HEIDELBERG UNIVERSITY

SEMINAR: APPLICATIONS OF TQFT'S SS 2017
LECTURE 7

MENELAOS ZIKIDIS

Abelian Anyons and Fractional Quantum Hall Effect

Contents

1 Motivation

The goal of this lecture is to build a bridge between the theory of Quantum Hall Effect, as we have seen it until now, and the Chern-Simons theories which will be discussed in the upcoming lectures. The pillars of the bridge are built out of the theory of braid statistics and the bridge will lead us directly into the core of Chern-Simons theory. Braid statistics can be found in the Fractional Quantum Hall effect, by introducing singular-like disturbances of the electron density of the quantum Hall fluid and looking at their behaviour under exchange processes.

We start by introducing the mathematics behind Braid-Statistics, their abelian representation theory and then we see how they fit in the theory of the fractional quantum Hall effect.

2 Exchange Statistics and Anyons

2.1 Geometric Braids

We define the **configuration space of N-particles** living on a manifold $\mathcal{M}$, which for our purposes may be connected and locally simply connected, by:

$$\mathcal{C}_N = {(\mathcal{M}^N - \Delta_N)}\big/_{G}$$

where

- $\Delta_N = \{(x_1, \cdots, x_N) | \exists i, j : x_i = x_j\}$, which we subtract, to prevent two or more particles from occupying the same point in our space-manifold.
- G is the group that describes permutation-symmetry among identical particles. For example if we have N-identical particles: $G = \mathfrak{S}_N$, the symmetric group of N-objects.

We give now a definition for the **N-Strand Braid Group on** $\mathcal{M}$ as:

$$\mathcal{B}_N(\mathcal{M}) := \pi_1(\mathcal{C}_N)$$

where an element of the braid group, $[\alpha] \in \mathcal{B}_N(\mathcal{M})$ is the class describing an exchange process, that begins and ends with the same particle configuration up to interchanges of indistinguishable particles. We may visualize the trajectories of $[\alpha(t)]$ in $M \times [0, 1]$. For $\mathcal{M} = \mathbb{R}^2$ we have for example such a process in $\mathcal{B}_4(\mathbb{R}^2)$: The strands are allowed to move in the xy-plane, but the end configuration is confined by the start configuration, as we just explained.

One could think of interpreting this strands as worldlines of particles, and try to understand how the path-integral formalism would react, briefly: evolution operators are

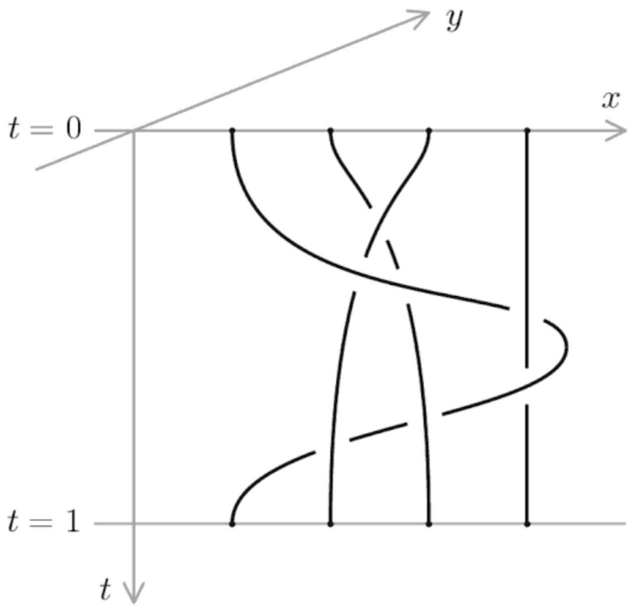

Figure 1: Example of a geometric braid in $\mathcal{B}_4(\mathbb{R}^2)$

given by unitary representations of $\mathcal{B}_N(\mathcal{M})$ and so the propagator splits into contributions from homotopically inequivalent path sectors, labeled by the classes in $\pi_1(\mathcal{C}_N)$. Schematically we have:

$$G(X_b, t_b; X_a, t_a) = \sum_{[\alpha]\in\pi_1(\mathcal{C}_N)} U([\alpha]) \cdot G^{[\alpha]}(X_b, t_b; X_a, t_a)$$

where $[\alpha]$ parametrizes a particle exchange process and $U([\alpha])$ is the operator representation of the statistics of the transformation.

