SEMINAR ON SUPERSYMMETRY IN GEOMETRY AND QUANTUM PHYSICS

GROMOV-WITTEN THEORY

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Contents

1. Introduction/Motivation

We will look at the A-model and will see how their correlation functions motivate the definition of Gromov-Witten invariants in mathematics. These invariants can be interpreted as counting holomorphic curves in a Kähler manifold X in a special, virtual way. But they are rather difficult to determine. In the B-model, it is much simpler to calculate the correlation functions, as this only involves wedging differential forms on the target space X and their integration. Since the two models are connected by mirror symmetry, the computation of the Gromov-Witten invariants gets dramatically simplified by linking the correlation functions of the two models.

Concerning the physical part, we mainly follow section 16.4.1 in the book "Mirror Symmetry" [1] by Hori et al. Moreover, the lecture notes [2] by Collinucci and Wyder as well as the paper [3] by Witten were used. This talk continues Robert's talk "Topological Twist of 2d Field Theories" [5] and is also related to the talk [6] presented by Fabio. For the mathematical part, i.e. the definition of the Gromov-Witten invariants, see chapter 21 ff. in "Mirror Symmetry" [1]. If this is your first contact with the topic, the book "Enumerative Geometry and String Theory" [4] by Katz can be recommended, as Katz explains in an easy, intuitive way, how the Gromov-Witten invariants arise from physics and why they are that important in mathematics.

2. The twisted A-model

Let (Σ, h) be a Riemann surface of genus g and (X, ω) be a Kähler manifold of complex dimension $d \coloneqq \dim_{\mathbb{C}} X$. In local coordinates we write the Kähler form as

$$\omega = g_{\bar{i}j} \mathrm{d}z^{\bar{i}} \mathrm{d}z^{j}. \tag{2.1}$$

Consider the 2-dimensional $\mathcal{N} = (2, 2)$ supersymmetric non-linear sigma model

$$\phi : \Sigma \longrightarrow X \tag{2.2}$$

introduced in Robert's talk "Topological Twist of 2d field theories" [5]. In his talk we also saw that twisting with respect to the R_V-symmetry yields a new theory called the A-model.

2.1. **Action and SUSY.** The action of the A-model is given by

$$\begin{aligned} S \quad &= \quad S_{\text{bosonic}} \quad + \quad S_{\text{fermionic}} \\ S_{\text{bosonic}} &= 2t \int_{\Sigma} \mathrm{d}^2 z \left(g_{i\bar{j}} \partial_z \phi^i \partial_{\bar{z}} \phi^{\bar{j}} + g_{i\bar{j}} \partial_{\bar{z}} \phi^i \partial_z \phi^{\bar{j}} \right) \\ S_{\text{fermionic}} &= 2t \int_{\Sigma} \mathrm{d}^2 z \left(\mathrm{i} g_{\bar{i}j} \rho_z^{\bar{i}} \mathrm{D}_{\bar{z}} \chi^j + \mathrm{i} g_{i\bar{j}} \rho_{\bar{z}}^{i} \mathrm{D}_z \chi^{\bar{j}} + \tfrac{1}{2} R_{i\bar{j}k\bar{l}} \rho_{\bar{z}}^{i} \rho_z^{\bar{j}} \chi^k \chi^{\bar{l}} \right) \end{aligned} \tag{2.3}$$

where t is a positive real parameter, which is actually only needed in section 4 and can be neglected most of the time by setting $t = 1$. We denote the canonical bundle of Σ by $K = (T^{1,0}\Sigma)^*$. After twisting, two of the four fermions have become fermionic vectors, i.e. they transform as vectors on the worldsheet Σ, but remain anti-commuting. The other two fermions get twisted to fermionic scalars.

$$\begin{aligned} &\left.\begin{aligned} \rho_z &\in \Gamma\big(\Sigma, K \otimes \phi^* T^{0,1} X\big) \\ \rho_{\bar{z}} &\in \Gamma\big(\Sigma, \overline{K} \otimes \phi^* T^{1,0} X\big) \end{aligned}\right\} \text{fermionic vectors} \\ &\left.\begin{aligned} \chi &\in \Gamma\big(\Sigma, \phi^* T^{1,0} X\big) \\ \bar{\chi} &\in \Gamma\big(\Sigma, \phi^* T^{0,1} X\big) \end{aligned}\right\} \text{fermionic scalars} \end{aligned} \tag{2.4}$$

