# Symmetry Breaking

The topic of the talk was symmetry breaking. Here the symmetry breaking is discussed separately in the cases:

**1 Quantum Mechanics**

**2 Statistical Mechanics**

**3 Quantum Field Theory**

The Statistical Mechanics part is treated with the most detail and only the quantum mechanical setting is discussed, as it is the only part where an adequate mathematical language was available to me. This language is that of quasi-local $C^*$ algebras and KMS states on such algebras. Definitions and a brief discussion can be found in the

**A Appendix**.

In contrast the section on Quantum Field Theory is brief and uses usual language found in theoretical physics textbooks.

## References


For the Quantum Mechanics section, the textbook
**Deligne et. al.,** *Quantum Fields and Strings: A course for Mathematicians Vol. 2*
was used.

For the Statistical Mechanics part the background was taken from
**O. Bratteli, D. Robinson,** *Operator Algebras and Quantum Statistical Mechanics, Vol. 1 & 2*
and the parts on Mermin-Wagner from an online write-up:
**B. Nachtergalle,** *The Mermin-Wagner Theorem*
link: `https://www.math.ucdavis.edu/~bxn/mermin-wagner.pdf`

For the Quantum Field Theory part, the books
**S. Coleman,** *Aspects of symmetry*
**M. Shifman,** *Advanced topics in Quantum Field Theory*
**S. Weinberg,** *Quantum Theory of Fields II*
were used.

# 1. Quantum Mechanics

In most cases in Quantum Mechanics the ground state is unique. A consequence is that the ground state will be invariant under any symmetries of the system, and thus any pertubation theory built up around the ground state will be invariant under the symmetry. In contrast this does not happen in classical mechanics, indeed if one looks at pertubation theory around a groundstate in classical mechanics one does not need to see any trace of the symmetry.

In this section we will prove for non-relativistic bosonic particle in one dimension that the ground state is unique. The proof ideas extend to any number of bosons on $\mathbb{R}^n$.
**Theorem.** *Let $H = -\frac{1}{2}\partial_x^2 + U(x)$ be a Schrödinger operator on $L^2(\mathbb{R})$ with $U(x) \to \infty$ as $x \to \pm\infty$ and let $E_0$ be the smallest eigenvalue of $H$, then there exists a unique (up to a factor) state $\psi \in L^2(\mathbb{R})$ s.t. $H\psi = E_0\psi$.*

The proof follows from a Lemma, together with the existence of a ground state:

**Lemma.** Any ground state wave function has constant sign.

The proof then follows since if $\psi, \phi$ are two ground states one has $\langle\psi|\,|\phi\rangle \neq 0$, since solutions must have support $\mathbb{R}$ (or be constant 0, this follows from the assumption of $U$ being continuous and the Picard Lindelöf theorem). This is a contradiction to the ground state space being more than 1 dimensional.

To prove the lemma note that a ground state minimises $\langle\psi|\,H\,|\psi\rangle$ on the sphere $\|\psi\| = 1$. If one writes $\psi = |\psi|(x)e^{i\varphi(x)}$, then equation to minimise is:

$$\int dx(\frac{1}{2}|\psi'|^2 + U(x)|\psi|^2) = \int dx(\frac{1}{2}|\psi|'^2 + \frac{1}{2}|\psi|^2\varphi'^2 + U(x)|\psi|^2) \tag{1}$$

By having $\varphi$ be constant outside of $|\psi| = 0$ one can obviously make $\langle\psi|\,H\,|\psi\rangle$ smaller. So the wave function has constant signs on the connected components of $\{x \mid \psi(x) \neq 0\}$. If one assumes this sign is not globally constant, then a variational argument gives that there exist states where $\langle\psi|\,H\,|\psi\rangle$ is lower. To see this note that the set $\{x \mid \psi(x) = 0\}$ consists of isolated points (from the Schrödinger equation) and then if $\psi$ has a sign change, the derivative of $|\psi|$ will make a jump at that point. If one then varies with a real valued $\eta$:

$$\int dx(\frac{1}{2}(|\psi|' + \epsilon\eta')^2 + U(x)(|\psi| + \epsilon\;\eta)^2)$$

One finds that the near one of the points $a$ where $|\psi|'$ jumps, after partial integration, that the integral is changed by:

$$\frac{\epsilon}{2}\eta(a)(|\psi|'(a^-) - |\psi|'(a^+)) - \frac{\epsilon}{2}\int |\psi|''\eta + \frac{\epsilon}{2}\int \eta|\psi|U(x) + \mathcal{O}(\epsilon)^2$$

By choosing an $\eta$ that is strongly localised near $a$ the integral terms become negligible, but with the correct sign the value of the first term can be made negative, so one could not have been in a situation where (1) was minimal.