2.2 Representations

We usually think of exchange statistics of our system to be given by 1-dimensional unitary representations of the group that prevails our statistics, namely the Braid group. Although higher representations are of great interest and give rise to theories with non-abelian statistics and superselection rules, we will stick for now to the abelian case and we will discuss those some other time[1].

We have sketched above how a braid in 2 dimensions may look like, and it turns out that we can find abelian braid statistics in their full beauty only in 2-d systems! Lets see why this is the case:

- For $\dim\mathcal{M} \geq 3$ the braid group reduces to its subgroup, the symmetric group, $\mathcal{B}_N(\mathcal{M}) = \mathfrak{S}_N$. This very general result was proven in Fadell[62],but at least in

[1] *Bill Evans: "Some Other Time"* - `https://www.youtube.com/watch?v=WV53dWisQBw`

$\mathbb{R}^3$ one may try to imagine that relative to the 2d case, we have enough degrees of freedom, so that we can homotopically unbraid the strands by lifting them in the third dimension, without cutting through any of them. That would mean that all configurations of worldlines producing the same permutation of particle-positions are homotopically equivalent and this reduces the braid group into the symmetric group.

If we consider the 1-dim representations of $\mathfrak{S}_N$, we have the trivial one, which gives rise to bosonic statistics and the alternating one which gives rise to fermionic statistics. These are exactly the statistics we expect to see in the 3 dimensional world we live in.

- So let us now constrain into 2-dim and set for simplicity $\mathcal{M} = \mathbb{R}^2$:

Here in our exchange statistics, "non-trivial"-"pure" braids statistics, will be playing a role. Before we proceed to the representation theory, we shall give a description of Artin's braid group in terms of generators and relations. $\mathcal{B}_N$ is the group of infinite order generated by halftwists:

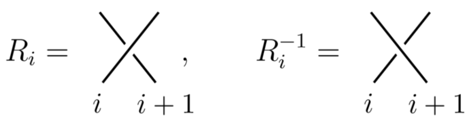

Figure 2: Halftwists-Generators

which obey the relations:

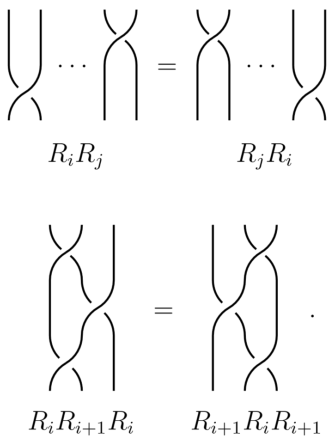

Figure 3: Halftwists-Generators

So the Artin braid group is given as:

$$\mathcal{B}_N = \langle \{R_i\}_{i=1,\cdots,N} | R_i R_j = R_j R_i \; \forall i,j : |i-j| \geq 2 \; ; \; R_i R_j R_i = R_j R_i R_j \rangle$$

One dimensional representations of $\mathcal{B}_N$ are in general of the form : $\mathcal{D}[R_j] = e^{i\theta} \, \forall j$ where θ can take *any* value in $[0, 2\pi)$ So we call these representations ***Any*-ons**, and F.Wilczek is the one we have to blame for the name...

In the early 90's we understood that the only reasonable physical manifestations of braid statistics are Chern-Simons theories, with a few exceptions of some particular $O(3)$ non-linear σ-models which share similar characteristics with the Chern-Simons theories (their Hopf-terms are equivalent to C.S.-theories), see for example Fröhlich[3]. But now it is time to go back and consider how this kind of statistics fits into QHE.

3 Fractional Quantum Hall Effect

3.1 Quasi-Holes

We consider excitations of the $\nu = \frac{1}{m}$- Laughlin wavefunction.

- The initial ground-state wavefunction we introduced in the last lecture by Roman, had the form:

$$\psi(z) = \prod_{k \leq l} (z_k - z_l)^m e^{-\sum_i^N \frac{|z_i|^2}{4 l_B^2}}$$

  where $l_B = \sqrt{\frac{\hbar}{B \cdot e}}$.