We neglect half of the supersymmetry and only look at the SUSY-trafos given by

$$\begin{aligned}
\delta\phi^i &= \mathrm{i}\alpha\chi^i \\
\delta\phi^{\bar{i}} &= \mathrm{i}\tilde{\alpha}\chi^{\bar{i}} \\
\delta\rho_z^{\bar{i}} &= -\alpha\partial_z\phi^{\bar{i}} - \mathrm{i}\tilde{\alpha}\chi^{\bar{k}}\Gamma^{\bar{i}}_{\bar{k}\bar{m}}\rho_z^{\bar{m}} \\
\delta\rho_{\bar{z}}^{i} &= -\tilde{\alpha}\partial_{\bar{z}}\phi^{i} - \mathrm{i}\alpha\chi^{k}\Gamma^{i}_{km}\rho_{\bar{z}}^{m} \\
\delta\chi^i &= 0 \\
\delta\chi^{\bar{i}} &= 0
\end{aligned} \tag{2.5}$$

where the two SUSY-parameters α and $\tilde{\alpha}$ are now fermionic scalars. This is one big advantage of the twisted model, because the scalars α and $\tilde{\alpha}$ can now easily been chosen to be non-vanishing everywhere, e.g. constant. Due to this stronger supersymmetry, it is much easier to evaluate correlation functions in the A-model, as we will see. Before twisting, the SUSY-parameters were sections of $K^{-1/2}$ or $\bar{K}^{-1/2}$. In many cases, a nowhere-vanishing section does not even exist for those bundles. The same holds for K^{-1} and $\bar{K}^{-1}$, whose spaces of sections would have contained the two neglected SUSY-parameters.

We will mostly look at SUSY-trafos with $\alpha = \tilde{\alpha}$. The corresponding BRST-operator will be denoted by

$$Q_A = \overline{Q}_+ + Q_- . \tag{2.6}$$

By setting $\alpha = \tilde{\alpha}$, we overlook the Hodge decomposition of the moduli space.

First, we rewrite the action S in (2.3) and get the following identity which holds modulo terms vanishing by the e.o.m. for ρ.

$$S \;\simeq\; \mathrm{i}t\int_\Sigma \mathrm{d}^2z\,\{Q_A, V\} + t\int_\Sigma \phi^*(\omega)\ , \tag{2.7}$$

where we have defined

$$V := g_{\bar{i}j}\left(\rho_z^{\bar{i}}\partial_{\bar{z}}\phi^j + \partial_z\phi^{\bar{i}}\rho_{\bar{z}}^{j}\right) . \tag{2.8}$$

Here, we see that we are dealing with a topological field theory, because the first term in the rewritten action is Q_A-exact and the second term only depends on ϕ up to homotopy. In relation to this, we know the energy-momentum tensor is Q_A-exact, i.e. $T_{\mu\nu} = \{Q_A, G_{\mu\nu}\}$ for some fermionic symmetric tensor $G_{\mu\nu}$, leading to correlation functions which are independent of the worldsheet metric $h_{\mu\nu}$.

2.2. **Localization.** From last semester we know that the path integral localizes to loci with vanishing variation of the fermions. In view of the SUSY-trafos (2.5)

$$\begin{aligned}
&\Rightarrow \quad \delta\rho_z^{\bar{i}} = 0 = \delta\rho_{\bar{z}}^{i} \\
&\Rightarrow \quad \partial_z\phi^{\bar{i}} = 0 = \partial_{\bar{z}}\phi^{i}\ , \quad \text{since } \alpha \text{ and } \tilde{\alpha} \text{ are independent} \\
&\Rightarrow \quad \phi : \Sigma \longrightarrow X \quad \text{is holomorphic} \\
&\quad\text{and} \qquad V = 0.
\end{aligned} \tag{2.9}$$

Thus, including localization we see that the important part of the action (2.7) is given by

$$t\int_\Sigma \phi^*(\omega) \;=\; t\int_\beta \omega \qquad \text{where} \qquad \beta := \phi_*[\Sigma] \in \mathrm{H}_2(X)\,. \tag{2.10}$$

