Supersymmetry in Geometry and Quantum Physics

Solitons with fractional fermion number

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Abstract

The aim of the present write-up is to discuss solitons in $\mathcal{N}=2$ supersymmetric Landau-Ginzburg models, focusing on a third-order superpotential. Previously, we discuss in depth the original discovery of fermion number fractionization as well as its intimate connections to index theory.

Contents

1 Introduction

There exists an intriguing topological and quantum mechanical aspect of fermion-soliton interactions: the resulting quantum states carry fermionic quantum numbers, which can have fractional or even irrational values [8].

In a quantum field theory the vacuum state is characterized by its trivial physical quantum numbers. Particularly, the vacuum fermion number must vanish. Superselection rules of field theories imply that local operators can always be regarded as carrying integer fermion numbers. Thus, every state accessible from the vacuum by the operation of local operators must have an integer fermion number, and only through nonlocal, nontrivial deformations of the quantum system can one expect to arrive at a state with a noninteger fermion number.

Among the possibilities of such a realization are the spontaneous breakdown of the fermion number symmetry, resulting in a superconducting phase with the fermion number being not well-defined, as well as to consider field theory models with topological solitons. The latter models have sectors characterized by their nontrivial topological quantum numbers that are disconnected from the vacuum sector by an infinite energy barrier. As a consequence of the nonlocal nature of the soliton configurations, the fermion number operator may now have noninteger eigenvalues.

The present notes are structured as follows: we start with a brief introduction to solitons in general, following the lines of the original paper by Jackiw and Rebbi [7]. A simple but illustrative example, the polymer story, will serve to explain basic principles behind fractional fermion numbers. We will proceed by pointing out in section 2 a remarkable link between fermion numbers and an open space generalization of the Atiyah-Patodi-Singer index theorem (APS). Finally, in section 3 we will discuss the actual goal of the talk, soliton states in $\mathcal{N}=2$ supersymmetric Landau-Ginzburg models.

The main resource for sections 1 and 2 was an excellent review by Niemi and Semenoff [8], while section 3 mainly draws from two insightful papers by Fendley and Intriligator [4, 5].

1.1 Soliton states

Following [2], solitons, obeying classical field equations of motion in the Minkowski space-time, are characterised by three central properties: (i) they are of permanent form, (ii) they are localized within a certain region, and (iii) upon interaction with other solitons they emerge unchanged from the collison, except for a phase shift. Hence, in that they are non-dispersive localised packets of energy moving uniformly, solitons resemble extended particles, even though they are solutions of non-linear *wave* equations. A correspondence between classical soliton solutions and extended-particle states of quantised field theories could be established in 1974-1975 by investigating fluctuations about the soliton, giving rise to whole series of excitated states as well. From this, also instantons arose as localized finite-action classical solutions of the Euclidean version of the field equations of any given model, which turned out to lead to tunnelling effects that can significantly affect the structure of the theories' vacuum states.

Typically, soliton and instanton solutions are characterised by some topological index, related to their behaviour at spatial infinity [9]. For solitons, this index turns out to be a conserved quantity which, in the quantised theory, becomes a conserved quantum number characterising the soliton state, while in the case of instantons, the existence of a non-zero topological index leads to the generation of a family of vacuum states, characterised by some vacuum angle θ.

1.2 Discovering fermion number fractionization

In principle, the fermion number is a conserved charge corresponding to the continous $U(1)$ phase invariance of a Lagrangian with Dirac fermions. It is a c-number that is odd under C or CP. Furthermore, in field theory models with a charge conjugation symmetry general considerations, as those mentioned before, indicate that the eigenvalues of the second quantised fermion number operator are always either integers or half-integers. Indeed, let N, N' be the fermion number of any two quantum states. Their difference in a given sector is always an integer,

$$N - N' = n, \tag{1}$$

and consequently, since under charge conjugation C for all states with fermion number N, a state with fermion number $-N$ must also exist, we find that $2N = n$, i.e.

$$N = \frac{1}{2}n. \tag{2}$$

In particular, if a single state with half-integer N exists, by (1) every state that can be built from the action of local operators on it must have half-integer fermion number.

The first to actually investigate such a state were Jackiw and Rebbi in 1976 [7]. We review their calculation here analogously to [8]. They considered 1+1 dimensional theories involving a scalar field Φ and a spinor field Ψ with a Hamiltonian

$$H = \int dx \left\{ \frac{1}{2}\Pi^2 + \frac{1}{2}\left(\frac{d}{dx}\Phi\right)^2 + V(\Phi) + \Psi^\dagger \left(-i\alpha\frac{d}{dx} + \beta\Phi\right)\Psi \right\}, \tag{3}$$

where Π is the canonical momentum conjugate to Φ and $V(\Phi)$ is the scalar field potential energy density. We assume $V(\Phi)$ to be a symmetric potential with doubly degenerate minimum, being given by

$$V(\Phi) = \frac{\lambda^2}{2\Phi_0^2}\left(\Phi^2 - \Phi_0^2\right)^2 = V(-\Phi). \tag{4}$$

The two Dirac matrices present in 1+1 dimensions can be represented by

$$\alpha = \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \beta = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{5}$$

We perform a *weak-coupling expansion* in the model (3) by first identifying the minima of the bosonic energy functional (for static fields):

$$E_B = \int dx \left\{ \frac{1}{2}\left(\frac{d}{dx}\Phi\right)^2 + V(\Phi) \right\}. \tag{6}$$

For a double-well potential as (4) is one E_B has several local minima. Soliton states can be found by minimizing (6) with an x-dependent Φ, subject to the condition of energy-finiteness. Generically, soliton states exist whenever the potential energy $V(\Phi)$ has *degenerate minima*. In the case of (4) the ground state is double degenerate, which implies the existence of a soliton interpolating between $-\Phi_0$ at $x = -\infty$ and Φ_0 at $x = +\infty$, and an antisoliton interpolating between Φ_0 at $x = -\infty$ and $-\Phi_0$ at $x = +\infty$.

Having found the minima of (6) we quantize the Dirac Hamiltonian treating Φ as a background field. Since we consider here an adiabatic approximation, we ignore the bosonic quantum fluctuations about the classical background configurations. This approximation is semiclassical in nature but, in the case of ground state quantum numbers, can be expected to brovide very accurate results unless the phase symmetry is broken.

In the ground state, the minimum of $V(\Phi)$, given by (4), is at $\Phi = \pm\Phi_0$ and the reflection symmetry $\Phi \leftrightarrow -\Phi$ is spontaneously broken. We find the Dirac Hamiltonian to reduce to the translation invariant free field Hamiltonian

$$\left(-i\alpha\frac{d}{dx} + \beta\Phi_0\right)u^{(\pm)}(x) = E^{\pm}u^{(\pm)}(x) = \pm\sqrt{k^2 + \Phi_0^2}\,u^{(\pm)}(x), \tag{7}$$

where charge conjugation is implemented by σ^3:

$$u_k^{(-)}(x) = \sigma^3 u_k^{(+)*}(x), \tag{8}$$

$$\sigma^3 = i\sigma^2\sigma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{9}$$

We introduce the usual eigenmode expansion for the second quantized field:

$$\Psi(t,x) \;=\; \int dk \left\{ u_k^{(+)}(x)\, e^{-iE_k t} a_k + \left(\sigma^3 u_k^{(+)*}(x) \right) e^{iE_k t} b_k^\dagger \right\}, \tag{10}$$

where $a_k, a_k^\dagger$ and $b_k, b_k^\dagger$ are the creation and annihilation operators for fermion and antifermions respectively. Upon imposing the standard equal-time anticommutation relations,

$$\left\{ \Psi(t,x), \Psi^\dagger(t,y) \right\} \;=\; \delta(x-y), \tag{11}$$

$$\left\{ \Psi(t,x), \Psi(t,y) \right\} \;=\; \left\{ \Psi^\dagger(t,x), \Psi^\dagger(t,y) \right\} \;=\; 0, \tag{12}$$

we find the conventional nonvanishing anticommutators

$$\left\{ a_k^\dagger, a_q \right\} \;=\; \left\{ b_k^\dagger, b_q \right\} = \delta(k-q). \tag{13}$$

The fermion number operator is defined as

$$Q \;=\; \int dx : \Psi^\dagger(x)\Psi(x) := \frac{1}{2}\int dx \left[\Psi^\dagger(x), \Psi(x) \right], \tag{14}$$

where we have implemented normal ordering using the standard Dirac commutator prescription. This has the virtue, that under C the fermion number is odd, $\Psi \to \sigma_3 \Psi^*$. We find furthermore

$$Q \;=\; \int dk \left\{ a_k^\dagger a_k - b_k^\dagger b_k \right\}. \tag{15}$$

The eigenstate spectrum can be built inter alia on the $\Phi = \pm\Phi_0$ vacuum, for which

$$Q\,|\Omega\rangle \;=\; 0, \tag{16}$$

$|\Omega\rangle$ being the vacuum state, holds.

