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Physical Applications of Topological Quantum Field Theory

The Cobordism Hypothesis

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Contents

Before it gets started, I would like to repeat our terminology. If there stands a manifold, it is meant to be a smooth compact oriented manifold, and usually it has the boundary. If there stands "$Cob(d)$", this shall denote the category, whose objects are given by $(d-1)$ closed manifolds, and the morphisms shall be given by $Hom(M,N) \cong \{B : \partial B = M^* \sqcup N\}$, where M^* denotes that the orientation of M has been switched. The composion law is obtained by gluing of bordisms.

1 Introduction

I would like to start this section by stating the original definition of d-dimensional TQFT, which is suggested by Atiyah.

Definition 1.0.1. *A TQFT (of dimension d) is a tensor functor*

$$Z : Cob(d) \to Vect_{\mathbb{C}} \ .$$

The expression "tensor functor" means that the following relation holds with units

$$Z(M \sqcup N) = Z(M) \otimes Z(N) \ ,$$
$$Z(\varnothing) = \mathbb{C} \ .$$

Example 1.0.2. *d=1.*

As we recall that the objects of the category $Cob(2)$ are simply closed 1-manifolds, where every closed 1-manifold can be decomposed in a finite disjoint union of circles. It would be enough to identify the value of $Z(S^1)$. Through the first two talks, we had characterized this value. Note that Z is a functor from the category $Cob(2)$ to the category $Vect_{\mathbb{C}}$ by Atiyah's definition. Then we had assigned $Z(S^1)$ to the Frobenius algebra A. i.e.

$$Z(S^1) = A \ .$$

We have quite well understood what happens in the case of the dimension $d = 2$. So the natural question is then, what if d is large? For some readers who might not be interested in the mathematics, this can easily be an useless thoughts. However, mathematics has its own rule for building a computational model where all logically possible computations work well without any contradictions.

2 Extended TQFT's

Example 2.0.1. *Let M be a closed d-manifold.*

Note that M can be seen as a bordism from an empty set $\varnothing$ to $\varnothing$ itself through M. This allows us to observe,

$$Z(M) = \cdot\lambda : Z(\varnothing) \simeq \mathbb{C} \to Z(\varnothing) \simeq \mathbb{C} \,,$$

since we need to assign a $\mathbb{C}$-linear maps. By abusing the notation, we may identify $Z(M) = \lambda$. Moreover, we wish this $Z(M)$ be an invariant for every closed d-manifold M, which associates a value to M. The golden goal with the cobordism hypothesis is computing these values by cutting a manifold into very small pieces. The Problem is, if we cut a d-manifold with submanifolds of codimension 1, this decomposition might be in $(d-1)$- sences not enough small. For $d = 2$ this cutting method worked quite well; we would always get some cups, caps, and pairs of pants. For higher d, we need more than that. If the cutting is still large in $(d-1)$- senses, we would require to cut those $(d-1)$ pieces with $(d-2)$-cutters, and so forth until we get pieces in the form of "points". This yields us to come up with an elaborate definition of TQFT. [BD95].

Definition 2.0.2. *(Sketch) An extended TQFT (of dimension d) is a following rule which associates,*

given some date	*data you associate*
closed d-manifolds	*complex numbers*
closed $(d-1)$-manifolds	*complex vector spaces*
bordisms between $(d-1)$-manifolds	*linear maps of corresponding vector spaces*
closed $(d-2)$-manifolds	*so-called $\mathbb{C}$-linear categories*
bordisms between $(d-2)$-manifolds	*$\mathbb{C}$-linear functors*

In other words, an extended TQFT is a tensor functor between "d-categories".

[BD95] At this point, we obtain a statement of the cobordism hypothesis informally:

(Baez-Dolan) Extended TQFT's are "easy" to describe/ construct/ classify.

We find out a special case where the extended TQFT's have no difference from the ordinary TQFT's. This is exactly the case where $d = 1$.

