Non-linear σ model

Renormalisation Group Flow

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1 Introduction

1.1 Motivation

The Non-linear Sigma Model originated in a quantum field theoretic context as a generalisation of the linear Sigma Model.
It includes non-linear couplings of scalar fields while maintaining the masslessness of the excitations. It therefore became especially important in String Theory where the massless excitations already give rise to the full theory as the interactions only depend on the global properties of the interacting strings. As a result, the NLσM is the most general model that one could employ in a bosonic String Theory. Even more importantly, the quantum field theoretical beta functions arising in a renormalisation group flow treatment of the NLσM will give rise to the Einstein field equations demonstrating how these could emerge in the limit of a String theoretic context (this will be shown in the following).

Simultaneously, the Sigma Model and its renormalisation group flow can also be studied as an abstract mathematical model with fields mapping from a curved surface into a curved target space.
It can then be brought into different contexts, also outside quantum field theory, yielding new applications like the Heisenberg Spin Torus which we will elaborate upon in the last section.

1.2 Structure of treatise

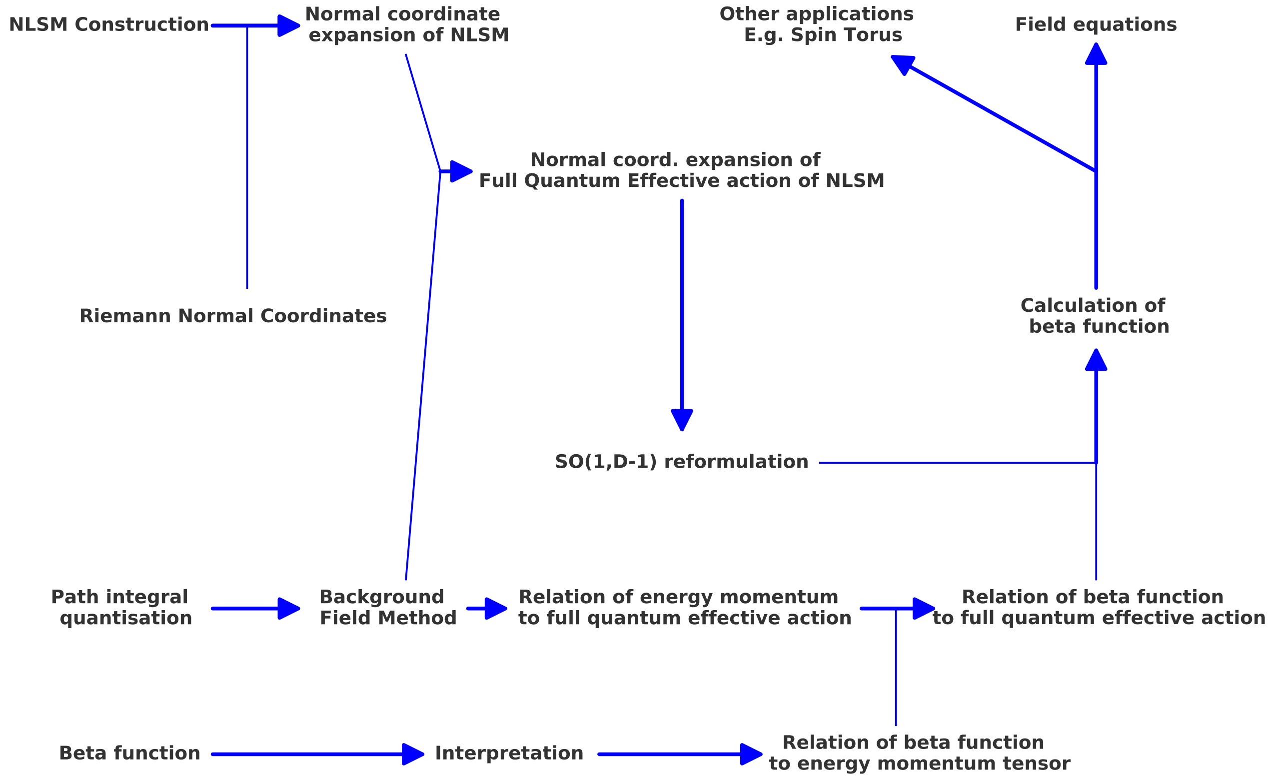
As the relationships of the topics we are going to investigate are quite intertwined, I decided to give a short overview in advance.

Basically, the combination of the four topics "NLσM", "Riemann normal coordinates", "Path integral quantisation" and "Beta functions" leads us to our end result. These are all themes existing in their own right independent of each other. However, we will

1. expand the NLσM in normal coordinates and combine that expansion with the background field method of path integral quantisation.
2. Simultaneously, we will introduce the beta function (corresponding to the response of the couplings to a variation in the energy scale) and its interpretation in order to point out its relationship with the energy momentum tensor.

3. The energy momentum tensor has also a relationship to the full quantum effective action which therefore leads to a relationship between the full quantum effective action and the beta functions.
4. Combining this relationship with the SO(1,D-1) formulation of our expanded effective action (named in 1.), we can compute the beta functions.
5. Finally the interpretation of the result will lead us to the Einstein field equations and further applications like "Spins on an elastic torus section".

The structure of this write up is depicted in diagrammatic form below for clarity.



2 NLσM (Non-Linear Sigma Model)

2.1 Construction

We want the most general action in a bosonic setting. For that, one can use the correspondence between vertex operators and variations of background fields given in [1].
Assuming a string state of squared mass $M^2 = -2 + 2p$, and defining the fields $E_{\mu_1 \cdots \mu_{2p}}(X)$ associated with these states, the employed correspondence gives the contribution to the action as

$$S_E[x, g] = \int_\Sigma d^2\sigma \sqrt{g}\; E_{\mu_1 \cdots \mu_{2p}}(X) P^{\mu_1 \cdots \mu_{2p}}(\partial, \partial^2 x, \cdots), \tag{1}$$

where $P^{\mu_1 \cdots \mu_{2p}}$ denotes a Polynomial in the derivatives of x.

For $p = 1$, one obtains a familiar massless string action.
For $p > 1$, it turns out that the resulting contributions are non-renormalizable on WS (so called world sheet Σ). Therefore these are considered *irrelevant* in a consistent theory and the most general bosonic action for a String Theory is thus made up by massless fields.
The result of putting these together (except for the tachyon which would produce negative states) is the non-linear sigma-model.