- By disturbing/ "creating a hole" in the electron density distribution at a point $\eta \in \mathbb{C}$, we get a new factor in the wavefunction:

$$\psi(z;\eta) = \prod_i^N (z_i - \eta) \prod_{k \leq l} (z_k - z_l)^m e^{-\sum_i^N \frac{|z_i|^2}{4 l_B^2}}$$

  We say that this wavefunction describes a quasi-hole at the point $\eta \in \mathbb{C}$.

- In the same manner we can create m-such quasi holes at points $\{\eta_j\}_{j=1,\cdots,m}$ and we obtain a wavefunction:

$$\psi = \prod_j^m \prod_i^N (z_i - \eta_j) \prod_{k \leq l} (z_k - z_l)^m e^{-\sum_i^N \frac{|z_i|^2}{4 l_B^2}}$$

- Now lets suppose that we bring all m quasi-holes at one point:

$$\psi(z;\eta) = \prod_i^N (z_i - \eta)^m \prod_{k \le l} (z_k - z_l)^m e^{-\sum_i^N \frac{|z_i|^2}{4l_B^2}}$$

What we observe is that the first product here describes the deficit of an electron at η, where η is a non- dynamical variable. Heuristically, one may say that each one of the m-quasi-holes behaves as $\frac{1}{m}$th of an electron and carries fractional charge $\frac{+e}{m}$, (where we take the charge of the electron to be $-e$).

- Interpreting this wavefunction in terms of the plasma analogy, which was also introduced in the last lecture, we obtain a potential for a quasi-hole/ impurity in the plasma:

$$\mathcal{U} = -m^2 \sum \log\left(\frac{|z_i - z_j|}{l_B}\right) + \frac{m}{4l_B^2} \sum_i^N |z_i|^2 - m \sum \log\left(\frac{|z_i - \eta|}{l_B}\right)$$

  Where the first two terms build the potential for the pure plasma, with no impurities, and the last term gives the particle-quasi hole interaction. As we have discussed last time, the electrons, as particles in the quantum Hall fluid carry charge $q = -m$, so the impurities carry charge: $1 = -\frac{q}{m}$. Therefore the effective charge missing from the fluid is $\frac{1}{m}$ of an electron. We will use the plasma analogy to see if there is any physical validity in the heuristic we have discussed in the last point.

3.2 The Berry-Connection in the plasma analogy

Let $|\eta\rangle = |\eta_1, \cdots, \eta_M\rangle$ be a state of M-quasi-holes, that is :

$$\langle z, \bar{z}|\eta\rangle = \prod_j^M \prod_i^N (z_i - \eta_j) \prod_{k \le l} (z_k - z_l)^m e^{-\sum_i^N \frac{|z_i|^2}{4l_B^2}}$$

which we normalize to: $|\psi\rangle = \frac{1}{\mathcal{Z}} |\eta\rangle$, with :

$$\mathcal{Z} := \langle \eta|\eta\rangle = \int \prod d^2 z_i \ \exp\left[\sum_{ij} \log\left(|z_i - z_j|^2\right) - \frac{1}{2l_B^2} \sum_i^N |z_i|^2 + m \sum_{ij} \log\left(|z_i - \eta_j|^2\right)\right]$$

Which plays the role of the partition function in the plasma analogy in the presence of impurities at points $\{\eta_j\}_j$. Now we compute for this model the holomorphic Berry-connection :

$$\mathcal{A}_\eta(\eta, \bar{\eta}) = -i \langle \psi| \frac{\partial}{\partial \eta} |\eta\rangle = \frac{i}{2\mathcal{Z}} \frac{\partial \mathcal{Z}}{\partial \eta} - \frac{i}{\mathcal{Z}} \langle \psi| \frac{\partial}{\partial \eta} |\eta\rangle$$

and we can rewrite this expression for the connection just in terms of $\mathcal{Z}$ and its derivatives by:

$$\frac{\partial \mathcal{Z}}{\partial \eta} = \frac{\partial}{\partial \eta} \langle \eta|\eta\rangle \xrightarrow{\langle \eta| \text{ is antihol.}} \frac{\partial \mathcal{Z}}{\partial \eta} = \langle \psi| \frac{\partial}{\partial \eta} |\eta\rangle$$