# 2. Statistical Mechanics

In this section we define what it means for symmetry breaking to occur in a quantum statistical setting and present a general theorem, which guarantees the absence of symmetry breaking in certain cases. The theorem is briefly discussed and applied to derive a slightly generalised Mermin Wagner theorem, which states that in 2d lattice systems continuous compact symmetries are not broken.

## 2.1. Symmetry breaking in statistical physics

We start by considering as an example a spin system on a finite lattice $\Lambda$, the problem is described in a Hilbert space $\mathcal{H} = \bigotimes_{x\in\Lambda} \mathcal{H}_x$. $\mathcal{H}_x$ are the Hilbert spaces of the individual spins. One looks at Hamiltonians of the form:

$$H_\Lambda = \sum_{x,y\in\Lambda} J(x,y)\ \vec{S}_x \cdot \vec{S}_y. \tag{2}$$

Here $J(x,y) \in \mathbb{C}$ and $\vec{S}_x$ the spin operators at $x$. The Hamiltonian determines a canonical statistical state $\omega_{\Lambda,\beta}$ and time evolution $\alpha_t$ acting on the algebra of bounded observables:

$$\alpha_t(A) = e^{iH_\Lambda t} A e^{-iH_\Lambda t}$$
$$\omega_{\Lambda,\beta}(A) = \frac{\mathrm{Tr}(Ae^{-\beta H_\Lambda})}{\mathrm{Tr}(e^{-\beta H_\Lambda})}$$

If a unitary map obeys $U^\dagger H U = H$ then it is called a symmetry of the time evolution. An example for the Hamiltonian (2) would be a rotation in spin space. From permutation invariance of the trace it follows that

$$\omega_{\Lambda,\beta}(U^\dagger A U) = \frac{\mathrm{Tr}(U^\dagger A U e^{-\beta H_\Lambda})}{\mathrm{Tr}(e^{-\beta H_\Lambda})} = \frac{\mathrm{Tr}(Ae^{-\beta U H_\Lambda U^\dagger})}{\mathrm{Tr}(e^{-\beta H_\Lambda})} = \omega_{\Lambda,\beta}(A)$$

and the statistical sate is also invariant under the symmetry, so such symmetries cannot broken in finite systems. One also notes that if the following limits exist then:

$$\lim_{\Lambda\to\infty} \omega_{\Lambda,\beta}(U^\dagger A U) = \lim_{\Lambda\to\infty} \omega_{\Lambda,\beta}(A) =: \langle A \rangle$$

and such a definition of the thermodynamic limit would not be sophisticated enough to describe symmetry breaking. In the method of quasi-averages one defines an alternative thermodynamic limit state by adding a symmetry breaking term to the Hamiltonian and letting it go to 0 in the thermodynamic limit:

$$H_\Lambda \to H_\Lambda + \vec{h} \sum_{x\in\Lambda} \vec{S}_x$$
$$\langle A \rangle := \lim_{h\to 0} \lim_{\Lambda\to\infty} \omega_{\Lambda,\beta,h}(A).$$

This would describe an alternative thermodynamic limit state. Here it is possible that in the thermodynamic limit $\langle U^\dagger A U \rangle \neq \langle A \rangle$ and one says the symmetry is broken. This method is however constructive and not exhausting if one wishes to show the absence of symmetry breaking.

The previous section serves as motivation for the following statements:

- Statistical states are described by functionals on the algebra of observables.
- Symmetry breaking does not occur in finite systems.
- The canonical statistical states have a compatibility relation with the time-evolution.
- There may not be a unique canonical statistical state in an infinite system.