The NLσM is a bosonic theory with the following symmetries:

(a) Diff(Σ) (Diffeomorphism (Reparameterisation) invariance on the world sheet)

(b) Diff(M) (Diffeomorphism invariance on the target space)

(c) $\mathrm{U}(1)_B$ gauge invariance, $B \to B + d\gamma$, $\gamma \in \Omega^{(1)}(M)$

The most general such action that can be constructed out of massless fields is then given by

$$\Rightarrow \quad S_\sigma = \frac{1}{8\pi l^2} \int d^2\sigma \; \sqrt{g} \left\{ (g^{ab} G_{\mu\nu}(X) + \epsilon^{ab} B_{\mu\nu}(X)) \partial_a X^\mu \partial_b X^\nu + \boxed{l} R^{(2)} \Phi(X) \right\} \tag{2}$$

with the metric $G_{\mu\nu}$ (a symmetric 2-tensor), $B_{\mu\nu} = -B_{\nu\mu}$, the so-called Kalb-Ramond gauge field, and Φ, the dilaton (which in String theory gives rise to a dynamical coupling of the String: $g_s = e^{\Phi(X)}$). g^{mn} and ϵ^{mn} are the symmetric and antisymmetric metrics on Σ respectively and $\sqrt{g} = \sqrt{\det g^{mn}}$. $R^{(2)}$ is the Riemann tensor on Σ. (l, the coupling constant, is sometimes enframed in blue for clarity.)

Some more information

(a) The name σ-model has its roots in the history of pions.

(b) The model is a generalisation of the Polyakov action to a model that includes a general metric and all possible fields of a bosonic theory.

(c) One could also apply the model to the study of field theoretic effects where the X- fields not necessarily embody space time coordinates any more.
In the last section we will see the example, where the "world sheet" is a magnetized torus and the fields are associated to magnetic fields having another dynamical topological shape.

2.2 Quantum Dynamics

To obtain the quantum dynamics of the system, one can use the path integral and introduce l^{-2}, the "String length" perturbation parameter instead of $\hbar^{-1}$. The full effective action Γ is defined as usual.

$$e^{W[J]} := \int DX\ e^{-l^{-2}S_\sigma[X]+X\cdot J}, \qquad \Gamma[\langle X\rangle] := J\cdot\langle X\rangle - W[J], \qquad J = \frac{\delta\Gamma}{\delta\langle X\rangle} \tag{3}$$

with the short hand notation $X\cdot J := \int d^2\sigma\sqrt{g}\ G_{\mu\nu}(X)\ X^\mu J^\nu$.

This is not exactly solvable. One thus needs perturbation theory.

The metric is non-flat $\Rightarrow$ One would like to retain the covariant formulation while renormalising. For that we will introduce the *Background Field Quantization Method*.

2.3 Background Field Quantization Method

As a first step, we will only rewrite the full quantum effective action in such a way that we obtain a perturbation series relating Γ and S. This can be done likewise in any QFT.

We shift the integration variable ($X \to X + \langle X\rangle$) while $D(X+\langle X\rangle) = DX$ is invariant.

$$\begin{aligned}\Rightarrow\ e^{-\Gamma[\langle X\rangle]} &= \int DX\ \exp\left(-l^{-2}\ S_\sigma[X+\langle X\rangle] + X\cdot\frac{\delta\Gamma[\langle X\rangle]}{\delta\langle X\rangle}\right)\\ &\overset{(X\to lX)}{=} \int D(lX)\ \exp\left(-l^{-2}\ S_\sigma[lX+\langle X\rangle] + lX\cdot\frac{\delta\Gamma[\langle X\rangle]}{\delta\langle X\rangle}\right)\end{aligned} \tag{4}$$

The last transformation ($X\to lX$) allows us to expand in a Laurent series:

$$\begin{aligned}\frac{1}{l^2}S[\langle X\rangle + lX] &= \frac{1}{l^2}S[\langle X\rangle] + \frac{1}{l}X\cdot\frac{\delta S(\langle X\rangle)}{\delta\langle X\rangle} + \tilde{S}[\langle X\rangle; X](l),\\ \Gamma\ (\langle X\rangle) &= \frac{1}{l^2}S[\langle X\rangle] + \tilde{\Gamma}[\langle X\rangle](l)\ \Rightarrow\ lX\cdot\frac{\delta\Gamma[\langle X\rangle]}{\delta\langle X\rangle} = \frac{1}{l}X\cdot\frac{\delta S[\langle X\rangle]}{\delta\langle X\rangle} + lX\cdot\frac{\delta\tilde{\Gamma}[\langle X\rangle](l)}{\delta\langle X\rangle}\end{aligned} \tag{5}$$

$\tilde{S}$ and $\tilde{\Gamma}$ denominate Taylor series expansions in l. We thus obtain a recursive relation between $\tilde{S}(l)$ and $\tilde{\Gamma}(l)$ in path integral form:

$$e^{-\tilde{\Gamma}[\langle X\rangle](l)} = \int D(lX)\ \exp\left(-\tilde{S}_\sigma[\langle X\rangle; X](l) + lX\cdot\frac{\delta\tilde{\Gamma}[\langle X\rangle](l)}{\delta\langle X\rangle}\right). \tag{6}$$

To proceed, we need to expand our action and thus its fields. However, to do this properly, our non-flat background will require us to introduce *normal coordinates*.

2.4 Riemann normal coordinates

The next step is to expand the metric around a background: $G_{\mu\nu}(X) = G_{\mu\nu}(X_0) + \cdots$.
This is possible by expanding the fields $X^\mu \to X_0^\mu + l\xi^\mu(X_0) + \frac{l^2}{2}(\xi^\mu)^2(X_0) + \cdots$ on which the metric depends. But addition of coordinates is not covariant under Diff(M), so one needs to do it with care.

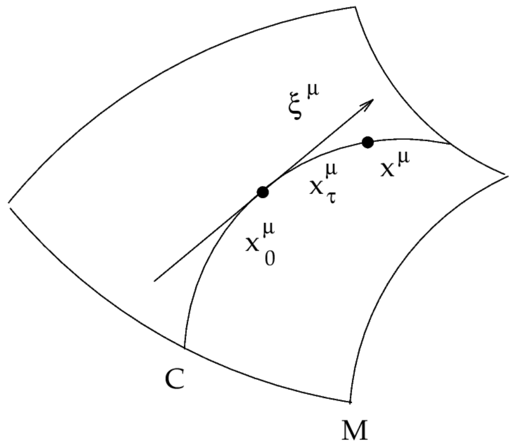


Illustration taken from [1].

