Supersymmetry in 2D

$\mathcal{N}$ = (2,2) Supersymmetry

Thomas Jan Mikhail

Contents

In this short write up, we will follow chapter 12 of [1] with a few minor details taken from [2].

1 Superfield Formalism

1.1 Superspace and Superfields

The *superspace* is spanned by the usual spacetime coordinates along with a number of fermionic (or Grassmannian) coordinates. In particular, in 2D $\mathcal{N} = (2, 2)$ supersymmetry we have the $\theta^+, \theta^-, \overline{\theta}^+, \overline{\theta}^-$ fermionic coordinates in addition to the usual x^μ with $\mu = 0, 1$. All together we have

$$\text{Superspace} = \{x^\mu, \theta^\pm, \overline{\theta}^\pm\}. \tag{1}$$

Due to the anticommuting nature of $\theta^\pm, \overline{\theta}^\pm$ we have $(\theta^\pm)^2 = 0 = (\overline{\theta}^\pm)^2$. The $\pm$ superscript stands for the chirality (or spin) under a Lorentz boost, i.e.

$$\begin{aligned} \theta^\pm &\longmapsto e^{\pm\gamma/2}\theta^\pm \\ \overline{\theta}^\pm &\longmapsto e^{\pm\gamma/2}\overline{\theta}^\pm \end{aligned} \tag{2}$$

where γ is the pseudorapidity. The corresponding transformation of the spacetime coordinates is given by the familiar

$$\begin{pmatrix} x^0 \\ x^1 \end{pmatrix} \longmapsto \begin{pmatrix} \cosh\gamma & \sinh\gamma \\ \sinh\gamma & \cosh\gamma \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix}. \tag{3}$$

Functions defined on the superspace are called *superfields*. Since $\theta^\pm, \overline{\theta}^\pm$ all square to zero we can Taylor expand superfields in to monomials in $\theta^\pm$ and $\overline{\theta}^\pm$, namely

$$\mathcal{F}(x^\mu, \theta^\pm, \overline{\theta}^\pm) = f_0(x^\mu) + \theta^\alpha f_\alpha(x^\mu) + \overline{\theta}^\alpha \tilde{f}_\alpha(x^\mu) + \theta^+\theta^- f_{+-}(x^\mu) + ... \tag{4}$$

where we sum over $\alpha = +, -$. The functions $f_0, f_\alpha, ...$ are called the components of the superfield. We see that a given superfields can have at most $2^4 = 16$ components. A superfield $\mathcal{F}$ is said to be bosonic if $[\mathcal{F}, \theta^\alpha] = 0$ and fermionic if $\{\mathcal{F}, \theta^\alpha\} = 0$.

Next we will introduce two set of differential operators. The first set of differential operators is given by

$$\mathcal{Q}_\pm := \frac{\partial}{\partial\theta^\pm} + i\overline{\theta}^\pm\partial_\pm, \qquad \overline{\mathcal{Q}}_\pm := -\frac{\partial}{\partial\overline{\theta}^\pm} - i\theta^\pm\partial_\pm, \tag{5}$$

where

$$x^{\pm} := x^0 \pm x^1, \qquad \partial_{\pm} := \frac{\partial}{\partial x^{\pm}} = \frac{1}{2}\left(\frac{\partial}{\delta x^0} \pm \frac{\partial}{\partial x^1}\right). \tag{6}$$

These will turn out to be a representation of the supersymmetry generators, the same way $-i\frac{\partial}{\partial x^1}$ is a representation of the generator of translation, i.e. the momentum operator $\hat{p}$. They satisfy the anticommuting relations

$$\{\mathcal{Q}_{\pm}, \overline{\mathcal{Q}}_{\pm}\} = -2i\partial_{\pm} \tag{7}$$

with all other anticommutators vanishing.

The second set consists of the operators

$$D_{\pm} := \frac{\partial}{\partial \theta^{\pm}} - i\overline{\theta}^{\pm}\partial_{\pm}, \qquad \overline{D}_{\pm} := -\frac{\partial}{\partial \overline{\theta}^{\pm}} + i\theta^{\pm}\partial_{\pm}, \tag{8}$$

which satisfy

$$\{D_{\pm}, \overline{D}_{\pm}\} = 2i\partial_{\pm}. \tag{9}$$

These can be thought of as covariant derivatives in the sense that if a superfield is invariant under supersymmetry, then the covariant derivative of that field is also invariant. This is a direct consequence of the fact that they anticommute with the supersymmetry generators

$$\begin{aligned} \{D_{\pm}, \mathcal{Q}_{\pm}\} &= 0, \qquad \{\overline{D}_{\pm}, \mathcal{Q}_{\pm}\} = 0, \\ \{D_{\pm}, \overline{\mathcal{Q}}_{\pm}\} &= 0, \qquad \{\overline{D}_{\pm}, \overline{\mathcal{Q}}_{\pm}\} = 0. \end{aligned} \tag{10}$$