Explicit soliton profiles can be found now as solutions to the classical equations of motion,

$$-\frac{d^2}{dx^2}\Phi(x) + V'(\Phi) \;=\; 0, \tag{17}$$

subject to the boundary conditions $\Phi(+\infty) = -\Phi(-\infty) = \pm\Phi_0$, which imply that these solitons interpolate between the two vacua. Specializing to the potential (4), the soliton solutions can be explicitely given by

$$\Phi(x) \;=\; \pm\Phi_0 \tanh\left(\Phi_0 (x - x_0)\right), \tag{18}$$

where x_0 denotes the position of the soliton. $+$ and $-$ refer to the soliton S and the antisoliton $\bar{S}$, respectively, both of which have finite energy, computed via (6),

$$E_S \;=\; \frac{4}{3}\lambda\Phi_0^2, \tag{19}$$

which we interpret as the mass of the soliton. To second quantize the fermions around the soliton, we need to solve the eigenvalue equation

$$H\psi_E(x) \;=\; \left(-i\alpha\frac{d}{dx} + \beta\Phi(x) \right) \begin{pmatrix} u \\ v \end{pmatrix} \;=\; E\psi_E(x) \;=\; \mathrm{sign}(E)\sqrt{k^2+\Phi_0^2} \begin{pmatrix} u \\ v \end{pmatrix}, \tag{20}$$

where $\Phi(x)$ denotes the soliton solution. Iterating the component form of this we find u and v to satisfy the following Schrödinger equations

$$\left(-\frac{d^2}{dx^2}-\Phi'(x)+\Phi^2(x)\right)u(x)=E^2u(x)\,,\qquad \left(-\frac{d^2}{dx^2}+\Phi'(x)+\Phi^2(x)\right)v(x)=E^2v(x)\,. \quad (21)$$

However, it suffices to solve only one of these equations. The charge conjugation symmetry

$$\psi_{-E}(x) \;=\; \sigma^3\psi_E^*(x) \quad (22)$$

again pairs positive and negative energy modes. Surprisingly, the situation markedly changed: in addition to the continuum modes, the Dirac Hamiltonian also admits a normalizable zero-energy eigenmode,

$$\psi_0(x) \;=\; N\begin{pmatrix}1\\0\end{pmatrix}\exp\left(-\int_0^x dy\,\Phi(y)\right), \quad (23)$$

which is charge self-conjugate. Hence, a new bound state appears at the mass gap center.

An example of a solution for u to (20) can be found in a closed form, if the soliton (18) is implemented: the continuum normalized energy eigenstates are

$$u(x) \;=\; -\frac{\Phi_0}{\Phi_0+ik}\left(\tanh\Phi_0x-\frac{ik}{\Phi_0}\right)e^{ikx}, \quad (24)$$

while the zero-energy mode is obtained by substituting (18) into (23). As before, the quantization happens by expansion in eigenmodes:

$$\Psi(t,x) \;=\; \psi_0(x)\,d+\int dk\left\{\psi_k(x)\,e^{-iE_kt}a_k+\left(\sigma^3\psi_k^*(x)\right)e^{iE_kt}b_k^\dagger\right\}. \quad (25)$$

Here, the operators $a_k^\dagger, a_k$ and $b_k^\dagger, b_k$ create and annihilate fermions and antifermions in the soliton sector. However, since d is associated with a zero-energy eigenmode, when operating on the soliton state it produces another state with the same energy. Hence, the ground state is doubly degenerate and in order to distinguish these states we label them as $|\pm,S\rangle$.

Still imposing anticommutation relations (11) we find

$$\left\{a_k,a_q^\dagger\right\} \;=\; \left\{b_k,b_q^\dagger\right\} \;=\; \delta(k-q)\,, \quad (26)$$

$$\left\{d,d^\dagger\right\} \;=\; 1, \quad (27)$$

with all other anticommutators vanishing. This implies, that the ground states $|\pm1,S\rangle$ form a two-dimensional representation of the algebra (27),

$$d\,|+,S\rangle \;=\; |-,S\rangle\,, \quad (28)$$

$$d^\dagger\,|-,S\rangle \;=\; |+,S\rangle\,, \quad (29)$$

$$d\,|-,S\rangle \;=\; d^\dagger\,|+,S\rangle \;=\; 0. \quad (30)$$

We obtain for the fermion number operator

$$Q \;=\; d^\dagger d+\int dk\left\{a_k^\dagger a_k-b_k^\dagger b_k\right\}-\frac{1}{2}. \quad (31)$$

This implies, that the ground states now carry a nonvanishing fermion number,

$$Q\,|\pm,S\rangle \;=\; \pm\frac{1}{2}\,|\pm,S\rangle\,. \quad (32)$$

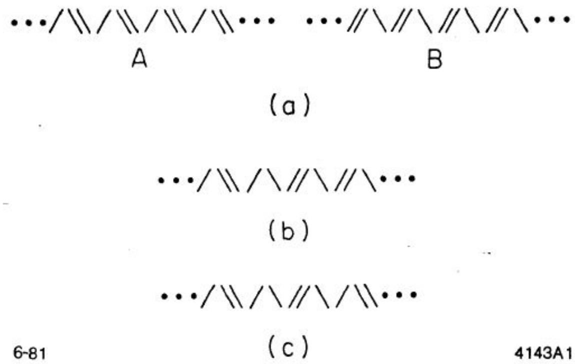

Fig. 1: (a) the two degenerate ground states for the electronic structure of polyacetylene, (b) an imperfection interpolating between the two ground states, and (c) a chain with two imperfections. Taken from [6].

The soliton state is doubly degenerate with each state carrying a *fractional fermion number* $\pm 1/2$. Furthermore, since other states in the soliton sector are obtained by the action of local operators on the states $|\pm, S\rangle$ the fermion number of all other states must also be a half-integer. In the system discussed here, Q is diagonal in the number representation, as a consequence of the orthogonality of positive and negative energy eigenfunctions of the Dirac Hamiltonian. Consequently, the fermion number has not only half integral expectation values, but also vanishing fluctuations.

In the context of conventional quantum field theory the appearance of a fractional fermion number is best understood as an effect of vacuum polarization, i.e. a modification of the fermionic Dirac sea by its interaction with solitons.

1.3 The polymer story

Even though Jackiw and Rebbi concluded in their famous paper [7],

> *"The existence of states with fermion number $\pm 1/2$ in a theory where all fundamental fields have integral fermion number is truly remarkable, yet the practical significance of this is in no way obvious."*,

fermion number fractionization has found an important phenomenological realization in the physics of linearly conjugated polymers, reviewed inter alia in [6]. There exist fractionally charged solitons on polyacetylene, a caricature model of which is shown in Fig. 1 (a). In the ground state we have alternating electronic single and double bonds, which may be arranged in two inequivalent, but degenerate forms A and B.

Upon implementation of an imperfection the configuration cannot be brought anymore to either pure A or pure B by any finite rearrangement of electrons, so it will relax to a topologically stable configuration, giving rise to a domain-wall soliton. Asymptotically in one direction the polymer is in phase A, whereas asymptotically in the opposite direction it is in phase B. In between, there must be a region of finite energy density where the bond alternation pattern interpolates between these two phases. This is the location of the soliton.

If we put two imperfections together as in Fig. 1 (b), we find a configuration which asymptotically begins and ends as A. Compared to the corresponding segment of pure A, it is missing one bond. If we add an electron to the two-imperfections strand, we can deform this configuration by a finite rearrangement into a pure A strand. Interpreting this, we see a two-soliton state is equivalent to the ground state, if we add an electron. Hence, by symmetry, each separated soliton must carry electron number 1/2 per spin degree of freedom. Indeed, we find that such domain-wall solitons in polymers display anomalous quantum numbers as a signature of fermion fractionization.

There has even been experimental evidence for this. In polyacetylene chains the charged solitons interact with each other to form a soliton band which can eventually merge with the band edge to create true metallic conductivity. The Polyacetylene can show conducting behaviour. The discovery and development of conducting polymers were reason enough to award the Nobel prize in chemistry in 2000 to Heeger, MacDiarmid and Shirakawa, which would not have been possible, if Jackiw and Rebbi had not came accross the phenomenon of fermion number fractionization.

2 Solitons in QFT and index theory

In the following section we will link solitons in quantum field theory to index theory, especially to indices of the respective fermion number operators. The Atiyah-Patodi-Singer index theorem (APS) and respective η-invariants will play a major role in their computation.

2.1 APS for manifolds with boundary revisited

In the Atiyah-Patodi-Singer index theorem the manipulation of characteristic classes plays an essential role. Characteristic classes allow for the distinction between inequivalent fiber bundles.

Characteristic classes

The description follows basically [3]. Let α be a complex $k\times k$ matrix and $P(\alpha)$ be a polynomial in the components of α. Then $P(\alpha)$ is called a characteristic polynomial, if

$$P(\alpha) \quad = \quad P\left(g^{-1}\alpha g\right)\ \forall g\in \mathrm{Gl}(k,\mathbb{C}). \tag{33}$$

If α has eigenvalues $\{\lambda_1,\dots,\lambda_k\}$, $P(\alpha)$ is a symmetric function of the eigenvalues. If $S_j(\lambda)$ is the jth symmetric polynomial, i.e.

$$S_j(\lambda) \quad = \quad \sum_{i_1<i_2<\dots<i_j}\lambda_{i_1}\lambda_{i_2}\dots\lambda_{i_j}, \tag{34}$$

$P(\alpha)$ becomes a polynomial in the $S_j(\lambda)$. If a matrix-valued 2-form Ω is substituted for the matrix α in an invariant polynomial, we find the following properties to hold: (i) $P(\Omega)$ is closed, (ii) $P(\Omega)$ has topologically invariant integrals.