Hence, it is reasonable to introduce for $\beta \in \mathrm{H}_2(X)$ the space $\mathcal{M}_\Sigma(X,\beta)$ of holomorphic maps $\phi : \Sigma \to X$ satisfying $\phi_*[\Sigma] = \beta$. Since we work in Euclidean signature, the partition function Z can now be rewritten in the following way

$$Z \;=\; \int \mathcal{D}\phi\mathcal{D}\chi\mathcal{D}\rho \; \mathrm{e}^{-S(\phi,\chi,\rho)} \;=\; \sum_{\beta\in \mathrm{H}_2(X)} \mathrm{e}^{-t\int_\beta \omega} \int_{\phi_*[\Sigma]=\beta} \mathcal{D}\phi\mathcal{D}\chi\mathcal{D}\rho \; \mathrm{e}^{-S_Q} \tag{2.11}$$

where $S_Q := \mathrm{i}t\int_\Sigma \mathrm{d}^2z\,\{Q_A, V\}$ is the first term appearing in the rewritten action (2.7). As we will see later, S_Q will modifiy the path integral measure when we restrict the domain of integration due to localization. See also §5 in [3], where Witten first gives an even stronger localization principle and then explains how the new measure arises as the one-loop determinants of the degrees of freedom being transverse to the fixed point locus of Q_A. He shows that the path integral localizes to the fixed point locus of Q_A, i.e.

$$\begin{aligned} &\delta\phi^i = \delta\phi^{\bar{i}} = \delta\rho^{\bar{i}}_z = \delta\rho^i_{\bar{z}} = \delta\chi^i = \delta\chi^{\bar{i}} = 0 \\ \overset{(2.5)}{\Rightarrow}\quad &\chi^i = 0 = \chi^{\bar{i}} \quad \text{and thus} \quad \partial_z\phi^{\bar{i}} = 0 = \partial_{\bar{z}}\phi^i \\ \Rightarrow\quad &\chi^i = 0 = \chi^{\bar{i}} \quad \text{and} \quad \phi : \Sigma \longrightarrow X \quad \text{is holomorphic} \end{aligned} \tag{2.12}$$

Here, we did not use the independence of α and $\tilde{\alpha}$, but still get the same results.

2.3. **Anomaly.**

Since in many cases the Grassmann integration over the zero modes of the fermions in the path integral is unsaturated, the result is zero. As we are interested only in non-trivial correlation functions, we do an anomaly calculation telling us which operators we should insert into the path integral in order to get a non-zero result. Because of

$$\mathrm{D}_z\bar{\chi} = (\mathrm{D}_{\bar{z}}\chi)^* \quad \text{and} \quad \mathrm{D}_z\rho_{\bar{z}} = (\mathrm{D}_{\bar{z}}\rho_z)^* \tag{2.13}$$

the number of χ zero modes is equal to number of $\bar{\chi}$ zero modes. We denote this number by l_χ. Analogue is true for ρ. l_ρ is defined as the number of ρ zero modes. Each fermion zero mode gives a contribution to the measure of the path integral. If we look at the action S in (2.3), we see that in each summand ρ_z is always paired with χ and same for $\rho_{\bar{z}}$ and $\bar{\chi}$. This balance is not satisfied when we look at the fermion zero modes in the measure. Thus, we have to compensate for the difference $k := l_\chi - l_\rho$ by adding appropriate operators. But first let us calculate this anomaly, which is described by the number k.

$$\begin{aligned}(2.14)\qquad l_\chi &= \dim\{\chi \mid \mathrm{D}_{\bar z}\chi = 0\} = \dim\{\bar\chi \mid \mathrm{D}_z\bar\chi = 0\}\\ &= \dim \mathrm{H}^0\big(\Sigma, \phi^*(T^{1,0}X)\big)\\ l_\rho &= \dim\{\rho_z \mid \mathrm{D}_{\bar z}\rho_z = 0\} = \dim\{\rho_{\bar z} \mid \mathrm{D}_z\rho_{\bar z} = 0\}\\ &= \dim \underbrace{\mathrm{H}^0\big(\Sigma, K\otimes\phi^*(T^{0,1}X)\big)}_{\underset{\text{Serre duality}}{\cong}\mathrm{H}^1(\Sigma,\phi^*(T^{1,0}X))}\end{aligned}$$

Now, we use the Atiyah-Singer index theorem for the bundle $\phi^*(T^{1,0}X)$ on Σ to compute the difference of the numbers of zero modes l_χ and l_ρ.

$$\begin{aligned}(2.15)\qquad k = l_\chi - l_\rho &= \int_\Sigma \mathrm{ch}\left(\phi^*(T^{1,0}X)\right)\mathrm{td}(T\Sigma)\\ &= \int_\Sigma \left(d + \phi^*\mathrm{c}_1(T^{1,0}X)\right)\left(1+\tfrac{1}{2}\mathrm{c}_1(T\Sigma)\right)\\ &= \int_\Sigma \phi^*\mathrm{c}_1(T^{1,0}X) \;+\; \tfrac{d}{2}\underbrace{\int_\Sigma \mathrm{c}_1(T\Sigma)}_{=2-2g} \qquad (\text{ Euler characteristic })\\ &= \int_\beta \mathrm{c}_1(X) + d(1-g)\end{aligned}$$

This number k tells us that the operators, we put into the path integral, should have exactly k more χ's than ρ_z's in total and k more $\bar\chi$'s than $\rho_{\bar z}$'s, if we would like to have a non-vanishing correlation function.