Suppose X_0^μ and X^μ are sufficiently close to each other such that there is a unique geodesic curve C parameterised by X_τ^μ on the manifold interpolating between them.
We set $X^\mu_{\tau=0} = X^\mu_0$ and $X^\mu_{\tau=l} = X^\mu$ and require

$$\frac{D}{D\tau}\dot{X}^\mu_\tau = \ddot{X}^\mu_\tau + \Gamma^\mu_{\nu\rho}(X_\tau)\dot{X}^\nu_\tau\dot{X}^\rho_\tau = 0. \tag{7}$$

Now let $\xi(X)$ be a section on the pullback tangent bundle $\xi(X) \in \Gamma(X^*TM)$. We want to choose X_τ such that we can extend $\xi(X)$ to $\xi(X,t)$ by requiring

$$\xi(X,\tau) = X^*\frac{\partial}{\partial\tau}, \qquad \nabla_\tau\xi(X,\tau) = 0 =: \nabla_\xi\xi \tag{8}$$

In the last step, we made a slight abuse of notation, because ξ is actually not a vector on M but on the WS, but this will be corrected when we pull back to Σ. [3]
Comparing to the above equation, one sees that this is fulfilled for $\xi(X,t) = \dot{X}_\tau$, that is we parallel transport $\xi(X,0)$ along X_τ. The tangent vector at $\tau = 0$ is $\xi^\mu := \dot{X}^\mu_0$

Now imagine we have an arbitrary vector $v = X^*\frac{\partial}{\partial\mu}$ on the pullback tangent bundle, then $[\partial_t, \partial_\mu] = 0$ (we use a torsion free connection here for demonstration) and we get by the same abusive notation

$$\nabla_\xi v = \nabla_v\xi \;\Rightarrow\; \nabla^2_\xi v = \nabla_\xi\nabla_v\xi = R(\xi,v)\xi \tag{9}$$

where the last term is the equation of geodesic deviation. In this manner, one can see how $R(\xi,v)$, the Riemannian curvature tensor, arises in the expansion.

We are ready to expand an arbitrary tensor on the pullback tangent bundle around X_0.

$$X^*T = X^*\left(T + l\nabla_\xi T + \frac{l^2}{2}\nabla^2_\xi T + \cdots\right) = X^* e^{l\nabla_\xi} T \tag{10}$$

or in index notation, we write the short hand $X = e^{l\xi}X_0$.

E.g. for our metric $X^*g(\partial_\mu, \partial_\mu) = X^*g(v,v)$, we obtain up to 2nd order:

$$\begin{aligned}
&\text{0th: } g(v,v), \qquad \text{1st: } \nabla_\xi g(v,v) = 2g(v,\nabla_\xi v) = 2g(v,\nabla_v\xi)\\
&\text{2nd: } \nabla_\xi 2g(v,\nabla_v\xi) = 2g(\nabla_v\xi,\nabla_v\xi) + 2g(v,\nabla_\xi\nabla_v\xi) = 2g(\nabla_v\xi,\nabla_v\xi) + 2g(v,R(\xi,v)\xi)\\
&\quad\ldots
\end{aligned}$$

In index notation $v^\mu = \partial_a X^\mu$, $(\nabla_v \xi)^\nu = D_\mu \xi^\nu$ and $g(u, R(v,w)\xi) = R_{\mu\nu\rho\lambda} u^\mu \xi^\nu v^\rho w^\lambda$ such that a scalar field would yield the expansion [1] (orders of l are marked in blue for later power counting)

$$\Phi(X) = \Phi(X_0) + l D_\kappa \Phi(X_0)\xi^\kappa + \frac{l^2}{2} D_\kappa D_\sigma \Phi(X_0)\xi^\kappa \xi^\sigma + \mathcal{O}(l^3) \tag{11}$$

while a second rank tensor could be written as

$$\begin{aligned} T_{\mu\nu}(X) &= T_{\mu\nu}(X_0) + l D_\kappa T_{\mu\nu}(X_0)\xi^\kappa + \\ &+ \frac{l^2}{2}\left\{ D_\kappa D_\lambda T_{\mu\nu}(X_0) - \frac{1}{3} R^\rho_{\kappa\mu\lambda} T_{\rho\nu}(X_0) - \frac{1}{3} R^\rho_{\kappa\nu\lambda} T_{\mu\rho}(X_0)\right\} \xi^\kappa \xi^\lambda + \mathcal{O}(l^3) \end{aligned} \tag{12}$$

D_κ: covariant derivative w.r.t. affine connection $\Gamma^\mu_{\nu\rho}$; $R^\rho_{\kappa\mu\lambda}$: associated Riemann curvature.

2.5 Set up perturbation

We now combine the last and the prelast section by noting that when expanding around a background field X_0, this field can be identified with the mean field used in equations around (4). That is, $\langle X \rangle = X_0$ and $X = e^{l\xi} X_0 \approx X_0 + l\xi X_0$ around X_0. The measure becomes to first order $D(le^{l\xi}X_0) \propto D\xi$. ($D\xi$ is more precisely defined through the functional measure $||\xi^\mu||^2 = \int_\Sigma d^2\sigma \sqrt{g}\, G_{\mu\nu}(X_0)\xi^\mu \xi^\nu$.)

A field can be thought of as an infinite dimensional vector such that the chain rule upon differentiation actually results in an integral. In particular, we obtain

$$\begin{aligned} D_l S[X_0 + l\xi X_0 + l^2 \xi^2 X_0^2/2 + \cdots]_{l=0} &= \int d^2\sigma \, \frac{\partial(l\xi(\sigma))}{\partial l} \frac{\delta S[e^{l\xi} X_0]}{\delta(l\xi)}\bigg|_{(l\xi)=0} \\ = \int d^2\sigma \sqrt{g}\, \xi \, \frac{1}{\sqrt{g}} \frac{\delta S[e^{\xi'} X_0]}{\delta \xi'}\bigg|_{\xi'=0} &=: \int d^2\sigma \sqrt{g}\, \xi^\mu \, S^{(1)}_\mu[X_0] =: \xi \cdot S^{(1)} \end{aligned}$$

with $S^{(1)}_\mu[X_0] = \frac{1}{\sqrt{g}} \frac{\delta S[e^{\xi'} X_0]}{\delta \xi'}\Big|_{\xi'=0}$. Using that, we can expand $S[X]$ properly

$$\Rightarrow \; S[X] = S[e^{l\xi} X_0] = S[X_0] + l \int d^2\sigma \sqrt{g}\, \xi^\mu \, S^{(1)}_\mu[X_0] + \tilde{S}[X_0, \xi] \tag{13}$$