The last statement can be used to find a proper definition of symmetry breaking in the statistical mechanics setting:

- One says a symmetry $\tau$ of the time-evolution is broken if any of the canonical statistical states are not invariant under the symmetry (ie $\omega \circ \tau \neq \omega$ for a state $\omega$). Otherwise one says the symmetry conserved or not broken.

The correct criterium for a state to be a canonical state is the KMS condition. The correct way to describe the algebra of observables "at infinity" (ie the algebra of the thermodynamic limit state) is via quasi-local algebras. These topics are briefly introduced and described in the appendix for the case that all operators are continuous (ie bounded).

## 2.2. Mermin Wagner Theorem

**Theorem.** *Let $\mathcal{A}$ be a a quasi-local $C^*$-Algebra, $\alpha_t$ a time evolution on $\mathcal{A}$ and $\tau$ a symmetry of $\alpha_t$, meaning $\tau \circ \alpha_t = \alpha_t \circ \tau$. Let $\delta := \mathrm{gen}(\alpha_t)$. If:*

***MWH1*** *$\tau$ is approximately inner, meaning there is a sequence of unitaries $U_n \in \mathcal{A}$ s.t. $\forall A \in \mathcal{A} \lim_{n\to\infty} \|\tau(A) - U_n^* A U_n\| = 0$ and $U_n \in \mathcal{D}(\delta)$*

***MWH2*** *one of the following holds:*

*(i) $\exists M \in \mathbb{R}$ s.t. $\|[\delta, U_n]\| \leq M$ for all $n$*

*(ii) $\|U_n^* \delta U_n + U_n \delta U_n^* - 2\delta\| \leq M$ for all $n$ and $\forall\beta$-KMS states $\omega$ one has $\omega \circ \tau^2 = \omega$ .*

*Then all $\beta$-KMS states are $\tau$ invariant.*

The proof of the theorem is not given.

**Remark.** The condition **MWH2** (ii) will be used to show that there is no breaking of continuous compact symmetries. If one considers for example an approximately inner symmetry action via $S^1$ and assumes the necessary bounds to hold, since $\tau_\pi^2 = \tau_{2\pi} = \mathrm{id}$, one then has for all $\beta$-KMS states:

$$\omega \circ \tau_{2\pi} = \omega \implies \omega \circ \tau_\pi = \omega \implies \omega \circ \tau_{\pi/2} = \omega \implies \omega \circ \tau_{\pi/4} = \omega \implies \ldots$$

So all states are invariant under symmetries of the form $\tau_\phi$ with $\phi \in \{2^{-n}\pi \mid n \in \mathbb{N}\}$ and thus also under compositions of such states, ie under angles $\phi = \sum_{n=0}^{N} a_n 2^{-n}\pi$, $a_n \in \mathbb{N}$. However such angles are dense in $S^1$, and since for all $A \in \mathcal{A}$ $(\omega \circ \tau_\phi)(A)$ is continuous in $\phi$ one has $(\omega \circ \tau_\phi)(A) = \omega(A)$ for all $\phi$. Thus $\omega \circ \tau_\phi = \omega$ for all $\phi \in S^1$.

The argument extends to any compact continuous symmetry by considering a generating set of compact 1 dimensional subgroups.

## 2.3. Application to 2d lattice gases

The former theorem can be used to show that compact symmetries in 2d lattice gases are not broken. To do this we consider a Hamiltonian of the form

$$H = \sum_{x,y \in \mathbb{Z}^2} J(x,y)\, \Phi_{x,y}. \tag{3}$$

Here $\Phi_{x,y} \in \mathcal{A}_{\{x,y\}}$, ie if the algebra is represented on a Hilbert space $\bigotimes_{x \in \mathbb{Z}^2} \mathcal{H}_x$ then $\Phi_{x,y}$ is a linear combination of operators of the form $\bigotimes_{z \in \mathbb{Z}^2} \Phi_z$ with $\Phi_z = \mathbb{1}$ for $z \notin \{x,y\}$. Wlog symmetry $\Phi_{x,y} = \Phi_{y,x}$ is assumed.