2.4. **Physical Operators.** Our physical operators are zero-form operators which are Q_A-exact and depend only on ϕ and χ. If we also used their derivatives or ρ, we would have to contract the appearing worldsheet indices using the worldsheet metric $h_{\mu\nu}$ leading to a Q_A-exact operator and hence zero correlation function. Let $\mathcal{O}$ be a physical operator inserted at the point $P \in \Sigma$. We can write the operator as

$$(2.16)\qquad \mathcal{O}(P) = \mathcal{O}_\omega(P) = \omega_{i_1,\dots,i_p,\bar j_1,\dots,\bar j_q}(\phi(P))\chi^{i_1}\dots\chi^{i_p}\chi^{\bar j_1}\dots\chi^{\bar j_q}$$

where $\omega_{\dots}$ is a smooth function on X being antisymmtric in its indices. Thus, we can interpret ω as a (p,q)-form on X. By (2.5) we get the proportionality

$$(2.17)\qquad \{Q_A, \mathcal{O}_\omega(P)\} \sim \frac{\partial\omega_{i_1,\dots,i_p,\bar j_1,\dots,\bar j_q}(\phi(P))}{\partial\phi^I}\chi^I\chi^{i_1}\dots\chi^{i_p}\chi^{\bar j_1}\dots\chi^{\bar j_q} = \mathcal{O}_{\mathrm{d}\omega}(P)$$

where $\mathrm{d}\omega$ is the exterior derivative of the differential form ω. Thus, we identify physical operators with closed differential forms on X according to the rule

$$(2.18)\qquad \phi^i \leftrightarrow z^i \qquad \phi^{\bar i} \leftrightarrow \bar z^{\bar i} \qquad \chi^i \leftrightarrow \mathrm{d}z^i \qquad \chi^{\bar i} \leftrightarrow \mathrm{d}\bar z^{\bar i}$$

and finally get the correspondence

$$(2.19)\qquad \{\text{ physical operators mod } Q_A\text{-exactness }\} \;\cong\; \mathrm{H}^\bullet_{\mathrm{dR}}(X)\,.$$

Representing the physical operator $\mathcal{O}_i$ by the closed form $\omega_i \in \Omega^{p_i,q_i}(X)$, the correlation function $\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle$ is only non-vanishing, if $\omega_1 \wedge \cdots \wedge \omega_n \in \Omega^{k,k}(X)$, i.e. $k = \sum p_i = \sum q_i$, as we have seen in the previous section.

2.5. **Correlation Functions.** For $i = 1, \dots, n$ let $\mathcal{O}_i$ be a physical operator inserted at the point P_i on the worldsheet Σ. Let ω_i be the closed form on X corresponding to $\mathcal{O}_i$. We will first compute the correlation function $\langle \mathcal{O}_1(P_1) \dots \mathcal{O}_n(P_n) \rangle$ under the assumption that $\mathcal{M}_\Sigma(X, \beta)$ is a smooth manifold and that there are no ρ zero modes, i.e. $l_\rho = 0$ and in particular $k = l_\chi \geqslant 0$. This is the so called *generic case.*

$$\begin{aligned} \langle \mathcal{O}_1(P_1) \dots \mathcal{O}_n(P_n) \rangle &= \sum_{\beta \in H_2(X)} \mathrm{e}^{-t \int_\beta \omega} \; \langle \mathcal{O}_1(P_1) \dots \mathcal{O}_n(P_n) \rangle_\beta \\ \text{with} \quad \langle \mathcal{O}_1(P_1) \dots \mathcal{O}_n(P_n) \rangle_\beta &:= \int\limits_{\phi_*[\Sigma]=\beta} \mathcal{D}\phi \mathcal{D}\chi \mathcal{D}\rho \; \mathcal{O}_1(P_1) \dots \mathcal{O}_n(P_n) \; \mathrm{e}^{-S_Q} \end{aligned} \tag{2.20}$$