So we have for the holomorphic and antiholomorphic Berry-connections:

$$\mathcal{A}_{\eta}(\eta, \bar{\eta}) = -\frac{i}{2\mathcal{Z}}\frac{\partial \mathcal{Z}}{\partial \eta} = \frac{i}{2}\frac{\partial \log(\mathcal{Z})}{\partial \eta}$$
$$\mathcal{A}_{\bar{\eta}}(\eta, \bar{\eta}) = \frac{i}{2}\frac{\partial \log(\mathcal{Z})}{\partial \bar{\eta}}$$

Now we are left with the dirty job to calculate $\frac{\partial \log(\mathcal{Z})}{\partial \eta}$. Instead of performing this task by brute force, we will use some physical intuition to simplify it. In the plasma analogy, there is an effect taking place, called screening. There, mobile particles of the quantum Hall fluid, cluster around the impurities, so that they effectively screen the quasi-hole-potential. This means that from great distances we can not tell the exact position of the quasi-holes. Therefore the free energy $\mathcal{F}$ of the system must be independent of the exact positions $\{\eta_j\}_j$. We know for the free energy that $\mathcal{F} \propto \log(\mathcal{Z})$, but for an equality to hold, there are two ingredients to be added:

1. The energy cost between the impurities and the constant background charge.
2. The Coulomb-interaction energy between the quasi-holes.

These considerations lead us to a "corrected" potential for the plasma with M-impurities:

$$\begin{aligned}\mathcal{U}' &= \mathcal{U} + \frac{1}{4l_B^2}\sum_i^M |\eta_i|^2 - \sum \log\left(\frac{|\eta_i - \eta_j|}{l_B}\right) \\ &= -m^2 \sum_{i,j} \log\left(\frac{|z_i - z_j|}{l_B}\right) + \frac{m}{4l_B^2}\sum_i^N |z_i|^2 - m\sum_{i,j}\log\left(\frac{|z_i - \eta_j|}{l_B}\right) \\ &\quad + \frac{1}{4l_B^2}\sum_i^M |\eta_i|^2 - \sum \log\left(\frac{|\eta_i - \eta_j|}{l_B}\right)\end{aligned}$$

We can also write down the new partition function, for the corrected potential:

$$\mathcal{C} := \int \prod_i dz_i e^{-\beta \cdot \mathcal{U}'(\{z_i\}_i, \{\eta_j\}_j)} = \exp\left(\frac{1}{2ml_B^2}\sum_i^M |\eta_i|^2 - \frac{1}{m}\sum \log\left(|\eta_i - \eta_j|^2\right)\right) \cdot \mathcal{Z}$$

From the screening-argument we have that the free energy, and therefore also the corrected partition function $\mathcal{C}$, are independent from the exact position of the impurities and this will allow us to calculate $\frac{\partial \log(\mathcal{Z})}{\partial \eta}$ easily, as:

$$\mathcal{Z} = \mathcal{C} \cdot \exp\left(-\frac{1}{2ml_B^2}\sum_i^M |\eta_i|^2 + \frac{1}{m}\sum \log\left(|\eta_i - \eta_j|^2\right)\right)$$

$$\Longrightarrow \begin{cases} \mathcal{A}_{\eta_i} = -\frac{i}{2m}\sum_{j\neq i}\frac{1}{\eta_i - \eta_j} + i\frac{\bar{\eta}_i}{4ml_B^2} & \text{Holomorphic Berry-Conncetion} \\ \mathcal{A}_{\bar{\eta}_i} = +\frac{i}{2m}\sum_{j\neq i}\frac{1}{\bar{\eta}_i - \bar{\eta}_j} - i\frac{\eta_i}{4ml_B^2} & \text{Anti-Holomorphic Berry-Conncetion}\end{cases}$$

We will use now the computed Berry-Connection to discuss explicitly the charge and the statistics of the quasi-holes.