Example 2.0.3. *[Tel12], [Tel16].* $d = 1$.
Given an (extended) TQFT

$$Z : Cob(1) \to Vect_{\mathbb{C}} .$$

There is a particularly interesting object in this category $Cob(1)$. Recall that the objects in $Cob(1)$ are closed 0-manifolds, namely points. In fact, there are two of our interesting objects, since our manifolds are oriented. Let $Z(+)$ is given by a $\mathbb{C}$-vector space X. From the involutionary axiom of Atiyah, we then get $Z(-) = X^\vee$. Thus follows

$$Z(\supset) = \; ev \; : \; X \otimes X^\vee \to \mathbb{C} \quad , \quad Z(\subset) = \; coev \; : \mathbb{C} \to X \otimes X^\vee .$$

Note that if X is finite dimensional, then $X^\vee$ is also finite dimensional. These maps implies some kind of relationship between adjoint objects and dual objects. We are about to consider the morphisms in this category $Cob(1)$. They are given by a closed 1-manifold, where every closed 1-manifold can be written as a disjoint union of connected 1-manifolds, i.e. circles. We may interpret a circle as a morphism from an empty set through two opposite oriented points and finally to the empty set again.

$$Z(\varnothing) \to Z(:) \to Z(\varnothing)$$

Note that the first map is given by the inclusion by unit, whereas the second one is given by the trace map. This implies the value of a circle $Z(S^1)$ is equal to $dim(X)$. Since we have a canonical isomorphism

$$X \otimes X^\vee \simeq End(X) \; ,$$

for all finite dimensional vector space X and due to the symmetry, that is, if we perform the morphism with the circle in opposite direction, our functor Z will assign the circle to $dim(X^\vee)$, we must require the condition that X be a finite dimensional.

3 TQFT's in higher dimensions

The very simple idea of the cobordism hypothesis is that a field theory should be determined essentially by what it does on a point. For higher dimensions there are two obstructions to realize the idea. Cleary we want to state : [L+09], [Tel16]

In all dimensions, an extended TQFT Z is determined by $Z(pt)$.

3.1 Orientation Issue

If we want to know what a field theory does on a complicated manifold in a large dimension, we should be able to assemble that from the local informations. Thus we "frame" the manifold. For $d = 1$, there is no difference between giving orientation and giving a frame, for large d, the difference comes out, simply because the manifolds must not have the trivial tangent bundles, in fact, they may have nontrivial tangent bundles. To address this, we require to modify the definition of our extended TQFT. [L+09], [Tel16].

Let us start with the definition of d-framing.

Definition 3.1.1. *Let M be a manifold of dimension m. Let $m \leq d$. A d-framing of M is a trivialization of its tangent bundle by adding a trivial bundle of rank $(d - m)$. That is, the bundle isomorphism Φ over M,*

$$\Phi : TM \oplus \mathbb{R}^{d-m} \cong \mathbb{R}^d \ .$$

Definition 3.1.2. *(Sketch) $Cob(d)^{fr}_{ext}$ is a d-category, whose*

objects	*0-dimensional manifolds with a d-framing*
morphisms	*d-framed bordisms between d-framed 0-manifolds*
2-morphisms	*d-framed bordisms between d-framed 1-manifolds*
$\cdots$	$\cdots$
d-morphisms	*d-framed bordisms between d-framed $(d-1)$-manifolds*

especially, d-morphisms in this category are equal to d-manifolds with corners upto the diffeomorphisms related to boundary.

Remark 3.1.3. *In the geometrical view, the extra $\mathbb{R}$ summands on the boundary ∂M are the inward or outward normals to ∂M, according to the direction of the bordism. This makes it clear that a manifold with corners can be read as a morphism in many ways. Moreover there are exactly two d-framed points $+, -$ in $Cob(d)^{fr}_{ext}$ upto the isomorphisms. They are distinguished by determinant sign of the framing $\mathbb{R}^d \cong \mathbb{R} \oplus T(pt)$.*

cobordism hypothesis, slightly modified for the sake of the orientation issue

Let $\mathcal{C}$ be any d-category with a tensor product. Then the followings are equal:

(i)

$$Z : Cob(d)^{fr}_{ext} \to \mathcal{C} \quad \mathcal{C}\text{-valued TQFT's },$$

(ii)

There exists an object X in $\mathcal{C}$, such that $X = Z(pt)$ holds.

Yet this is not correct even with the case $d = 1$. Thus arises the second issue at this point, which is strongly related to the second assertion.

3.2 Second Issue

It is not surprising that not every object in $\mathcal{C}$ can appear as $Z(pt)$. As we have seen in the first section, we assign a point to a finite dimensional vector space X under Atiyah's definition. Similarly enough, if we consider an arbitrary category $\mathcal{C}$, we require some kind of "finiteness". This gives us almost complete statement of cobordism hypothesis, yet is still vague in some senses.

cobordism hypothesis, [L+09].