From the SUSY-trafos (2.5) we know $\delta \phi^I \sim \chi^I$. We want ϕ to stay holomorphic, so we should also take χ holomorphic, i.e. $\mathrm{D}_{\bar{z}} \chi = 0$. Hence, we see that the χ zero modes span the tangent space of $\mathcal{M}_\Sigma(X, \beta)$.

$$\begin{aligned} T_\phi \mathcal{M}_\Sigma(X, \beta) &= \{ \chi \mid \mathrm{D}_{\bar{z}} \chi = 0 \} \\ \dim_{\mathbb{C}} \mathcal{M}_\Sigma(X, \beta) &= l_\chi = k. \end{aligned} \tag{2.21}$$

As discussed before, localization causes the path integral in $\langle \mathcal{O}_1(P_1) \dots \mathcal{O}_n(P_n) \rangle_\beta$ to reduce to an integral over the finite dimensional space $\mathcal{M}_\Sigma(X, \beta)$. The new measure is given by integration of the infinitely many non-zero modes, giving just one due to the cancellation of bosonic and fermionic determinants. Define the evaluation maps

$$\mathrm{ev}_i : \mathcal{M}_\Sigma(X, \beta) \to X, \;\; \phi \mapsto \phi(P_i). \tag{2.22}$$

In view of (2.16) and by the definition of the pullback of a differential form, we can write the correlation function after localization as

$$\langle \mathcal{O}_1(P_1) \dots \mathcal{O}_n(P_n) \rangle_\beta = \int\limits_{\mathcal{M}_\Sigma(X,\beta)} \mathrm{ev}_1^*(\omega_1) \wedge \ldots \wedge \mathrm{ev}_n^*(\omega_n) =: \langle \omega_1, \dots, \omega_n \rangle_\beta \tag{2.23}$$

Next, we look at what is called the *non-generic case.* We allow for ρ zero modes, but want their number l_ρ to be constant along $\mathcal{M}_\Sigma(X, \beta)$, which we still assume to be a smooth manifold. Thus, the ρ zero modes form a vector bundle $\mathcal{V}$ of rank l_ρ over $\mathcal{M}_\Sigma(X, \beta)$ with fiber $\mathrm{H}^0\big(\Sigma, K \otimes \phi^*(T^{0,1}X)\big)$. As k never depends on $\phi \in \mathcal{M}_\Sigma(X, \beta)$, the number of χ zero modes $l_\chi = l_\rho + k$ does not vary along $\mathcal{M}_\Sigma(X, \beta)$ either. Like in the generic case, $T_\phi \mathcal{M}_\Sigma(X, \beta)$ is spanned by the χ zero modes and its dimension is given by

$$\dim_{\mathbb{C}} \mathcal{M}_\Sigma(X, \beta) = l_\chi = l_\rho + k. \tag{2.24}$$

During localization, the integration over the infinitely many non-zero modes in the quadratic approximation gives one as before. But regarding the zero modes, another part of the action survives, since there are ρ zero modes now.

$$S_0 = t\int_\Sigma \mathrm{d}^2z\left(\tfrac{1}{2}\rho^i_{\bar z}R_{i\bar j k\bar l}\chi^k\chi^{\bar l}\rho^{\bar j}_z - \tfrac{1}{4}\rho^i_{\bar z}\chi^{\bar k}\partial_z\phi^l R_{i\bar j\bar k l}G^{z\bar z\,\bar j i}\chi^k\partial_{\bar z}\phi^{\bar l}R_{\bar j i k\bar l}\rho^{\bar j}_z\right) \tag{2.25}$$

Here, the first term is already part of the original action S in (2.3). The second term arises from completing the square of the bosonic nonzero modes, which is necessary, because the covariant derivative of χ in the action S involves a term containing ϕ.

$$D_{\bar z}\chi^k = \partial_{\bar z}\chi^k + \partial_{\bar z}\phi^i\Gamma^k_{ij}\chi^j \tag{2.26}$$