The individual Hilbert spaces $\mathcal{H}_x$ in the representation do not need to be the same, one could for example be considering spins of different magnitudes at different sites. One considers a uniform bound on $\|\Phi_{x,y}\|$ and a bound on the range of $J(x,y)$ via:

$$\sup_{x \in \mathbb{Z}^2} \sum_{y \in \mathbb{Z}^2} J(x,y)|x-y|^2 < \infty.$$

The restriction on $J(x,y)$ implements the 2 dimensionality of the lattice; with no restrictions on $J$ one could biject any lattice into $\mathbb{Z}^2$ and push the $J$ forward from it.

As stated before, if the consideration holds for any symmetry group action of $S^1$ it will hold for any continuous compact symmetry group, so we will consider the case $S^1$. We assume the symmetry to be "local" in the sense that it is represented on $\bigotimes_{x \in \mathbb{Z}^2} \mathcal{H}_x$ via $\otimes_{x \in \mathbb{Z}^2} U_x$ with unitaries $U_x$ generated by hermitian operators $X_x$.

Further one assumes the generators to be uniformly bounded by some constant $\|X_x\| \leq G$.

In this case **MWH1** can be easily verified by defining $\Lambda_m := [-m,m]^2 \subseteq \mathbb{Z}^2$ and then taking $U_m(\phi)$ as:

$$U_m(\phi) := \bigotimes_{x \in \Lambda_{2m}} U_x(\phi_m(x)), \qquad \phi_m(x) := \begin{cases} \phi & x \in \Lambda_m \\ 2 - \frac{\min |x_1|,|x_2|}{m} & \phi \in \Lambda_{2m} - \Lambda_m \\ 0 & \phi \notin \Lambda_{2m} \end{cases} \tag{4}$$

For any local $A \in \mathcal{A}$ one has that $U_m^* A U_m$ eventually becomes the action of the symmetry, and thus for any quasi-local $A \in \mathcal{A}$ one has $\lim_{m\to\infty} \|U_m^* A U_m - U(A)\| = 0$, so the symmetry is approximately inner.

The only work is to show that **MWH2** (ii) holds. Here we consider:

$$\|U_m(\phi_m)HU_m^*(\phi_m) + U_m^*(\phi_m)HU_m(\phi_m) - 2H\| \leq \sum_{x,y} |J(x,y)| \; \|\Delta_{x,y}\| \tag{5}$$

In the sum at least one of $x, y$ lies in $\Lambda_{2m}$. Further one has defined:

$$\begin{aligned}\Delta_{x,y} := {} & e^{i(\phi_m(x)X_x + \phi_m(y)X_y)}\Phi_{x,y}\, e^{-i(\phi_m(x)X_x+\phi_m(y)X_y)} \\ & + e^{i(\phi_m(x)X_x + \phi_m(y)X_y)}\Phi_{x,y}\, e^{-i(\phi_m(x)X_x+\phi_m(y)X_y)} - 2\Phi_{x,y}\end{aligned} \tag{6}$$

In order to calculate this one uses the formula

$$e^A B e^{-A} = \sum_{n=0}^{\infty} \frac{1}{n!} \operatorname{ad}_A^n(B) \tag{7}$$

which holds in complete normed algebras, as can be seen from $\operatorname{ad}_a = L_a - R_a$ with $L_a, R_a$ the left and right multiplication with $a$. Then

$$e^a B e^{-a} = \exp(L_a)(\exp(-R_a)(B)) = \exp(L_a - R_a)(B) = \exp(\operatorname{ad}_a)(B)$$

since $L_a$ and $R_a$ commute. So

$$\Delta_{x,y} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \operatorname{ad}^n_{\phi_m(x)X_x+\phi_m(y)X_y} \Phi_{x,y} + \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \operatorname{ad}^n_{\phi_m(x)X_x+\phi_m(y)X_y} \Phi_{x,y} - 2\Phi_{x,y} \tag{8}$$

$$= 2\sum_{n=1}^{\infty} \frac{(-1)^n}{2n!} \operatorname{ad}^{2n}_{\phi_m(x)X_x+\phi_m(y)X_y} \Phi_{x,y} \tag{9}$$

