Variation of Morse theory.

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June 28, 2017

1 Introduction.

Our main aim is to understand, proof and explore the inmidiate consequences of the following theorem:

Theorem 1 (Folk Theorem) *Let $F : Bord^{SO}_{\langle 1,2\rangle} \to Vect_{\mathbb{K}}$ be a Topological Quantum Field Theory (TQFT). Then $F(S^1)$ is a commutative Frobenius algebra.*

Conversly: Let A be a commutative Frobenius algebra, then exist a TQFT $F_A : Bord^{SO}_{\langle 1,2\rangle} \to Vect_{\mathbb{K}}$ such that $F_A(S^1) = A$

2 Bordisms.

In the sequel we will work with compact manifolds without boundary, which will be called *closed manifolds* for short, e.g: S^n, $S^n \times S^m$, or the union of finitelly many closed manifolds. As a particular example we can think in S^1 and $S^1 \cup S^1$ (see fig 1)

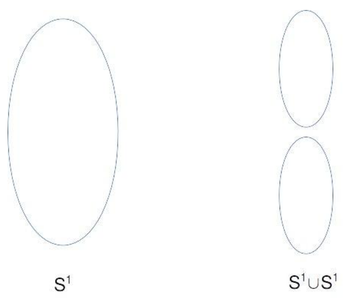


Figure 1: Examples of closed 1-manifolds.

Indeed those manifolds are similar, however they are not diffeomorphic. A new object will be usefull to express this fact.

Definition 1 (Bordism) *Let Y_0 and Y_1 be closed n-manifolds. A bordism is a tuple $(X, p, \theta_0, \theta_1)$ such that:*

- *[·] X is a compact $n+1$-manifold with boundary*
- *[·] $p : \partial X \to \{0, 1\}$ is a partition of the boundary*
- *[·] The maps $\theta_0 : [0, 1) \times Y_0 \to X$, $\theta_1 : (-1, 0] \times Y_1 \to X$ are embeddings.*
- *[·] $\theta_i(0, Y_i) = p^{-1}(i)$ for $i = 1, 2$*

For short, a bordism between two manifolds will be denoted as $X : Y_0 \to Y_1$

Example 1 *Consider $Y_0 = \{p\}$, $Y_1 = \{q\}$ two 0-manifolds of one point. A bordism between them could be*

$$X = [0,1]$$
$$p(0) = 0, \quad p(1) = 1$$
$$\theta_0 : [0,1) \times \{p\} \to [0,1] \quad \theta_0(x,p) = \frac{x}{3}$$
$$\theta_1 : (-1,0] \times \{p\} \to [0,1] \quad \theta_1(x,p) = 1 + \frac{x}{3}$$

Its easy to see that, in this case, the bordism X corresponds to a line joining the two manifolds.

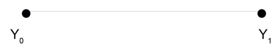


Figure 2: Example of 1-bordism between identical 0-manifolds

Example 2 *Consider $Y_0 = S^1$ and $Y_1 = S^1 \cup S^1$. Those can be ambedded in $\mathbb{R}^3$ as $Y_0 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 - 1 = 0 \wedge z = 0\}$ and $Y_1 = \{(x, y, z) \in \mathbb{R}^3 | [(x-2)^2 + y^2 - 1][(x+2)^2 + y^2 - 1] \wedge z = 1\}$.A possible bordism between them:*

$$X = \{(x, y, z) \in \mathbb{R}^3 | z[(x-2)^2 + y^2 - 1][(x+2)^2 + y^2 - 1] + (1-z)[x^2 + y^2 - 1] = 0 \wedge 0 \leq z \leq 1\}$$
$$p(x, y, z) = z \quad \forall (x, y, z) \in \partial X$$

Graphicaly it corresponds to a pair of pants, see fig 3. The maps θ_i correspond to bands near the boundaries.

