Quantum Cohomology of $\mathbb{CP}^n$

Seminar: Supersymmetry in Geometry and Quantum Physics

Ricardo Waibel

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Contents

1 Introduction

1.1 General

These notes will introduce the concept of quantum cohomology, a technique that has made it possible to prove deep results in enumerative geometry. It has been a long standing problem to find the correct number of curves going through a number of general points, basically higher degree analogues of the fact that there is only one straight line going through two distinct points. It should be noted, that throughout these notes we will work over the complex numbers.

It has also been long known [3], that there is only one smooth conic passing through five general points in the plane, up to projective equivalence. For the next degree, rational plane cubics, it took till 1848 when Steiner [3] determined that there are 12 going through 8 general points in the plane. Then Zeuthen [3] showed that there 620 different plane rational quartics through 11 points in 1873. But a general formula was elusive till 1994. The general problem may be phrased as:

> Determine the number N_d of rational curves of degree d passing through $3d-1$ points in general position in the complex projective plane. [3]

The word rational in this context comes from algebraic geometry and basically means that the curves is birationally equivalent to the line $\mathbb{CP}^1$. The degree d refers to the degree of the homogeneous polynomials specifying it. For the exact definition of the degree check [3].

These notes are heavily based on the book 'An Invitation to Quantum Cohomology' [3], thus except at very important or ambiguous parts the reference will be omitted.

These enumerative geometry questions are usually phrased in projective space, so not the highest generality for definitions will be sought. We will mostly work with a target space $\mathbb{CP}^n$, n denoting the complex dimension of the space. One fact that will be used is the form of the cohomology groups of $\mathbb{CP}^n$ and its ring structure. The groups can be computed via the Mayer-Vietoris sequence and an induction argument. This works for all coefficient rings, so the most general (integers) has been chosen. The ring structure (cup-product) is induced (for the de Rham complex) by the normal wedge-product of differential forms. The more general cup-product on the simplicial cohomology will have the same structure, so the following results can be entirely phrased over the integers. Alternatively, use supersymmetric quantum mechanics to obtain the same result.

Fact. *The cohomology groups of* $\mathbb{CP}^n$ *are given by*

$$H^k(\mathbb{CP}^n) \cong \begin{cases} \mathbb{Z} & \text{for } k \text{ even and } 0 \leq k \leq 2n \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Let h *be a generator of* $H^2(\mathbb{CP}^n)$*, the hyperplane class. Then there is a ring isomorphism*

$$H^*(\mathbb{CP}^n) \cong \mathbb{Z}[h] \Big/ (h^{n+1}). \tag{2}$$

1.2 Moduli Spaces

Definition (Moduli spaces)**.** *Generally speaking, moduli spaces arise in classifying chosen geometric objects up to a chosen equivalence. The space needs to be in natural bijection with those equivalence classes, and provide some structure (from algebraic geometry: variety or scheme). Some moduli spaces that are important for the notes will be defined here. The notion of equivalence here is projective equivalence.*

- $\mathcal{M}_{0,m}$*: This space can have two interpretations in our context. Firstly, it is the space for classifying* m*-tuples up to projective equivalence. For a quadruple* (p_1, p_2, p_3, p_4) *there exists a unique automorphism* ϕ *of* $\mathbb{CP}^1$*, such that* $\phi(p_1) = 0$*,* $\phi(p_2) = 1$ *and* $\phi(p_3) = \infty$*. Then the point* $\lambda(p_4)$*, the* cross ratio *of the quadruple is the image of* p_4 *under* ϕ*. Now, two* m*-tuples* $(p_1, \ldots, p_m)$*,* $(p'_1, \ldots, p'_m)$ *are called projectively equivalent if all the cross ratios*

$$\lambda(p_1, p_2, p_3, p_i) = \lambda(p'_1, p'_2, p'_3, p'_i) \tag{3}$$

match $(i = 4, \ldots, m)$*. So we already found a description for the two spaces* $\mathcal{M}_{0,3}$ *and* $\mathcal{M}_{0,4}$*, namely that all triples are projectively equivalent and that all quadruples are characterized by their cross ratio. Thus,*

$$\mathcal{M}_{0,3} = \{pt.\} \tag{4}$$
$$\mathcal{M}_{0,4} = \mathbb{CP}^1 \backslash \{0, 1, \infty\}. \tag{5}$$

Then the moduli space $\mathcal{M}_{0,m}$ *can be described via these easier spaces.*

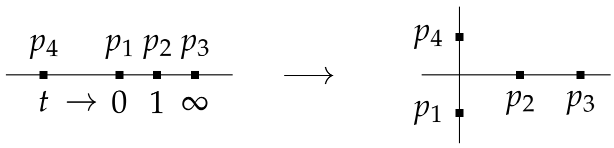
$$\mathcal{M}_{0,m} \simeq \underbrace{\mathcal{M}_{0,4} \times \cdots \times \mathcal{M}_{0,4}}_{m-3 \text{ times}} \backslash \bigcup \textit{diagonals} \tag{6}$$

Now, to the second interpretation: An m-pointed smooth rational curve $(C, p_1, \ldots, p_m)$ *is a projective smooth rational curve C equipped with a choice of m distinct marks. An isomorphism between two such curves is an isomorphism* $\varphi : C \xrightarrow{\sim} C'$ *such that* $\varphi(p_i) = p_i' \ \forall i = 1, \ldots, m$. *It then turns out, that* $\mathcal{M}_{0,m}$ *is the moduli space for classifying m-pointed smooth rational curves up to isomorphism* ($m \geq 3$).