The outcome of the action of $\delta := \epsilon_+\mathcal{Q}_- - \epsilon_-\mathcal{Q}_+ - \overline{\epsilon}_+\overline{\mathcal{Q}}_- + \overline{\epsilon}_-\overline{\mathcal{Q}}_+$ on superfields is

$$\begin{aligned} e^{i\delta}\mathcal{F}(x^{\pm}, \theta^{\pm}, \overline{\theta}^{\pm}) &\approx \left(1 + i\epsilon_+\mathcal{Q}_- - i\epsilon_-\mathcal{Q}_+ - i\overline{\epsilon}_+\overline{\mathcal{Q}}_- + i\overline{\epsilon}_-\overline{\mathcal{Q}}_+\right)\mathcal{F}(x^{\pm}, \theta^{\pm}, \overline{\theta}^{\pm}) \\ &= \Bigg[1 + i\epsilon_+\frac{\partial}{\partial\theta^-} - i\epsilon_-\frac{\partial}{\partial\theta^+} + i\overline{\epsilon}_+\frac{\partial}{\partial\overline{\theta}^-} - i\overline{\epsilon}_-\frac{\partial}{\partial\overline{\theta}^+} \\ &\qquad + \left(\epsilon_-\overline{\theta}^+ + \overline{\epsilon}_-\theta^+\right)\partial_+ + \left(-\epsilon_+\overline{\theta}^- - \overline{\epsilon}_+\theta^-\right)\partial_-\Bigg]\mathcal{F}(x^{\pm}, \theta^{\pm}, \overline{\theta}^{\pm}) \\ &\approx \mathcal{F}(x^{\pm} \pm \epsilon_{\mp}\overline{\theta}^{\pm} \pm \overline{\epsilon}_{\mp}\theta^{\pm},\ \theta^{\pm} \mp i\epsilon_{\mp},\ \overline{\theta}^{\pm} \mp i\overline{\epsilon}_{\mp}) \end{aligned} \tag{11}$$

where everything has been evaluated to first order. Alluding to the comment after equation (5) we see that supersymmetry transformations correspond to translations in the spacetime and fermionic coordinates.

Apart from translation, one can also define rotations in the fermionic direction. Let us therefore introduce *vector R-rotations* and *axial R-rotations* defined respectively by

$$e^{i\alpha F_V} : \mathcal{F}(x^\pm, \theta^\pm, \overline{\theta}^\pm) \longmapsto e^{i\alpha q_V} \mathcal{F}(x^\pm, e^{-i\alpha}\theta^\pm, e^{i\alpha}\overline{\theta}^\pm), \tag{12}$$

$$e^{i\alpha F_A} : \mathcal{F}(x^\pm, \theta^\pm, \overline{\theta}^\pm) \longmapsto e^{i\alpha q_A} \mathcal{F}(x^\pm, e^{\mp i\alpha}\theta^\pm, e^{\pm i\alpha}\overline{\theta}^\pm). \tag{13}$$

Here, the quantities q_V and q_A are the *vector R-* and *axial R-charge* of the superfield $\mathcal{F}$.

Finally, let us define the following fields, which will prove to be particularly useful in constructing supersymmetric actions.

A *chiral superfield* satisfies

$$\overline{D}_\pm \Phi(x^\pm, \theta^\pm, \overline{\theta}^\pm) = 0. \tag{14}$$

The most general chiral superfield can be written as

$$\Phi(x^\pm, \theta^\pm, \overline{\theta}^\pm) = \phi(y^\pm) + \theta^\alpha \psi_\alpha(y^\pm) + \theta^+\theta^- F(y^\pm) \tag{15}$$

where $y^\pm := x^\pm - i\theta^\pm\overline{\theta}^\pm$. One can check that the product and the sum of two chiral superfields is also a chiral superfield. On the other hand, if the field satisfies

$$D_\pm \overline{\Phi}(x^\pm, \theta^\pm, \overline{\theta}^\pm) = 0 \tag{16}$$

it is said to be an *anti-chiral superfield.* As the notation suggests, chiral and anti-chiral fields are complex conjugate of each other. Keep in mind that for any two Grassmann numbers η, θ we have $\overline{\eta\theta} = \overline{\theta}\overline{\eta}$.

Furthermore, a *twisted chiral superfield* obeys

$$\overline{D}_+ U(x^\pm, \theta^\pm, \overline{\theta}^\pm) = 0 = D_- U(x^\pm, \theta^\pm, \overline{\theta}^\pm). \tag{17}$$

The most general twisted chiral field has the form

$$U(x^\pm, \theta^\pm, \overline{\theta}^\pm) = v(\tilde{y}^\pm) + \theta^+ \overline{\chi}_+(\tilde{y}^\pm) + \overline{\theta}^- \chi_-(\tilde{y}^\pm) + \theta^+\overline{\theta}^- E(\tilde{y}^\pm) \tag{18}$$

where $\tilde{y}^{\pm} := x^{\pm} \mp i\theta^{\pm}\overline{\theta}^{\pm}$.

Lastly, the *twisted anti-chiral superfield* satisfies

$$D_{+}\overline{U}(x^{\pm},\theta^{\pm},\overline{\theta}^{\pm}) = 0 = \overline{D}_{-}\overline{U}(x^{\pm},\theta^{\pm},\overline{\theta}^{\pm}) \tag{19}$$

where again U and $\overline{U}$ are conjugate.

As a final comment, the components of these fields are said to form a multiplet. Therefore, for example, the chiral multiplet consists of the components of the chiral field and similarly for the other fields.