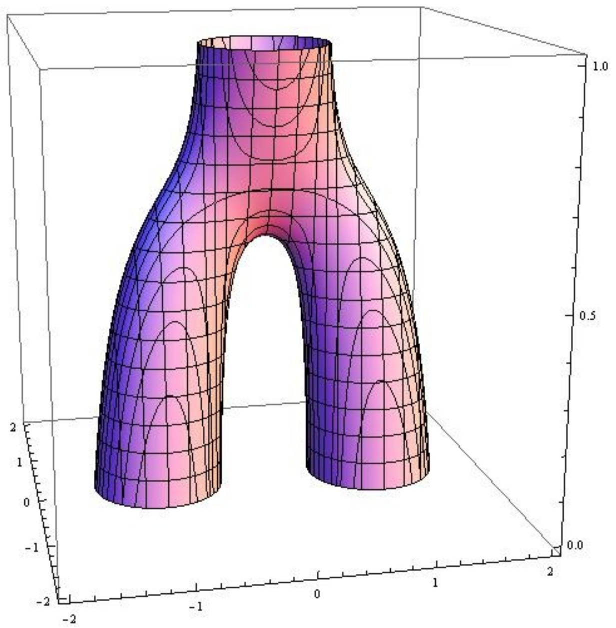


Figure 3: Example of 2-bordism between the manifolds of fig 1. The image do not correspond exactly to the equations presented in the text, but has been streched in the x, y plane.

Through this work we will not work with bordism of dimension superior than 2. The graphical interpretation will be helpful to avoid long mathematical descriptions of the bordisms.

We will say that to manifolds Y_0, Y_1 are *bordant* ($Y_0 \sim Y_1$) if there is a bordism $X : Y_0 \to Y_1$. Bordisms define an equivalence relation between manifolds.

- Is reflexive: $\forall Y$, $Y \sim Y$ since $X = [0, 1] \times Y$ with $p(i, Y) = i$ and $\theta_0(i, Y) = (\frac{i}{3}, Y)$, $\theta_1(i, Y) = (1 + \frac{i}{3}, Y)$ is a bordism.
- Is symmetric: $\forall Y, Y'$ $Y \sim Y' \Rightarrow Y' \sim Y$. Given a bordism $X : Y \to Y'$ we can define $X' = X$, $p' = 1 - p$, $\theta'_0(t, Y') = \theta_1(-t, Y')$, $\theta'_1(t, Y) = \theta_0(-t, Y)$ as a bordism $X' : Y' \to Y$
- Is transitive: $\forall Y, Y', Y''$ $Y \sim Y' \wedge Y' \sim Y'' \Rightarrow Y \sim Y''$. Indeed, given two bordisms $X : Y \to Y'$ and $X' : Y' \to Y''$ we can define a new bordism such the new manifold is the union of the other two and the maps are chosen by picking form the previous bordisms the right choice depending on the point[1].

[1]In the literature, an analogous definition of bordism is heavily used: A bordism between two manifolds Y and Y' is a new manifold X such that ∂X is the disjoint union of Y and Y', i.e. $\partial X = Y \bigsqcup Y' = (Y \times 0) \cup (Y' \times 1)$. With this definition, the proof of transitivity is straightforward.

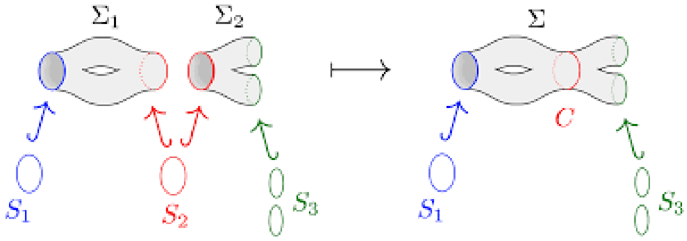


Figure 4: Bordism relation is transitive. Graphic visualitation of gluing bordisms.

Cobordant equivalence is broader that diffeomorphic equivalence. Is easy to proof that diffeomorphic manifolds are cobordant but the converse is not true in general.

In addition to this cobordant equivalence based uniquely in the topology, we could endow this relation with more characteristic. As an example we could talk about *oriented bordism* between oriented manifolds: Two manifolds are cobordant with an oriented bordism if they are cobordant in the previous sense and also the bordism is oriented in such a way that the maps θ_i preserve the orientation.