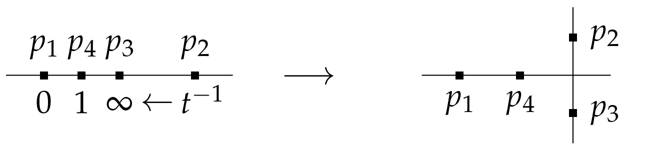
- $\overline{\mathcal{M}}_{0,m}$: *One of the issues with the space* $\mathcal{M}_{0,m}$ *is that it is non-compact. Obvious compactifications could be* $(\mathbb{CP}^1)^{m-3}$ *or* $\mathbb{CP}^{m-3}$. *But this will lead to issues with limits. As an example [3] consider the two families of quadruples*

$$C_t = (0, 1, \infty, t), \qquad D_t = (0, t^{-1}, \infty, 1). \tag{7}$$

In the cases $t \neq 0, 1, \infty$, *those are families of 4-pointed smooth rational curves. They also have the same cross ratio t, thus they are isomorphic. But the limit at* $t = 0$ *involves coincident points:* $C_0 = (0, 1, \infty, 0)$ *and* $D_0 = (0, \infty, \infty, 1)$. *These two configurations are not projectively equivalent, which is undesirable. The right way to approach this is to allow* trees of projective lines, *that is a connected curves with at most double points and a notion of stability, to ensure arithmetic genus zero, or equivalently, that each irreducible component is automorphism free. In the case of the example, this creates a new irreducible component for the limit* $t \to 0$. *First for* C_t:



Then for D_t:



Now these two configurations are equivalent as stable m-pointed rational curves, as desired. Then, the associated moduli space for classifying these is the smooth projective variety $\overline{\mathcal{M}}_{0,m}$ ($m \geq 3$), *and* $\mathcal{M}_{0,m}$ *is a dense subset. In the case of* $\mathcal{M}_{0,4}$, *the compactification can be realized as a blowup of* $\mathbb{CP}^1 \backslash \{0, 1, \infty\}$ *in the three points* $0, 1, \infty$.

- $\mathcal{W}(n, d)$: *Now we are turning to the main object we wanted to study in the first place, namely rational curves. As they can always be parametrized by the projective line* $\mathbb{CP}^1$, *investigate maps* $\mu \colon \mathbb{CP}^1 \to \mathbb{CP}^n$. *Giving such a map of degree d is equivalent to specifying* $n + 1$ *binary forms (functions defined by homogeneous polynomials in two* *variables) of degree d, nowhere vanishing simultaneously [3]. This condition defines a set $\mathcal{W}(n,d)$. There are several issues with this space, for starters it is not compact, and different reparametrizations of the same curve are distinct objects.*

- $\mathcal{M}_{0,m}(\mathbb{CP}^n,\, d)$: *To avoid these redundancies, one considers again projective equivalence, that is we go to the space $\mathcal{M}_{0,0}(\mathbb{CP}^n,\, d) = \mathcal{W}(n,d) \big/ \mathrm{Aut}(\mathbb{CP}^1)$. An issue here is, that automorphisms can exist. To suppress those, one considers again marked curves. Those m-pointed rational maps $\mu\colon C \to \mathbb{CP}^n$ are defined as maps from C, a tree of projective lines, with m distinct marked smooth points of C (i.e. the marked points are not the intersections points of irreducible components). The associated moduli space to classifying those is for $m \geq 3$:*

$$\mathcal{M}_{0,m}(\mathbb{CP}^n,\, d). \tag{8}$$

- $\overline{\mathcal{M}}_{0,m}(\mathbb{CP}^n,\, d)$: *The last issue we have to overcome is that $\mathcal{M}_{0,m}(\mathbb{CP}^n,\, d)$ is not compact. This is fixed by considering* Kontsevich stable *m-pointed rational maps. The property of being stable is equivalent to having only a finite number of automorphisms. The moduli space associated to those maps is $\overline{\mathcal{M}}_{0,m}(\mathbb{CP}^n,\, d)$, which is a projective normal irreducible variety. Its dimension is given by*

$$\dim(\overline{\mathcal{M}}_{0,m}(\mathbb{CP}^n,\, d)) = nd + n + d + m - 3. \tag{9}$$

Now some tools and properties of these moduli spaces will be introduced shortly, so that later the underlying ideas of the proof can be communicated:

- There are *forgetful morphisms* (for $m \geq 3$)

$$\overline{\mathcal{M}}_{0,m}(\mathbb{CP}^n,\, d) \to \overline{\mathcal{M}}_{0,m}, \tag{10}$$
$$\overline{\mathcal{M}}_{0,m+1}(\mathbb{CP}^n,\, d) \to \overline{\mathcal{M}}_{0,m}(\mathbb{CP}^n,\, d), \tag{11}$$
$$\overline{\mathcal{M}}_{0,m+1} \to \overline{\mathcal{M}}_{0,m}. \tag{12}$$

  The first just forgets the map μ, as the underlying tree of projective lines has m marks. Unstable components need to be stabilized, but that is always possible. The two other morphisms just forget one of the $m+1$ marks on the curve. Again the curve might need to be stabilized.