We now have all the tools to construct supersymmetric actions.

1.2 Supersymmetric Actions

We are looking for actions which are invariant under the transformation

$$\delta := \epsilon_{+}\mathcal{Q}_{-} - \epsilon_{-}\mathcal{Q}_{+} - \overline{\epsilon}_{+}\overline{\mathcal{Q}}_{-} + \overline{\epsilon}_{-}\overline{\mathcal{Q}}_{+}. \tag{20}$$

The following three types of actions all fulfil this condition.

1. The *D-term*

   This functional is of the form

   $$\int d^2x d^4\theta K(\mathcal{F}_i) := \int d^2x d\theta^{+} d\theta^{-} d\overline{\theta}^{+} d\overline{\theta}^{-} K(\mathcal{F}_i), \tag{21}$$

   where K is an arbitrary differentiable function of the superfields $\mathcal{F}_i$. Any such functional is invariant under (20). To see this let us check the variation of the D-term induced by $\epsilon_{+}\mathcal{Q}_{-}$ as an example.

   $$\int d^2x d^4\theta \epsilon_{+}\mathcal{Q}_{-}K(\mathcal{F}_i) = \int d^2x d^4\theta \epsilon_{+} \left(\frac{\partial}{\partial\theta^{-}} + i\overline{\theta}\partial_{-}\right) K(\mathcal{F}_i). \tag{22}$$

   The $d^4\theta$ integral picks out the coefficient of $\theta^4 := \theta^{+}\theta^{-}\overline{\theta}^{+}\overline{\theta}^{-}$ in the Taylor expansion of K. Since the first term in the integral does not contain θ^{-} we see that this term vanishes. The second is just a total derivative and therefore vanishes as well. The same arguments apply to the $\epsilon_{-}, \overline{\epsilon}_{\pm}$ cases.

2. The *F-term*

   F-terms are given by functionals of the form

$$\int d^2xd^2\theta W(\Phi_i) := \int d^2xd\theta^- d\theta^+ W(\Phi_i)\Big|_{\overline{\theta}^{\pm}=0} \tag{23}$$

   with W an arbitrary holomorphic function of the chiral superfields Φ_i. The same arguments as for the D-term can be used to show that the variations proportional to $\epsilon_\pm$ vanish. For the $\overline{\epsilon}_\pm$ terms, notice first that $\overline{\mathcal{Q}}_\pm = \overline{D}_\pm - 2i\theta^\pm\partial_\pm$. The first term now vanishes because the fields Φ_i are chiral and the second term is again a total derivative. [1]

3. The *Twisted F-term*

   This final action is of the form

$$\int d^2xd^2\widetilde{\theta}\, \widetilde{W}(U_i) := \int d^2xd\overline{\theta}^- d\theta^+ \widetilde{W}(U_i)\Big|_{\theta^-=0=\overline{\theta}^+} \tag{24}$$

   where $\widetilde{W}$ is an arbitrary function of the twisted anti-chiral fields U_i. The invariance of this term under δ can be shown using similar arguments as for the F-term.

[1] The name F-term is due to the fact that the integral picks out the F-component of $W(\Phi_i)$. Remember that the product and the sum of two chiral superfields is also a chiral field. Therefore any holomorphic function of chiral fields is also chiral and can thus be expanded into the form (15). We see that the integral (23) over the fermionic degrees extracts the coefficient of $\theta^+\theta^-$ in the expansion of W which is exactly the F-component of W. We will see later that supersymmetry acting on superfields induces a transformation on its components. In particular, the transformation of the F-component of a chiral superfield is a total derivative. Thus, an action build from this component is automatically invariant under supersymmetry (a fact which we explicitly checked by calculating the variation of the action under supersymmetry). Similarly, for a generic superfield, the coefficient of θ^4, typically called $D(x)$, transforms as a total derivative and can therefore be used to build supersymmetric actions. This is exactly what we have done in the D-term, which also explains the origin of its name.

2 An Example: Theory of a Chiral Superfield

Let us now give an example to illustrate the power of the machinery developed in the previous chapter.

Consider a theory composed of a D-term and a F-term. Of interest to us will be the D-term

$$S_{Kin} = \int d^2x d^4\theta \overline{\Phi}\Phi \tag{25}$$

with Φ a chiral field. The suggestive subscript Kin stands for kinetic. In order to treat this term we will further expand equation (15) into

$$\begin{aligned} \Phi &= \phi(y^{\pm}) + \theta^{\alpha}\psi_{\alpha}(y^{\pm}) + \theta^{+}\theta^{-}F(y^{\pm}) &(26)\\ &= \phi(x^{\pm}) - i\theta^{+}\theta^{-}\partial_{+}\phi(x^{\pm}) + .. &(27)\end{aligned}$$

exposing all the θ-dependance and collect the terms proportional to θ^4 in $\overline{\Phi}\Phi$. After integrating over the fermionic coordinates we will be left with coefficient of this term.