Partial integration turns the Christoffel symbol into the Riemann curvature tensor appearing twice in the second term of S_0. After another partial integration, the Laplacian $D_{\bar z}D_z$ is part of the bosonic kinetic term. When completing the corresponding square, we need the inverse of $D_{\bar z}D_z$ appearing as the "Green's function" $G^{z\bar z}$ in S_0. In order to perform the path integral with the remnant action S_0 over the fermionic zero modes, one can endow the vector bundle $\mathcal{V}$ of ρ zero modes with an Hermitian inner product $(\cdot\,,\cdot)$, such that

$$S_0 = (\rho, F_{\mathcal{V}}\rho), \tag{2.27}$$

where $F_{\mathcal{V}}$ is the curvature of an Hermitian connection on $\mathcal{V}$. Here, we identify χ with one-forms on $\mathcal{M}_\Sigma(X,\beta)$, such that $F_{\mathcal{V}}$ becomes indeed a two-form on $\mathcal{M}_\Sigma(X,\beta)$, as S_0 is purely quadratic in χ. The integration of $\mathrm{e}^{-S_0} = \mathrm{e}^{-(\rho,F_{\mathcal{V}}\rho)}$ over ρ modes yields the Pfaffian of $F_{\mathcal{V}}$, which is proportional to the Euler class of the bundle $\mathcal{V}$, i.e. $\mathrm{Pf}(F_{\mathcal{V}}) \sim \mathrm{e}(\mathcal{V})$. Hence, the correlation function can be written as

$$\langle\mathcal{O}_1(P_1)\ldots\mathcal{O}_n(P_n)\rangle_\beta = \int\limits_{\mathcal{M}_\Sigma(X,\beta)} \mathrm{e}(\mathcal{V})\wedge \mathrm{ev}_1^*(\omega_1)\wedge\ldots\wedge\mathrm{ev}_n^*(\omega_n) \tag{2.28}$$

The Euler class $\mathrm{e}(\mathcal{V})$ is represented by an (l_ρ, l_ρ)-form. Therefore, the integration over the $(l_\rho + k)$-dimensional manifold $\mathcal{M}_\Sigma(X,\beta)$ can only be non-vanishing, if $k = \sum p_i = \sum q_i$, as we have already discussed at the end of the previous section.

3. Gromov-Witten Invariants

The correlation functions, we computed in the previous section, are interesting invariants of the symplectic manifold (X,ω) and hence motivate the Gromov-Witten invariants in the following way. Let $n\in\mathbb{N}_0$ and $\beta\in \mathrm{H}_2(X)$. We introduce the moduli space $\mathcal{M}_{g,n}(X,\beta)$ of holomorphic maps $\phi:\Sigma\to X$ satisfying $\phi_*[\Sigma]=\beta$. This means we look at the quotient space of all tuples $(\Sigma, P_1,\ldots,P_n,\phi)$, where Σ is a closed Riemann surface of genus g and P_i are distinct points in Σ, modulo a certain equivalence relation defined as follows

$$\begin{aligned} &\mathcal{M}_{g,n}(X,\beta) := \{(\Sigma,P_1,\ldots,P_n,\phi)\mid \phi:\Sigma\longrightarrow X \text{ holom.},\ \phi_*[\Sigma]=\beta\}/\sim \\ &\qquad \text{where}\quad (\Sigma,P_1,\ldots,P_n,\phi)\sim(\Sigma',P_1',\ldots,P_n',\phi') \\ &\qquad \iff\quad \exists\ \psi:\Sigma\to\Sigma' \text{ biholomorphic, such that} \\ &\qquad\qquad \phi=\phi'\circ\psi \quad\text{and}\quad \psi(P_i)=P_i'\ . \end{aligned} \tag{3.1}$$

Unfortunenately, $\mathcal{M}_{g,n}(X,\beta)$ is in general a very singular space having the structure of a Deligne-Mumford stack. At least locally one can think of a stack as the quotient of a scheme by a finite group. In this way, it is an algebraic generalization of an orbifold. But in contrast to an orbifold, a stack has in general a non-trivial isotropy group at a generic point and there is no notion of dimension any more. In order to make contact with the previous section, we first assume that $\mathcal{M}_{g,n}(X,\beta)$ is a smooth manifold. Compared with $\mathcal{M}_\Sigma(X,\beta)$, the holomorphic structure of the underlying Riemann surface Σ can now vary. If the genus $g \geqslant 2$, the moduli space of holomorphic structures on a closed Riemann surface of genus g has dimension $3g-3$. For every point P_i, we add to the surface, the dimension increases by one. Thus, with $\mathcal{M}_{g,n}$ shorthand for $\mathcal{M}_{g,n}(\mathrm{pt},0)$ we get

$$\dim_{\mathbb{C}} \mathcal{M}_{g,n} = 3g - 3 + n \ , \tag{3.2}$$

We assume, we are in the generic case $\dim \mathcal{M}_\Sigma(X,\beta) = k$, defined in the previous section. For every structure $(\Sigma, P_1, \ldots, P_n)$ there are holomorphic maps in $\mathcal{M}_\Sigma(X,\beta)$, i.e. locally $\mathcal{M}_{g,n}(X,\beta) \mathrel{\widehat{=}} \mathcal{M}_\Sigma(X,\beta) \times \mathcal{M}_{g,n}$ and therefore

$$\begin{aligned} \dim \mathcal{M}_{g,n}(X,\beta) &= \dim \mathcal{M}_\Sigma(X,\beta) + \dim \mathcal{M}_{g,n} \\ &= \int_\beta \mathrm{c}_1(X) + (d-3)(1-g) + n \end{aligned} \tag{3.3}$$