- In the case of $\overline{\mathcal{M}}_{0,m}(\mathbb{CP}^n,\, d)$ an element can be represented as $(C; p_1, \ldots, p_m; \mu)$, where C is the tree of projective lines, $(p_1, \ldots, p_m)$ are the distinct marks and

$\mu\colon C \to \mathbb{CP}^n$ is a map of degree d. From that data one can construct the *evaluation maps* ν_i given by

$$\begin{aligned} \nu_i\colon \overline{\mathcal{M}}_{0,m}(\mathbb{CP}^n, d) &\to \mathbb{CP}^n \\ (C; p_1, \ldots, p_m; \mu) &\mapsto \mu(p_i). \end{aligned} \tag{13}$$

Additionally, define the "total" evaluation map

$$\begin{aligned} \underline{\nu}\colon \overline{\mathcal{M}}_{0,m}(\mathbb{CP}^n, d) &\to \mathbb{CP}^n \times \cdots \times \mathbb{CP}^n \\ (C; p_1, \ldots, p_m; \mu) &\mapsto (\mu(p_1), \ldots, \mu(p_m)) \,. \end{aligned} \tag{14}$$

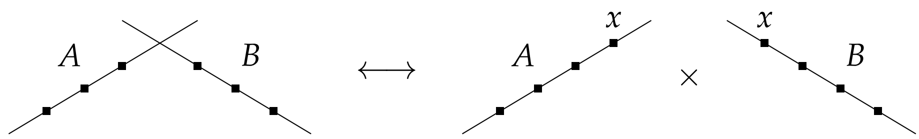
It should be noted, that this operation is compatible with the process of forgetting a mark. These maps will prove useful in pulling back differential forms from $\mathbb{CP}^n$ to the moduli space.

- Another important concept are *boundary divisors*. Those are the closure of a given labelled configuration in $\overline{\mathcal{M}}_{0,m}$, that have codimension 1. To illustrate this, denote by S the marking set $\{p_1, \ldots, p_m\}$. There is a boundary divisor for each partition $S = A \cup B$ with A and B disjoint and $|A| \geq 2$, $|B| \geq 2$. Then a general point in $D(A|B)$ is a curve with two irreducible components, with the marks of A on one component, and the marks of B on the other.

  Firstly, there is a canonical isomorphism which is responsible for the recursion occurring later

$$D(A|B) \simeq \overline{\mathcal{M}}_{0,A\cup\{x\}} \times \overline{\mathcal{M}}_{0,B\cup\{x\}}, \tag{15}$$

  given by:



  Secondly, consider the forgetful map $\overline{\mathcal{M}}_{0,m} \to \overline{\mathcal{M}}_{0,4}$ for $m \geq 4$. On $\overline{\mathcal{M}}_{0,4}$ there are only three boundary divisors, and all of them linearly equivalent as $\overline{\mathcal{M}}_{0,4} \simeq \mathbb{CP}^1$. Pulling that relation back via the forgetful map gives the relation

$$\sum_{\substack{i,j\in A\\ k,l\in B}} D(A|B) = \sum_{\substack{i,k\in A\\ j,l\in B}} D(A|B) = \sum_{\substack{i,l\in A\\ j,k\in B}} D(A|B) \tag{16}$$

in $\overline{\mathcal{M}}_{0,m}$.

Thirdly, this can also be done in $\overline{\mathcal{M}}_{0,m}(\mathbb{CP}^n,\, d)$, by distributing the degree d as well on the two irreducible components as d_A and d_B. Then composition of the two forgetful maps $\overline{\mathcal{M}}_{0,m}(\mathbb{CP}^n,\, d) \to \overline{\mathcal{M}}_{0,m} \to \overline{\mathcal{M}}_{0,4}$ produces the following *fundamental relation*

$$\sum_{\substack{A\cup B\\ i,j\in A\\ k,l\in B\\ d_A+d_B=d}} D(A|B) = \sum_{\substack{A\cup B\\ i,k\in A\\ j,l\in B\\ d_A+d_B=d}} D(A|B) = \sum_{\substack{A\cup B\\ i,l\in A\\ j,k\in B\\ d_A+d_B=d}} D(A|B). \tag{17}$$

1.3 Gromov-Witten Invariants

It is then shown in [3], that indeed counting these stable maps does count rational curves. For this counting intersection theory is used. The intersection numbers are basically the Gromov-Witten invariant, which will be defined after that. There are technical details which need to be considered for the integration, check [3] for those.