and the $n = 0$ and all odd $n$ terms in the sum cancel each other. Further one splits

$$\phi_m(x)X_x + \phi_m(y)X_y = \frac{1}{2}(\phi_m(x) + \phi_m(y))(X_x + X_y) + \frac{1}{2}(\phi_m(x) - \phi_m(y))(X_x - X_y)$$

all these terms commute, so one can separate the adjoints in (9):

$$\Delta_{x,y} = 2\sum_{n=1}^{\infty} \frac{(-1)^n}{2n!} \left( \frac{1}{2}(\phi_m(x) + \phi_m(y)) \operatorname{ad}_{X_x+X_y} + \frac{1}{2}(\phi_m(x) - \phi_m(y)) \operatorname{ad}_{X_x-X_y} \right)^{2n} \Phi_{x,y} \tag{10}$$

However $\operatorname{ad}_{X_x+X_y} \Phi_{x,y} = 0$, since this is the action of the symmetry generator on the $x, y$ part of the Hamiltonian. Since the two adjoints in the above equation commute, the $\operatorname{ad}_{X_x+X_y}$ term drops out entirely. One is left with

$$\|\Delta_{x,y}\| \leq 2\sum_{n=1}^{\infty} \frac{1}{2^n 2n!} |\phi_m(x) - \phi_m(y)|^{2n} \| \operatorname{ad}_{X_x-X_y} \|^{2n} \|\Phi_{x,y}\| \tag{11}$$

Note $\left\|\mathrm{ad}_{X_x - X_y}\right\| \leq 2G$. From (4) one finds that $|\phi_m(x) - \phi_m(y)| \leq \frac{|x-y|}{m}|\phi|$ and $|\phi_m(x) - \phi_m(y)| \leq |\phi|$. This allows one to split off and bound one $(\phi_m(x) - \phi_m(y))^2$ term in (11) via $\frac{|x-y|^2}{m^2}\ |\phi|$ and the rest by $|\phi|$ to get

$$\|\Delta_{x,y}\| \leq 2\frac{|x-y|^2}{m^2}\sum_{n=1}^{\infty}\frac{1}{2n!}(|\phi|G)^{2n}\|\Phi_{x,y}\| \tag{12}$$

Putting this back into (5) gives

$$\|....\| \leq 2\sum_{x,y}|J(x,y)|\frac{|x-y|^2}{m^2}\ e^{|\phi|\ G}\|\Phi_{x,y}\| \tag{13}$$

$$\leq 4\sum_{x\in\Lambda_{2m}, y\in\mathbb{Z}^2}|J(x,y)|\frac{|x-y|^2}{m^2}\ e^{|\phi|\ G}\|\Phi_{x,y}\| \tag{14}$$

$$\leq \sup_{x\in\mathbb{Z}^2}\sum_{y\in\mathbb{Z}^2}|J(x,y)|\,|x-y|^2\cdot\text{const.} \tag{15}$$

which verifies **MWH2** (ii), and thus all $\beta$-KMS states are invariant under the action of the symmetry.

# 3. Quantum Field Theory

In Quantum Field Theory every breaking of a global continuous symmetry is associated with a massless boson coupled to the symmetry generator. This is known as the Goldstone theorem. In this section the standard argument for the theorem is presented at the level of rigour found in physics textbooks. A theorem by Coleman states in rough analogy to the Mermin Wagner theorem that massless bosons cannot exist in a 2 dimensional QFT, meaning that symmetry breaking cannot occur.

## 3.1. Goldstone theorem

In Quantum Field Theory one associates a global continuous symmetry with a current $j^\mu$ that is preserved, ie

$$\partial_\mu j^\mu = 0.$$

In the following we will assume the current to be hermitian for simplicity. The generator of the current is given by $Q = \int d^{D-1}x\, j^0(x)$, from $\partial_\mu j^\mu = 0$ it follows that this time independent. If $\phi(x)$ is a local field, $[\phi, Q] =: \chi$ also defines a local field.