The recursive relation from above then yields in proper notation

$$e^{-\tilde{\Gamma}[X_0]} = \int D\xi^\mu \, \exp\left(-\tilde{S}_\sigma[e^{l\xi} X_0] + l\xi^\mu \cdot \tilde{\Gamma}^{(1)}_\mu[X_0]\right). \tag{14}$$

From here, one can proceed to list the specific expansion terms, $\tilde{S} = \tilde{S}_0 + l\tilde{S}_1 + l^2 \tilde{S}_2 + \mathcal{O}(l^3)$. These are again constructed by expanding the individual fields employing the normal coordinate procedure.

$$\begin{aligned} \tilde{S}_0 &= \frac{1}{8\pi}\int_\Sigma d^2\sigma \sqrt{g}\, \{ g^{mn} \mathcal{D}^*_m \xi^\mu \mathcal{D}^*_n \xi^\nu G_{\mu\nu}(X_0) + (g^{mn} - \epsilon^{mn}) \mathcal{R}_{\mu\nu\rho\sigma} \partial_m X^\mu_0 \partial_n X^\rho_0 \xi^\nu \xi^\sigma \} \\ \tilde{S}_1 &= \frac{1}{8\pi}\int_\Sigma d^2\sigma \, \sqrt{g} \, \left\{ \frac{1}{3} \epsilon^{mn} H_{\mu\nu\rho} \xi^\mu \mathcal{D}^*_n \xi^\nu \mathcal{D}^*_m \xi^\rho \right\} \\ \tilde{S}_2 &= \frac{1}{8\pi}\int_\Sigma d^2\sigma \, \sqrt{g} \, \left\{ \left(\frac{g^{mn} R_{\mu\nu\rho\sigma}}{3} - \frac{\epsilon^{mn} \mathcal{R}_{\mu\nu\rho\sigma}}{2} \right) \xi^\nu \xi^\rho \mathcal{D}^*_n \xi^\mu \mathcal{D}^*_m \xi^\sigma + 2R^{(2)} D_\mu D_\nu \Phi(X_0) \xi^\mu \xi^\nu \right\} \end{aligned} \tag{15}$$

$R^{(2)}$ still denominates the Ricci scalar on the world sheet while the other R and $\mathcal{R}$ come from the normal coordinate expansion of the target space. $\mathcal{D}^*_m\xi^\mu$ is the covariant derivative with torsion on TM pulled back to Σ:

$$\mathcal{D}^*_m\xi^\mu := D^*_m\xi^\mu + \frac{1}{2}H^\sigma_{\nu\rho}g_{mp}\epsilon^{pq}\partial_q x^\nu\xi^\rho \tag{16}$$

while

$$\mathcal{R}_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \frac{1}{2}D_\rho H_{\sigma\mu\nu} - \frac{1}{2}D_\sigma H_{\rho\mu\nu} + \frac{1}{4}H_{\rho\mu\alpha}H_{\sigma\nu}{}^\alpha - \frac{1}{4}H_{\sigma\mu\alpha}H_{\rho\nu}{}^\alpha \tag{17}$$

(because the covariant derivatives with torsion on TM are $\mathcal{D}_\mu\xi^\nu := D_\mu\xi^\nu + \frac{1}{2}H^\sigma_{\mu\rho}\xi^\rho$.)

2.6 Reformulation as SO(1,D-1) gauge theory

However the above expansion describes fields propagating in an arbitrary background which is not known how to deal with.
A resolution is to absorb the metric into new parameters, because then the process amounts to renormalising a flat metric with transformed coordinates.

To facilitate that, we introduce a **Vielbein** (an orthonormal frame) which we denote by $e_\mu{}^a$, ($a \in\{0,\cdots,D-1\}$). It is a basis of $\Gamma(TM) = \mathcal{V}(M)$. And importantly, it has an inverse denoted by $e_a{}^\mu$. The relevant quantities are then given by

$$\begin{aligned}
&e_a{}^\mu e^b_\mu = \delta^b_a, \qquad e^a_\mu e_a{}^\nu = \delta^\nu_\mu, \qquad (e^*)_m{}^a := \partial_m X^\mu_0 e_\mu{}^a(X_0), \qquad \xi_\mu = e_\mu{}^a\xi_a\\
&G_{\mu\nu} = e_\mu{}^a e_\nu{}^b\eta_{ab}, \qquad \eta_{ab} = \mathrm{diag}\,(-1,+1,\cdots,+1)\\
&D_\mu e_\nu{}^a := \partial_\mu e_\nu{}^a - \Gamma^\kappa_{\mu\nu}e_\kappa{}^a + \omega_\mu{}^a{}_b e_\nu{}^b\\
&\mathcal{D}_\mu e_\nu{}^a := D_\mu e_\nu{}^a + \frac{1}{2}H^\rho_{\mu\nu}e_\rho{}^a, \qquad \mathcal{D}_a = e_a{}^\mu\mathcal{D}_\mu\\
&H_{\mu\nu\rho} = e_\mu{}^a e_\nu{}^b e_\rho{}^c, \qquad \mathcal{R}_{\mu\nu\rho\lambda} = e_\mu{}^a e_\nu{}^b e_\rho{}^c e_\lambda{}^d\mathcal{R}_{abcd}
\end{aligned} \tag{18}$$

In particular, these include the SO(1,D-1) invariant flat metric η^{ab} and we get

$$G_{\mu\lambda}\xi^\mu x^\nu = e_\mu{}^a e_\nu{}^b\eta_{ab}e^\mu{}_c\xi^c e^\nu{}_d\xi^d = \delta^a_c\delta^b_d\eta_{ab}\xi^c\xi^d = \eta_{ab}\xi^a\xi^b \tag{19}$$

which we can use to rewrite the above expansion and which we will also use later to obtain the beta function coefficients.