Let $\mathcal{C}$ be any d-category with a tensor product. Then the followings are equal:

(i)

$$Z : Cob(d)^{fr}_{ext} \to \mathcal{C} \quad \mathcal{C}\text{-valued TQFT's },$$

(ii)

There exists a "fully dualizable" object X in $\mathcal{C}$, such that $X = Z(pt)$ holds.

To summarize, the cobordism hypothesis tells us that giving a fully dualizable object in any category $\mathcal{C}$ is equivalent to give an extended TQFT to be found on a framed manifold.

4 Cobordism Hypothesis in general Dimension

As the dimension gets higher, the more elaborate tools are required. Most of definitions and examples consists of only humble sketch. Readers interested in details, please look up the literatures, such as [FHLT09], [L+09], [Tel12], and [Tel16]

4.1 Higher Category

Definition 4.1.1. *A (strict) d-category $\mathcal{C}$ consists of the following data:*

(i) A collection of objects,

(ii) for every pair of objects there is a collection $Hom_{\mathcal{C}}(X, Y)$ given by $(d-1)$ category,

(iii) the composition law is constructed by a $(n-1)$-functor,

$$functor : Hom_{\mathcal{C}}(X, Y) \times Hom_{\mathcal{C}}(Y, Z) \to Hom_{\mathcal{C}}(X, Z)\ ,$$

(iv) there holds the associativity and exist units.

Example 4.1.2. *The fundamental 2-groupoid $\pi_{\leq 2}X$ of a topological space X is a 2-category.*

objects	*points of X*
morphims	*paths in X*
2-*morphisms*	*homotopies between paths upto homotopic equivalence*
composition law	*concatenation upto homotopic equivalence*

The concatenation is not associative in strong sense as path, but it is associative upto the homotopic equivalence.

Example 4.1.3. *The fundamental d-groupoid $\pi_{\leq d}X$ is a d-category.*

objects	*points of X*
1-*morphisms*	*paths in X*
2-*morphisms*	*homotopies between paths*
3-*morphisms*	*homotopies between homotopies*
$\cdots$	$\cdots$
d-morphisms	*d-fold homotopies upto homotopic equivalence*

Definition 4.1.4. *A d-groupoid is a d-category where all of its k-morphisms are invertible at all levels. i.e. Every morphism is an isomorphism.*

Let us consider the direct limit of the fundamental d-groupoid of X,

$$\lim_{d\to\infty} \pi_{\leq d}X = \pi_{\leq\infty}X ,$$

to introduce the notion of infinite categories. I used an abused notation above only to denote, or to deliver the concept that we consider the case where the top level d goes to infinity. An ∞-groupoid is an ∞-category, whose morphisms are invertible at all levels. The (∞, d)-category is defined similarly. [Bae97].

Definition 4.1.5. *An* <u>(∞, d)*-category*</u> *is a higher category where all k-morphisms are invertible for $k > d$. i.e. An (∞, d)-category $\mathcal{C}$ consists of the following data:*

(i) a collection of objects,

(ii) for every pair of objects there exists an $(\infty, d-1)$ category of morphisms between them,

(iii) the composition law holds for morphisms,

(iv) and the associativity holds upto isomorphism.

Definition 4.1.6. *We define $Bord_d$ as the (∞, d)-category of bordisms, whose*

objects	0*-manifolds*
1*-morphisms*	1*-manifolds with boundary*
2*-morphisms*	2*-manifolds with corners*
$\cdots$	$\cdots$
d*-morphisms*	n*-manifolds with corners*
$(d+1)$*-morphisms*	*diffeomorphisms between* d*-manifolds*
$(d+2)$*-morphisms*	*isotopies between* $(d+1)$*-morphisms*
$\cdots$	
upto ∞	

Note that bordisms between k-manifolds are equivalent to $(k+1)$-manifolds with boundary/ corners. $Bord_d^{fr}$ is defined with d-framed manifolds. Moreover, $Bord_d$ is a symmetric monoidal (∞, d)-category. i.e. there exists a symmetric tensor product.

4.2 Cobordism Hypothesis, Jacob Jurie

In this section, I would like to introduce the statement of the cobordism hypothesis as stated in [L+09], which has now become a theorem of Jacob Lurie, by using the notion of (∞, d)-category.