The Chern form of a complex vector bundle E over a manifold M is obtained by substituting the curvature 2-form Ω into the invariant polynomical $\mathrm{Det}(1+\alpha)$. We define the *total Chern form* as

$$c(\Omega) \quad = \quad \mathrm{Det}\left(I+\frac{i}{2\pi}\Omega\right) \;=\; 1+c_1(\Omega)+c_2(\Omega)+\dots, \tag{35}$$

with the individual Chern forms $c_j(\Omega)$ being polynomials of degree j in Ω,

$$c_0=1, \qquad c_1=\frac{i}{2\pi}\mathrm{Tr}\Omega, \;\; \dots \tag{36}$$

The sum in (35) is always finite: $c_j=0$ for $2j>n=\dim M$. Since $P(\Omega)$ is closed, any homogeneous polynomial in the expansion of an invariant polynomial $P(\Omega)$ is closed: $dc_j(\Omega)=0$. Thus, the Chern forms $c_j(\Omega)$ define the $2j$th cohomology classes. This cohomology class, which we denote by $c_j(E)$, is independent of the connection because $P(\Omega)-P(\Omega')$, Ω' being the curvature of another connection on the manifold, is exact for any characteristic polynomial.

Upon diagonalizing Ω as

$$g^{-1}\frac{i\Omega}{2\pi}g \quad = \quad \mathrm{diag}\left(\lambda_1,\ldots,\lambda_k\right), \tag{37}$$

the Chern class becomes

$$c\left(\Omega\right) \quad = \quad \prod_{i=1}^{k}\left(1+\lambda_i\right). \tag{38}$$

Another occurring characteristic polynomial is the *Chern character*, defined as

$$\mathrm{ch}\left(\Omega\right) \quad \equiv \quad \mathrm{Tr}\exp\left(\frac{i\Omega}{2\pi}\right) = \sum_{i=1}^{k}\exp\left(\lambda_i\right), \tag{39}$$

as well as the *Todd class*, defined as

$$\mathrm{Td}\left(\Omega\right) \quad \equiv \quad \prod_i \frac{\lambda_i}{1-\exp\left(-\lambda_i\right)}. \tag{40}$$

However, the Euler class is assigned to the tangent bundle TM of the manifold M,

$$e\left(TM\right) \quad = \quad \prod_{i=1}^{k/2}\lambda_i. \tag{41}$$

Index theorem for manifolds without boundary

The index theorem states the existence of a relationship between the analytic properties of differential operators on fiber bundles and the topological properties of the fiber bundles themselves.

In the following M is defined to be a compact smooth manifold without boundary of dimension n. Let E,F be vector bundles over M and $D: C^\infty\left(E\right)\to C^\infty\left(F\right)$ be a first-order differential operator. We choose local bundle coordinates for E and F, with $\{x_i\}$ being local coordinates on M. This allows us to decompose D,

$$D \quad = \quad a_j\left(x\right)\frac{\partial}{\partial x_j}+b, \tag{42}$$

where a_j,b are matrix-valued. D is said to be an *elliptic operator*, if $E=F$ and if its Fourier transform is invertible for the Fourier-transform variable $k\neq 0$. Similarly, higher-order elliptic operators can be defined.

The *index* of an operator, determined by the number of zero-frequency solutions to a generalized Laplace's equation, is expressed in terms of the characteristic classes of the fiber bundles involved. It can be defined as

$$\mathrm{index}\left(D\right) \quad \equiv \quad \dim\ker D - \dim\,\ker\, D^\dagger. \tag{43}$$

Let $\{E_p\}$ be a finite sequence of vector bundles over M and let $D_p: C^\infty\left(E_p\right)\to C^\infty\left(E_{p+1}\right)$ be a sequence of differential operators, that is furthermore a complex. Let $D_p^\dagger: C^\infty\left(E_{p+1}\right)\to C^\infty\left(E_p\right)$ be the adjoint map and let

$$\Delta_p \quad = \quad D_p^\dagger D_p + D_{p-1}D_{p-1}^\dagger \tag{44}$$

be the associated Laplacian. We define the complex to be elliptic, i.e. Δ_p to be an elliptic operator on $C^\infty(E_p)$. We denote the elliptic complex by $(E,D) = (\{E_p\},\{D_p\})$. Its cohomology groups can be defined as

$$H^p(E,D) = \ker D_p/\operatorname{image} D_{p-1}. \tag{45}$$

Since each cohomology class contains a *unique harmonic representative*, we find to have an isomorphism

$$H^p(E,D) \cong \ker\Delta_p. \tag{46}$$

Thus, the index of an elliptic omplex (E,D) may be defined as

$$\operatorname{index}(E,D) \equiv \sum_p (-1)^p \dim H^p(E,D) \tag{47}$$

$$= \sum_p (-1)^p \dim\ker\Delta_p. \tag{48}$$

We now roll up the complex and construct a convenient two-term elliptic complex with the same index as a given complex (E,D). Let

$$F_0 = \bigoplus_p E_{2p}, \qquad F_1 = \bigoplus_p E_{2p+1} \tag{49}$$

be the even and odd bundles, respectively. We consider operators

$$A = \bigoplus_p \left(D_{2p} + D^\dagger_{2p-1}\right), \qquad A^\dagger = \bigoplus_p \left(D^\dagger_{2p} + D_{2p-1}\right), \tag{50}$$

where $A : C^\infty(F_0) \to C^\infty(F_1), A^\dagger : C^\infty(F_1) \to C^\infty(F_0)$. Associated Laplacians are

$$\Box_0 = A^\dagger A = \bigoplus_p \Delta_{2p}, \qquad \Box_1 = AA^\dagger = \bigoplus_p \Delta_{2p+1}. \tag{51}$$

Consequently, we find

$$\operatorname{index}(F,A) = \dim\ker\Box_0 - \dim\ker\Box_1 \tag{52}$$

$$= \sum_p (-1)^p \dim\ker\Delta_p = \operatorname{index}(E,D). \tag{53}$$

Now, we are able to describe the general index theorem. Let (x,k) be local coordinates for $T^*(M)$ and choose the "symplectic orientation" $dx_1 \wedge dk_1 \wedge \cdots \wedge dx_n \wedge dk_n$. Let $D(M)$ be the unit disk bundle in $T^*(M)$ defined as $D(M) = \left\{(x,k) : |k|^2 \leq 1\right\}$, and $S(M)$, the unit sphere bundle, be its boundary. We take two copies $D_\pm(M)$ of the unit disk bundles and glue them together along their common boundary $S(M)$ to define a new fiber bundle $\Psi(M)$ over M with fiber S^n, the *compactified tangent bundle*, and the orientation on $\Psi(M)$ chosen to be that of $D_+(M)$. $\rho : \Psi(M) \to M$ be the projection onto M, $\rho_\pm$ be the restrictions of ρ to the "hemisphere bundles" $D_\pm(M)$, $\rho_\pm : D_\pm(M) \to M$.

Subsequently, we consider the pullback bundles

$$F_+ = \rho^*_+(F_0) \text{ over } D_+(M), \qquad F_- = \rho^*_-(F_1) \text{ over } D_-(M). \tag{54}$$

We define the symbol bundle $\Sigma(A)$ as F_+ and F_- glued together using a transition function $\tilde{A}(x,k) = \sigma_L(A)(x,k)$ mapping from F_+ to F_- over $S(M) = D_+(M) \cap D_-(M)$. Define $\operatorname{td}(M) \equiv \operatorname{Td}(TM)$ and

$\mathrm{ch}(\Sigma(A))$ be the Chern character of the symbol bundle. The *Atiyah-Patodi-Singer index theorem* states, that

$$\mathrm{index}(E,D) \quad = \quad \mathrm{index}(F,A) \; = \; \int_{\Psi(M)} \mathrm{ch}(\Sigma(A)) \wedge \rho^* \mathrm{td}(M), \tag{55}$$

where in the integrand only those terms of dimension $2n = \dim\Psi(M)$ are included, the others vanish.

For the four classical elliptic complexes, this formula reduces to

$$\mathrm{index}(E,D) \quad = \quad (-1)^{n(n+1)/2} \int_M \mathrm{ch}\left(\bigoplus_p (-1)^p E_p\right) \frac{\mathrm{td}(M)}{e(M)}\bigg|_{\mathrm{vol}}, \tag{56}$$

which we encountered already in the previous talk on APS.

Index theorem for manifolds with boundary

So far, the index theorem holds only for bundles with base manifolds M which are closed and compact without boundary. As shown in [3] topological indices may change upon the manifolds having nonempty boundaries. We will extend APS here to base manifolds M which may have nonempty boundaries or which, for M noncompact, can be treated as limiting cases of manifolds with boundary.

First, we must investigate the boundary conditions which determine the spectra of the operators. However, there exist topological obstructions to finding good local boundary conditions, such that for a general index theorem for a manifold with boundary we must consider non-local boundary conditions as well. Atiyah, Patodi and Singer discovered, that appropriate non-local boundary conditions could indeed be used to formulate an index theorem for elliptic complexes over manifolds with boundary.