The expansion (15) in these new parameters then reads:

$$\begin{aligned}
\tilde{S}_0 &= \frac{1}{8\pi}\int d^2\sigma\sqrt{g}\left\{g^{mn}\mathcal{D}^*_m\xi^a\mathcal{D}^*_n\xi^b\eta_{ab} + (g^{mn}-\epsilon^{mn})\mathcal{R}_{abcd}(e^*)_m{}^a(e^*)_n{}^c\xi^b\xi^d\right\},\\
\tilde{S}_1 &= \frac{1}{8\pi}\int_\Sigma d^2\sigma\sqrt{g}\left\{\frac{1}{3}\epsilon^{mn}H_{abc}\xi^a\mathcal{D}^*_n\xi^b\mathcal{D}^*_m\xi^c\right\}\\
\tilde{S}_2 &= \frac{1}{8\pi}\int_\Sigma d^2\sigma\sqrt{g}\left\{\left(\frac{g^{mn}\mathcal{R}^{\mathcal{H}}{}_{abcd}}{3} - \frac{\epsilon^{mn}\mathcal{R}_{abcd}}{2}\right)\xi^b\xi^c\mathcal{D}^*_n\xi^a\mathcal{D}^*_m\xi^d + 2R^{(2)}D_aD_b\Phi(X_0)\xi^a\xi^b\right\}
\end{aligned} \tag{20}$$

with $\mathcal{R}^{\mathcal{H}}{}_{abcd} := \mathcal{R}_{abcd} - \frac{1}{4}H_{caf}H_{db}{}^f$.
Before moving on to the Feynman graphs, it is a good moment to introduce the β-*function*.

3 β-function

3.1 Introduction and Interpretation

The above terms are an order by order expansion and the calculation of their diagrams will lead to a regularisation procedure on the worldsheet Σ.
This regularisation introduces an energy scale μ into our description of the underlying physical processes because each loop actually corresponds to taking more remote interactions into account.
The β-function is *defined* as the response of a coupling to a variation of that energy scale, that means e.g. for our metric:

$$\boxed{\beta := \mu \frac{\partial}{\partial \mu} G_{\mu\nu}(X, \mu)} \tag{21}$$

By the De-Broglie relation, these energy variations in turn can be physically interpreted as a length scale in a quantum theory:

$$l = \frac{\hbar}{p} \tag{22}$$

By going from high to low energy/momentum scales, one actually considers long range effects, i.e. the mean effect of all the excitations and interactions of the fields in the area under consideration. Consequently, varying the renormalisation scale μ can be described by the metaphor of a microscope zooming out of microscopic length scales to macroscopic phenomena.

Sometimes, there is employed even another interpretation in terms of time scales: As the path integral also includes the X^0-component, one can in some cases think of the present state of the renormalisation scale as the interactions that already reached the position one is looking at. With other words: Not fully integrating out a renormalisation scale corresponds to looking at the system after a finite time.

Here we have a particularly interesting situation because the coupling we are looking at is the metric itself. That means that the curvature of long range interactions (low momenta) will look different from the ones at high energy (where Γ can be approximated with S).
It might even happen that e.g. the probability for a scattering at high energy and strong curvature is equivalent to the one of a low energy scattering at low curvature.

3.2 Relation of β-function to trace of energy momentum tensor

We know that a conformal transformation $g_{ab} \to e^{\phi} g_{ab}$ leaves our action invariant ($S[X, e^{\phi} g] = S[X, g]$) such that the energy momentum tensor is invariant as well:

$$T^{ab}[X, g] := \frac{4\pi}{\sqrt{g}} \frac{\delta S[X, g]}{\delta g_{ab}} \;\Rightarrow\; T^{ab}[X, e^{\phi} g] = \frac{4\pi}{\sqrt{g} e^{-2\phi/2}} \frac{\delta S[X, g]}{\delta (e^{\phi} g_{ab})} = T^{ab}[X, g] \tag{23}$$

From there we deduce that it's trace has to vanish if we want it to describe the same physics before and after a Weyl transformation.

$$T_m^{\ m} = g_{ab} T^{ab} \to e^{-\phi} g_{ab} T^{ab} \overset{(!)}{=} g_{ab} T^{ab} \;\Rightarrow\; T_m^{\ m} \overset{(!)}{=} 0 \tag{24}$$

After Quantisation however, the Faddev - Popov procedure will produce ghosts that lead to negative norm states that can only be controlled by requiring that the Noether charge of the BRST symmetry fulfills the physical state condition $Q_B^2 \overset{(!)}{=} 0$.

This in turn is only possible if the Virasoro algebra has no anomalies. The Virasoro generators are the fourier modes of the energy momentum tensor. Therefore, if the anomaly has to vanish in a consistent string theory and the classical energy momentum tensor is traceless, the trace of the quantised energy momentum tensor should also vanish for consistency.
But vanishing of the trace is quivalent to retaining Weyl invariance as shown above. Therefore absence of the Weyl anomaly is a physical consistency condition.

Polchinsky showed [4] that local conformal/Weyl invariance is equal to scale invariance in two dimensions. But the response to the scale change is by definition the beta function.
Therefore the coefficients of the trace of the energy momentum tensor are the beta functions.

$$T_m^{\ m} = \partial_m x_0^\mu \partial_n x_0^\nu (\beta^G_{\mu\nu} g^{mn} + \beta^B_{\mu\nu} \epsilon^{mn}) + \beta^\Phi R^{(2)} \tag{25}$$

As we will see later, the consistency condition $T_m^{\ m}$ will lead to recovering equations similar to Einsteins field equations. However, I want to make the remark that in a non string-theoretic context, Weyl invariance would not be a consistency condition for the NLσM and hence, for other physical models like the Heisenberg spin torus, the trace of the energy momentum tensor and thus the beta function could be **non-vanishing**.

3.3 Expansion in a general Weyl-anomalous background

We start with a general expansion of the β-function quoting [1].

$$\beta(X) = \sum_{p=0}^{\infty} l^{2p} \beta^{(p)}(X) \tag{26}$$

The coefficients $\beta^{(p)}$ are restricted by the following symmetries:

1. local in X
2. independent of g
3. of Σ dimension 0
4. of M dimension $2 + 2p$ for β^G, β^B and $2p$ for β^Φ
5. Diff(M) tensors
6. $U(1)_B$ invariant
7. invariant under $\Phi \to \Phi$ + const.