Definition. *The Gromov-Witten invariant of degree d associated with the classes* $\gamma_1, \dots, \gamma_m \in H^*(\mathbb{CP}^n)$ *is*

$$I_d(\gamma_1 \cdots \gamma_m) := \int\limits_{\overline{\mathcal{M}}_{0,m}(\mathbb{CP}^n,d)} \underline{\nu}^*(\gamma_1, \dots, \gamma_m). \tag{18}$$

$$\tag{19}$$

Properties 1. *Here are three important properties of Gromov-Witten invariants:*

(1) These numbers are only non-zero when the sum of the codimensions of the γ_i *is equal to the dimension of* $\overline{\mathcal{M}}_{0,m}(\mathbb{CP}^n,\, d)$*. The term codimension is a slight abuse of notation, consult [3] for details.*

(2) They are invariant under the permutation of the γ_i*.*

(3) They are linear in each of their arguments.

As a corollary to a more general statement (4.1.5 in [3]) from intersection theory in $\overline{\mathcal{M}}_{0,m}(\mathbb{CP}^n,\, d)$ we get a connection between the sought after numbers N_d and the Gromov-Witten invariants.

Corollary 2. *For* $\mathbb{CP}^2$ *we have*

$$I_d(\underbrace{h^2 \cdots h^2}_{3d-1 factors}) = N_d. \tag{20}$$

Here we can give a reason, why $3d - 1$ general points is the right amount: We have the condition, that the codimensions must sum to the dimension of the moduli space. Thus in the present case we have

$$2d + 2 + d + m - 3 = 2m \tag{21}$$
$$\Rightarrow m = 3d - 1. \tag{22}$$

Now follow several lemmas concerning more properties of the Gromov-Witten invariants.

Lemma 3. *For* $d = 0$*, only those Gromov-Witten invariants with* $m = 3$ *and* $\sum codim = n$ *are non-zero. In that case, we have*

$$I_0(\gamma_1 \cdot \gamma_2 \cdot \gamma_3) = \int (\gamma_1 \cup \gamma_2 \cup \gamma_3) \cap [\mathbb{CP}^n]. \tag{23}$$

Lemma 4. *For* $m < 3$*, only*

$$I_1(h^n \cdot h^n) = 1 \tag{24}$$

is non-zero. This corresponds to the fact that there is a unique line passing through two general points.

Lemma 5. *If one of the* γ_i *is the fundamental class* $1 = h^0 \in H^0(\mathbb{CP}^n)$*, then the only non-zero Gromov-Witten invariant occurs with* $d = 0$ *and* $m = 3$*.*

Lemma 6. *If one of the* γ_i *is the hyperplane class* $h \in H^2(\mathbb{CP}^n)$*, then we have the divisor equation:*

$$I_d(\gamma_1 \cdots \gamma_m \cdot h) = I_d(\gamma_1 \cdots \gamma_m) \cdot d \tag{25}$$

Thus in the case of $\mathbb{CP}^2$, it is enough to consider the $I_d(h^2 \cdots h^2) = N_d$ to compute all the Gromov-Witten invariants. One can prove an even stronger result. Firstly, because of the isomorphism in equation (15) one can prove the splitting lemma (4.3.2 in [3]), with the important corollary:

Corollary 7. *The following recursion holds:*

$$\int_{D(A,B;d_A,d_B)} \nu_1^*(\gamma_1) \cup \cdots \cup \nu_m^*(\gamma_m) = \sum_{e+f=n} I_{d_A}(\prod_{a\in A} \gamma_a \cdot h^e) \cdot I_{d_B}(\prod_{b\in B} \gamma_b \cdot h^f) \tag{26}$$

This in turn can the be used to prove (with several other lemmas)

Theorem 8 (Reconstruction for $\mathbb{CP}^n$)**.** *All the Gromov-Witten invariants for $\mathbb{CP}^n$ can be computed recursively, and the only necessary initial value is $I_1(h^n \cdot h^n) = 1$.*

2 Quantum Cohomology of $\mathbb{CP}^n$

After this rather lengthy introduction/reminder, start with the definition of the quantum cohomology in the case of $\mathbb{CP}^n$. First, a rather easy but very useful concept needs to be introduced.

2.1 Generating Functions

For a sequence of numbers $\{N_k\}_{k=0}^{\infty}$ define the *(exponential) generating function* as a formal power series

$$F(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} N_k. \tag{27}$$

Its derivative $F_x = \frac{\mathrm{d}}{\mathrm{d}\,x} F$ is the generating function for the sequence $\{N_{k+1}\}_{k=0}^{\infty}$, which can be easily seen by explicit calculation:

$$\begin{aligned} F_x &= \sum_{k=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}\,x} \frac{x^k}{k!} N_k = \sum_{k=1}^{\infty} \frac{k \cdot x^{k-1}}{k!} N_k = \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!} N_k \\ &= \sum_{k=0}^{\infty} \frac{x^k}{k!} N_{k+1} \end{aligned} \tag{28}$$

This gives a hint, that generating functions can be used to switch between differential equations and recursions of numbers. Another relation we will need later is the product of two generating functions: Let $\{f_k\}_{k=0}^{\infty}$, $\{g_k\}_{k=0}^{\infty}$ be two sequences of numbers and let F, G be their generating functions. Then their product $(F \cdot G)$ is the generating function for the sequence

$$h_k = \sum_{i=0}^{k} \binom{k}{i} f_i g_{k-i}. \tag{29}$$

Again, this can be easily verified directly.

