Supersymmetry in Geometry and Quantum Physics

The Coleman-Mandula Theorem

Sascha Leonhardt

04.05.2016

1 Introduction

The Coleman-Mandula Theorem is a no-go theorem stating, that given some physically reasonable assumptions, the symmetry group of the S-matrix is a direct product of the Poincaré group and an internal symmetry group. We will define these terms in the following section. The theorem was originally proved in Coleman and Mandula's paper "All Possible Symmetries of the S Matrix" in 1967 (see [2]). We however, in section 5, will look at the proof by Weinberg in [5], which rearranges the original proof and directly considers symmetry (Lie) algebras and not the groups themselves, but the last step going back to groups is easy. We will look at the exact statement of the theorem in section 3.

In section 4 we will study Witten's kinematic argument why the theorem should be true: Any additional symmetry beyond Poincaré in a relativistically covariant theory overconstrains the elastic scattering amplitudes and allows non-zero amplitudes only for discrete scattering angles. The assumption of analyticity of scattering angles therefore rules out such symmetries (see [6]).
Section 5 will be the rigorous proof and finally in section 6 we will shortly list possible loopholes of the theorem.

This is an introduction to a supersymmetry seminar, so you might wonder why we are looking at classical Lie algebras.
The theorem tells us, that all generators of internal symmetries (gauge symmetries in the standard model) will commute with the generators of the Poincaré group (space-time symmetries) or as Coleman and Mandula put it: "We prove a new theorem on the impossibility of combining space-time and internal symmetries in any but a trivial way." (see [2])
This means that for a unified theory of Gravity and Gauge interactions, we are forced to look at a loophole of this theorem which is to consider superalgebras or general graded algebras. So as long as we do not want to give up field theory as a framework, we must consider supersymmetry. Because of this, the theorem was the starting point for physicist to try and implement SUSY into their theories and the general study of SUSY from a mathematical perspective.
In 1975 Haag, Lopuszanski and Sohnius generalised the theorem to constrain possible SUSYs in their paper "All Possible Generators of Supersymmeties of the S-Matrix" (see [3]).

2 Relativistic Field Theory and Scattering Theory

We will start with a short summary of important terms in field theory.

We will usually look at our symmetry acting on states. These states form the Hilbert space

$$\mathcal{H} = \bigoplus_n \mathcal{H}^{(n)}, \tag{1}$$

where $\mathcal{H}^{(n)}$ is the n-particle Hilbert space

$$\mathcal{H}^{(n)} \simeq \bigotimes^n \mathcal{H}^{(1)}. \tag{2}$$

The S-matrix is a unitary operator on $\mathcal{H}$ that maps "in" to "out" states

$$\begin{gathered} |\alpha, in\rangle = S\,|\alpha, out\rangle\,, \\ S_{\beta\alpha} = \langle \beta, out\,|\alpha, in\rangle = \langle \beta, in|\,S\,|\alpha, in\rangle\,, \end{gathered} \tag{3}$$

where $S_{\alpha\beta}$ is called the S-matrix element. The "in" and "out" states are elements of Hilbert spaces as above, which are non-interacting in the sense that the states are looked at at past and future infinity, where they are far apart from each other and can be thought of as asymptotically free. Any interaction in perturbative scattering theory is then captured by the isomorphism S. So the matrix elements $S_{\alpha\beta}$ are what physically describes interaction.

This brings us to the definition of a symmetry transformation as a unitary operator U, that fulfills
(1) U turns one-particle states into one-particle states

$$U\,|\text{1-particle}\rangle = |\text{1-particle'}\rangle\,; \tag{4}$$

(2) U acts on many-particle states as if they were tensor products of one-particle states

$$U\,|\alpha, \beta\rangle = (U \otimes \mathbb{1} + \mathbb{1} \otimes U)\,|\alpha\rangle \otimes |\beta\rangle\,; \tag{5}$$

(3) U commutes with S

$$[S, U] = 0, \tag{6}$$

such that "the physics is unchanged"

$$\begin{gathered} S_{\beta\alpha} = \langle \beta, out\,|\alpha, in\rangle = \langle \beta, in|\,SU^\dagger U\,|\alpha, in\rangle \\ = \langle \beta, in|\,U^\dagger SU\,|\alpha, in\rangle = \langle \beta', in|\,S\,|\alpha', in\rangle = S_{\beta'\alpha'}. \end{gathered} \tag{7}$$

Note that the associated element of the algebra (connected by the exponential map) is a Hermitian operator.