The consideration of these contraints results in the following expansion for our massless interaction couplings $G_{\mu\nu}, B_{\mu\nu}$ and Φ. We neglect everything that has more than 2 derivatives.

$$\begin{aligned} \beta^G_{\mu\nu} &= a_1 R^G_{\mu\nu} + a_2 G_{\mu\nu} + a_3 G_{\mu\nu} R^G + a_4 H_{\mu\rho\sigma} H_\nu{}^{\rho\sigma} + a_5 G_{\mu\nu} H_{\rho\sigma\tau} H^{\rho\sigma\tau} \\ &\quad + a_6 D_\mu D_\nu \Phi + a_7 G_{\mu\nu} D^2 \Phi + a_8 G_{\mu\nu} D^\rho \Phi D_\rho \Phi \\ \beta^B_{\mu\nu} &= b_1 D^\kappa H_{\kappa\mu\nu} + b_2 D^\kappa \Phi H_{\kappa\mu\nu} \\ \beta^\Phi &= c_0 + l^2 \left\{ c_1 R^G + c_2 D^2 \Phi + c_3 D^\kappa \Phi D_\kappa \Phi + c_4 H_{\rho\sigma\tau} H^{\rho\sigma\tau} \right\} \end{aligned} \tag{27}$$

(These are just the most general terms that can be constructed to 2nd order. The coefficients still have to be determined by comparison with the actual calculations of the coefficients of $T_m{}^m$.)

4 Calculation of β-function of NLσM

Having set up the definition and interpretation of the β-function, we can proceed with the construction of Feynman diagrams and their contribution to the trace of the energy momentum tensor $T_m{}^m$. However, until now, we have not made a connection between $T_m{}^m$ and the non-linear σ model. The only thing we are given so far is the full quantum effective action Γ. Consequently, we need to find a way to relate Γ with $T_m{}^m$.

This can be done in the following way: Assume we are performing a Weyl transformation ($g^{ab} \to e^{-2\phi} g^{ab} \approx (1-2\phi) g^{ab} \;\Rightarrow\; \delta_W g_{ab} = -2\phi g_{ab}$) and calculate the variation of the path integral under that transformation:

$$\begin{aligned} -\delta_W \Gamma = \delta_W e^W[0] = \delta_W Z[X] &\overset{\text{(chain rule)}}{=} \int DX\; e^{-S} \left(-\int d^2\sigma \sqrt{g}\; \delta_W g_{ab}(\sigma) \frac{1}{\sqrt{g}} \frac{\delta S}{\delta g_{ab}} \right) \\ &= -\frac{1}{4\pi} \int d^2\sigma \sqrt{g}\; \delta_W g_{ab}(\sigma) \int DX\; e^{-S}\; T^{ab} \\ &\overset{(\delta_W g_{ab} = -2\phi g_{ab})}{=} \frac{1}{2\pi} \int d^2\sigma \sqrt{g}\; \phi\; \langle T_m{}^m \rangle \end{aligned}$$

$$\Rightarrow \boxed{\delta_W \Gamma[X_0] = -\frac{1}{2\pi} \int_\Sigma d^2\sigma\; \sqrt{g}\; \phi\; T_m{}^m} \tag{28}$$

The last equation holds in the quantum theory as an "operator equation", that is, $T_m{}^m$ is now an expectation value taking into account all quantum interactions (associated with the path integral).

Combining (28) and (25), we obtain the important correspondence between β-functions and Γ:

$$\delta_W \Gamma = -\frac{1}{2\pi} \int_\Sigma d^2\sigma\; \sqrt{g}\; \phi\; \left\{ \partial_m x_0^\mu \partial_n x_0^\nu (\beta^G_{\mu\nu} g^{mn} + \beta^B_{\mu\nu} \epsilon^{mn}) + \beta^\Phi R^{(2)} \right\} \;. \tag{29}$$

4.1 Coefficient determination

One can structure the calculations according to loop orders of the coefficients of the expansion (27).

We investigate the first order l-dependence of the graphs contributing to each coefficient. In (12) is shown that a 2-tensor goes with one factor of l (that is, $G_{\mu\nu}, R_{\mu\nu}, G_{\mu\nu} R, H_{\mu\lambda\rho} H_\nu{}^{\lambda\rho}, \cdots$ all contribute one factor of l). A scalar field does so likewise (11) but Φ is implemented in the action (2) with an additional factor of l and therefore tree level contributions of Φ are of the same order as 1-loop contributions of $G_{\mu\nu}$ and $B_{\mu\nu}$[1]. Consequently, their respective coefficients have one order of l less for each Φ appearing behind them. The powers of l then amount to the loop order of our renormalisation scheme.

As a_8 would go with $1/l$, it would correspond to a (-1)-loop which is not possible in our scheme. We conclude $a_8 = 0$. The loop orders of the other coefficients are then determined by counting according to the above explanation:

$$\begin{aligned} &0-\text{ loop:} \quad a_6, a_7, \quad b_2, \quad c_3 \\ &1-\text{ loop:} \quad a_1, \cdots, a_5, \quad b_1, \quad c_0, c_2 \\ &2-\text{ loop:} \quad c_1, c_4. \end{aligned} \tag{30}$$

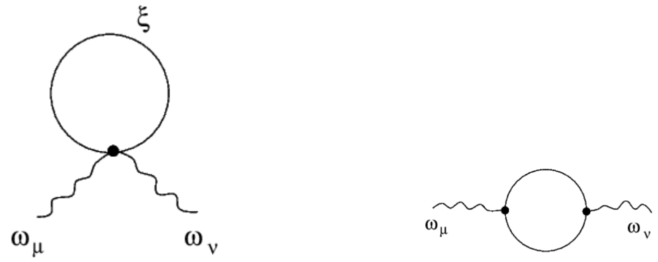
The coefficients do not depend on the fields. Thus one can calculate $a_2 = 0$ and $c_0 = D/6$ by setting $\Phi = 0 = B_{\mu\nu}$ on the level of the lagrangian. Then one can reintroduce them to calculate the rest. One should actually also include the contributions of the Faddev-Popov ghosts arising during a path integral quantisation. Their effect is however limited to an addition of $-26/6$ to β^Φ ([1],[2]), that is $c_0 \to (D-26)/6$.

The other 0-loop coefficients are obtained from tree-level calculations: $a_6 = 1, a_7 = 0, b_2 = 1/2, c_3 = 2$.