One says the symmetry is broken if $Q\,|\text{vac}\rangle \neq 0$, or if there exists a local field so that $\langle\text{vac}|\,\chi\,|\text{vac}\rangle \neq 0$. In other words the vacuum state is not invariant under the symmetry. If we assume the symmetry is broken and consider the corellator:

$$\Pi^\mu(q) = -\int d^D x\, e^{iqx}\, \langle\text{vac}|\, T(j^\mu(x), \phi(0))\, |\text{vac}\rangle$$

One notes that

$$iq_\mu \Pi^\mu = -i \int d^D x\, (\partial_\mu e^{iqx})\, \langle\text{vac}|\, T(j^\mu(x), \phi(0))\, |\text{vac}\rangle$$

Out of habit one would now like to use partial integration in order to bring the derivative over to the $j^\mu$ term. But one must note that $T(j^\mu(x), \phi(0))$ is not continuous at $x^0 = 0$, so one gets boundary terms. If one splits the the integral over $x^0$ into $(-\infty, 0)$ and $(0, \infty)$, then the term is smooth on both regions. Only the boundary terms from the $x^0$ integrations survive, and in this case one drops the infinity boundary terms. The result is then:

$$\begin{aligned} iq_\mu \Pi^\mu &= -i\, \langle\text{vac}| \int e^{i\vec{q}\vec{x}} d^{D-1}x\, (j^0(x)\phi(0) - \phi(0) j^0(x)) - \int d^D x\, e^{iqx} T(\partial_\mu j^\mu(x), \phi(0))\, |\text{vac}\rangle \\ &= -i\, \langle\text{vac}| \int d^{D-1}x\, e^{i\vec{q}\vec{x}}\, \left[j^0(x), \phi(0)\right]\, |\text{vac}\rangle \end{aligned}$$

being careless with limit $q \to 0$ (like we were careless with the boundary at infinity) and pulling it into the integral then gives:

$$\lim_{q\to 0} iq_\mu \Pi^\mu = -i\, \langle\text{vac}|\, [Q, \phi(0)]\, |\text{vac}\rangle$$

and $\Pi^\mu$ has a $q^\mu/q^2$ singularity, which is a term that one expects only from a massless boson coupled to both $j^\mu$ and $\phi$.

**Remark.** If symmetry breaking occurs classically, (that is the Lagrangian is invariant under a continuous symmetry but no classical ground state is), then one can already see the Goldstone boson on a perturbation level. As an example consider scalar fields $\phi_1, .., \phi_n$ in a

$$(\vec{\phi}^2 - a^2)^2$$

term in the Lagrangian. Choose a vector $\vec{\phi}_0$ with modulus $a$ and write $\vec{\phi} = \vec{\phi}_0 + \overrightarrow{\text{Long}} + \overrightarrow{\text{Trans}}$, where $\overrightarrow{\text{Trans}}$ is orthogonal to $\vec{\phi}_0$ and $\overrightarrow{\text{Long}}$ parallel. The term in the Lagrangian becomes:

$$((\vec{\phi}_0 + \overrightarrow{\text{Long}} + \overrightarrow{\text{Trans}})^2 - a^2)^2 = (2\,\vec{\phi}_0 \cdot \overrightarrow{\text{Long}} + \overrightarrow{\text{Trans}}^2 + \overrightarrow{\text{Long}}^2)^2$$

and the only $\overrightarrow{\text{Trans}}^2$ term is an interaction term so the transversal modes all correspond to massless fields.

## 3.2. Coleman theorem

The Coleman theorem states that massless bosons are not well defined in a 2 dimensional QFT. Specifically if in 2 dimensions the greens function in momentum space should look like:

$$\int d^2x\, e^{ipx} \langle \text{vac}| T(\phi(x), \phi(0)) |\text{vac}\rangle = \frac{i}{p^2}.$$

Then in coordinate space one has:

$$\langle \text{vac}| T(\phi(x), \phi(0)) |\text{vac}\rangle = \int \frac{d^2p}{2\pi}\, e^{-ipx}\, \frac{i}{p^2}$$

which has an infra-red divergence in neighbourhoods of 0. This divergence is more drastic and problematic than the usual infinities one finds all over Quantum Field Theory; the theory cannot be made well defined, although it is not entirely clear to me why. Regularisation of the integral via a mass term would result in a Greens function that behaves as:

$$G(x) \simeq \ln(x)$$

and signals from the origin would be amplified in detectors far away.