2.2 Gromov-Witten Potential

As a reminder, we have a basis for $H^*(\mathbb{CP}^n)$ as a graded ring given by

$$\{h^0, h^1, \ldots, h^{n-1}, h^n\}. \tag{30}$$

Then by linearity of the I_d, the possible input classes for the Gromov-Witten invariants are

$$(h^0)^{\cdot a_0}(h^1)^{\cdot a_1} \cdots (h^n)^{\cdot a_n}, \tag{31}$$

parametrized by $\mathbf{a} = (a_0, \ldots, a_n) \in \mathbb{N}^{n+1}$ (the dot in the exponent denotes the i-fold insertion into I_d, not an i-fold product with itself).

Then, to get rid of the parameter d, define the *"collected" Gromov-Witten invariants*:

$$I(\gamma_1 \cdots \gamma_m) = \sum_{d=o}^{\infty} I_d(\gamma_1 \cdots \gamma_m) \tag{32}$$

It should be noted, that no information is lost, as at most one term in the sum on the right hand side is non-zero. This is because the sum of codimensions must equal the dimension of the moduli space. Thus:

$$\sum_i \operatorname{codim}(\gamma_i) = nd + n + d + m - 3 \tag{33}$$

$$\Rightarrow d = \frac{\sum_i \operatorname{codim}(\gamma_i) - n - m + 3}{n+1} \tag{34}$$

Finally, define the *Gromov-Witten potential* Φ as the generating function for the numbers $I((h^0)^{\cdot a_0}(h^1)^{\cdot a_1} \cdots (h^n)^{\cdot a_n})$:

$$\Phi(x_0, \ldots, x_n) = \sum_{a_0, \ldots, a_n} \frac{x_0^{a_0} \cdots x_n^{a_n}}{a_0! \cdots a_n!} I((h^0)^{\cdot a_0}(h^1)^{\cdot a_1} \cdots (h^n)^{\cdot a_n}) \tag{35}$$

Now, for practical purposes, rewrite this in multi-index notation. For that, use $\mathbf{x} = (x_0, \ldots, x_n)$ and $\mathbf{a} = (a_0, \ldots, a_n)$ together with the rules $\mathbf{x}^{\mathbf{a}} = x_0^{a_0} \cdots x_n^{a_n}$, $\mathbf{a}! = a_0! \cdots a_n!$ and $\mathbf{h}^{\mathbf{a}} = (h^0)^{\cdot a_0}(h^1)^{\cdot a_1} \cdots (h^n)^{\cdot a_n}$ to write Φ as

$$\Phi(\mathbf{x}) = \sum_{\mathbf{a}} \frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{a}!} I(\mathbf{h}^{\mathbf{a}}). \tag{36}$$

Another formal manipulation is needed for later. Interpret the formal variables $\{x_0,$

$\ldots, x_n\}$ as generic coordinates on $\mathbb{CP}^n$ with respect to the chosen basis. Thus, a general element $\gamma \in H^*(\mathbb{CP}^n)$ can be written as

$$\gamma = \sum_{i=0}^{n} x_i h^i. \tag{37}$$

Now, the formal variables in Φ can be hidden and the following relation holds:

$$\Phi = I(\exp(\gamma)) = \sum_{m=0}^{\infty} \frac{1}{m!} I(\gamma^{\cdot m}) \tag{38}$$

This can be seen by applying I to the following equation and using its linearity.

$$\begin{aligned} \exp(\gamma) &= \exp(\sum_{i=0}^{n} x_i h^i) = \prod_{i=0}^{n} \exp(x_i h^i) \\ &= \prod_{i=0}^{n} \left(\sum_{a_i=0}^{\infty} \frac{x_i^{a_i}}{a_i!} (h^i)^{\cdot a_i} \right) = \sum_{\mathbf{a}} \frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{a}!} \mathbf{h}^{\mathbf{a}} \end{aligned} \tag{39}$$

At last, the partial derivatives of the Gromov-Witten potential will prove to be interesting. Taking a partial derivative with respect to x_i is the same as inserting an additional h^i, i.e.

$$\Phi_i = \sum_{\mathbf{a}} \frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{a}!} I(\mathbf{h}^{\mathbf{a}} \cdot h^i) \tag{40}$$

and

$$\Phi_{ijk} = \sum_{\mathbf{a}} \frac{\mathbf{x}^{\mathbf{a}}}{\mathbf{a}!} I(\mathbf{h}^{\mathbf{a}} \cdot h^i \cdot h^j \cdot h^k). \tag{41}$$