Let us start outlining the general nature of the APS index theorem. We begin by considering a classical elliptic complex (E,D) over a manifold M with nonempty boundary ∂M. For simplicity, one may assume $\{E\}$ being rolled up to a 2-term complex, $D : E_0 \to E_1$. In order to fomulate the index theorem, we require analytic information on the boundary in addition to the purely topological information which sufficed in the case without boundary.

For the time being we assume that M admits a product metric

$$ds^2 \quad = \quad f(\tau_0)\, d\tau^2 + g_{ij}(\tau_0, \theta_k)\, d\theta^i d\theta^j \tag{57}$$

on the boundary, where $\tau = \tau_0$ defines the boundary manifold ∂M. We construct from D a Hermitian operator whose eigenfunctions ϕ are subject to the boundary condition

$$\phi \sim e^{-kr}, \qquad k > 0 \tag{58}$$

near the boundary. Cohomology classes $H^p(E,D,\partial M)$ are defined and the corresponding index is taken to be

$$\mathrm{index}(E,D,\partial M) \quad = \quad \sum_p (-1)^p H^p(E,D,\partial M). \tag{59}$$

The extended APS index theorem for manifolds with boundary takes the general form

$$\mathrm{index}(E,D,\partial M) \quad = \quad V[M] + S[\partial M] + \xi[\partial M]. \tag{60}$$

Here $V[M]$ is the integral over M of the same characteristic classes as in the $\partial M = \emptyset$ case. Thus, V is computable from the curvature alone. $S[\partial M]$ is the integral over ∂M of the Chern Simons form and only present if one uses a metric on M which does not become a product metric on the boundary. S

is computable from the connection, the curvature, and the second fundamental form determined by a choice of the normal to the boundary. However, we will not describe S in more detail here. For the interested reader we may refer to [3] for more details. $\xi[\partial M] = c\eta[M]$ is proportional the η-invariant of the boundary determined by the eigenvalues of the tangential part of D restricted to the boundary ∂M.

The η-invariant

We again consider our 2-term elliptic complex from the above. We choose $\partial/\partial\tau$ to represent the outward normal derivative on ∂M and write D as

$$D \;=\; A\cdot\partial + B\frac{\partial}{\partial\tau} = B\left(B^{-1}A\cdot\partial + \frac{\partial}{\partial\tau}\right), \tag{61}$$

where A and B are matrices and $A\cdot\partial$ represents the tangential part of D. Whereas D itself might not have a true eigenvalue spectrum since in general $E_0 \neq E_1$, the operator

$$\hat{D} \;=\; B^{-1}A\cdot\partial|_{\partial M} \tag{62}$$

maps $E_0 \to E_0$ on ∂M and does indeed have a well-defined spectrum. By $\{\lambda_i\}$ we denote the eigenvalues of the tangential operator $\hat{D}$ acting on ∂M.

The η-invariant of APS may finally be defined as

$$\eta_D[s,\partial M] = \sum_{\{\lambda_i\},\,\lambda_i\neq 0} \mathrm{sign}(\lambda_i)\,|\lambda_i|^{-s}, \qquad s > n/2. \tag{63}$$

Despite the apparent singularities at $s=0$, it can be shown that this expression possesses a regular analytic extension to $s=0$, which defines the *η-invariant*,

$$\eta_D[\partial M] \;\equiv\; \eta_D[s=0,\partial M]. \tag{64}$$

If the elliptic operator D of interest admits zero eigenvalues, then one must be careful to account for them in the definition of η_D. The correct prescription is to add $h_D(\partial M)$,which is the dimension of the space of function harmonic under $\hat{D}$,

$$\eta_D \;\to\; \eta_D + h_D. \tag{65}$$

Intuitively, η_D counts the asymmetry between the number of positive and negative eigenvalues on the boundary. Furthermore, η_D is independent of the scale of the metric and consequently independent of the numerical values of the $\{\lambda_i\}$.

2.2 Linking fermion numbers to index theory

Topological indices popping up

We shall now compute the fermion number in 1+1 dimensions for a background soliton, which has the additional complication that it does not exhibit a conjugation symmetry. This calculation gives rise to topological indices and thus shows up the intimate connection between fermion numbers and index theory.

We start by defining two elliptic differential operators $D, D^\dagger$ by

$$D = -\frac{d}{dx} + \varphi(x), \qquad D^\dagger = \frac{d}{dx} + \varphi(x), \tag{66}$$

assuming the background field $\varphi(x)$ to have a soliton profile, $\varphi(x \to \pm\infty) \to \pm\varphi_0$. We may write the fermion Hamiltonian as

$$H = \begin{pmatrix} m & D \\ D^\dagger & m \end{pmatrix} = H_0 + m\sigma^3. \tag{67}$$

Since $\{H_0, \sigma^3\} = 0$ we conclude that $H^2 = H_0^2 + m^2 \geq m^2$, such that consequently all eigenvalues of H,

$$H\psi_E = H\begin{pmatrix} u \\ v \end{pmatrix} = E\begin{pmatrix} u \\ v \end{pmatrix} = E\psi_E, \tag{68}$$

satisfy $E^2 \geq m^2$. The components of (68) read

$$D^\dagger u = (E+m)\,v, \qquad Dv = (E-m)\,u, \tag{69}$$

which upon iteration result in

$$DD^\dagger u = \left(E^2 - m^2\right)u, \qquad D^\dagger Dv = \left(E^2 - m^2\right)v. \tag{70}$$

Notice, that (69) implies that every solution of (70) yields two solutions of (68) in the case of $E \neq \pm m$ and one solution if $E = \pm m$.

If $m \neq 0$, no constant 2×2 matrix anticommutes with H, such that this Hamiltonian does not admit of any norm-preserving symmetry such as charge conjugation symmetry relating its positive and negative energy eigenmodes. However, due to

$$\frac{1}{2}\left[H^2, \sigma^3\right] = \left\{H, \frac{1}{2}\left[H, \sigma^3\right]\right\} = 0, \tag{71}$$

we conclude, that the operator

$$\frac{1}{2}\left[H, \sigma^3\right] = \begin{pmatrix} 0 & -D \\ D^\dagger & 0 \end{pmatrix} \tag{72}$$

maps all positive energy eigenmodes of H not annihilated by D or $D^\dagger$ to its negative energy eigenmodes and vice versa.

If we take u to be a properly normalized solution of (70) with $E \neq \pm m$, then

$$\psi_E(x) = \sqrt{\frac{E+m}{2E}}\begin{pmatrix} u \\ v \end{pmatrix}, \tag{73}$$

while similarly, if v is properly normalized with $E \neq \pm m$, then

$$\psi_E(x) = \sqrt{\frac{E-m}{2E}}\begin{pmatrix} u \\ v \end{pmatrix}. \tag{74}$$

Upon application of (72) to these wavefunctions we obtain

$$\tilde{\psi}_E(x) = \frac{1}{2}\left[H, \sigma^3\right]\psi_E(x) = \sqrt{\frac{E+m}{2E}}\begin{pmatrix} (m-E)\,u \\ D^\dagger u \end{pmatrix} = -\mathrm{sign}(E)\sqrt{E^2 - m^2}\,\psi_{-E}(x), \tag{75}$$

and if we assume $\psi_E(x)$ being (continuum) normalized to unity, we find that $\tilde{\psi}_E(x)$ is normalized to

$$\int dx\, \tilde{\psi}^\dagger_{E_1}(x)\,\tilde{\psi}_{E_2}(x) = \left(E_1^2 - m^2\right)\delta\left(E_1 - E_2\right). \tag{76}$$

Thus, the symmetry operator (72) annihilates all zero modes of D and $D^\dagger$ and maps all other bound states and continuum state solutions of (68) into the corresponding bound and continuum states with the opposite energy. However, we do not expect in general that a mapping like (75) preserves the density of continuum states.