One can show that this formula is also true for $g = 0, 1$. If we are not in the generic case, but $\mathcal{M}_{g,n}(X,\beta)$ is still non-singular, there are obstructions to deformations of the holomorphic map ϕ, see chapter 24.4 in [1]. These obstructions constitute a vector bundle Ob over $\mathcal{M}_{g,n}(X,\beta)$ with fiber $\mathrm{Ob}(\Sigma, P_1, \ldots, P_n, \phi)$, similar to the bundle $\mathcal{V}$ of ρ zero modes before. Then one defines the virtual fundamental class by caping the fundamental class of $\mathcal{M}_{g,n}(X,\beta)$ with the Euler class of Ob.

$$[\mathcal{M}_{g,n}(X,\beta)]^{\mathrm{vir}} := [\mathcal{M}_{g,n}(X,\beta)] \cap \mathrm{e}(\mathrm{Ob}) \tag{3.4}$$

This is a homology class of degree equal to the dimension calculated in eqn (3.3)

$$\mathrm{vdim}\, \mathcal{M}_{g,n}(X,\beta) := \int_\beta \mathrm{c}_1(X) + (d-3)(1-g) + n \ , \tag{3.5}$$

where this number is called the virtual dimension of $\mathcal{M}_{g,n}(X,\beta)$. For cohomology classes $\omega_i \in \mathrm{H}^\bullet_{\mathrm{dR}}(X)$, the *Gromov-Witten invariants* are then defined as

$$\langle \omega_1, \ldots, \omega_n \rangle^X_{g,\beta} \ := \int\limits_{[\mathcal{M}_{g,n}(X,\beta)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\omega_1) \wedge \ldots \wedge \mathrm{ev}_n^*(\omega_n) \ , \tag{3.6}$$

where the i-th evaluation map is now given as

$$\mathrm{ev}_i : \mathcal{M}_{g,n}(X,\beta) \to X, \ [\Sigma, P_1, \ldots, P_n, \phi] \mapsto \phi(P_i). \tag{3.7}$$

If $\mathcal{M}_{g,n}(X,\beta)$ fails to be smooth, there exists in general neither a fundamental class $[\mathcal{M}_{g,n}(X,\beta)]$ nor a vector bundle like Ob. Nevertheless, one can always define a virtual fundamental class $[\mathcal{M}_{g,n}(X,\beta)]^{\mathrm{vir}}$ in the homology of $\mathcal{M}_{g,n}(X,\beta)$ in degree $\mathrm{vdim}\, \mathcal{M}_{g,n}(X,\beta)$, such that the Gromov-Witten invariants can be defined and computed as above. One often combines the Gromov-Witten invariants for different β into a total Gromov-Witten invariant

$$\begin{aligned} \langle\omega_1,\dots,\omega_n\rangle_g^X &:= \sum_{\beta\in \mathrm{H}_2(X)} \mathrm{e}^{-t\int_\beta \omega}\ \langle\omega_1,\dots,\omega_n\rangle_{g,\beta}^X \\ &= \sum_{\beta\in \mathrm{H}_2(X)} \langle\omega_1,\dots,\omega_n\rangle_{g,\beta}^X\ q^\beta, \end{aligned} \tag{3.8}$$

where we have adopted the shorthand notation $q^\beta := \mathrm{e}^{-t\,\omega\cdot\beta} = \mathrm{e}^{-t\int_\beta\omega}$, motivated by the formal identity $q^{\beta_1+\beta_2} = q^{\beta_1}q^{\beta_2}$.