2.3 Classical and Quantum Product

The classically we have

$$\int_{\mathbb{CP}^n} h^i \cup h^j = \begin{cases} 1 & \text{for } i + j = n \\ 0 & \text{otherwise} \end{cases} \tag{42}$$

or, more generally

$$h^i \cup h^j = h^{i+j}. \tag{43}$$

This can then be rewritten using the zero-degree Gromov-Witten invariants.

$$h^i \cup h^j = \sum_{e+f=n} I_0(h^i \cdot h^j \cdot h^e) h^f \tag{44}$$

This holds, because the zero-degree invariant is given by (lemma 3)

$$I_0(h^i \cdot h^j \cdot h^e) = \int_{\mathbb{CP}^n} h^i \cup h^j \cup h^e. \tag{45}$$

So, the degree-zero invariants appear as the structure constants for the classical product on the cohomology. The idea is then, to define a *quantum product*, using all the Gromov-Witten invariants

$$h^i \star h^j := \sum_{e+f=n} \Phi_{ije} h^f. \tag{46}$$

This defines a product on $H^*(\mathbb{CP}^n) \otimes_{\mathbb{Z}} \mathbb{Q}[[x]]$, the *(large) quantum cohomology ring* (the authors in [3] chose rational coefficients).

The easiest property to see is that this product is commutative, as the Φ_{ijk} are invariant under permutation of the (i, j, k). Additionally, the fundamental class h^0 is the identity for the product $\star$. This can be seen via

$$h^0 \star h^i = \sum_{e+f=n} \Phi_{0ie} h^f = \sum_{e+f=n} \left(\int_{\mathbb{CP}^n} h^0 \cup h^i \cup h^e \right) h^f = h^i, \tag{47}$$

using lemma 5 and equation (42). The last property is lengthy and hard to prove, but it will prove to be quite fundamental.

Theorem 9. *The quantum product is associative, i.e.*

$$(h^i \star h^j) \star h^k = h^i \star (h^j \star h^k). \tag{48}$$

Proof. Only a brief sketch will be given here. Start by writing down the associativity in terms of the Gromov-Witten potential:

$$\begin{aligned} (h^i \star h^j) \star h^k &= \sum_{e+f=n} \sum_{l+m=n} \Phi_{ije} \Phi_{fkl} \, h^m \\ &= \sum_{e+f=n} \sum_{l+m=n} \Phi_{jke} \Phi_{fil} \, h^m = h^i \star (h^j \star h^k) \end{aligned} \tag{49}$$

Then, as the h^m are linearly independent, this is equivalent to having

$$\sum_{e+f=n} \Phi_{ije}\Phi_{fkl} = \sum_{e+f=n} \Phi_{jke}\Phi_{fil} \tag{50}$$

for every i, j, k, l. Those are the so-called *WDVV equations* after Witten, Dijkgraaf, Verlinde and Verlinde. This equation in turn is proven using the fundamental equivalence of boundary divisors, as detailed in equation (17). So in other words, all this boils down to the equivalence of the two boundary divisors

$$D(p_1p_2|p_3p_4) = D(p_2p_3|p_1p_4), \tag{51}$$

or as a diagram:

□

All of this can also be brought into a different, broader context. Let X be a projective homogeneous variety. Then consider the vector space $V = H^*(X, \mathbb{C})$ as a smooth manifold. Let $T_0, \ldots, T_m$ be a basis of that and $\partial_0, \ldots, \partial_m$ be the corresponding vector fields. Then one can define the *Poincaré metric* $g_{ij} = \langle \partial_i | \partial_j \rangle$ and as well a (formal) connection via its Christoffel symbols $A_{ij}^f = \sum_e \Phi_{ije} g^{ef}$. Then associativity of the quantum product is equivalent to the connection being flat [3]. Making this integrability a condition leads to the concept of a *Frobenius manifold*.

3 Kontsevich's Formula

Split the Gromov-Witten potential into a degree-zero and a positive-degree part

$$\Phi = \Phi^{\mathrm{cl}} + \Gamma. \tag{52}$$

Then the classical part Φ^{cl} is given by

$$\Phi^{\mathrm{cl}} = \sum_{m=0}^{\infty} \frac{1}{m!} I_0(\gamma^{\cdot m}) = \sum_{i,j,k} \frac{1}{3!} I_0(h^i \cdot h^j \cdot h^k), \tag{53}$$

using again lemma 3 to considerably simplify the expression. Then as expected, the third derivatives, the structural constants, are given by the degree-zero Gromov-Witten invariants

$$\Phi^{\mathrm{cl}}_{ijk} = I_0(h^i \cdot h^j \cdot h^k). \tag{54}$$

The quantum part Γ cannot be simplified any further and is given by

$$\Gamma = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{d>0} I_d(\gamma^{\cdot m}) =: \sum_{m=0}^{\infty} \frac{1}{m!} I_+(\gamma^{\cdot m}). \tag{55}$$