We shall now compute the expectation value of the fermion number operator in the soliton sector. First, we introduce the second quantized fermion field operator,

$$\Psi(x) = \sum_n \psi_n^{(+)}(x)\, a_n + \sum_n \psi_n^{(-)}(x)\, b_n^\dagger + \int dk \left\{ \psi_k^{(+)}(x)\, a_k + \psi_k^{(-)} b_k^\dagger \right\}, \tag{77}$$

where the $\psi_n^{(\pm)}(x)$ denote positive and negative energy bound state solutions of (68) and the $\psi_k^{(\pm)}(x)$ denote continuum solutions, respectively. The continuum operators $a_k, b_k^\dagger$ are defined analogously to those in section 1, and $a_n, b_n^\dagger$ are creation and annihilation operators for the fermion and and antifermion bound states. The usual anticommutation relations are imposed to hold, such that

$$\begin{aligned} \left\{a_m, a_n^\dagger\right\} &= \left\{b_m, b_n^\dagger\right\} = \delta_{mn} && (78)\\ \left\{a_k, a_q^\dagger\right\} &= \left\{b_k, b_q^\dagger\right\} = \delta(k-q). && (79) \end{aligned}$$

Substituting (77) and its Hermitian conjugate into (14) yields

$$N = \frac{1}{2}\int dx\, \langle 0| \left[\Psi^\dagger(x), \Psi(x)\right] |0\rangle = -\frac{1}{2}\int dx \int dE\, \psi_E^\dagger(x)\, \psi_E(x)\, \mathrm{sign}(E), \tag{80}$$

where $|0\rangle$ denotes the (eventually degenerate) ground state of the theory, and the ψ_E are defined as above. We rewrite this as

$$\begin{aligned} N = &-\frac{1}{2}\int_{-\infty}^{\infty} dx \sum_{E=\pm|m|} \psi_E^\dagger(x)\, \psi_E(x)\, \mathrm{sign}(E) \\ &-\frac{1}{2}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{dk}{2\pi} \sum_{\pm E} \psi_E^\dagger(x)\, \psi_E(x)\, \mathrm{sign}(E), \end{aligned} \tag{81}$$

where the first term is the sum over the zero modes of D and $D^\dagger$, while the second term is an integral over the continuum states with

$$E = \pm\sqrt{k^2 + m^2 + \varphi_0^2}. \tag{82}$$

Notice, that the normalizable eigenstates with energy $E \neq \pm m$ have disappeared in this formula. Substituting (73) into the second term we obtain

$$\begin{aligned} N = &-\frac{1}{2}\mathrm{sign}(m)\,\mathrm{index}(H_0) - \frac{1}{2}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{dk}{2\pi} \sum_{\pm E} \mathrm{sign}(E) \left\{ \frac{2+m}{2E} u_E^\dagger(x)\, u_E(x) \right. \\ &\left. + \frac{1}{2E(E+m)} \left(D^\dagger u_E(x)\right)^\dagger \left(D^\dagger u_E(x)\right) \right\}, \end{aligned} \tag{83}$$

where the index of H_0 is defined analogously to its definition in section 2.1 as

$$\mathrm{index}(H_0) = \dim\ker D^\dagger - \dim\ker D, \tag{84}$$

here the index acquires values of $+1\,(-1)$ for a soliton (antisoliton). Note that here we defined the index of H_0 as the negative of the definition in section 2.1, (43). We further integrate (83) by parts and introduce asymptotic forms of the continuum solutions to (70),

$$u(x) = \begin{cases} e^{ikx} + R_{DD^\dagger} e^{-ikx} & \text{if } x \to -\infty \\ T_{DD^\dagger} e^{ikx} & \text{if } x \to +\infty. \end{cases} \tag{85}$$

We thus arrive at

$$N = -\frac{1}{2}\operatorname{sign}(m)\operatorname{index}(H_0) - \int_{-\infty}^{\infty}\frac{dk}{2\pi}\sum_{\pm E}\frac{\varphi_0\operatorname{sign}(E)}{4E(E+m)}\left(|T|^2+|R|^2+1\right). \tag{86}$$

Using unitarity, $|T_{DD^\dagger}|^2 + |R_{DD^\dagger}|^2 = 1$, this integral can be evaluated, such that we finally find the fermion number to be

$$N = -\frac{1}{\pi}\arctan\left(\frac{\varphi_0}{m}\right). \tag{87}$$

In the limit $m \to 0$ this reduces indeed to the Jackiw and Rebbi case (32), i.e.

$$N = -\frac{1}{2}\operatorname{sign}(m). \tag{88}$$

Thus we find, that in the absence of a conjugation symmetry relating positive and negative energy eigenmodes of the Dirac Hamiltonian, the fermion number generally acquires a nontrivial contrinuum state contribution.

The spectral density and η-invariants

This paragraph serves to relate the spectral density to the fermion number following the review [8]. Contrary to the preceding calculation, the boundaries may result in significant difficulties when dealing with inter alia topological solitons such as magnetic monopoles, since the soliton field may have a nontrivial topology on the boundary. We will here formulate the problem in an open space, avoiding such difficulties.

Consider a Hermitian Dirac operator H with eigenfunctions

$$H\psi_\lambda(\kappa,x) = \lambda(\kappa)\psi_\lambda(\kappa,x), \tag{89}$$

that form a complete set of states. We assume there to be N parameters $\kappa_i : 1,\ldots,N$ labelling eigenfunctions and eigenvalues. Some of these parameters may be continous, some of them may be discrete, and we normalize the eigenfunctions by

$$\sum\!\!\!\!\!\!\!\int d\kappa\,\psi_\lambda(\kappa,x)\,\psi_\lambda^\dagger(\kappa,y) = \delta(x-y), \tag{90}$$

$$\int dx\,\psi_\lambda^\dagger(\kappa,x)\,\psi_\lambda(\tilde{\kappa},x) = \delta(\kappa-\tilde{\kappa}). \tag{91}$$

Momentarily, assume H to not have zero eigenvalues. Define space-dependent ξ- and η-functions (cf. section 2.1) as

$$\xi_H(s,x,y) = \sum\!\!\!\!\!\!\!\int d\kappa\,\psi_\lambda(\kappa,x)\,\psi_\lambda^\dagger(\kappa,y)\,|\lambda(\kappa)|^{-s}, \tag{92}$$

$$\eta_H(s,x,y) = \sum\!\!\!\!\!\!\!\int d\kappa\,\psi_\lambda(\kappa,x)\,\psi_\lambda^\dagger(\kappa,y)\operatorname{sign}(\lambda(\kappa))\,|\lambda(\kappa)|^{-s}. \tag{93}$$

Here, s is a complex number, and in the $s \to 0$ limit both, ξ_H and η_H exist in the sense of distributions with support on the x,y space.

Now we perform a variable-change by means of using λ, α_j, $j = 1,\ldots,N-1$ instead of the κ_i, where the α_j's are some conveniently chosen parameters. The parameter space integration measure changes to $d\kappa = J(\lambda,\alpha)\,d\lambda\,d\alpha$ with $J(\lambda,\alpha)$ a Jacobian determinant. We introduce the density matrix ρ_H,

$$\rho_H(\lambda,x,y) = \sum\!\!\!\!\!\!\!\int d\alpha\,J(\lambda,\alpha)\,\psi_\lambda(\alpha,x)\,\psi_\lambda^\dagger(\alpha,y). \tag{94}$$

After substituting (94) into (92) and (93), taking a trace over spinor indices and integrating over the space we get the ξ-function

$$\zeta_H(s) = \int dx\,dy\,\delta(x-y)\,\mathrm{tr}\left[\int d\lambda\,\rho_H(\lambda,x,y)\,|\lambda|^{-s}\right]. \tag{95}$$

To obtain a meromorphic $\zeta_H(s)$ with a finite $s \to 0$ limit we redefine it by subtracting a suitable comparison Hamiltonian expression, i.e.

$$\zeta_H(s) = \int dx\,dy\,\delta(x-y)\,\mathrm{tr}\left[\int d\lambda\,\left(\rho_H(\lambda,x,y) - \rho_{H_0}(\lambda,x,y)\right)|\lambda|^{-s}\right]. \tag{96}$$

Similarly, we may obtain the finite $s \to 0$ limit having η-invariant

$$\eta_H(s) = \int dx\,dy\,\delta(x-y)\,\mathrm{tr}\left[\int d\lambda\,\rho_H(\lambda,x,y)\,\mathrm{sign}(\lambda)\,|\lambda|^{-s}\right], \tag{97}$$

which we in general encountered already in section 2.1 when we dealt with the open space extension of APS. We wish to find a spectral density $\rho_H(\lambda)$ with the properties

$$\zeta_H(s) = \int_{-\infty}^{\infty} d\lambda\,\rho_H(\lambda)\,|\lambda|^{-s}, \tag{98}$$

$$\eta_H(s) = \int_{-\infty}^{\infty} d\lambda\,\rho_H(\lambda)\,\mathrm{sign}(\lambda)\,|\lambda|^{-s}. \tag{99}$$

Upon introducing the even and odd parts of $\rho_H(\lambda)$,

$$\rho_H(\lambda) = \frac{1}{2}\left(\rho_H(\lambda)+\rho_H(-\lambda)\right) + \frac{1}{2}\left(\rho_H(\lambda)-\rho_H(-\lambda)\right) \equiv \tau_H(\lambda)+\sigma_H(\lambda), \tag{100}$$

we can rewrite these properties to find

$$\zeta_H(s) = 2\int_0^{\infty} d\lambda\,\tau_H(\lambda)\,\lambda^{-s}, \tag{101}$$

$$\eta_H(s) = 2\int_0^{\infty} d\lambda\,\sigma_H(\lambda)\,\lambda^{-s}. \tag{102}$$

By inverting these so-called Mellin transformations we obtain

$$\tau_H(\lambda) = \frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty} ds\,\zeta_H(s)\,|\lambda|^{s-1}, \tag{103}$$

$$\sigma_H(\lambda) = \frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty} ds\,\eta_H(s)\,|\lambda|^{s-1}, \tag{104}$$

which we shall interpret as the definitions of $\tau_H(\lambda)$ and $\sigma_H(\lambda)$. We may thus redefine the spectral density by

$$\rho_H(\lambda) \equiv \frac{1}{4\pi i}\int_{c-i\infty}^{c+i\infty} ds\,\{\zeta_H(s)+\mathrm{sign}(\lambda)\,\eta_H(s)\}\,|\lambda|^{s-1}, \tag{105}$$

which we can expect to be well-defined on the real-line by a proper subtraction description.