We will treat bordism as a fundamental object to build a TQFT. In order to make a complete treatment we require notions of category theory

3 Cathegory theory.

Here we provide a very brief excursion through cahegory theory. This branch of Mathematics is very broad so we will cover solely the most relevant topics.

Definition 2 (Category) *A category is a pair $C = (C_0, C_1)$. The elements of C_0 are called objects while elements of C_1 are called morphisms. They must satisfy:*

- $\forall Y_0, Y_1 \in C_0 \quad f : Y_0 \to Y_1 \Rightarrow f \in C_1$.
- $\forall Y \in C_0 \exists f \equiv id_Y \in C_1 | id_Y : Y \to Y$
- *Exists a composition law* $\circ$ *such that* $f : Y_0 \to Y_1, g : Y_1 \to Y_2 \quad \forall f, g \in C_1 \Rightarrow g \circ f : Y_0 \to Y_2, g \circ f \in C_1$.
- $\circ$ *is asociative.*
- $\forall f \in C_1, f : Y \to Z \Rightarrow id_Z \circ f = f \circ id_Y$.

Some examples of cathegories:

$Set =$ (Sets, Functions)

$Ab =$ (Abelian groups, Homomorphisms)

$Vect_{\mathbb{K}} =$ (Linear spaces over a field $\mathbb{K}$, Linear aplications)

Definition 3 *Functor A functor F between cathegories C and D is a pair of maps $F = (F_0, F_1)$ such that $F_0 : C_0 \to D_0$, $F_1 : C_1 \to D_1$ satisfying:*

- $F(id_X) = id_{F(X)} \quad \forall X \in C_0$
- $F(f \circ g) = F(g) \circ F(f) \quad \forall f, g \in C_1$

Some examples of functors:

- Consider the map $F : Ab \to Set$ that associate at each abelian group the set of its elements and to each homomorphism the corresponding function.
- Given a category, consider the functor which maps each object to itself and each morphism to itself. This is called identity functor.

With the identity functor and defining the composition of functors naturally, we can define the category of cathegories $Cat =$ (Cathegories, Functors). In the same spirit we could go further. Consider $Hom(C, D)$ the category of functors between C and D. Functors act as object while morphism between functors, called natural transformations, are defined abstractly.

Definition 4 *Simetric Monoidal Cathegory/Symmetric monoidal structure Given a category C a symmetric monoidal structure over C is a choice of a functor $\otimes : C \times C \to C$ which is symmetric with respect to its arguments and associative, and an object called unit tensor 1_C which satisfies $Y \otimes 1_C = 1_C \otimes Y = Y \quad \forall Y \in C_0$.*

An example of a symmetric monoidal category is $Vect_{\mathbb{K}}$ with the following structure: $\otimes$ as the usual tensor product of linear spaces and as the unit tensor $1_C = \mathbb{K}$ the field as a linear space over itself.

Provided with this tool we can propose the following object $Bord_{\langle n-1,n \rangle}$:

- Objects: Closed $n-1$-manifolds
- Morphisms: Set of (equivalence classes under diffeomorphism equivalence of) bordisms between manifolds
- Composition: Gluing bordisms Taking in account that for every manifold Y, $[0,1] \times Y$ is a bordism from the manifold to itself, this will be the identity morphism for each object, and $Bord_{\langle n-1,n \rangle}$ is a category.
- Functor $\otimes$: Disjoint union of manifolds.
- Tensor unit: Empty $n-1$-manifold

Analogously we could define $Bord^{\chi(n)}_{\langle n-1,n \rangle}$, the symmetric monoidal category of bordisms between manifolds with a $\chi(n)$ structure. In particular $Bord^{SO}_{\langle n-1,n \rangle}$ is the category of oriented bordisms.

As a particular example we can consider $Bord_{\langle 1,2 \rangle}$. An object will be a finite union of circles. The identity morphism then is a colection of cilinders The tensor unit is the empty 1-manifold

Definition 5 *Symmetric monoidal Functor A symmetric monoidal functor is a functor between symmetric monoidal categories wich preserve the symmetric monoidal structure. I.e. given two symmetric monoidal categories $(C, \otimes_C, 1_C)$ and $(D, \otimes_D, 1_D)$, a functor $F : C \to D$ is a symmetric monoidal functor if and only if $F(1_C) = 1_D$ and $F(Y \otimes_C Y') = F(Y) \otimes_D F(Y') \quad \forall Y, Y' \in C$.*

A TQFT is a symmetric monoidal functor $F : Bord_{\langle n-1,n \rangle} \to Vect_{\mathbb{K}}$ between the symmetric monoidal categories we have defined so far. In particular, $F(\emptyset) = \mathbb{K}$ Since the image of the unit tensor should be the unit tensor the second category.