The S-matrix and therefore the theory is called Lorentz-invariant if it possesses a symmetry group that contains the Poincaré group as a subgroup (up to isomorphism). Then the states in $\mathcal{H}^{(1)}$ are of the form $|\alpha, p, n\rangle$, where p is four momentum, n is the spin and α are all the other quantum numbers arising from the rest of the symmetry group, where we used the definition of the Hilbert space as the representation space of the symmetry group.

We call B an *internal* symmetry transformation if it commutes with the generators of the Poincaré group and therefore only acts on particle type indices α.

What is actually of interest is the connected S-matrix T, which is the non-trivial part of S

$$S = 1 - i(2\pi)^4 \delta^{(4)}(P_\mu - P'_\mu)T. \tag{8}$$

Usually one does not care to write T rather than S, but in some steps, where it is of importance we will make the distinction by actually writing T.

Lastly, we state the fact that generators A of a symmetry are integral operators of the form

$$A\,|p,n\rangle = \sum_{n'} \int d^4p' \Big(\mathscr{A}(p',p)\Big)_{n'n} |p',n'\rangle \,, \tag{9}$$

where the kernel $\mathscr{A}$ is a matrix-valued (we will justify below why the indices n and n' are discrete and finite) distribution in momentum space. (Note, if $[A,P]=0$ then $\mathscr{A} = \delta^{(4)}(p-p')a(p)$ for some matrix-valued function $a(p)$.)

3 The Coleman-Mandula Theorem

We now cite the original theorem from [2].

Theorem: Let $\mathcal{G}$ be a connected symmetry group of the S-matrix, and let the following five conditions hold:

1. (Lorentz invariance.) $\mathcal{G}$ contains a subgroup locally isomorphic to $\mathcal{P}$.
2. (Particle-finiteness.) All particle types correspond to positive-energy representations of $\mathcal{P}$. For any finite M, there are only a finite number of particle types with mass less than M.
3. (Weak elastic analyticity.) Elastic-scattering amplitudes are analytic functions of center-of-mass energy, s, and invariant momentum transfer, t, in some neighborhood of the physical region, except at normal thresholds.
4. (Occurence of scattering.) Let $|p\rangle$ and $|p'\rangle$ be any two one-particle momentum eigenstates, and let $|p,p'\rangle$ be the two-particle state made from these. Then

$$T\,|p,p'\rangle \neq 0, \tag{10}$$

   except perhaps for certain isolated values of s. Phrased briefly, at almost all energies, any two plane waves scatter.
5. (An ugly technical assumption.) The generators of $\mathcal{G}$, considered as integral operators in momentum space, have distributions for their kernels. More precisely: There is a neighborhood of the identity in $\mathcal{G}$ such that every element of $\mathcal{G}$ in this neighborhood lies on some one-parameter group $g(t)$. Further if x and y are any two states in $\mathcal{D}$, then

$$\frac{1}{i}\frac{d}{dt}(x, g(t)y) = (x, Ay), \tag{11}$$

   exists at $t=0$ and defines a continuous function of x and y, linear in y and antilinear in x.

Then, $\mathcal{G}$ is locally isomorphic to the direct product of an internal symmetry group and the Poincaré group.

As already alluded to in the beginning, we will not work with a symmetry group, but rather its generators, i.e. its algebra and therefore already implementing assumption (5) of Coleman and Mandula to some extent. The statement of the theorem then reads that for any symmetry generators B not in the Poincaré algebra and given the assumptions stated above we must have

$$[B, \mathcal{P}] = 0, \tag{12}$$

that is to say B commutes with all generators of the Poincaré algebra $\mathcal{P}$ (we will use the same symbol for the group and the algebra).

4 Short, Non-Rigorous Kinematic Proof

The argument by Witten in the introduction, that any additional symmetry beyond Poincaré overconstrains scattering amplitudes can be made quantitative as follows.
This calculation is taken from a lecture on supersymmetry (see [1]).