4. Quantum Cohomology

The Gromov-Witten invariants give rise to a new product on cohomology

$$*\colon \mathrm{H}^\bullet_{\mathrm{dR}}(X)\times \mathrm{H}^\bullet_{\mathrm{dR}}(X) \longrightarrow \mathrm{H}^\bullet_{\mathrm{dR}}(X)\,. \tag{4.1}$$

Let $\omega,\eta\in \mathrm{H}^\bullet_{\mathrm{dR}}(X)$ be cohomology classes and $(T^k)_k$ be a basis for the cohomology $\mathrm{H}^\bullet_{\mathrm{dR}}(X)$ with dual basis $(T_k)_k$ regarding the intersection pairing $(\cdot\,,\cdot)$, i.e.

$$\big(T^i,T_j\big) := \int_X T^i\wedge T_j = \delta^i{}_j\ , \tag{4.2}$$

where the intersection pairing $(\cdot\,,\cdot)$ is non-degenerate due to Poincaré duality on the manifold X. Then we define the *(small) quantum product* as

$$\omega * \eta := \sum_k \langle\omega,\eta,T_k\rangle_g^X\, T^k\ . \tag{4.3}$$

Note the dependence on the genus g. There is also a big quantum product involving not just the 3-pt functions but all correlation functions and hence more information. We know $\mathcal{M}_{g,n}(X,\beta)$ consists of classes of holomorphic maps ϕ satisfying $\phi_*[\Sigma]=\beta$. As the integral of the Kähler form ω over a holomorphic curve is non-negative, we have

$$\int_\beta\omega = \int_\Sigma \phi^*\omega \geqslant 0\ . \tag{4.4}$$

Hence, if we take the real parameter t to infinity, we get

$$\langle\omega,\eta,T_k\rangle = \sum_{\beta\in \mathrm{H}_2(X)} \langle\omega,\eta,T_k\rangle_\beta\ \mathrm{e}^{-t\int_\beta\omega} \xrightarrow[t\to\infty]{} \langle\omega,\eta,T_k\rangle_0\ , \tag{4.5}$$

where we have neglected the X and the g in the indices. Thus, only the $\beta\!=\!0$-term of the GW invariant survives this limit. Now assume $g=0$, i.e. Σ is a Riemann sphere. Then by some version of Liouville's theorem, a holomorphic map ϕ with $\phi_*[\Sigma]=0$ has to be the constant map. Consequently, $\mathcal{M}_{0,n}(X,0)\cong\mathcal{M}_{0,n}\times X$. Since we want to compute the 3-point function, we take $n=3$ and look for complex structures on the Riemann sphere with 3 punctures. By the uniformization theorem there exists a unique complex structure on the sphere. The automorphisms on the Riemann sphere are given by Möbius transformations, which are uniquely determined by specifying the image of 3 distinct points. This tells us that $\mathcal{M}_{0,3}\cong \mathrm{pt}$, such that the evaluation map ev_i becomes the identity on X and $[\mathcal{M}_{0,3}(X,0)]^{\mathrm{vir}} = [X]$. In summary, the zeroth summand of the GW invariant simplifies to

$$\langle \omega, \eta, T_k \rangle_0 = \int_X \omega \wedge \eta \wedge T_k \tag{4.6}$$

$$\implies \quad \omega \cup \eta = \omega \wedge \eta = \sum_k \left(\int_X \omega \wedge \eta \wedge T_k \right) T^k = \sum_k \langle \omega, \eta, T_k \rangle_0 \, T^k$$

Hence, if $g = 0$, we get the usual cup product in cohomology in the limit $t \to \infty$.

$$\omega * \eta \quad \xrightarrow[t \to \infty]{} \quad \omega \cup \eta \tag{4.7}$$

In this sense, the $\beta \neq 0$-terms in the quantum product $*$ can be interpreted as quantum corrections to the usual cup product $\cup$, when the parameter t is large.

5. Interpretation

Let $\omega_i \in \mathrm{H}^\bullet_{\mathrm{dR}}(X)$ be cohomology classes. Then their Poincaré dual classes in homology can be represented by complex submanifolds Z_i in general position. In special cases, e.g. in the generic or non-generic case we discussed before, the GW invariant counts the number of holomorphic curves of given genus g in X representing a prescribed homology class β and intersect the submanifolds Z_i non-trivially.

$$\langle \omega_1, \ldots, \omega_n \rangle^X_{g,\beta} = \# \left\{ \phi : \Sigma \to X \text{ holomorphic} \;\middle|\; \begin{matrix} \phi_*[\Sigma] = \beta \\ \phi(\Sigma) \cap Z_i \neq \varnothing \end{matrix} \right\} \tag{5.1}$$

However, in general this counting takes place in some virtual way, as the GW invariants do not need to yield integral numbers, but can be truly rational.

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