For the 1-loop level, we employ the above derived background field method. To this end, we use (29) which enables us to compare the equations (15) and (27).

$$\begin{aligned} \Rightarrow \quad & -\frac{1}{2\pi}\int_\Sigma d^2\sigma\sqrt{g}\ \phi\ [\partial_m X_0^\mu \partial_n X_0^\nu] \left(\tilde\beta^G_{\mu\nu}(X_0) g^{mn} + \tilde\beta^B_{\mu\nu}(X_0)\epsilon^{mn}\right) \\ &= \delta_W \left\langle -\frac{1}{8\pi}\int_\Sigma d^2\sigma\sqrt{g}\ \mathcal{R}_{\mu\rho\nu\lambda}(X_0)\ [\partial_m X_0^\mu \partial_n X_0^\nu]\ \xi^\rho\xi^\lambda (g^{mn} - \epsilon^{mn}) \right\rangle \end{aligned} \tag{31}$$

We use a tilde on the beta functions to denote the contributions to this order only.



One - loop - Illustration taken from [1]

Splitting the symmetric and antisymmetric tensor parts results in

$$\begin{aligned} \tilde\beta^G_{\mu\nu}(X_0)\ \phi &= \frac{1}{8}\delta_W \langle (\mathcal{R}_{\mu\rho\nu\lambda} + \mathcal{R}_{\nu\rho\mu\lambda})\ \xi^\rho\xi^\lambda \rangle \\ \tilde\beta^B_{\mu\nu}(X_0)\ \phi &= -\frac{1}{8}\delta_W \langle (\mathcal{R}_{\mu\rho\nu\lambda} - \mathcal{R}_{\nu\rho\mu\lambda})\ \xi^\rho\xi^\lambda \rangle. \end{aligned} \tag{32}$$

[1] This is actually a physical consistency condition because the dilaton would otherwise contribute non-vanishing terms to the tree level which could not be cancelled by anything else. Therefore, to have an anomaly free tree level and simultaneously keep the possibility of dynamics in the string coupling $g_s = e^{\Phi(X)}$, the only possibility is to implement Φ with an additional factor of l in the lagrangian.

(Recall that ϕ comes from the Weyl trf.). Now comes the moment, where we need to employ the Vielbein reparameterisation elaborated in section 2.6. Specifically, we use equation (19) to obtain

$$\begin{aligned}\tilde{\beta}^G_{\mu\nu}(X_0)\ \phi &= \frac{1}{8}\left(\mathcal{R}_{\mu a\nu b} + \mathcal{R}_{\nu a\mu b}\right)\delta_W\langle\xi^a\xi^b\rangle \\ \tilde{\beta}^B_{\mu\nu}(X_0)\ \phi &= -\frac{1}{8}\left(\mathcal{R}_{\mu a\nu b} - \mathcal{R}_{\nu a\mu b}\right)\delta_W\langle\xi^a\xi^b\rangle\end{aligned} \tag{33}$$

Crucially, this enables us to evaluate the mean value: $\boxed{\delta_W\langle\xi^a\xi^b\rangle = 2\cdot 2\phi\ \eta^{ab} \propto \eta^{ab}}$.
Using equation (17), we thus finally get the contributions of this order:

$$\begin{aligned}\tilde{\beta}^G_{\mu\nu}(X_0) &= \frac{1}{2}R^G_{\mu\nu} - \frac{1}{8}H_{\mu\alpha\beta}H_\nu{}^{\alpha\beta} \\ \tilde{\beta}^B_{\mu\nu} &= -\frac{1}{4}D^\alpha H_{\alpha\mu\nu}.\end{aligned} \tag{34}$$

We also want to obtain the rest of the contributions to first order of β^Φ. As mentioned below (30), c_0 and c_3 are already determined and thus only c_2, the coefficient of D^2, remains to be obtained to first order. Looking back at (20), we fortunately see, that there is only one term proportional to D^2. As a result, we get (using once more (29))

$$\begin{aligned}-\frac{1}{2\pi}\int_\Sigma d^2\sigma\sqrt{g}\ \phi\ \tilde{\beta}^\Phi R^{(2)} &= \delta_W\left\langle -\frac{1}{8\pi}\int_\Sigma d^2\sigma\sqrt{g}\ 2R^{(2)}D_aD_b\Phi(X_0)\xi^a\xi^b\right\rangle \\ \Rightarrow\ \tilde{\beta}^\Phi\ \phi &= \frac{1}{2}D_aD_b\delta_W\langle\xi^a\xi^b\rangle = 2D^2\Phi(X_0)\ \phi\end{aligned} \tag{35}$$

Comparison with (27) therefore yields $c_2 = 2$. The coefficients up to one loop order are now all determined:

$$\begin{aligned}&0-\text{ loop:}\quad a_6 = 1, a_7 = 0,\quad b_2 = 1/2,\quad c_3 = 2 \\ &1-\text{ loop:}\quad a_1 = 1/2, a_2 = 0, a_3 = 0, a_4 = -1/8, a_5 = 0,\quad b_1 = -1/4,\quad c_0 = D/6, c_2 = 2\end{aligned} \tag{36}$$

According to equation (30), c_1 and c_4 remain to be calculated.
The last step is therefore to compute the 2-loop diagrams depicted below:

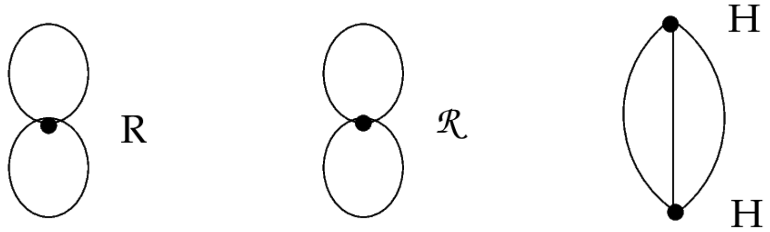


Illustration taken from ([1])

I will not do the calculation here but rather quote the result from the literature [1]: $c_1 = -\frac{1}{2}$, $c_4 = +\frac{1}{24}$.

This finally leads to our **end result for the β-functions of the NLσM**:

$$\begin{aligned}\beta^G_{\mu\nu} &= \frac{1}{2}R^G_{\mu\nu} - \frac{1}{8}H_{\mu\alpha\beta}H_\nu{}^{\alpha\beta} + D_\mu D_\nu\Phi, \\ \beta^B_{\mu\nu} &= -\frac{1}{4}D^\alpha H_{\alpha\mu\nu} + \frac{1}{2}D^\alpha\Phi H_{\alpha\mu\nu}, \\ \beta^\Phi &= \frac{D-26}{6} + l^2\left\{2D_\alpha\Phi D^\alpha\Phi - 2D_\alpha D^\alpha\Phi - \frac{1}{2}R^G + \frac{1}{24}H^2\right\}.\end{aligned} \tag{37}$$

4.2 Discussion

The above beta functions denote the reaction of the metric, the antisymmetric field and the dilaton coupling to a change of the energy scale.