Assume there is a symmetry generator $Q_{\mu\nu}$, such that $[Q_{\mu\nu}, P_\rho] \neq 0$, $Q_{\mu\nu} \neq J_{\mu\nu} \in \mathrm{so}(1,3)$ and $Q_{\mu\nu}$ symmetric and traceless.
From its symmetry and tracelessness we know that

$$\langle p| Q_{\mu\nu} |p\rangle \propto p_\mu p_\nu - \frac{1}{4}\eta_{\mu\nu}p^2. \tag{13}$$

It will act on one-particle states as a tensorproduct, so for orthonormal states $|p^1\rangle$ and $|p^2\rangle$ we have

$$\langle p^1, p^2| Q_{\mu\nu} |p^1, p^2\rangle = \langle p^1| Q_{\mu\nu} |p^1\rangle + \langle p^2| Q_{\mu\nu} |p^2\rangle . \tag{14}$$

Now, for an elastic scattering $p^1, p^2 \longrightarrow q^1, q^2$ with $Q_{\mu\nu}$ conserved we have

$$\langle p^1, p^2| Q_{\mu\nu} |p^1, p^2\rangle = \langle q^1, q^2| Q_{\mu\nu} |q^1, q^2\rangle \tag{15}$$

and therefore using 13 and 14

$$p^1_\mu p^1_\nu + p^2_\mu p^2_\nu = q^1_\mu q^1_\nu + q^2_\mu q^2_\nu. \tag{16}$$

Since for elastic scattering we know that $p^1_\mu + p^2_\mu = q^1_\mu + q^2_\mu$ we can see that making the ansatz

$$q^1_\mu = p^1_\mu + a_\mu, q^2_\mu = p^2_\mu + b_\mu, \tag{17}$$

that the momentum change has to be $a_\mu = -b_\mu$. Replacing the q's above this finally gives

$$a_\mu(p^1_\nu - p^2_\nu) + a_\nu(p^1_\mu - p^2_\mu) + 2a_\mu a_\nu = 0. \tag{18}$$

If this is to hold for arbitrary p's we see that we have $a_\mu = 0$ and therefore we can only have trivial scattering.

5 Proof

We now want to prove the theorem rigorously on an infinitesimal level: If $\mathcal{G}$ is a symmetry group of the S-matrix in a relativistic field theory, and the assumptions (2) to (4) hold, then the generators of $\mathcal{G}$ consist only of the generators of the Poincaré group $\mathcal{P}$ and the generators of internal symmetries. Assumption (5) will only be needed to argue why we can work with elements of an algebra. The main steps of this proof are following [5] with some of the calculations that Weinberg skipped carried out.

5.1 The subalgebra $\mathcal{B}$

Note again that we will use the same symbol for groups and their algebras. We begin with the subalgebra $\mathcal{B}$ of $\mathcal{G}$ consisting of symmetry generators B_α who commute with the four-momentum operator P_μ

$$[B_\alpha, P_\mu] = 0. \tag{19}$$

These act on multiparticle states as on a tensor product of one-particle states, i.e.

$$\begin{aligned} B_\alpha \left|p, m; q, n; \ldots\right\rangle = \sum_{m'} \Big(b_\alpha(p)\Big)_{m'm} \left|p, m'; q, n; \ldots\right\rangle + \\ \sum_{n'} \Big(b_\alpha(q)\Big)_{n'n} \left|p, m'; q, n; \ldots\right\rangle + \ldots . \end{aligned} \tag{20}$$

The b_α are, for any given momentum p, finite (assumption (2)) Hermitian (symmetry generators have to be Hermitian, such that their exponentials are unitary operators) representation matrices, m, n, etc. are discrete indices labelling spin and particle type. The generators obey a Lie algebra (!) relation

$$[B_\alpha, B_\beta] = i \sum_\gamma C^\gamma_{\alpha\beta} B_\gamma. \tag{21}$$

We will, from now on, use sum convention for greek indices. From this we can calculate by acting on a state

$$\begin{aligned} [B_\alpha, B_\beta] \left|p, m\right\rangle = \cdots = i \sum_{m'} C^\gamma_{\alpha\beta} \Big(b_\gamma(p)\Big)_{m'm} \left|p, m'\right\rangle \\ \Rightarrow [b_\alpha(p), b_\beta(p)] = i C^\gamma_{\alpha\beta} b_\gamma(p). \end{aligned} \tag{22}$$