The F-term we will consider is

$$S_W = \int d^2x d^2\theta W(\Phi) + \text{c.c.} \tag{28}$$

Performing the $d^2\theta$ integral we are left with

$$W(\Phi)|_{\theta^2} = W'(\phi)F + W''(\phi)\psi_{+}\psi_{-}. \tag{29}$$

where the prime denotes differentiation with respect to the argument of the function, in this case ϕ. A similar expression holds for the complex conjugate term. Putting it all together the total action is given by

$$\begin{aligned} S = S_{Kin} + S_W = \int d^2x \Big(&|\partial_0\phi|^2 - |\partial_1\phi|^2 - |W'(\phi)|^2 + i\overline{\psi}_{-}(\partial_0 + \partial_1)\psi_{-} \\ &+ i\overline{\psi}_{+}(\partial_0 - \partial_1)\psi_{+} - W''(\phi)\psi_{+}\psi_{-} - \overline{W}''(\overline{\phi})\overline{\psi}_{-}\overline{\psi}_{+}\Big) \end{aligned} \tag{30}$$

where we have used integration by parts. Also, in this expression, the term $|W'(\phi)|^2$ has been added and subtracted to complete the square, producing a term $|F + \overline{W}'(\overline{\phi})|^2$. Since the action contains no kinetic term for the field F (making it an auxiliary field) we can integrate it out, using its equation of motion $F = -\overline{W}'(\overline{\phi})$. This explains why the field F does not appear in the action.

Now, a number of comments are in order here. First, notice that we have recovered the action of a complex scalar fields ϕ with potential $|W'(\phi)|^2$ and the action for a left and a right chirality Weyl spinor, along with the interaction terms $W''(\phi)\psi_+\psi_-, \overline{W}''(\overline{\phi})\overline{\psi}_-\overline{\psi}_+$ (which include the fermion mass term and a Yukawa coupling term). By construction, this theory is invariant under supersymmetry.

Secondly, the scalar field and the spinors are not arbitrary. Supersymmetry imposes restrictions which can be seen by the fact that the the potential of the scalar and the mass term of the spinor are related through the function $W(\phi)$. In particular, the two fields have equal mass. To see this, assume we can Taylor expand the function W into

$$W(\phi) = \frac{1}{2}m\phi^2 + \frac{1}{3}g\phi^3 + ... \tag{31}$$

with m and g real. The potential of the scalar is then given by $|W'(\phi)|^2 = m^2|\phi|^2 + ...$ We can now go back and see in retrospect why we ommitted the constant and first order term in (31). The usual arguments in Quantum Field Theory are that the linear term vanishes because we want to place our field at a minimum (in order to have a stable vacuum) and the constant term is irrelevan since it does not enter into the equations of motion. Now the mass squared of a scalar field is given my $m_\phi^2 = d^2|W'(\phi)|^2/d\phi^2|_{\phi=0} = m^2$. On the other hand, the mass of the fermions is given by $m_\psi = W''(\phi)|_{\phi=0} = m$.[2]

Now, the action of δ on a superfield induces a transformation on the components. Let us see how this works for the chiral field. Observe that if Φ is a chiral field, then $\delta\Phi$ is also chiral. This is a consequence of the fact that the D's and the $\mathcal{Q}$'s anticommute since we have $\overline{D}_\pm\delta\Phi = \delta\overline{D}_\pm\Phi = 0$. We can therefore define

$$\delta\Phi = \delta\phi + \theta^\alpha\delta\psi_\alpha + \theta^+\theta^-\delta F, \tag{32}$$

calculate the action of δ on Φ and identify the components.

Consider $\epsilon_\pm$ cases. We need to calculate the effect of $\epsilon_\pm\mathcal{Q}_\mp$ on Φ. Since for any function $H(y^\pm)$

$$\begin{aligned}\mathcal{Q}_\pm H(y^\pm) &= \left(\frac{\partial}{\partial\theta^\pm} + i\overline{\theta}^\pm\partial_\pm\right) H\left(x^\pm - i\theta^\pm\overline{\theta}^\pm\right) \\ &= -i\overline{\theta}^\pm H'(y^\pm) + i\overline{\theta}^\pm H'(y^\pm) = 0\end{aligned} \tag{33}$$

[2]Furthermore, W'' will also produce the Yukawa coupling $-2g\phi\psi_+\psi_-$, which is related to the scalar field self coupling $-g^2|\phi|^4$, coming from the expansion of $|W'(\phi)|^2$. This is the source of the so called miraculous cancellation of divergent terms in 4D SUSY perturbation theory.

we have

$$\begin{aligned}\mathcal{Q}_\pm\Phi &= \mathcal{Q}_\pm\left[\phi(y^\pm) + \theta^\alpha\psi_\alpha(y^\pm) + \theta^+\theta^- F(y^\pm)\right] \\ &= \psi_\pm(y^\pm) \pm \theta^\mp F(y^\pm).\end{aligned} \tag{34}$$

The next step is to demand

$$\begin{aligned}\left(\epsilon_+\mathcal{Q}_- - \epsilon_-\mathcal{Q}_+\right)\Phi &= \epsilon_+\psi_- - \epsilon_-\psi_+ - \epsilon_+\theta^+ F - \epsilon_-\theta^- F \\ &\overset{!}{=} \delta\phi + \theta^+\delta\psi_+ + \theta^-\delta\psi_- + \theta^+\theta^-\delta F\end{aligned} \tag{35}$$

from which we read off

$$\begin{aligned}&\delta\phi = \epsilon_+\psi_- - \epsilon_-\psi_+, \\ &\delta\psi_\pm = \epsilon_\pm F, \\ &\delta F = 0.\end{aligned} \tag{36}$$