Theorem 4.2.1. *[L+09]. Let $\mathcal{C}$ be a symmetric monoidal (∞, d)-category. Then it holds*

$$Fun^{\otimes}(Bord_d^{fr}, \mathcal{C}) \simeq \{\textit{fully dualizable objects of } \mathcal{C}\} ,$$

$$Z \mapsto Z(pt) .$$

If $\mathcal{C} = Vect_{\mathbb{C}}$, an object $V \in \mathcal{C}$ is fully dualizable if and only if V is finite dimensional. Its dual object is given by $V^{\vee} = Hom(V, \mathbb{C})$. Let V be finite dimensional, in particular $V \otimes V^{\vee} \cong End(V) \cong V^{\vee} \otimes V$ in this case. Then the following maps are compatible;

$$V \xrightarrow{id \otimes coev} V \otimes V^{\vee} \otimes V \xrightarrow{ev \otimes id} V$$

$$V^{\vee} \xrightarrow{coev \otimes id} V^{\vee} \otimes V \otimes V^{\vee} \xrightarrow{id \otimes ev} V^{\vee}$$

Note that those maps are identity maps. This compatibility generalizes the fully dualizable condition.

Definition 4.2.2. *[L+09]. In an arbitrary symmetric monoidal (∞, d)-category $\mathcal{C}$, an object $X \in \mathcal{C}$ is fully dualizable, if there exists an object $X^{\vee}$ in $\mathcal{C}$ such that there exist the ev map and coev map, for which the above compatibility holds upto the homotopy.*

Remark 4.2.3. *If $\mathcal{C}$ is a symmetric monoidal $(\infty, 1)$-category, an object $X \in \mathcal{C}$ is fully dualizable if it is dualizable.*

The above compatibility can be considered with the notion of adjoint functors. By studying adjoint functors, we may generalize and precisely understand the compatibility.

Definition 4.2.4. *Let $\mathcal{C}, \mathcal{D}$ be categories. Functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ are said to be adjoint to one another if there exists natural bijection*

$$Hom_{\mathcal{D}}(Fc, d) \simeq Hom_{\mathcal{C}}(c, Gd)\ .$$

It is said for F to be left adjoint, for G to be right adjoint, where c denotes an object in $\mathcal{C}$, d an object in $\mathcal{D}$.

Taking $d = Fc$, we can construct a map

$$\Theta :\ Hom_{\mathcal{D}}(Fc, d) \to Hom_{\mathcal{C}}(c, Gd)\ ,$$

$$id_{Fc} \mapsto \Big(c \mapsto (G \circ F)(c)\Big)\ .$$

In other words, we have a natural transformation

$$u :\ id_{\mathcal{C}} \to G \circ F\ .$$

We can use u to recover maps.

$$\begin{array}{ccc} Hom_{\mathcal{D}}(Fc, d) & \longrightarrow & Hom_{\mathcal{C}}(G \circ Fc, Gd) \\ & & \Big\downarrow u \\ & & Hom_{\mathcal{C}}(c, Gd) \end{array}$$

To guarantee the bijectivity of our map Θ, we find the inverse map. We get the map

$$v :\ F \circ G \to id_{\mathcal{D}}\ ,$$

by taking $c = Gd$ just as we got the natural transform u. Moreover, v induces the inverse map of Θ. Then, how should u and v be related? Similar as the very first observation with ev and $coev$ maps, we may consider that "adjoints are like duals".

$$F \xrightarrow{id \times u} F \circ G \circ F \xrightarrow{v \times id} F$$

$$G \xrightarrow{u \times id} G \circ F \circ G \xrightarrow{id \times v} G\ .$$

Definition 4.2.5. *Given $\mathcal{C}$ an (∞, d)-category, we say that $f : X \to Y$, and $g : Y \to X$ are adjoint, if there exists $u : id \to g \circ f$, and $v : f \circ g \to id$ such that the above relation holds upto isomorphisms.*

Definition 4.2.6. *[L$^+$09]. An (∞, d)-category $\mathcal{C}$ has adjoints if*

- *always holds if $d = 1$,*
- *for $d \geq 2$ every morphism in $\mathcal{C}$ has a left and right adjoints,*
- *if $d > 2$, $Hom_{\mathcal{C}}(X, Y)$ has adjoints, i.e. there exists adjoints upto the top level.*

Definition 4.2.7. *[L$^+$09]. Let $\mathcal{C}$ be a symmetric monoidal (∞, d)-category. $\mathcal{C}$ has duals if*

(i) $\mathcal{C}$ has adjoints,

(ii) every object in $\mathcal{C}$ has a dual.

Note that $Bord_{\mathcal{C}}^{fr}$ has duals, where they are obtained by changing the direction of the framing). But, what if $\mathcal{C}$ does not have duals? For any $(\infty, d)^{\otimes}$-category $\mathcal{C}$, there exists "largest" subcategory $\mathcal{C}^{fd}$ in $\mathcal{C}$, such that every object in $\mathcal{C}^{fd}$ are fully dualizable.