In String Theory: As elaborated in section 3.2, physical consistency demands an anomaly free energy momentum tensor and the consequence for these beta functions in String Theory is therefore that they should all vanish identically (for Weyl invariance to be retained).
One can show [2] that all three equations can vanish simultaneously in 26 dimensions (that is, that the imposed condition is consistent with the restrictions imposed by the set of equations).
The symmetries of the individual beta function terms pointed out in [6] were used by [2] to state (on p. 44) that imposing $\beta^G_{\mu\nu} = 0 = \beta^B_{\mu\nu} = 0 = \beta^\Phi$ is equivalent to the following more suggestive set of equations (by using identities like the trace of the first eq. to elimate coefficients of the 3rd etc.)

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}G_{\mu\nu}R &= \boxed{\frac{1}{4}\left[H_{\mu\nu} - \frac{1}{6}G_{\mu\nu}H^2\right] -2\nabla_\mu\nabla_\nu\Phi + 2G_{\mu\nu}\nabla^2\Phi =: \Theta^{\mu\nu}}\,, \\ \nabla^\lambda H_{\lambda\mu\nu} &= 2\nabla^\lambda\Phi H_{\lambda\mu\nu} \\ \nabla^2\Phi - 2(\nabla\Phi)^2 &= -\frac{1}{2}H^2 \end{aligned} \tag{38}$$

The above defined quantity $\boxed{\Theta^{\mu\nu}}$ can be shown to be a symmetric 2-tensor. Furthermore, **only** upon use of the equations of motion, $\boxed{\Theta^{\mu\nu}}$ is conserved, thus fulfilling the requirements for an energy momentum tensor of our target space. [2]
Only these further steps then enable us to view the beta functions additionally as a tool of String Theory to reproduce something that looks similar to the **Einstein field equations** (the physics normally associated with the energy momentum tensor still needs to be associated properly with H and Φ).

In general: However, in general the above functions do not have to vanish if Weyl invariance is not a physical consistency condition. This can e.g. be the case if the X- fields are not viewed as space time coordinates and we will, to round up this write up, briefly show the example of the Heisenberg Spin Torus in the last section to demonstrate that.

5 Heisenberg Spins on an elastic Torus

Reference ([7]) points out, that if one defines $n^i(x)$ to be the direction of magnetisation, then the magnetic energy $\mathcal{H}_{\text{magn}}$ on a curved surface S, in curvilinear coordinates, is given by the **non-linear sigma model**. In particular, their target space is not space time but the order parameter manifold.

$$\mathcal{H}_{\text{magn}} = J\int_S d^2x\ \sqrt{g(x)}g^{ab}(x)\partial_a n^i(x)\partial_b n^j(x)h_{ij(x)} \tag{39}$$

J: Coupling between spin-continuum, g^{ab}: Metric on curved surface, h_{ij}: Metric on target space.

This is actually interpreted as the continuum limit of the Heisenberg Hamiltonian for classical spins. The surfaces of a continuum of spins could e.g. be a magnetic material or a magnetorheological fluid.

In contrast to the NLσM discussed above, the "physical" space is given by the physical coordinates the magnetic field lives on. n^i are not spacetime coordinates but rather magnetic scalar field components.

Benoit and Dandoloff [7] then go on to specify the metric of S to that of a torus and parameterize

$$\rho = R + r\cos\phi, \qquad z = r\sin\phi. \tag{40}$$

This is the parameterisation of a rigid torus for which they carry through an analysis before relaxing the condition of rigidity. This will then in the end result in the possibility of obtaining geometric deformations of the supporting surface induced by the fields thus leading to the description of a novel effect, namely a *global shrinking with swellings* of the torus.
In particular they state the form of the metric g^{ab} in peripolar coordinates (ξ, η) and the form of the metric h_{ij} in polar coordinates (θ, ϕ) on the Heisenberg sphere:

$$g = \frac{a}{(\cosh b - \cos\eta)^2}[\sinh^2 b\; d\xi \otimes d\xi + d\eta \otimes d\eta], \qquad h = d\theta \otimes d\theta + \sin^2\theta\; d\phi \otimes d\phi. \tag{41}$$

They go on to classify the spin configurations according to their homotopy class and consider only toroidal symmetric configurations. From there, they extract Euler-Lagrange equations and obtain the soliton configuration of their model. Additionally, they deduce that the geometry of the support manifold leads to the non-satisfaction of the Bogomol'nyi's inequality which results in effects of *geometrical frustration.*

To bring elasticity into their model they add to the NLSM hamiltonian the elastic energy consisting of the bending of the support

$$\mathcal{H}_{el} = \frac{1}{2}k_c \int_S d^2x\; \sqrt{g}(H - H_0)^2, \qquad \text{with } \; k_c : \text{bending rigidity} \tag{42}$$

The resulting e.o.m. of the overall Hamiltonian relate to Lamé's equation **which occurs in several physical contexts**. They finally give a deformation function Λ based on the Lamé function solutions $\mathcal{L}$:

$$\Lambda(\xi) = \mathcal{L}(q_\eta \sinh b_0\; \xi | 1 + \tilde{m}; A, B, j) \tag{43}$$

and deduce for small deformations that the soliton tries to increase the eccentric angle b while the bending rigidity tends to "attract" it to the spontaneous eccentric angle b_0. This results in a physical interpreation of the geometrical deformation of the surface. Introducing the inner and outer radius as $\underline{R} = (R - r)$ and $\overline{R} = (R + r)$ respectively, their relative dilation is given by

$$\frac{\overline{R}}{\underline{R}} = \frac{\tanh b_0/2}{\tanh b/2} \tag{44}$$

As a conclusion, increasing the eccentric angle b leads to a global shrinking whereas a local swelling arises where the spins twist.

The derivations made above are quite general and would also be valid for other physical associations with continuous fields. The torus of a continuum of spins is an example of how the NLσM can be associated to different physics outside quantum field theory.
It is therefore useful to investigate the model as such just looking at its mathematical structure in order to have all ways open for possible applications.

To work out the abstract structure of a mathematical theory in all clarity and to associate it properly with physical reality is the only way to get a full understanding of what the essence of the objects we are looking at and thinking about is.

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