde{m},m}$$

Thus, $f_{m+1,m+1}$ is the only LLL-state that $\Psi(T) = f_{m,m+1}$ has overlap with and so $f_{m,m+1}$ will be a superposition of higher energy states. If by some unspecified physical process $\Psi(T) = f_{m,m+1}$ drops into $f_{m+1,m+1}$, this will shift our particle from $r \sim \sqrt{m}$ to $r \sim \sqrt{m+1}$ as required. This shows that T is exactly the time needed for an electron to take the position of its predecessor. In other words exactly one electron is transferred from the inner to the outer ring of the annulus as $t \longrightarrow t + T$ and so we obtain an electric current $I = \frac{e}{T}$ driven by the inductive voltage $U_{ind} = \frac{\Phi_0}{T}$. For the resistivity the dependence on T drops out and we end up with

$$R = \frac{U}{I} = \frac{\Phi_0}{e} =: R_0 \tag{21}$$

Remember that we had chosen $\Phi_\star T = 1$ and so by $2\pi\, \Phi_\star t = \Phi_0\, \frac{t}{T}$
Φ_0 is fixed as a universal constant[13]

$$\Phi_0 = 2\pi$$

or if we recover electron charge and Planck's constant:

$$\Phi_0 = 2\pi \frac{\hbar}{e}$$

What we have shown is that as the flux increases by one unit of Φ_0 exactly one electron is transferred from the inner to the outer ring - no matter on which time scale this happens.

2.2 The Integer Quantum Hall Effect

In the preceding section we have made sense of

Assumption 1 There is a universal constant

$$R_0 = \frac{2\pi\,\hbar}{e^2} = \frac{\Phi_0}{e}$$

known as the *von-Klitzing constant*
such that each Landau level contributes

$$\boxed{I = \frac{1}{R_0}\, U}$$

to the total current.

From Section 1.2 we know that Landau levels refer to the discrete energy spectrum

$$E_n = B\big(n + \frac{1}{2}\big) \qquad n \in \mathbb{N}$$

Electrons will occupy these energy levels according to a Fermi distribution, so at low temperatures there will be a sharp Fermi edge μ such that all Landau levels with $E \leq \mu$ are completely filled whereas those with $\mu < E$ remain empty. Let N denote the number of Landau levels below μ. By Assumption 1 the Hall resistance is then given by

$$\boxed{R = \frac{U}{I} = \frac{R_0}{N} \qquad N \in \mathbb{N}}$$

[13] Of course, we are technically free to choose the flux $\Phi_\star$ and this will determine the velocity of electrons by $T = \frac{\Phi_0}{2\pi\,\Phi_\star} = \frac{1}{\Phi_\star}$

With

Assumption 2 The chemical potential μ doesn't depend on B.

we are in a position to explain the experimental results:

$$\text{As long as} \qquad \frac{\mu}{(n+1)+1/2} < B \leq \frac{\mu}{n+1/2}$$

there are $n+1$ Landau levels[14] below μ, so the resistance is $R = \frac{R_0}{n+1} = const.$

Within this interval we find $B_{n+1} = \frac{\mu}{(n+1/2)+1/2} = \frac{\mu}{n+1}$
so the Hall resistance comes in steps but will be guided by a proportionality

$$R = \frac{R_0}{\mu}\, B$$

as shown in figure 8.

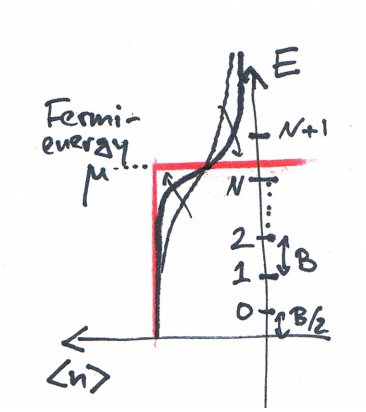


Figure 7: Fermi distribution for low temperatures

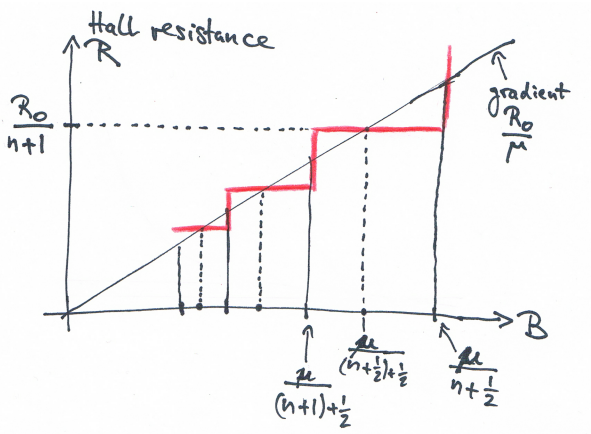


Figure 8: Quantized Hall resistance of the integer QHE

2.3 Outlook towards the Fractional QHE

The Quantum Hall effect becomes *fractional* when Landau levels are only partly filled due to interactions of the electrons. The multi-particle state of N non-interacting electrons filling up the lowest Landau level in symmetric gauge (cf. Section 1.2) is given by a *Slater determinant*

$$\begin{vmatrix} {z_1}^0 & {z_2}^0 & \dots & {z_N}^0 \\ z_1 & z_2 & \dots & z_N \\ \vdots & \vdots & & \vdots \\ {z_1}^{N-1} & {z_2}^{N-1} & \dots & {z_N}^{N-1} \end{vmatrix} = \prod_{i<j}(z_i - z_j)$$

where we have suppressed the $e^{-|z|^2}$ factors. This is a special case of the **Laughlin wavefunction**

$$\Psi_m = \prod_{i<j}(z_i - z_j)^m$$

for filling fraction $m = 1$.
As we will see in the next lecture, Ψ_m is associated with a reduced electron density $\rho \longrightarrow \frac{\rho}{m}$ resulting in a reduced current $I \sim \rho v \longrightarrow \frac{I}{m}$
So our Hall plateaux will appear at

$$R = R_0\, \frac{m}{n}$$

instead of $R = R_0\, \frac{1}{n}$.

[14] including n=0

References

[1] Tong, D. (2016) The Quantum Hall Effect *TIFR Infosys Lectures*
Available at: http://www.damtp.cam.ac.uk/user/tong/qhe.html

[2] Yoshioka, D. (2002) *The Quantum Hall Effect*. Springer

[3] Straumann, N. (2001) *Quantenmechanik*. Springer

[4] Nayak, C. (2004) Quantum Condensed Matter Physics. *Lecture notes*
Available at: http://www.physics.ucla.edu/ ∼ nayak/many_body.pdf

[5] Witten, E. (2016) Three Lectures on Topological Phases of Matter
arXiv:1510.07698v2 [cond-mat.mes-hall]