# A. Appendix

## A.1. Quasi-local $C^*$ algebras

$C^*$ algebras are modelled on closed sub-algebras of $\mathcal{L}(\mathcal{H})$, the algebra of linear bounded linear operators on a Hilbert space $\mathcal{H}$, which are invariant under the $^\dagger$ operation. Briefly:

**Definition.** A $C^*$ algebra is a complete normed $\mathbb{C}$ algebra together with an involution $^*$ (meaning $(AB)^* = B^*A^*$, $(\lambda A)^* = \bar{\lambda}A^*$ and $A^{**} = A$) so that:

$$\|A\|^2 = \|A^*A\| \tag{16}$$

In physics these algebras describe bounded observables in quantum systems. (Also, every $C^*$ algebra is isomorphic to a sub-algebra of $\mathcal{L}(\mathcal{H})$ for some Hilbertspace $\mathcal{H}$.)

In statistical physics and quantum field theory one often considers finite size systems, which are easy to work with, and takes some sort of limit where the size becomes infinite. In order to work with the infinite system one would like an algebra of observables “at infinity”, which includes the finite size sub-algebras. One way to describe this is via quasi-local algebras, to that end we define what it means for a directed set to have an orthogonality relation:

**Definition.** A directed set $I$ is said to posses and orthogonality relation if there is a relation $\perp$ between two elements of $I$ so that:

(i) if $\alpha \in I$ there exists a $\beta \in I$ with $\alpha \perp \beta$.

(ii) if $\alpha \leq \beta$ and $\beta \perp \gamma$ then $\alpha \perp \gamma$.

(iii) if $\alpha \perp \beta$ and $\alpha \perp \gamma$ then there exists a $\delta \in I$ s.t. $\alpha \perp \delta$ and $\delta \geq \beta, \gamma$.

As a reminder, a directed set is a partially ordered set so that for any two $\alpha, \beta \in I$ there exists a $\gamma \in I$ with $\gamma \geq \alpha, \beta$. In this write-up we consider $I$ for example to be the finite sub-sets of an infinite lattice $\Lambda$, where $A \leq B$ if $A \subseteq B$ and $A \perp B$ if $A \cap B = \emptyset$. One could also consider bounded open sub-sets of some infinite volume space for example.

In a quasi-local algebra one has $C^*$ algebras $\mathcal{A}_\alpha$ associated to each $\alpha \in I$ so that $\mathcal{A}_\beta \supseteq \mathcal{A}_\alpha$ whenever $\alpha \geq \beta$, and the local algebras $\mathcal{A}_\alpha$ and $\mathcal{A}_\beta$ are “independent” whenever $\alpha \perp \beta$. To be precise:

**Definition.** A quasi-local $C^*$ algebra is a $C^*$ algebra $\mathcal{A}$ together with a net $\{\mathcal{A}_\alpha\}_{\alpha \in I}$ of sub-algebras of $\mathcal{A}$ s.t. $I$ has an orthogonality relation $\perp$ and the following hold:

(i) If $\alpha \leq \beta$ then $\mathcal{A}_\alpha \subseteq \mathcal{A}_\beta$.

(ii) $\mathcal{A} = \overline{\bigcup_{\alpha \in I} \mathcal{A}_\alpha}$, where the closure is taken wrt the norm.

(iii) The algebras $\mathcal{A}_\alpha$ have a common identity.

(iv) If $\alpha \perp \beta$ then $[\mathcal{A}_\alpha, \mathcal{A}_\beta] = \{0\}$.