3.3 Fractional Charge

Let us choose a generic quasi-hole from our system at a point η and we take that hole around a closed path $\mathcal{P}$, which does not enclose any other quasi-hole. We calculate the geometric phase of the process with respect to the Berry-Connection:

$$e^{i\gamma} = \exp\left(-i\oint_{\mathcal{P}} \mathcal{A}_{\eta} d\eta - i\oint_{\mathcal{P}} \mathcal{A}_{\bar{\eta}} d\bar{\eta}\right)$$

Under the above assumption for the chosen path, the relevant part of the connection reduces to:

$$\mathcal{A}_{\eta_i} = i\frac{\bar{\eta}_i}{4ml_B^2} \ ; \ \mathcal{A}_{\bar{\eta}_i} = -i\frac{\bar{\eta}_i}{4ml_B^2}$$

Integration of the form:

$$\oint_{\mathcal{P}} \bar{\eta} d\eta = A$$

gives us just the area A of the enclosed surface. So we obtain in that case, a geometric phase of:

$$\gamma = \frac{e \cdot \Phi_o}{m \cdot \hbar} \cdot A = \frac{e \cdot \Phi}{m \cdot \hbar}$$

where $\frac{e \cdot \Phi_o}{m \cdot \hbar}$ comes from $\frac{1}{4ml_B^2}$ as we have per definition $l_B = \sqrt{\frac{\hbar}{B \cdot e}}$.

This looks somehow familiar! Lets take a step back an remember, the basic example for a geometric phase, which was considered in the context of the Aharonov-Bohm model. There we calculated the Berry-phase of the process, of taking a particle of charge q around a path α, enclosing a magnetic flux Φ. Where we have taken the Berry connection to be proportional to the electromagnetic gauge potential **A**, namely:

$$\mathcal{A}(\mathbf{x}) = \frac{e}{\hbar} \cdot \mathbf{A}.$$

Which gave as a phase:

$$e^{i\gamma'} = \exp\left(-i\frac{e}{\hbar}\oint_{\alpha} \mathbf{A}(\mathbf{x}) d\mathbf{x} \implies \gamma' = \frac{q \cdot \Phi}{\hbar}\right).$$

By comparing this result with the one obtained from the quantum Hall fluid we have:

$$\gamma' = \gamma \implies q = \frac{e}{m}$$

which is result showing an actual physical manifestation of the heuristic, of "fractional charge", we discussed before just by looking at the form of the Laughlin wavefunction.

3.4 Fractional Statistics

Now we describe the second case we left open in the discussion above, namely what happens if we take a quasi-hole around a path $\tilde{\mathcal{P}}$ that encloses another quasi-hole. So lets take a quasi-hole η_1 around a closed loop, that encloses the quasi-hole η_2. Now we have also a contribution of the first term appearing in the Berry-connection, and that is exactly the term describing the statistics of quasi-holes:

$$e^{i\gamma} = \exp\left(-\frac{1}{2m} \underbrace{\oint_{\tilde{\mathcal{P}}} \frac{d\eta_1}{\eta_1 - \eta_2}}_{=-2\pi i} + \frac{1}{2m} \underbrace{\oint_{\tilde{\mathcal{P}}} \frac{d\bar{\eta}_1}{\bar{\eta}_1 - \bar{\eta}_2}}_{=2\pi i} \right) = \underbrace{e^{\frac{2\pi i}{m}}}_{\text{statistics}}$$

Looking at the exchange statistics of a wavefunction describing two particles, we have under position exchange:

- $\psi(r_1, r_2) = e^{i\pi\theta}\psi(r_2, r_1)$ after one exchange and
- $\psi(r_1, r_2) = e^{2\pi i\theta}\psi(r_1, r_2)$ after exchanging the positions twice.