Following the same steps for the $\bar{\epsilon}_\pm$ and putting it all together one finds

$$\begin{aligned}&\delta\phi = \epsilon_+\psi_- - \epsilon_-\psi_+, \\ &\delta\psi_\pm = \pm 2i\bar{\epsilon}_\mp\partial_\pm\phi + \epsilon_\pm F, \\ &\delta F = -2i\bar{\epsilon}_+\partial_-\psi_+ - 2i\bar{\epsilon}_-\partial_+\psi_-.\end{aligned} \tag{37}$$

In what follows we will be needing the variations of the components of the anti chiral-field as well. These are obtained by taking the complex conjugate of the above variations

$$\begin{aligned}&\delta\overline{\phi} = -\bar{\epsilon}_+\overline{\psi}_- + \bar{\epsilon}_-\overline{\psi}_+, \\ &\delta\overline{\psi}_\pm = \mp 2i\epsilon_\mp\partial_\pm\overline{\phi} + \bar{\epsilon}_\pm\overline{F}, \\ &\delta\overline{F} = -2i\epsilon_+\partial_-\overline{\psi}_+ - 2i\epsilon_-\partial_+\overline{\psi}_-.\end{aligned} \tag{38}$$

Since the classical system has a symmetry we can exploit Noether's Theorem and obtain conserved currents and charges. After further sprinkling around the word "super" we have the four *supercurrents*

$$G^0_\pm = 2\partial_\pm\overline{\phi}\psi_\pm \mp \overline{\psi}_\mp\overline{W}'(\overline{\phi}), \qquad G^1_\pm = \mp 2\partial_\pm\overline{\phi}\psi_\pm - \overline{\psi}_\mp\overline{W}'(\overline{\phi}), \tag{39}$$
$$\overline{G}^0_\pm = 2\overline{\psi}_\pm\partial_\pm\phi \pm \psi_\mp W'(\phi), \qquad \overline{G}^1_\pm = \mp 2\overline{\psi}_\pm\partial_\pm\phi \pm \psi_\mp W'(\phi),$$

and by integrating the supercurrents over the spatial volume the four *supercharges*

$$Q_\pm = \int dx^1 G^0_\pm, \qquad \overline{Q}_\pm = \int dx^1 \overline{G}^0_\pm. \tag{40}$$

The supercharges are in one to one correspondence to the fermionic coordinates. Under a Lorentz boost they transform as

$$Q_\pm \longmapsto e^{\pm\gamma/2}Q_\pm, \qquad \overline{Q}_\pm \longmapsto e^{\pm\gamma/2}\overline{Q}_\pm. \tag{41}$$

Let us now demonstrate the derivation of the supercurrents, taking G^μ_- as an example. This supercurrent corresponds to the $\epsilon_+\mathcal{Q}_-$ term in the variation (20). The Lagrangian is given by

$$\begin{aligned}\mathcal{L} = {} & |\partial_0\phi|^2 - |\partial_1\phi|^2 - |F(\overline{\phi})|^2 + i\overline{\psi}_-(\partial_0+\partial_1)\psi_- + i\overline{\psi}_+(\partial_0-\partial_1)\psi_+ \\ & + \overline{F}'(\phi)\psi_+\psi_- + F'(\overline{\phi})\overline{\psi}_-\overline{\psi}_+\end{aligned} \tag{42}$$

where we have traded W for F using $F = -\overline{W}'$. The equations of motion for the fields ϕ, ψ_+ and ψ_- are given by

$$\partial_\mu\partial^\mu\phi + \overline{F}(\phi)F'(\overline{\phi}) + F''(\overline{\phi})\overline{\psi}_-\overline{\psi}_+ = 0, \tag{43}$$
$$i(\partial_0 - \partial_+)\psi_+ - F'(\overline{\phi})\overline{\psi}_- = 0, \tag{44}$$
$$i(\partial_0 + \partial_+)\psi_- - \overline{F}'(\phi)\overline{\psi}_+ = 0, \tag{45}$$

along with their complex conjugate for the fields $\overline{\phi}, \overline{\psi}_-$ and $\overline{\psi}_+$.

Now, by writing the variation of a given field φ_i as $\delta\varphi_i =: \epsilon_+\Delta\varphi_i$ and the shift by a total derivative of the Lagrangian as $\delta\mathcal{L} =: \epsilon_+\partial_\mu V^\mu$ the current is given via Noether's procedure as

$$G^\mu_+ = \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\varphi_i\right)}\Delta\varphi_i - V^\mu \tag{46}$$

where we sum over all fields labelled i.