To find a relation between the spectral density and the fermion number we introduce the second quantized Dirac fields: at time $t=0$ the (ζ-function regulated) second quantized field has the expansion

$$\Psi_s(x) \;=\; \sum\!\!\!\!\!\!\int d\kappa \left\{u_\kappa(x)\,a_\kappa + v_\kappa(x)\,b_\kappa^\dagger\right\} |\lambda(\kappa)|^{-s/2}\,, \tag{106}$$

where $u_\kappa(x)$ and $v_\kappa(x)$ are the positive and negative energy solutions to the eigenvalue equation (89). The operators $a_\kappa, b_\kappa^\dagger$ and their Hermitian conjugates are annihilation and creation operators for the fermion and antifermion bound states, respectively. Insisting on the equal-time anticommutator

$$\left\{\Psi(t,x)\,,\Psi^\dagger(t,y)\right\} \;=\; \zeta_H(s,x,y) \tag{107}$$

with all other anticommutators vanishing we find as usual

$$\left\{a_\kappa, a_\mu^\dagger\right\} \;=\; \left\{b_\kappa, b_\mu^\dagger\right\} = \delta(\kappa-\mu)\,. \tag{108}$$

We now wish to introduce a normal ordered fermion number operator: demanding that the fermion number operator is bilinear in the quantized fields, its most general form is

$$Q_s \;=\; \int dx : \Psi_s^\dagger(x)\,\Psi_s(x): \tag{109}$$

$$\;=\; A\int dx \left[\Psi_s^\dagger(x)\,,\Psi_s(x)\right] + B\int dx \left\{\Psi_s^\dagger(x)\,,\Psi_s(x)\right\}, \tag{110}$$

where A and B are c-numbers and $A+B=1/2$. Substituting (106) and using completeness of the eigenfunctions, we find

$$Q_s \;=\; 2A\int d\kappa \left\{a_k^\dagger a_k - b_k^\dagger d_k\right\} |\lambda(\kappa)|^{-s} - A\eta_H(s) + B\zeta_H(s)\,. \tag{111}$$

To fix A, B we require the fermion number being odd under charge conjugation. Since $a_\kappa^\dagger a_\kappa \leftrightarrow b_\kappa^\dagger b_\kappa$ under charge conjugation, this is possible only if $B=0$. Thus, the c-number part of the (regulated) fermion number operator is

$$N_s \;=\; -\frac{1}{2}\eta_H(s)\,. \tag{112}$$

The ground state fermion number now finally reads

$$N \;=\; \lim_{s\to 0} N_s \;=\; -\frac{1}{2}\eta_H(0)\,, \tag{113}$$

and the fermion number of all other states differs from this only by integers as mentioned before.

To conclude this section, upon combination of (113) and (102) we obtain the following relation between the spectral density and the fermion number,

$$N \;=\; -\lim_{s\to 0}\int_0^\infty d\lambda\,\sigma_H(\lambda)\,\lambda^{-s}. \tag{114}$$

Hence, the fermion number of the ground state is (essentially) the Mellin transformation of the odd spectral density evaluated at $s=0$.

2.3 The general case

The foregoing computations of the fermion number have always been performed in 1+1 dimensions. However, the employed techniques are useful in higher dimensions only, if the background field has spherical symmetry and if we restict ourselves to partial wave analysis. In this section we briefly want to examine the general case as reviewed in section 15 of [8].

We consider an arbitrary Dirac Hamiltonian of the form

$$H = i\gamma^0\gamma^i\partial_i + Q(x), \tag{115}$$

where the γ^μ's are the (Minkowski space) Dirac matrices and $Q(x)$ includes all background fields. Promote $Q(x)$ to a one-parameter family of background fields $Q(x;\tau)$ with $\tau \in (-\infty, +\infty)$ such that

$$\lim_{\tau\to+\infty} Q(x;\tau) = Q(x), \qquad \lim_{\tau\to-\infty} Q(x;\tau) = Q_0(x), \tag{116}$$

where $Q_0(x)$ is a background field defining a comparison Hamiltonian. We then consider a one-parameter family of Dirac Hamiltonians H_τ, such that in the $\tau \to +\infty$ limit we recover our original Hamiltonian. We further define

$$\not{D}_z = \begin{pmatrix} 0 & D \\ D^\dagger & 0 \end{pmatrix} + z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \equiv \not{D} + z\Gamma^c, \tag{117}$$

where z is a positive constant and

$$D = i\gamma^0(\partial_\tau + H_\tau), \qquad D^\dagger = i(\partial_\tau - H_\tau)\gamma^0, \tag{118}$$

$$\Gamma^0 \equiv \begin{pmatrix} 0 & \gamma^0 \\ \gamma^0 & 0 \end{pmatrix}, \qquad \Gamma^i \equiv \begin{pmatrix} 0 & i\gamma^i \\ i\gamma^i & 0 \end{pmatrix}. \tag{119}$$

A short calculation reveals that the index of $\not{D}$ can be computed as

$$\mathrm{index}(\not{D}) = -\lim_{z\to 0+} \eta(\not{D}_z), \tag{120}$$

where analogously to section 2.2 we defined the η-invariant with the following regularization:

$$\eta(\not{D}_z) = \lim_{\beta\to 0+} \int_{-\infty}^{\infty} d\lambda\, \rho_{\not{D}_z}(\lambda)\, \mathrm{sign}(\lambda)\, e^{-\beta|\lambda|}. \tag{121}$$

This can be rewritten with $\sigma \equiv \sqrt{\omega^2 + z^2}$ as

$$\eta(\not{D}_z) = \frac{1}{2\pi} \lim_{\beta\to 0+} \int_{-\infty}^{\infty} d\omega\, e^{-i\beta\omega} \frac{z}{\omega^2+z^2} \left\{ 2T_{d+1} + \oint_{R=\infty} dS^i\, \mathrm{tr}\langle x| i\Gamma^i\Gamma^c \frac{1}{\not{D}_z + i\sigma} |x\rangle \right\}, \tag{122}$$

with T_{d+1} being the pertinent Pontryagin index of the operator $\not{D}_z$ on the $d+1$-dimensional Euclidean space obtained by adding an imaginary time τ to the d-dimensional subspace of the original Minkowski-space.

The surface integral on the right-hand side of (122) gives rise to three different contributions,

$$\begin{aligned} \eta(\not{D}_z) &= T_{d+1} + \frac{1}{2\pi} \lim_{\beta\to 0+} \int_{-\infty}^{\infty} d\omega\, e^{-i\beta\omega} \frac{z}{\omega^2+z^2} \{ I_1(z) + I_2(z) + I_3(z) \} && (123) \\ &\equiv T_{d+1} + \Delta_1 + \Delta_2 + \Delta_3, && (124) \\ I_1(z) &= \int_{\tau=\infty} d^d x\, \mathrm{tr}\langle x| i\Gamma^0\Gamma^c \frac{1}{\not{D} + i\sigma} |x\rangle, && (125) \\ I_2(z) &= -\int_{\tau=-\infty} d^d x\, \mathrm{tr}\langle x| i\Gamma^0\Gamma^c \frac{1}{\not{D} + i\sigma} |x\rangle, && (126) \\ I_3(z) &= \oint_{R_\tau=\infty} d\sigma^\mu\, \mathrm{tr}\langle x| i\Gamma^\mu\Gamma^c \frac{1}{\not{D} + i\sigma} |x\rangle. && (127) \end{aligned}$$

Explicit computations of the I_i reveal, that $\Delta_1 = \frac{1}{2}\eta_H + \dim \ker H$, $\Delta_2 = -\frac{1}{2}\eta_{H_0}$ with H_0 being a the initial Hamiltonian $H_0 = i\gamma^0\gamma^i\partial_i + Q_0(x)$, and $\Delta_3 = \frac{1}{2}\eta\left(\mathrm{Re}\,(M)\right)$ with Re denoting the real part of M, which is further defined as

$$M \equiv -\gamma_T \cdot \partial_T + i\gamma^0\partial^0 + i\gamma^0 Q. \tag{128}$$

$(\gamma_T \cdot \partial_T)$ is the part of $\gamma^i\partial^i$ tangential to the asymptotic spatial surface.

We thus find that the fermion number corresponding to the Hamiltonian H is given by

$$N \equiv -\frac{1}{2}\eta_H = T_{d+1} + \mathrm{index}\,(\not{D}) - \frac{1}{2}\eta_{H_0} + \frac{1}{2}\eta\left(\mathrm{Re}\,(M)\right). \tag{129}$$

Therefore, we can compute N from the index of $\not{D}$ (calculable from $\eta(\not{D}_z)$), the anomaly contribution T_{d+1}, which is the standard chiral anomaly corresponding to the Dirac operator $\not{D}$, the comparison Hamiltonian contribution η_{H_0} (often H_0 is chosen such that this vanishes), and $\eta\left(\mathrm{Re}\,(M)\right)$, that is the flux of fermions into the system through spatial surfaces.