4 Morse theory.

Morse theory is the perfect tool to evaluate bordisms. The aim of the theory is to study the topology of a manifold via differentiable functions defined on it. The classical example is to evaluate the difference between a sphere and a torus. Consider both manifolds centered at the origin of coordinates observe the behaviour of the height function (i.e. the topology of the leve-sets). In both manifolds the set levels start as empty sets, evolving to a point and to a set diffeomorphic to S^1. After a certain point the level sets of the sphere continue to be diffeomorphic to the circle, while the torus gives a different behaviour.

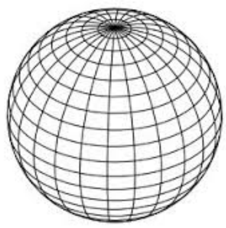


Figure 5: Level sets of a torus and a sphere.

Let $\mathcal{M}$ be a smooth n-manifold, and $f : \mathcal{M} \to \mathbb{R}$ a smooth function. We will call $p \in \mathcal{M}$ a critical point with critical value $f(p) = c$ if $df\big|_p = 0$

At a critical point, the hessian $d^2 f\big|_p : T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$ is a well defined bilinear form. If the for is non-degenerate,

p will be called non-degenerate critical point.

Lemma 1 *Morse Lemma If p is a non-degenerate critical point, then exists a local coordinate system in which the function takes the form*

$$f = \sum_{i=1}^{r} (x^i)^2 - \sum_{i=r+1}^{n} (x^i)^2 + c \tag{1}$$

$n - r$ is called the index of p and is independent of the local system of coordinates

Proof 1 *Without lose of generality we can take $f(p) = 0$ (otherwise redefine $f \to f - c$). Consider a system of coordinates centered in p. By Taylor expanding at firs order:*

$$f(x^1, \ldots, x^n) = \sum_{i=1}^{n} x^i \left. \frac{\partial f}{\partial x^i} \right|_0 = \sum_{i=1}^{n} x^i g_i(x^1, \ldots, x^n) \tag{2}$$

Since p is a critical point, $g_i(0) = 0$. Expand g_i in power series

$$g_i(x^1, \ldots, x^n) = \sum_{k=1}^{n} x^k h_{ik} \tag{3}$$

Therefore:

$$f(x^1, \ldots, x^n) = \sum_{i=1}^{n} \sum_{k=1}^{n} x^i x^k h_{ik} = \sum_{i,k} x^i x^k \frac{1}{2}(h_{ik} + h_{ki}) = \sum_{i,k} x^i x^k H_{ik} \tag{4}$$

where $H_i k$ is now symmetric. Furthermore, H_{ik} is the hessian of f at 0 which is non-degenerate by assumption, so no eigenvalue is 0. Diagonalize the hessian and reescale the eigenvectors in order to normalize the eigenvalues to 1, . The basis of eigenvectors is the local coordinate system. The number of positive and negative eigenvalues is independent of the coordinates as a result of the Sylvester criterion.

A function is called Morse function if it lacks of degenerate critical points. If that is the case then critical points appear isolated. .

Lemma 2 *Isolation of critical points If $f : \mathcal{M} \to \mathbb{R}$ is a Morse function, then for each critical point exist a neighbourhood in which no other critical point can be found.*

Proof 2 *Let $\phi : U \subset \mathcal{M} \to \mathbb{R}^n$ be a chart centered in a critical point $p \in U$. Define:*

$$\begin{gathered} g(x) : \phi(U) \to \mathbb{R}^m \\ g(x) = \left(\tfrac{\partial (f \circ \phi^{-1}}{\partial x^1}(x), \cdots, \tfrac{\partial (f \circ \phi^{-1}}{\partial x^n}(x) \right) \Rightarrow g(0) = 0 \end{gathered} \tag{5}$$

Since f is Morse $\left. dg \right|_0 = \left. d^2 f \right|_0$ is non-singular. The inverse function theorem guarantees that g is an injective diffeomorphism in a neighbourhood of 0, therefore $g(x) = 0 \Leftrightarrow x = 0$. Exists a neighbourhood arround p whose only critical point is p.