Definition 4.2.8. *[L$^+$09]. An object $X \in \mathcal{C}$ is fully dualizable if it belongs to $\mathcal{C}^{fr}$.*

By the equivalence of ∞-groupoids, we get the following statement, while we drop all non-invertible morphisms out.

$$Fun^{\otimes}(Bord_d^{fr}, \mathcal{C}) \simeq Fun^{\otimes}(Bord_d^{fr}, \mathcal{C}^{fd}) \simeq \{\text{objects in } \mathcal{C}^{fd}\} .$$

Example 4.2.9. *$d = 1$. $(\infty, 1)$-category $\mathcal{C}$: throw out all objects which doesn't have duals. Thus, an object $X \in \mathcal{C}$ is fully dualizalble if it is dualizable.*

Example 4.2.10. *[L$^+$09]. $d = 2$. An object $X \in \mathcal{C}$ is fully dualizable, if and only if*

- *X is dualizable,*
- *ev, $coev$ both have adjoints.*

Note that one can obtain ev^{adj} by having ${}^{adj}coev$.

Recall that d-framing for a n-dimensional manifold M is an isomorphism,

$$TM \oplus \mathbb{R}^{d-n} \cong \mathbb{R}^d \ .$$

We may observe that the orthogonal linear group $O(d)$ acts on $\mathbb{R}^d$, and consequently on d-framing of any manifolds. This defines an action

$$O(d) \curvearrowright Bord_d^{fr} \ .$$

From the cobordism hypothesis $Fun^{\otimes}(Bord_d^{fr}, \mathcal{C}) \simeq \{\text{objects in } \mathcal{C}^{fd}\}$, we obtain an action, [L+09],

$$O(d) \curvearrowright \{\text{objects in } \mathcal{C}^{fd}\} \ ,$$

seeing $\{\text{objects in } \mathcal{C}^{fd}\}$ as a topological space. More explicitly with the case $d = 1$, let an $(\infty, 1)$-category $\mathcal{C}$ have duals. Then every object $X \in \mathcal{C}$ has a dual $X^\vee$. The above observation tells us that the map $X \mapsto X^\vee$ gives an action of $O(1)$ on $\{\text{objects in } \mathcal{C}\}$. Now suppose that G is an arbitrary topological group with a representation $G \to O(d)$. Then we can define the notion of G-structure on a n-dimensional manifold, i.e. the principle G-bundle $P \to M$.

$$P \times {}^{\mathbb{R}^n}\!/_{G} \simeq TM \oplus \mathbb{R}^{d-n} \ .$$

[L+09]. Using this G-structure we can define $Bord_d^G$. Few examples are $Bord_d^{SO(d)} \simeq Bord_d$, $Bord_d^{\{0\}} \simeq Bord_d^{fr}$, and $Bord_d^{Spin(d)}$, which are spin manifolds. For an arbitrary category $\mathcal{C}$, G acts on the space of objects in $\mathcal{C}$ if $\mathcal{C}$ has duals.

Remark 4.2.11. *Few remarks about the proof of the cobordism hypothesis. I would state just a sketch of the proof, for more details please look up the paper [L+09]. It uses the induction on d. For $d = 1$, I believe we had enough discussions. So consider an functor $Z_0 : Bord_{d-1} \to \mathcal{C}$, where the target category $\mathcal{C}$ is as usual. Let Z_0 be giving an object $X \in \mathcal{C}$ which is an $SO(d-1)$-fixpoint on the space $\{$objects in $\mathcal{C}\}$. We will extend Z_0 to be a TQFT $Z : Bord_d \to \mathcal{C}$. For that we require to supply a piece of informations essentially namely,*

$$Z(D^d) = Z_0(S^{d-1} \mapsto 1_{\mathcal{C}}) \ .$$

To see the ball D^d as a bordism $S^{d-1} \to \varnothing$ is an $SO(d)$-equivariant way, satisfying an non-degeneracy condition. To sum this up, giving $Z : Bord_2 \to \mathcal{C}$, for example, is equivalent to give

(a) $Z_0 : Bord_1 \to \mathcal{C}$ as the restriction of Z,

(b) $Z(D^2) = \eta : Z_0(S^1) \to 1_{\mathcal{C}}$, where η should invariant under $SO(2)$ action, i.e. under S^1, such that η is non-degenerate.

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