Treating the action as a function of ϕ, ψ_+, ψ_- and their complex conjugate only, the variations which will be of interest to us are

$$\begin{aligned} \delta\phi &= \epsilon_+\psi_-, & \delta\overline{\phi}&, \\ \delta\psi_+ &= \epsilon_+ F, & \delta\psi_- &= 0, \\ \delta\overline{\psi}_+ &= 0, & \delta\overline{\psi}_- &= \epsilon_+ 2i\partial_-\overline{\phi}. \end{aligned} \tag{47}$$

From these we can calculate

$$\delta F = F'\delta\overline{\phi} = 0, \qquad \delta\overline{F} = \overline{F}'\delta\phi = \epsilon_+\overline{F}'\psi_-. \tag{48}$$

Notice that, by use of the equation of motion (44), these are consistent with (37) and (38). The second ingredient we will need is the function V^μ, which we can find by explicitly calculating the variation of the action. We have

$$\begin{aligned} \delta\mathcal{L} = \partial_0(\delta\phi)\partial_0\overline{\phi} - \partial_1(\delta\phi)\partial_1\overline{\phi} - F(\overline{\phi})\delta\overline{F}(\phi) + i\delta\overline{\psi}_-(\partial_0+\partial_1)\psi_- + i\overline{\psi}_+(\partial_0-\partial_1)\delta\psi_+ \\ + \delta\overline{F}'(\phi)\psi_+\psi_- + \overline{F}'(\phi)\delta\psi_+\psi_- + F'(\overline{\phi})\delta\overline{\psi}_-\overline{\psi}_+. \end{aligned} \tag{49}$$

After plugging in all the corresponding variations we are left with

$$\delta\mathcal{L} = \epsilon_+\left(-\partial_0\overline{\phi}\partial_1\psi_- + \partial_1\overline{\phi}\partial_0\psi_-\right) \tag{50}$$

which we can rewrite as

$$\delta\mathcal{L} = \epsilon_+\Bigg(\partial_0\left(\partial_1\overline{\phi}\psi_-\right) - \partial_1\left(\partial_0\overline{\phi}\psi_-\right)\Bigg). \tag{51}$$

From this we can read off

$$T^0 = \partial_1\overline{\phi}\psi_-, \qquad T^1 = -\partial_0\overline{\phi}\psi_-. \tag{52}$$

Finally, with this now at hand along with the variations (47) we can go back to (46) and conclude

$$
\begin{aligned}
G^0_- &= (\partial_0\overline{\phi})\psi_- + (-i\psi_+)F - \partial_1\overline{\phi}\psi_- \\
&= 2\partial_-\overline{\phi}\psi_- - i\overline{\psi}_+F \\
&= 2\partial_-\overline{\phi}\psi_- + i\overline{\psi}_+\overline{W}' \\
G^1_- &= (-\partial_1\overline{\phi})\psi_- + (i\overline{\psi}_+)F - (-\partial\overline{\phi}\psi_-) \\
&= 2\partial_+\overline{\phi}\psi_- + i\overline{\psi}_+F \\
&= 2\partial_-\overline{\phi}\psi_- - i\overline{\psi}_+\overline{W}'
\end{aligned}
\tag{53}
$$

where we have used $F(\overline{\phi}) = -\overline{W}'(\overline{\phi})$.

Now, our action contains more symmetry. We have seen that the only relevant terms for the D-term is the $\theta^4 = \theta^+\theta^-\overline{\theta}^+\overline{\theta}^-$ and for the F-term the $\theta^2 = \theta^+\theta^-$ (and its complex conjugate). Both are invariant under $\theta^\pm \longmapsto e^{\mp i\alpha}\theta^\pm, \overline{\theta}^\pm \longmapsto e^{\pm i\alpha}\overline{\theta}^\pm$, which is precisely the axial R-rotation defined earlier. Via Noether's procedure again we have the *axial R-current*

$$
J^0_A = \overline{\psi}_+\psi_+ - \overline{\psi}_-\psi_-, \qquad J^1_A = -\overline{\psi}_+\psi_+ - \overline{\psi}_-\psi_-
\tag{54}
$$

and the *axial R-charge*

$$
F_A = \int dx^0 J^0_A.
\tag{55}
$$

Under the axial R-symmetry the supercharges transform as

$$
Q_\pm \longmapsto e^{\mp i\alpha}Q_\pm, \qquad \overline{Q}_\pm \longmapsto e^{\pm i\alpha}\overline{Q}_\pm.
\tag{56}
$$

If and only if the potential $W(\phi)$ has a *vector R-charge* equal to 2, then the action (30) is also vector R-symmetric. The *vector R-current*, J^μ_V and vector R-charge, F_V are again given by Noether's Theorem.

Under the vector R-symmetry the supercharges transform as

$$
Q_\pm \longmapsto e^{-i\alpha}Q_\pm, \qquad \overline{Q}_\pm \longmapsto e^{i\alpha}\overline{Q}_\pm.
\tag{57}
$$

As a final remark, observe that if in all the above formulae we had exchanged $\theta^- \longleftrightarrow -\overline{\theta}^-$ we would have ended up with a theory of a twisted chiral superfield. This is evident from looking at the definitions of the D's and the (twisted) chiral field.