As a corolary, if the manifold $\mathcal{M}$ is compact, then the number of zeros must be finite since otherwise we can choose a sequence of critical points $\{p_n\}$ that converges to some p. Since partial derivatives are continuous $\lim_{n\to\infty} \left. df \right|_{p_n} = \left. df \right|_p = 0$. So p is a critical point but cannot be isolated, which contradicts the previous lemma.

Morse functions are very well-behaved. Their study is relevant because the set of Morse functions is dense in the space of smooth function.

Lemma 3 *Morse functions are a dense set. Given any smooth function f and $\varepsilon > 0$, $\exists g$ Morse function such that* $\sup\{|f - g|\} < \varepsilon$.

Proof 3 *Consider $g\left(\frac{\partial f}{\partial x^1}(x), \cdots, \frac{\partial f}{\partial x^n}(x)\right)$ and $a = (a_1, \ldots, a_n)$, a regular value of g. By Sard's theorem, the set of regular values of a is dense in $\Im(f)$ since the set of critical values has Lebesgue measure zero.*

Let $f_a = f(x) - \sum_i x^i a_i \Rightarrow df_a|_p = 0$ is a non-degenerate critical point of f_a, so it is a Morse function.

Conclusion: We can almost always find a Morse function of our convenience or, at least, a sequence of Morse functions that converges to a function we want to study.

The power of Morse functions comes form Morse theorem, which will allow us to dissect a manifold (in our case a bordism) in easier parts to analyse.

Theorem 2 *Morse's Theorem Consider* $f : X \to \mathbb{R}$ *a Morse function such that all its critical values are different. Let* $X_{a',a''} = f^{-1}([a', a''])$ *where* a', a'' *are regular values of* f*.*

- *If* $[a', a'']$ *contains no critical values, then* $X_{a',a''}$ *is isomorphic to* $[a', a''] \times f^{-1}(a)$ *for* $a \in [a', a'']$*.*
- *If there is a critical value* c*, then* $X_{a',a''}$ *is obtained from* $X_{a',c-\varepsilon}$ *by attaching an* n*-dimensional* r*-handle, where* r *is the index of the critical point.*

So if we find one of this Morse functions, that we call excelent functions, we can reconstruct the complete manifold solely from its behaviour at critical points.

From previous examples, lets take a look of 2-dimensional manifolds, in particular to bordisms in $Bord_{\langle 1,2 \rangle}$. In this case we are interested in Morse functions that take diferent critical values, independent of the topology. The only way to be sure of that is consider bordisms that only have one critical point. We will call this bordisms elementary bordisms

Figure 6: Elementary 2-bordisms.

Using excellent Morse functions, any bordism can be expressed as a composition of elementary bordisms. Consider a function with a set of critical values $c_1 < c_2 < \cdots < c_{N-1}$ between to non-critical values a_0, a_1

$$\begin{array}{rcl} X & = & X_N \circ \cdots \circ X_1 \\ X_1 & = & f^{-1}(a_0, c_1 - \varepsilon) + handle \\ X_2 & = & f^{-1}(c_1 + \varepsilon, c_2 - \varepsilon) + handle \\ & \cdots & \\ X_1 & = & f^{-1}(c_{N-1} + \varepsilon, a_1) + handle \end{array} \tag{6}$$

Seems trivial that using two Morse functions that can be transformed via a continuous path should produce the same descomposition. Fortunatelly the set of excellent functions is also dense in the space of smooth functions, although is not connected. However the following theorem provides a scape route

Theorem 3 *Cerf theorem Between two excellents functions* f_0, f_1 *always exist a path* f_t *such that* f_t *is excellent except forfinitelly many values of* t*.*

At this values of t, the function could have:

- The function is excelent everywhere except at a single point p in which $d^2 f\big|_p = 0$ but $d^3 f\big|_p \neq 0$. This is refered as type α.
- The function is excelent everywhere except of two critical non-degenerate points which have the same critical value. This is refered as type β.