Regard, that (129) is equivalent to the Atiyah-Patodi-Singer index theorem for the Hamiltonian H as given in (60), assuming the analytical indices being computed as performed here. The ξ-term in the theorem directly corresponds to the η-terms in (129), the corresponding V-term is the fermion number N itself and the anomaly contribution T_{d+1}, and S vanishes on the boundary.

3 Solitons in integrable $\mathcal{N} = 2$ supersymmetric models

We will in the following show that there are solitons with fractional fermion number in integrable $\mathcal{N} = 2$ supersymmetric models according to [4]. We will start by examining their soliton structure with an effective Landau-Ginzburg description, followed by a section on the effective superpotentials characterizing minimal models perturbed in the least relevant chiral primary field, which give rise to Chebyshev polynomial superpotentials. Subsequently, a proposed scattering theory for the $\mathcal{N} = 2$ supersymmetric minimal models perturbed in the least relevant operator is reviewed, following closely [4]. Finally, we review the relation of these models in the Landau-Ginzburg description to supersymmetric sigma-models as done in [5].

3.1 $\mathcal{N} = 2$ soliton structure

We consider a two-dimensional $\mathcal{N} = 2$ theory with an effective Landau-Ginzburg description characterized by some superpotential $W(X)$, where X is a chiral superfield consisting of a complex boson and a Dirac fermion. Due to nonrenormalization theorems many properties of those models are exactly calculable and the Landau-Ginzburg description becomes very powerful: many properties of $\mathcal{N} = 2$ theory follow solely from its Landau-Ginzburg potential, without requiring knowledge of the kinetic term [5].

Since the bosonic part of the potential is $|W'(X)|^2$, the vacua are the points in the complex X plane where $dW = 0$. The solitons X_{ij} are the finite energy solutions to the equations of motion connecting the ith and the jth vacua: $X(\sigma = -\infty) = X^{(i)}, X(\sigma = +\infty) = X^{(j)}$, while it has to be emphasized that not all such kinks are to be regarded as fundamental solitons. Kinks are solitons with a permanent profile.

In the soliton sector corresponding to a soliton X_{ij} with mass m and rapidity θ, the $\mathcal{N} = 2$ superalgebra reads

$$\begin{aligned} Q_+^2 = Q_-^2 = \bar{Q}_+^2 = \bar{Q}_-^2 &= \left\{Q_+, \bar{Q}_-\right\} = \left\{Q_-, \bar{Q}_+\right\} = 0, \\ \left\{Q_+, \bar{Q}_+\right\} = 2\Delta W, &\qquad \left\{Q_-, \bar{Q}_-\right\} = 2(\Delta W)^*, \\ \left\{Q_+, Q_-\right\} = 2me^{\theta}, &\qquad \left\{\bar{Q}_+, \bar{Q}_-\right\} = 2me^{-\theta}, \end{aligned} \tag{130}$$

where $\Delta W = W\left(X^{(j)}\right) - W\left(X^{(i)}\right)$. It follows from this algebra that we find a Bogomolny mass bound $m \geq |\Delta W|$ in the theory. The basic supermultiplet irreducible representation of (130) for solitons saturating this mass bound, which will be the only type of solitons in consideration here, is a doublet consisting of solitons $u(\theta)$ and $d(\theta)$ with

$$Q^- |u(\theta)\rangle = \sqrt{2m} e^{\theta/2} |d(\theta)\rangle, \qquad \bar{Q}^+ |u(\theta)\rangle = \omega\sqrt{2m} e^{-\theta/2} |d(\theta)\rangle,$$
$$Q^+ |d(\theta)\rangle = \sqrt{2m} e^{\theta/2} |u(\theta)\rangle, \qquad \bar{Q}^- |d(\theta)\rangle = \omega^* \sqrt{2m} e^{-\theta/2} |u(\theta)\rangle, \tag{131}$$

where $\omega = \Delta W/m$. All other actions annihilate the states.

The supersymmetry is defined on multi-particle states in the usual manner. Since Q is fermionic, one picks up phases when Q is brought through a particle with fermion number. For example, bringing Q through a fermion results in a minus sign. Since we consider solitons with eventually fractional charges, we need to generalize this notion to

$$Q^\pm |e_1 e_2\rangle \quad = \quad \left|\left(Q^\pm e_1\right) e_2\right\rangle + e^{\pm i\pi e_1} \left|e_1\left(Q^\pm e_2\right)\right\rangle, \tag{132}$$

where the action on one soliton $(Q^\pm e)$ is defined as in (131). The charges $\bar{Q}^\mp$ act with the same phases as $Q^\pm$. Using notation analogous to the one used for coproducts in a quantum group, (132) reads

$$\Delta\left(Q^\pm\right) \quad = \quad Q^\pm \otimes 1 + e^{\pm i\pi F} \otimes Q^\pm, \tag{133}$$

with F again being the fermion-number operator. This similarity with the quantum-group action is not a coincidence, since $\mathcal{N}=2$ supersymmetry is a special case of a quantum group. In here, the fractional fermion number is crucial for obtaining the correct soliton content and S-matrices.

The theory we investigate has a conserved $U(1)$ charge F corresponding to the fermion number. The generators $Q^\pm$ have fermion number ± 1, whereas $\bar{Q}^\pm$ corresponds to fermion number ∓ 1. As shown in section 2, in the soliton sectors the fermion number operator F generally picks up an additive constant piece, leading to fractional fermion numbers. Using inter alia the index theorem techniques developed in section 2 it can be shown that the fractional part of the fermion number in a soliton sector X_{ij} of the theory is given by

$$f \quad = \quad -\frac{1}{2\pi}\left(\Im \ln W''(X)\right)\Big|_{X^{(i)}}^{X^{(j)}}. \tag{134}$$

In the soliton doublet of interest $u(\theta)$ has fermion number e, while $d(\theta)$ has fermion number $e-1$, which is why they have not been labelled as boson and fermion. We will often label the solitons by their fermion number.

3.2 Chebyshev superpotentials

In 1991, when our main reference for this section [4] was published, there was considerable evidence that the $\mathcal{N}=2$ minimal models perturbed in the least relevant chiral primary field (the $\Phi_{(1,3)}$ perturbation) are integrable, such that in a collision all momenta are conserved individually, and that the n-body S-matrix factorizes into a product of two-body ones. Subsequently, the enormous number of constraints this implies means that the exact S-matrix can often be conjectured. In the perturbed theory we find the action

$$S \quad \mapsto \quad S + \lambda \int d^2x\, \Phi_{(1,3)}. \tag{135}$$

The effective superpotentials characterizing these perturbed theories are given by Chebyshev polynomials,

$$W_{k+2}\left(X=2\cos\theta\right) \quad = \quad \frac{2\cos\left(k+2\right)\theta}{k+2}, \tag{136}$$

such that $W\left(X\right)=X^{k+2}/\left(k+2\right)-X^{k}+\ldots$ For convenience, we have set the perturbing parameter to one. However, powers of this parameter can be put back in by charge counting. These perturbed theories are intimately connected with $SU\left(2\right)_k$, noting for example, that the chiral ring structure constants are the fusion rules of $SU\left(2\right)_k$. Note, that for $k=1$ the Chebyshev superpotential

$$W_3\left(X\right) \quad = \quad X^3/3+\beta X \tag{137}$$

directly relates to the Lagrangian considered by Jackiw and Rebbi, which can be easily computed from (3). From (137) we find for the bosonic potential with ϕ being the scalar field component,

$$-V\left(\phi\right) \quad = \quad \left|\frac{\partial W\left(\phi\right)}{\partial\phi}\right|^2 = \left(\phi^2+\beta\right)^2, \tag{138}$$

which up to prefactors equals the double-well potential considered in section 1.2, 4). The very same we find for the Yukawa term, since

$$-\frac{\partial^2 W\left(\phi\right)}{\partial\phi^2}\psi_+\psi_- - \frac{\partial^2\bar{W}\left(\bar{\phi}\right)}{\partial\bar{\phi}^2}\bar{\psi}_-\bar{\psi}_+ \quad = \quad -2\phi\,\psi_+\psi_- - 2\bar{\phi}\,\bar{\psi}_-\bar{\psi}_+, \tag{139}$$

where $\psi_\pm$ and $\bar{\psi}_\pm$ denote fermionic fields.

The vacua of our theory (136) are the $k+1$ solutions to $dW\left(X\right)=0$,

$$X^{(r)}=2\cos\frac{\pi r}{k+2}, \qquad r=1,\ldots,k+1. \tag{140}$$

Hence, all vacua lie on a line in the X-plane, which is characteristic for the $\mathcal{N}=0$ analog theories as well, whereas in that case there are also fermions.

As mentioned in [4], one can show that there exists a fundamental soliton connecting each of the adjacent critical points (140). Consequently, the spectrum consists of k solitons $X_{r(r+1)}$ for $r=1,\ldots,k$ and their k antisolitons $X_{(r+1)r}$. Any other possible soliton will break apart into two or more of these solitons. If we take the solitons of consideration to saturate the mass bound $m\geq|\Delta W|$ and use the value of the superpotential (136) at the critical points,

$$W\left(X^{(r)}\right) \quad = \quad \frac{2\left(-1\right)^r}{k+2}, \tag{141}$$

we see that our fundamental solitons connecting adjacent vacua all have equal mass $m=|\Delta W|=4/\left(k+2\right)$.