3 $\mathcal{N} = (2,2)$ Supersymmetric Quantum Field Theories

In the absence of anomalies, symmetries of the classical theory get passed down to the quantum theory. Of course in any Poincaré invariant theory we will have the charges H, P, M, i.e. the Hamiltonian, the Momentum and the Lorentz boost which correspond to the generators of translations in time, space and rotations in spacetime. In our case, we also have the generators of supersymmetry, the four supercharges, $Q_\pm, \overline{Q}_\pm$ which generate the transformation δ, i.e.

$$\delta\mathcal{O} = [\hat{\delta}, \mathcal{O}] \tag{58}$$

where $\hat{\delta} := i\epsilon_+ Q_- - i\epsilon_- Q_+ - i\bar{\epsilon}_+ \overline{Q}_- + i\bar{\epsilon}_- \overline{Q}_+$.

If in addition our theory is invariant under vector- and axial R-symmetry we also have the corresponding generators F_V and F_A.

These generators satisfy the $\mathcal{N} = (2,2)$ *supersymmetry algebra*

$$Q_\pm^2 = 0 = \overline{Q}_\pm^2, \tag{59}$$

$$\{Q_\pm, \overline{Q}_\pm\} = H \pm P, \tag{60}$$

$$\{\overline{Q}_+, \overline{Q}_-\} = 0, \qquad \{Q_+, Q_-\} = 0, \tag{61}$$

$$\{Q_-, \overline{Q}_+\} = 0, \qquad \{Q_+, \overline{Q}_-\} = 0, \tag{62}$$

$$[iM, Q_\pm] = \mp Q_\pm, \qquad [iM, \overline{Q}_\pm] = \mp \overline{Q}_\pm, \tag{63}$$

$$[iF_V, Q_\pm] = -iQ_\pm, \qquad [iF_V, \overline{Q}_\pm] = +i\overline{Q}_\pm, \tag{64}$$

$$[iF_A, Q_\pm] = \mp iQ_\pm, \qquad [iF_A, \overline{Q}_\pm] = \pm i\overline{Q}_\pm. \tag{65}$$

As a matter of fact, equations (61) and (62) can be relaxed to the form

$$\{\overline{Q}_+, \overline{Q}_-\} = Z, \qquad \{Q_+, Q_-\} = Z^* \tag{66}$$

$$\{Q_-, \overline{Q}_+\} = \widetilde{Z}, \qquad \{Q_+, \overline{Q}_-\} = \widetilde{Z}^*. \tag{67}$$

Here, Z and $\widetilde{Z}$ are central charges, i.e. they commute with all the symmetry generators. If F_V is conserved then Z must vanish and if F_A is conserved then $\widetilde{Z}$ must vanish. [3]

[3] Given the set of generators one can uniquely determine the supersymmetry algebra by demanding that it be closed. One could then proceed in defining the superspace as the manifold corresponding to the Lie supergroup of the supersymmetry algebra - which we shall call super Poincaré - quotiented out by the

4 Statement of Mirror Symmetry

Remarkably the algebra itself turns out to have a symmetry. Specifically, it is invariant under a $\mathbb{Z}_2$ outer automorphism

$$
\begin{aligned}
Q_- &\longleftrightarrow \overline{Q}_+ \\
F_V &\longleftrightarrow F_A \\
Z &\longleftrightarrow \widetilde{Z}.
\end{aligned}
\tag{70}
$$

Two $\mathcal{N} = (2,2)$ supersymmetric theories are said to be mirror to each other, if they are equivalent Quantum Field Theories where the isomorphism of the Hilbert spaces transforms the generators of $\mathcal{N} = (2,2)$ according to the above transformation.

Due to the exchange of the vector and axial R-charge, if a theory has broken (unbroken) vector R-symmetry, then the mirror theory has broken (unbroken) axial R-symmetry.

Moreover, chiral superfields are mapped to twisted chiral superfields and visa versa. To show this consider following relations which hold for the components of a chiral field

$$
\begin{aligned}
[\overline{Q}_\pm, \phi] &= 0 \\
\psi_\pm &= [iQ_\pm, \phi] \\
F &= \{Q_+, [Q_-, \phi]\}
\end{aligned}
\tag{71}
$$

and

$$
\begin{aligned}
[\overline{Q}_+, v] = 0 &= [Q_-, v] \\
\overline{\chi}_+ = [iQ_+, v], &\quad \chi_- = -[i\overline{Q}_-, v] \\
E &= -\{Q_+, [\overline{Q}_-, v]\}
\end{aligned}
\tag{72}
$$

---

manifold generated by the Lorentz group. In particular, an element of the supergroup can be written as

$$g = \exp\left(i(\omega M + x^\mu P_\mu + \theta^\alpha Q_\alpha + \theta^\alpha \overline{Q}_\alpha)\right) \tag{68}$$

where $P^0 = H$ and $P^1 = P$. We then have

$$\text{superspace} = \frac{\text{super Poincaré}}{\text{Lorentz}} = \frac{\{\omega, x^\mu, \theta^\pm, \overline{\theta}^\pm\}}{\{\omega\}} = \{x^\mu, \theta^\pm, \overline{\theta}^\pm\}. \tag{69}$$

This is true for all dimensions.

which hold for the components of the twisted chiral field. One could use these relations to construct the respective superfield. Given a field ϕ, for example, one can construct a chiral field using the second and third relation in (71). It is evident from the above relations, that the exchange of $Q_- \longleftrightarrow \overline{Q}_+$ results in the exchange of chiral superfields with twisted chiral superfields.