We can easily convince ourselves, that in 2 dimensions, exchanging two particles twice, is the same as taking one particle around the other, in braid-notation:

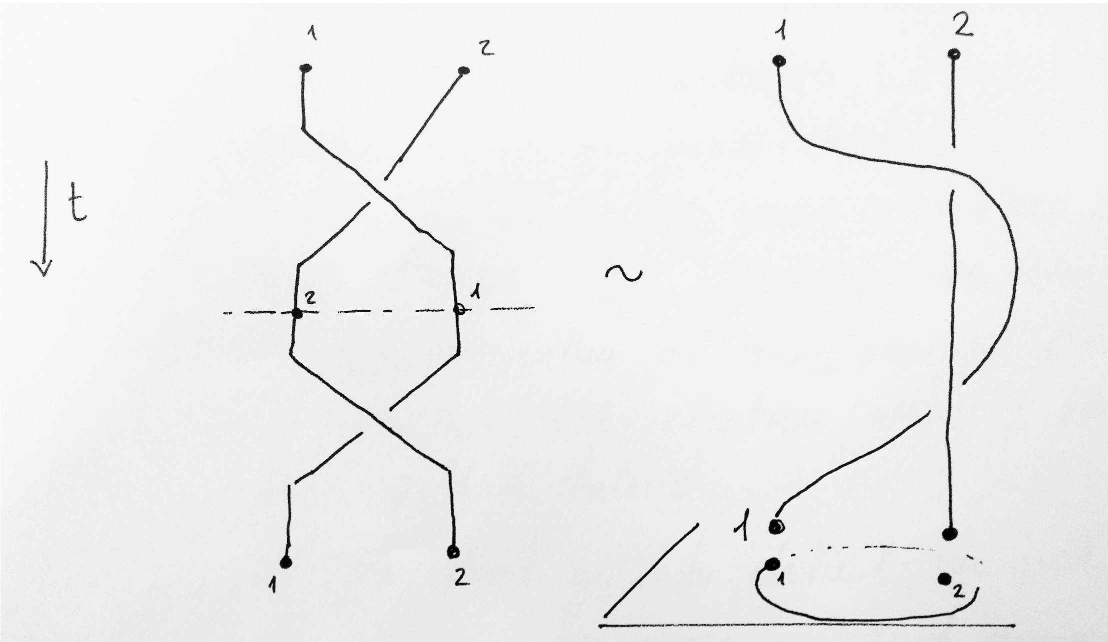

Figure 4: Particle exchange-twice

But we have already calculated the phase for a quasi-hole taking a closed path once around an other one, so we have:

$$e^{2\pi i\alpha} = e^{\frac{2\pi i}{m}} \implies \alpha = \frac{1}{m}$$

This means that for a fully filled Landau level, m=1, the quasi-holes are of fermionic nature, but in general, for a fractional quantum Hall state, the quasi-holes are anyons.

3.5 Ground State Degeneracy

Here we will discuss a property of the Fractional quantum Hall effect which first becomes apparent when the system lives on a compact manifold. Namely, that the number of ground states depends on the topology of the ambient manifold!

Remark:Until now we have introduced only the concept of quasi holes, but these have dual-cousins, the **quasi-particles** . These respect the exact same statistics but have charge $\frac{-e}{m}$ and increase the electron density in the quantum Hall fluid, and therefore they decrease the relative angular momentum. We will not go into detail but we just mention that we may produce pairs of quasi-holes/quasi-particles, which then can be again annihilated.

Let's consider now the following process taking place on the torus T^2: We produce a quasi-hole/quasi-particle pair on T^2 and we may think of these as actual punctures on the surface of the torus. The topology of the punctured torus is given by:

$$\pi_1(T^2 - pt) = \mathbb{Z} \star \mathbb{Z}$$

following the Seifert-van Kampen recipe. Analogously we get higher products for more punctures... This amalgamated product is per construction non-abelian.

- Take an operation T_1 to be the production of a pair, then taking the quasi- particle around the meridian cycle once and annihilating it with the quasi-hole.
- Analogously, take the operation T_2 to be the production of a pair, then taking the quasi- particle around the horizontal cycle once and annihilating it again with the quasi-hole.