In both cases the situation can be solved. Every bordism can be descomposed into elementary bordisms and analysed independently of the Morse function considered.

5 Folk theorem.

To prove the theorem we need to define Frobenious algebra.

Definition 6 *Frobenious algebra A Frobenius algebra is a finite dimensional algebra over a field $\mathbb{K}$ equipped with a bilinear form $\sigma : A \times A \to \mathbb{K}$ such that $\sigma(a \cdot b, c) = \sigma(a, b \cdot c)$*

Example 3 *A Frobenius algebra can be constructed as follow: Take the linear space $\mathbb{R}^3$ and equip the wedge product as the algebra operation. $(\mathbb{R}^3, \wedge)$ is now a 3-dimensional algebra. To achieve a Frobenius algebra use the standard inner product as a bilinear form, $A = (\mathbb{R}^3, \wedge, \cdot)$.*

We can now prove the theorem.

Theorem 4 (Folk Theorem) *Let $F : Bord^{SO}_{\langle 1,2 \rangle} \to Vect_{\mathbb{K}}$ be a Topological Quantum Field Theory (TQFT). Then $F(S^1)$ is a commutative Frobenius algebra.*

Conversly: Let A be a commutative Frobenius algebra, then exist a TQFT $F_A : Bord^{SO}_{\langle 1,2 \rangle} \to Vect_{\mathbb{K}}$ such that $F_A(S^1) = A$

Proof 4 *Given a TQFT $F : Bord^{SO}_{\langle 1,2 \rangle} \to Vect_{\mathbb{K}}$ we define $A = F(S^1)$ and prove that is a Frobenius algebra constructing explicitelly the bilinear form as we did in the previous example.*

Consider the elementary bordisms which involve S^1, and recall that $F(\emptyset) = \mathbb{K}$

We can identify:

- *An operation $u : \mathbb{K} \to A$ (fig 6, inferior left panel)*
- *An oposite operation $\tau : A \to \mathbb{K}$ (fig 6, inferior right panel)*
- *A composition law $m : A \otimes A \to A$ (fig 6, superior left panel)*
- *The inverse of the previous operation, which will not play a relevant rol. (fig 6, superior right panel)*

Then (A, m) is an algebra while $(A, m, \tau \circ m)$ is a Frobenius algebra since F is a symmetric monoidal functor, and hence associative.

Int he opposite direction we should construct the TQFT: consider a Frobenius algebra A and take $Y = S^1$. Via a similar argument as before, we can define the operations in the Frobenius algebra as values of different elementary bordisms. Because for an arbitrary bordism is a composition of elementary bordisms via an excellent function, we use the descomposition to define F_A. The result should be independent of the function chosen. In virtue of Cerf's theorem we should only check this fact under a transition of excelent functions through a function of Type α or β.

For a type α singularity we have a function which have a cubic singularity, represented in the figure 7. The transition between one Morse function to another builds the bordism $id_A : A \to A$. We can decompose this bordism as $id = m \circ u$; i.e $m(\phi, u(k)) = \phi$, so u is the identity for the multiplication. In this way, functions of type α can be analysed via Morse functions, and knowing the behaviour of the Frobenius algebra, its behaviour is independent of the choice of the function. For a type β singularity we have f_t a Morse function with two critical points with the same critical value for an specific value of t_{crit}. However the function is excellent in a neighbourhood of this value: evaluating the bordism for $t < t_{crit}$. For each elementary bordism the number of conected components in the boundary change by 1, therfore we only have four posibilities:

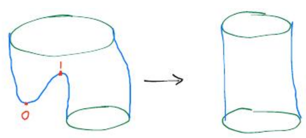
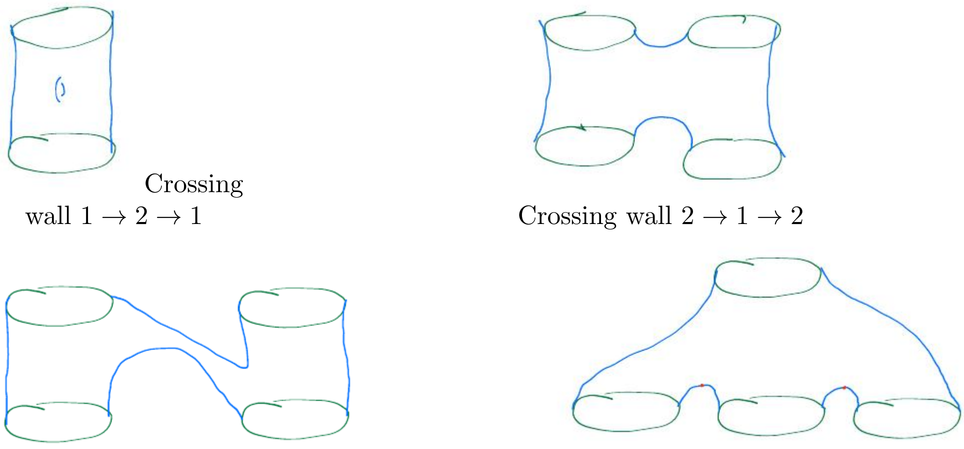


Figure 7: Example of 2-bordism via a non-excelent function before a wall crossing type α

- $1 \to 2 \to 1$ *We can express this by the composition $id_A = m \circ m^*$*
- $2 \to 1 \to 2$ *We can express this by the composition $id_A = m^* \circ m$*
- $2 \to 3 \to 2$ *We can express this by the composition $id_A = (m \otimes id) \circ (id \otimes m^*)$*
- $3 \to 2 \to 1$ *We can express this as $m(\phi_1, m(\phi_2, \phi_3))$ or $m(m(\phi_1, \phi_2), \phi_3)$, and it reduced to the fact that the*

algebra is associative.



Crossing wall $1 \to 2 \to 1$

Crossing wall $2 \to 1 \to 2$

Crossing wall $2 \to 3 \to 2$

Crossing wall $3 \to 2 \to 1$

Figure 8: TQFT built via composition of elementary bordisms.

Since all the cases can be calculated through the rules of the Frobenius algebra, the TQFT is completely determined, independently of the choice of the Morse function..

6 Physical interpretation.

Using the recipe from the previous section we can build a toy model of a TQFT to give a physical interpretation.

Assume that a closed d-dimensional spacelike manifold Σ can be associated with a Hilbert space $\mathcal{H}_\Sigma$, span of the states obteined after quantizing the theory in $\Sigma \times \mathcal{R}$, as well a similar one Σ' with its corresponding Hilbert space.

From the theorem we know that in a $d+1$-manifold $\mathcal{M}$ with boundaries Σ, Σ', exists a map $\Phi_\mathcal{M} : \mathcal{H}_\Sigma \to \mathcal{H}'_\Sigma$. this $\Phi_\mathcal{M}$ is the evolution operator of the TQFT. The Morse function chosen corresponds to a choice of "time slicing". The independency of the choice of the Morse faction reflects the relativity principle of the TQFT: every observer, chosen its Morse function, will realize the same TQFT.

Some comments are worth to notice:

- if $Sigma = \emptyset$ $\mathcal{H}_\Sigma = \mathbb{C}$. Quantum mechanics can be recovered.
- If $Sigma$ is the disjoint union of manifolds $\Sigma_1 \sqcup \Sigma_2$, the hilbert space $\mathcal{H}_\Sigma = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2}$, causal disconected manifolds will have independent dynamics.
- If $\mathcal{M}$ has to boundary components that can be identified, they can be composed, and as a result we will have the well known trace relation:

$$\Phi_{\mathcal{M}'} = Tr(\Phi_\mathcal{M}) = \sum_{v \in \mathcal{H}_\Sigma} \Phi_\mathcal{M}(v, v) \tag{7}$$

- When we treat oriented bordisms, if several boundary components are isomorphic, exist a natural action of the permutation group, reflecting a bosonic statistic, or a graded action, giving a fermionic nature.

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