Each of these solitons is a supermultiplet: the doublet discussed in the last subsection. Applying (134), we find that the soliton $u_{r(r+1)}\left(\theta\right)$ has fermion number 1/2, and $d_{r(r+1)}\left(\theta\right)$ has fermion number -1/2. The corresponding $2k$ antisolitons have opposite fermion numbers. The $k=1$ directly gives rise to the Jackiw and Rebbi case [7]: the Dirac equation has one zero mode in the presence of a soliton, and supersymmetry requires that there is at least one. Effectively, it turns out that there is exactly one. Upon quantization, this Dirac zero mode results in a doublet of states, and the charge-conjugation symmetry requires them to have charges $\pm1/2$.

3.3 The *S*-matrices

$X^3/3-\beta X$ (sine-Gordon at its $\mathcal{N}=2$ point)

The perturbed theory (136) with $k=1$ corresponds to the sine-Gordon model, with coupling at the $\mathcal{N}=2$ point: $\beta^2=(2/3)\,8\pi$ in the conventional normalization. The $\mathcal{N}=2$ symmetry is a special case of the general quantum group symmetry of the sine-Gordon model at any coupling. Even though the $X^3/3-\beta X$ model is the "same" as the sine-Gordon at this coupling, we find the action of supersymmetry in the two descriptions being very different. The difference arises from the fact that sine-Gordon and the $X^3/3-\beta X$ are different local projections of the same theory, analogously to the Ising model, where the spectrum can be either a free fermion or a strongly-interacting boson with *S*-matrix $S=-1$. They can be mapped onto each other by a non-local Jordan-Wigner transformation, but they most definitely are not identical. The same type of behaviour happens in our $\mathcal{N}=2$ model. The standard sine-Gordon *S*-matrix at this point does not have an obvious supersymmetry, it must be realized nonlocally on the sine-Gordon solitons. Even though the *S*-matrix in this section is formally the same as that of sine-Gordon at the appropriate coupling, the one we investigate does in fact allow a local action of supersymmetry.

Starting from fermion-number conservation the *S*-matrix for the process

$$|J(\theta_1)\,\bar{K}(\theta_2)\rangle \to |L(\theta_2)\,\bar{M}(\theta_1)\rangle\,, \tag{142}$$

corresponding to a soliton X_{12} colliding with a soliton X_{21}, has to be of the following form,

$$\begin{array}{cc} & \begin{array}{cc} u\bar{u} & d\bar{d} \end{array} \\ \begin{array}{c} u\bar{u} \\ d\bar{d} \end{array} & \left(\begin{array}{cc} c & b \\ b & c \end{array}\right) \end{array} \qquad \begin{array}{cc} & \begin{array}{cc} u\bar{d} & d\bar{u} \end{array} \\ \begin{array}{c} d\bar{u} \\ u\bar{d} \end{array} & \left(\begin{array}{cc} 0 & a \\ a & 0 \end{array}\right) \end{array}\,, \tag{143}$$

where as before u and d carry charges $\pm1/2$, respectively, with the antiparticles having opposite charges. Upon demanding, that the *S*-matrix commutes with the supersymmetry generators with actions (131) and (133), the *S*-matrix elements are required to be

$$a = Z(\theta)\cosh\frac{\theta}{2}, \tag{144}$$

$$b = Z(\theta)\,i\sinh\frac{\theta}{2}, \tag{145}$$

$$c = Z(\theta)\,. \tag{146}$$

Crossing, unitarity and the stipulation that there be no extra bound states fixes $Z(\theta)$ to read

$$Z(\theta) = \frac{1}{\cosh\theta/2}\exp\left(\frac{i}{4}\int_{-\infty}^{\infty}\frac{dt}{t}\,\frac{\sin\theta}{\cosh^2(\pi t/2)}\right). \tag{147}$$

It may be remarked, that it is crucial in this derivation that $\Delta W/m$ is 1 for the X_{12}- and -1 for the X_{21}-soliton.

This *S*-matrix is the same as that of the sine-Gordon model at $\beta^2=(2/3)\,8\pi$, if we identify u and $\bar{d}$ with the soliton of that model and d, $\bar{u}$ with the antisoliton, whereas the fermion number in the local description becomes the topological charge in sine-Gordon.

$\mathcal{N}=2$ minimal models with $\Phi_{(1,3)}$ perturbation

As previously mentioned, the soliton structure resulting from the Chebyshev superpotential is that of the corresponding $\mathcal{N}=0$ minimal model with additional $\mathcal{N}=2$ structure. However, it is not an immediate consequence that the S-matrices are a direct product of the $\mathscr{N}=0$ soliton S-matrix and a $\mathscr{N}=2$ part, because the representation of the supersymmetry algebra depends on $\Delta W/m$, and this can depend on which solitons are being scattered. For this reason, the S-matrices for the $\mathscr{N}=2$ minimal models perturbed by the most relevant operator cannot be a tensor product. But, with the $\Phi_{(1,3)}$, least-relevant perturbation, $\Delta W/m$ just alternates between ± 1, i.e. $\Delta W/m=(-1)^r$ for $X_{r(r+1)}$, and thus all of the two-dimensional supermultiplets obey the same supersymmetry algebra as the $X^3-\beta X$ model discussed previously. From the points of supersymmetry, every scattering process for all Chebyshev superpotentials looks the same. Thus, the S-matrix for (136) is a direct product,

$$S_k^{\mathscr{N}=2}(\theta) \quad = \quad S_k^{\mathscr{N}=0}(\theta)\otimes S_{k=1}^{\mathscr{N}=2}(\theta), \tag{148}$$

where $S_k^{\mathscr{N}=0}$ is the S-matrix for the massive scattering theory coming from the $\mathscr{N}=0$ minimal model $SU(2)_1\otimes SU(2)_k/SU(2)_{k+1}$ perturbed as in (135), and the S-matrix $S_{k=1}^{\mathscr{N}=2}$ for the supersymmetric part is the one discussed in the previous subsection. Again, the S-matrix allows a local action of supersymmetry, whereas the sine-Gordon S-matrix allows a nonlocal action. Both choices give rise to the same thermodynamics. However, the Landau-Ginzburg analysis gives the Chebyshev potential and therefore the soliton structure directly.

3.4 Relating LG to supersymmetric CP^{n-1} sigma-models

Supersymmetric sigma-models on CP^{n-1} are closely related to the perturbed Landau-Ginzburg theory [5],

$$W(X) \quad = \quad \frac{X^{n+1}}{n+1}-\beta X, \tag{149}$$

which we similarly encountered before. Since CP^{n-1} is Kähler, the supersymmetric sigma-model has $\mathscr{N}=2$ supersymmetry, but since CP^{n-1} is not Ricci flat due to the non-vanishing first Chern class, the sigma-model is not conformally invariant. The supersymmetric CP^{n-1} sigma-model is, however, integrable. While integrability of the non-supersymmetric CP^{n-1} sigma-models is spoiled for $n\neq 2$ by anomalies, such anomalies cancel in the supersymmetric case. To see a connection with the perturbed LG theory note that if we denote the Kähler form of CP^{n-1} by X, then the classical cohomology ring of CP^{n-1} is generated by powers of X with the ideal X^n, where powers are wedge products. Instantons modify this to the quantum deformed cohomology ring generated by powers of X with the modified ideal $X^n=\beta=\exp(-S_{inst})$, where $S_{inst}=R+i\theta$ is the holomorphic instanton action. This is precisely the ring of the deformed theory (149). Although it appears, that there are no solitons in supersymmetric CP^{n-1}, the fundamental particles are indeed solitons resulting from a spontaneously broken $\mathbb{Z}_n$ symmetry, just as the solitons we encountered before interpolate between vacua corresponding to a spontaneously-broken $\mathbb{Z}_n$ symmetry [5].

However, naturally the two theories are different. For example, the UV limit of the CP^{n-1} sigma-model has central charge $3(n-1)$, whereas the UV limit of (149) is the minimal model with central charge $3(n-1)/(n+1)$. In the IR limit both theories are massive, flowing to $c=0$. It was further found, that the differential equations for the metric on the space of Ramond ground states as a function of β are the same for the two theories with the only difference laying in the boundary conditions at $\beta\to 0$.

The soliton content of the $CP^{n-1}=SU(n)/SU(n-1)\otimes U(1)$ model exhibits the $SU(n)$ structure: for $r=1,\ldots,n-1$ there are $n!/r!(n-r)!$ solitons corresponding to the fundamental weight representation Λ_r of $SU(n)$. Each of these solitons is a doublet $(u_{r\alpha},d_{r\alpha})$ under the $\mathscr{N}=2$ supersymmetry with $r=1,\ldots,n-1$, $\alpha=1,\ldots,n!/r!(n-r)!$. The fermion numbers of the $\mathscr{N}=2$ multiplets turn out to be multiples of $1/n$.

Finally, we want to mention that the supersymmetric CP^{n-1} sigma-model S-matrix also exhibits a tensor product structure,

$$S^{\mathcal{N}=2}(\theta) \quad = \quad S^{\mathcal{N}=0}(\theta) \otimes S^{\mathcal{N}=2}[n](\theta) \,. \tag{150}$$

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