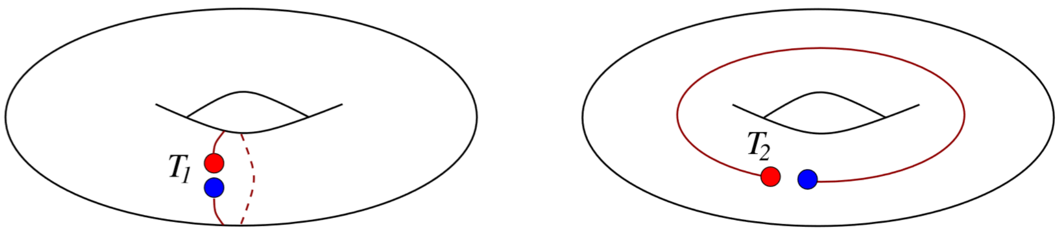

Figure 5: Operations T_1 and T_2

Because $\pi_1(T^2 - \{pt, ..\})$ is non-abelian, we have for the commutator of the generators T_1 and T_2 that $[T_1, T_2] = T_1 \cdot T_2 \cdot T_1^{-1} \cdot T_2^{-1} \neq 1$.

We can now interpret the commutator as the process taking the quasi-particle around the quasi-hole. This is explained by the following sketch of the fundamental polytope of the punctured torus:

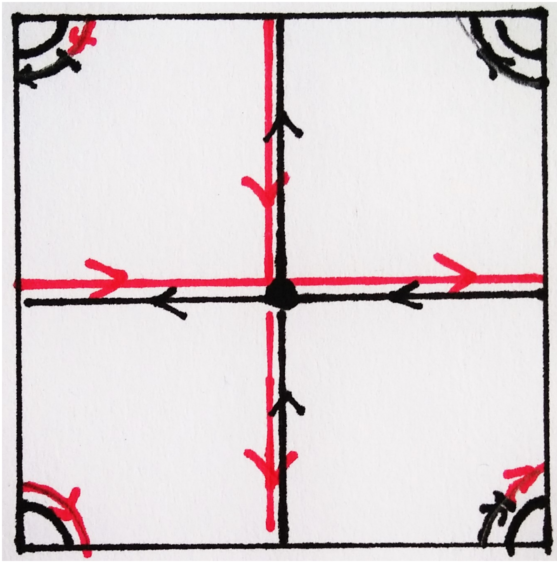

Figure 6: Fundamental polytope and cycles

Here we sliced-open the torus exactly at the point where the quasi-hole was and "we sliced the quasi-hole in four ". Then we performed the process described by the commutator on the quasi-hole, which lies at first in the middle of the polytope, and deformed-retracted all paths taken, up to the quasi-hole-singularity. Under this interpretation of the commutator and from the statistics of quasi-holes and quasi-particles, which we have already computed explicitly, we know that for a ground state $|\Omega\rangle$ must hold:

$$[T_1, T_2]\,|\Omega\rangle = e^{\frac{2\pi i}{m}}\,|\Omega\rangle\,.$$

So we obtain an algebra generated by T_1 and T_2 and respecting the commutation relation above:

$$\Xi = \langle T_1, T_2 | [T_1, T_2] = e^{\frac{2\pi i}{m}} \rangle\,.$$

What we also observe is that such an algebra of operators cannot be realized on a single ground-state $\implies$ Ground-state degeneracy! In particular the lowest dimensional representation of Ξ is m-dimensional and is isomorphic to:

$$T_1\,|n\rangle = e^{\frac{2\pi i}{m}}\,|n\rangle \;\; ; \;\; T_2\,|n\rangle = |n+1\rangle\,.$$

References

[1] *Non-Abelian Anyons and Topological Quantum Computation* -Sankar Das Sarma, Michael Freedman, Chetan Nayak, Steven H. Simon, Ady Stern

[2] *Quantum Groups,Quantum Categories and Quantum Field Theory*- J.Fröhlich,T.Kerler

[3] *Superselection Structure and Statistics in Three- Dimensional Quantum Field Theory* - J.Fröhlich, F.Gabbiani and P.A.Marchetti

[4] *Non-Abelian Anyons and Interferometry*-Parsa Hassan Bonderson

[5] *Lectures on the Quantum Hall Effect* -D.Tong

[6] *Topological Study of Field Theories* - C.T. Ann

[7] *Fractional Statistics and the Quantum Hall Effect*-D.Arovas, J.R.Schrieffer and F.Wilczek

[8] *Braid Groups (Graduate Texts in Mathematics)* -C.Kassel, V.Turaev