There exists a more general formulation of (iv), if one has an automorphism $\sigma$ on $\mathcal{A}$ s.t. $\sigma^2 = \mathrm{id}$, then one can split $A \in \mathcal{A}$ into odd and even parts, $A = A^+ + A^-$, $A^\pm := \frac{A \pm \sigma(A)}{2}$, where $\sigma(A^+) = A^+$ and $\sigma(A^-) = -A^-$. So $\mathcal{A}$ also splits into a direct sum of odd and even parts of $\sigma$: $\mathcal{A} = \mathcal{A}^+ \oplus \mathcal{A}^-$. If further $\sigma(\mathcal{A}_\alpha) = \mathcal{A}_\alpha$, then one takes:

(iv)' if $\alpha \perp \beta$ then $[\mathcal{A}^+_\alpha, \mathcal{A}^+_\beta] = \{0\}$, $\{\mathcal{A}^-_\alpha, \mathcal{A}^-_\beta\} = \{0\}$ and $[\mathcal{A}^+_\alpha, \mathcal{A}^-_\beta] = \{0\}$.

Here $\sigma$ splits the algebra into fermionic and bosonic parts. However we will not use this in the write-up.

## A.2. KMS states

KMS states are a certain class of states, that is positive norm one functionals, on a $C^*$ algebra that obey a compatibility relation with a time-evolution. In order to write this down we first define the notion of positive elements and that of a state.

**Definition.** A hermitian element $A$ (meaning $A^* = A$) of a $C^*$ algebra $\mathcal{A}$ is called positive if the following equivalent statements hold:

(i) There exists a hermitian $B \in \mathcal{A}$ so that $A = BB$.

(ii) There exists an element $B \in \mathcal{A}$ so that $A = B^*B$.

(iii) The spectrum $\sigma(A)$ (the set $\lambda \in \mathbb{C}$ for which $\lambda - A$ is not invertible in the unitisation of $\mathcal{A}$) is a subset of $\mathbb{R}_{\geq 0}$.

For example in $\mathrm{Mat}_{n\times n}(\mathbb{C})$ an element is positive iff it is hermitian and all eigenvalues are positive. In $C^*$ algebras of the form $C_0(X)$ an element is positive iff it only takes positive values.

**Definition.** A functional $\omega : \mathcal{A} \to \mathbb{C}$ is a state if $\omega(A) \geq 0$ for all positive $A \in \mathcal{A}$ and $\|\omega\| = 1$.

In the previous example one notes from the Riesz representation theorem that the functionals on $C_0(X)$ correspond to measures on $X$, and that positive norm one functionals correspond to positive mass one measures, ie states on the $C^*$ algebra $C_0(X)$ correspond to probability measures on $X$.

In general for every state $\omega$ there exists a Hilbert space $\mathcal{H}$, a representation $\pi_\omega : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ and a vector $\Omega \in \mathcal{H}$ so that $\omega(A) = \langle \Omega | \, \pi_\omega(A) \, | \Omega \rangle$.

**Definition.** A $C^*$ dynamical system on $\mathcal{A}$ is a group homomorphism $\mathbb{R}$ to the automorphism group of $\mathcal{A}$ that is strongly continuous. This means the map $t \mapsto \alpha_t(A)$ as a map $\mathbb{R} \to \mathcal{A}$ is continuous for all $A \in \mathcal{A}$ with the norm topology on $\mathcal{A}$. A stronger assumption (and the relevant case in for the write-up) is that $\alpha : \mathbb{R} \times \mathcal{A} \to \mathcal{A}$ is continuous in norm topology.

We remember that for finite systems the canonical statistical state was given by:

$$\omega_\beta(A) = \frac{Tr(Ae^{-\beta H})}{\mathrm{Tr}(e^{-\beta H})}.$$

It is simple to see that this state satisfies:

$$\omega_\beta \left(A\, \alpha_{i\beta}(B)\right) = \omega_\beta(BA).$$

In the finite dimensional case the statistical state is actually determined by this relation. For a $C^*$ algebra and a dynamic $\alpha_t$, the set on which $\alpha$ extends to an entire analytic function is dense. With this in mind the KMS condition is formulated:

**Definition.** A state $\omega$ on $\mathcal{A}$ is called a $\beta$-KMS state wrt a dynamic $\alpha$ if

$$\omega(A\alpha_{i\beta}(B)) = \omega(BA)$$

holds for all $A, B$ in a norm dense sub-algebra of $\mathcal{A}$.

States that satisfy this condition are identified as the physically relevant states in thermodynamic limits.