5 $\mathcal{N} = (1,1)$ Supersymmetry

As a final topic, we will discuss supersymmetric theories with half as many supercharges in this section. Since the supercharges are in one-to-one correspondence with the fermionic coordinates it is clear that restricting the superspace to a subspace will result in decreasing the number of supercharges. In particular, for a $\mathcal{N} = (1,1)$ supersymmetry the corresponding subspace is given by the following identification

$$\begin{aligned} e^{-i\nu^+}\theta^+ + e^{i\nu^+}\overline{\theta}^+ &= 0 \\ e^{-i\nu^-}\theta^- + e^{i\nu^-}\overline{\theta}^- &= 0 \end{aligned} \tag{73}$$

for arbitrary but fixed $\nu_\pm$. Define

$$\theta^\pm =: ie^{i\nu_\pm}\theta_1^\pm. \tag{74}$$

Plugging this in to the identification (73), we see that the $\theta_1^\pm$ are real. We will now follow the same steps as in the analysis of the $\mathcal{N} = (2,2)$ supersymmetry and therefore present only the key steps.

Let us start by defining the set of differential operators, namely

$$\mathcal{Q}_\pm^1 := e^{i\nu_\pm}\mathcal{Q}_\pm + e^{-i\nu_\pm}\overline{\mathcal{Q}}_\pm = -i\frac{\partial}{\partial\theta_1} + 2\theta_1^\pm\partial_\pm \tag{75}$$

$$D_\pm^1 := e^{i\nu_\pm}D_\pm + e^{-i\nu_\pm}\bar{D}_\pm = -i\frac{\partial}{\partial\theta_1} - 2\theta_1^\pm\partial_\pm. \tag{76}$$

These operators obey

$$\{\mathcal{Q}_\pm^1, \mathcal{Q}_\pm^1\} = -4i\partial, \qquad \{\mathcal{Q}_+^1, \mathcal{Q}_-^1\} = 0 \tag{77}$$

$$\{D_\pm^1, D_\pm^1\} = 4i\partial, \qquad \{D_+^1, D_-^1\} = 0 \tag{78}$$

$$\{\mathcal{Q}_\alpha^1, \mathcal{Q}_\beta^1\} = 0 \tag{79}$$

and are consistent with the above identification, (73). A general superfield $\mathbf{\Phi}$ can be Taylor expanded into

$$\mathbf{\Phi} = \phi + i\theta_1^+\psi_+ + i\theta_1^-\psi_- + \theta_1^+\theta_1^- f. \tag{80}$$

Functionals of the form

$$\int d^2 d^2\theta_1 \mathbf{F} := \int d^2 d\theta_1^+ d\theta_1^- \mathbf{F}\left(\mathbf{\Phi}_i, D_\pm^1\mathbf{\Phi}_i, ...\right) \tag{81}$$

are invariant under $\delta^1 := i\epsilon_-^1\mathcal{Q}_+^1 - i\epsilon_+^1\mathcal{Q}_-^1$.

Consider now the action,

$$S = \int d^2x d^2\theta_1 \left(\frac{1}{2}D_-^1\mathbf{\Phi}D_+^1\mathbf{\Phi} + ih(\mathbf{\Phi})\right). \tag{82}$$

By performing the $\theta_1^\pm$ integrals we get

$$\begin{aligned} S = \int d^2x \Bigg(&\frac{1}{2}(\partial_0\phi)^2 - \frac{1}{2}(\partial_1\phi)^2 - \frac{1}{2}\left(h'(\phi)\right)^2 \\ &+ \frac{i}{2}\psi_-(\partial_0+\partial_1)\psi_- + \frac{i}{2}\psi_+(\partial_0-\partial_1)\psi_+ - ih''(\phi)\psi_+\psi_- \Bigg) \end{aligned} \tag{83}$$

where we've integrated by parts and integrated out the auxiliary field f. Again we get the action for a scalar field with potential $U(\phi) = \frac{1}{2}\left(h'(\phi)\right)^2$ and that of a left and right chirality Weyl spinor with a Yukawa-like interaction term.

After quantisation we get the Noether supercharges, $Q_\pm^1$ which generate the supersymmetry transformation. They satisfy the algebra

$$\{Q_\pm^1, Q_\pm^1\} = 2\left(H \pm P\right), \qquad \{Q_+^1, Q_-^1\} = 0. \tag{84}$$

Similarly one could obtain a $\mathcal{N} = (0, 2)$ supersymmetric theory. The subspace which would result in such a theory is

$$\theta^- = 0 = \overline{\theta}^-. \tag{85}$$

Following again the same steps one arrives at a quantum field theory with two supercharges, Q_+ and $\bar{Q}_+$ which satisfy the algebra

$$\{Q_+, \bar{Q}_+\} = H + P, \qquad Q_+^2 = 0 = \bar{Q}_+^2. \tag{86}$$

References

[1] Kentaro Hori et al. "Mirror symmetry, Clay Mathematics Monographs 1". In: *American Mathematical Society, Providence, RI* 1 (2003).

[2] Sven Krippendorf, Fernando Quevedo, and Oliver Schlotterer. "Cambridge Lectures on Supersymmetry and Extra Dimensions". In: *arXiv preprint arXiv:1011.1491* (2010). http://arxiv.org/abs/1011.1491.