Then the quantum product splits into a classical and a quantum part:

$$h^i \star h^j = \sum_{e+f=n} \left(I_0(h^i \cdot h^j \cdot h^e) + \Gamma_{ije}\right) h^f \tag{56}$$

$$= h^i \cup h^j + \sum_{e+f=n} \Gamma_{ije}\, h^f \tag{57}$$

As we are ultimately interested in counting rational planar curves, we are now going to specify to $\mathbb{CP}^2$. The multiplication between elements is given by

$$\begin{aligned}
h^1 \star h^1 &= h^2 + \Gamma_{111}\, h^1 + \Gamma_{112}\, h^0 \\
h^1 \star h^2 &= \Gamma_{121}\, h^1 + \Gamma_{122}\, h^0 \\
h^2 \star h^2 &= \Gamma_{221}\, h^1 + \Gamma_{222}\, h^0 \\
h^0 \star h^1 &= h^1 \\
h^0 \star h^2 &= h^2.
\end{aligned}$$

Then use the associativity constraint to derive a relation between the Γ:

$$(h^1 \star h^1) \star h^2 = h^1 \star (h^1 \star h^2) \tag{58}$$

Simply multiplying this out gives the relation

$$\begin{aligned} &\Gamma_{221}\, h^1 + \Gamma_{222}\, h^0 + \Gamma_{111}(\Gamma_{121}\, h^1 + \Gamma_{122}\, h^0) + \Gamma_{112}\, h^2 \\ &= \Gamma_{121}(h^2 + \Gamma_{111}\, h^1 + \Gamma_{122}\, h^0) + \Gamma_{112}\, h^1 \end{aligned} \tag{59}$$

Then comparing the h^0-coefficients gives the (after checking all other constraints) unique relation between the Γ:

$$\Gamma_{222} + \Gamma_{111}\Gamma_{122} = \Gamma_{121}\Gamma_{112} = \Gamma_{112}\Gamma_{112} \tag{60}$$

Now it can be argued, that x_0 and x_1 in $\gamma = \sum_{i=0}^{2} x_i h^i$ can be set to zero without loosing information. The argument for x_0 is that Gromov-Witten invariants involving the fundamental class are only non-zero in degree zero, and these are not a part of Γ. For x_1 remember the divisor equation from lemma 6, which basically says that those Gromov-Witten invariants can be reduced to cases without h^1 without loosing information. Thus set $x = x_2$ and look at the third derivatives of Γ:

$$\Gamma_{ijk} = \sum_{m=0}^{\infty} \frac{1}{m!} I_+(\gamma^{\cdot m} \cdot h^i \cdot h^j \cdot h^k) \tag{61}$$

$$= \sum_{m=0}^{\infty} \frac{x^m}{m!} I_+((h^2)^{\cdot m} \cdot h^i \cdot h^j \cdot h^k) \tag{62}$$

The crucial observation is then, that Γ_{ijk} is the generating function for the numbers $\{I_+((h^2)^{\cdot m} \cdot h^i \cdot h^j \cdot h^k)\}_{m=0}^{\infty}$. Thus it is plausible that there is a recursive relation between these numbers (and hopefully also between the N_d) instead of the differential equation (60).

Using equation (29) for the product of two generating functions gives the recursive relation

$$\begin{aligned} &I_+((h^2)^{\cdot m}\, h^2\, h^2\, h^2) + \sum_{i=0}^{m} \binom{m}{i} I_+((h^2)^{\cdot i}\, h^1\, h^1\, h^1)\, I_+((h^2)^{\cdot (m-i)}\, h^1\, h^2\, h^2) \\ &= \sum_{i=0}^{m} \binom{m}{i} I_+((h^2)^{\cdot i}\, h^1\, h^1\, h^2)\, I_+((h^2)^{\cdot (m-i)}\, h^1\, h^1\, h^2). \end{aligned} \tag{63}$$

Then apply a formal manipulation to the sum to obtain

$$\begin{aligned} I_+((h^2)^{\cdot m}\, h^2\, h^2\, h^2) + \sum_{m_A+m_B=m} \frac{m!}{m_A!m_B!} I_+((h^2)^{\cdot m_A}\, h^1\, h^1\, h^1)\, I_+((h^2)^{\cdot m_B}\, h^1\, h^2\, h^2) \\ = \sum_{m_A+m_B=m} \frac{m!}{m_A!m_B!} I_+((h^2)^{\cdot m_A}\, h^1\, h^1\, h^2)\, I_+((h^2)^{\cdot m_B}\, h^1\, h^1\, h^2). \end{aligned} \tag{64}$$

Now use that the codimensions of the inputs must sum up to the dimension of the moduli space. Do this for the five appearing Gromov-Witten invariants:

(1) Here the moduli space is $\overline{\mathcal{M}}_{0,m+3}(\mathbb{CP}^2, d)$, which has dimension $3d + 2 + m$. The sum of the codimensions is $2m + 6$, thus

$$m = 3d - 4. \tag{65}$$

Accordingly we get

$$I_+((h^2)^{\cdot(3d-4)}\, h^2\, h^2\, h^2) = I_+((h^2)^{\cdot(3d-1)}) = N_d, \tag{66}$$

using corollary 2 from intersection theory.

(2) This time the moduli space in question is $\overline{\mathcal{M}}_{0,m_A+3}(\mathbb{CP}^2, d_A)$ of dimension $3d_A + 2 + m_A$. This gives $m_A = 3d_A - 1$ and with the help of lemma 6 we get

$$I_+((h^2)^{\cdot(3d_A-1)}\, h^1\, h^1\, h^1) = N_{d_A} d_A^3. \tag{67}$$

(3) In this case we have $m_B = 3d_B - 3$ and thus

$$I_+((h^2)^{\cdot(3d_B-3)}\, h^1\, h^2\, h^2) = N_{d_B} d_B. \tag{68}$$

(4) Here we have $m_A = 3d_A - 2$, which gives

$$I_+((h^2)^{\cdot(3d_A-2)}\, h^1\, h^1\, h^2) = N_{d_A} d_A^2. \tag{69}$$

(5) This case is equivalent to the last one. Thus $m_B = 3d_B - 2$ and

$$I_+((h^2)^{\cdot(3d_B-2)}\, h^1\, h^1\, h^2) = N_{d_B} d_B^2. \tag{70}$$

So, just on the basis of the associativity and the enumerative properties of the Gromov-Witten invariants we have shown the following long sought-after theorem:

Theorem 10 (Kontsevich 1994)**.** *The following recursive relation holds for N_d, the number of rational curves of degree d passing through $3d-1$ general points in the plane.*

$$\begin{aligned} N_d + \sum_{d_A+d_B=d} \frac{(3d-4)!}{(3d_A-1)!(3d_B-3)!} d_A^3 N_{d_A} d_B N_{d_B} \\ = \sum_{d_A+d_B=d} \frac{(3d-4)!}{(3d_A-2)!(3d_B-2)!} d_A^2 N_{d_A} d_B^2 N_{d_B} \end{aligned} \tag{71}$$

The only input needed is $N_1 = 1$.

4 Small Quantum Cohomology Ring

Now in this last chapter, we want to illustrate a version of the quantum product which arose first in the context of mathematical physics. For that go back to general $\mathbb{CP}^n$. Instead of using the full information content of the Gromov-Witten invariants, use information only from divisor classes, that is those with codimension 1. Thus consider only $\gamma = x_1 h^1$ in the Gromov-Witten potential. Then we can use the divisor equation from lemma 6 to simplify the third derivatives of Φ:

$$\begin{aligned}\Phi_{ijk} &= \sum_{m=0}^{\infty} \frac{x_1^m}{m!} \sum_{d\geq 0} I_d(h^{\cdot m} \cdot h^i \cdot h^j \cdot h^k) \\ &= \sum_{m=0}^{\infty} \frac{x_1^m}{m!} \sum_{d\geq 0} d^m I_d(h^i \cdot h^j \cdot h^k)\end{aligned} \tag{72}$$

For dimensional reasons we need $nd + n + d = i + j + k$, which is only possible for $d = 0$ (classical) and $d = 1$ (quantum part). Setting $q = \mathrm{e}^x$ gives a nice formula for the quantum product

$$h^i \star h^j = \begin{cases} h^{i+j} & \text{for } i+j \leq n \\ q\, h^{i+j-n-1} & \text{for } n < i+j \leq 2n. \end{cases} \tag{73}$$

So, classically we had the ring isomorphism $H^*(\mathbb{CP}^n) \cong \mathbb{Z}[h] \big/ (h^{n+1})$. This calculation then gives the *small quantum cohomology ring*, which for $\mathbb{CP}^n$ is isomorphic to

$$\mathbb{Z}[h,q] \big/ (h^{n+1} - q). \tag{74}$$

Here one can also take a more physical approach as in the original paper by Witten. Here the idea is to look at twisted $N = 2$ supersymmetric sigma models in two dimensions with a target manifold X which is Kähler. Then the cohomology of the supersymmetry Q can be identified with the cohomology of X. Considering furthermore the ring structure, these rings are called the *chiral rings*, see [4] for more. The classical version of the chiral ring is the cohomology ring, but there are quantum corrections. Thus the chiral ring of the sigma model is called the quantum cohomology of the Kähler manifold X [1].

Now specify again to the case of $\mathbb{CP}^n$. Consider the sigma model $\mathbb{CP}^1 \to \mathbb{CP}^n$ as in [2]. This theory is twisted with an A-twist and it flows in the infrared to the Landau-Ginzburg model [1]. It has $n+1$ vacua with a mass gap. So set the field σ simply equal to their expectation value. This is given by (formula 4.7 in [2])

$$\sigma = \Lambda \mathrm{e}^{\frac{2\pi \mathrm{i} k}{n+1}}, \quad k = 1, \dots, n. \tag{75}$$

So again we have recovered the same idempotents as before and we have the relation $\sigma^{n+1} = \Lambda^{n+1} \mathrm{e}^{2\pi \mathrm{i} k}$. [1] offers another calculation of the small quantum cohomology ring for the case of Fano hypersurfaces. It should be noted that for $\mathbb{CP}^n$ the small quantum cohomology carries no enumerative interpretation, but it is useful for more complex spaces, as it is easier to calculate than the full version [3].

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