So we can see that the $b_\alpha(p)$ also form a Lie algebra.
We now want to use a theorem, proved in [4], chapter 15.2, that any Lie algebra of finite Hermitian matrices like the $b_\alpha(p)$ must be a direct sum of a compact semi-simple Lie algebra and U(1) algebras. But this can only be applied to the algebra of interest, $\mathcal{B}$, if there is an isomorphism between the B_α and $b_\alpha(p)$, which is not yet apparent (up to now it is only a homomorphism as can be seen in the shared commutation relations / algebra).
What do we need to look at? For a given momentum p we have the map

$$B_\alpha \to b_\alpha(p). \tag{23}$$

This could be degenerate in the sense that $b_\alpha(p) = b_\beta(p)$ for $\alpha \neq \beta$. To ensure that this is not the case we want to proof that if there exist coefficients c^α such that the $b_\alpha(p)$ are not linearly independent, then also the B_α are not linearly independent, i.e.

$$\{B_\alpha\} \simeq \{b_\alpha(p)\} \Leftrightarrow \left(c^\alpha b_\alpha(p) = 0 \Rightarrow c^\alpha B_\alpha = 0\right), \tag{24}$$

for some momentum p and some coefficients c^α. To make it clearer: We want to show that the map is injective by showing that for any element of the image that is zero, already the preimage has to be zero, i.e. that the kernel of the map is zero.

Since B_α is completely characterized by its representation on the Hilbert space, $b_\alpha(k)$ for all momenta k, above statement of linear dependence of B_α is equivalent to $c^\alpha b_\alpha(k) = 0 \ \forall k$. At this point note, that by our particle finiteness assumption there will only be finitely many different masses in the spectrum of the theory and we only have to look at those momenta that are on mass-shell, i.e. those momenta for which $k^2 = m^2$ for some mass in the spectrum.

So let us try to prove this, by looking at two-particle states:

$$
\begin{aligned}
B_\alpha |p,m;q,n\rangle &= \sum_{m',n'} \Big(b_\alpha(p,q)\Big)_{m'n',mn} |p,m';q,n'\rangle\,, \\
\Big(b_\alpha(p,q)\Big)_{m'n',mn} &= \Big(b_\alpha(p)\Big)_{m'm}\ \delta_{n'n} + \Big(b_\alpha(q)\Big)_{n'n}\ \delta_{m'm}.
\end{aligned}
\tag{25}
$$

Since B_α are symmetry generators we have for a elastic 2-2-scattering $(p,q \longrightarrow p'q')$ with $p+q = p'+q'$, p, q, p', q' all on mass-shell, $p^2 = p'^2$ and $q^2 = q'^2$,

$$
\langle p',m';q',n'|\,[B_\alpha, S]\,|p,m;q,n\rangle = 0, \tag{26}
$$

which with quite a lot of work we can rewrite (using orthonormality of states of different particle type and spin) to

$$
b_\alpha(p',q')T(p',q';p,q) = T(p',q';p,q)b_\alpha(p,q). \tag{27}
$$

Here note that b_α and T are both (finite) matrices:

$$
\begin{aligned}
&\sum_{m''n''} \Big(b_\alpha(p',q')\Big)_{m'n',m''n''} \Big(T(p',q';p,q)\Big)_{m''n'',mn} \\
&= \sum_{m''n''} \Big(T(p',q';p,q)\Big)_{m'n',m''n''} \Big(b_\alpha(p,q)\Big)_{m''n'',mn}.
\end{aligned}
\tag{28}
$$

Assumptions (3) and (4) tell us that S (we will now go with the more common notation by using S rather than T) is non-singular and non-zero for almost all momenta (this "almost" will stick around for some time now, but we will only mention it again later), so it is invertible and above equation simply is a similarity tranformation of matrices

$$
b_\alpha(p',q') = S(p',q';p,q)\ b_\alpha(p,q)\ S^{-1}(p',q';p,q). \tag{29}
$$

This means that if there are coefficients c^α such that $c^\alpha b_\alpha(p,q) = 0$ we have

$$
c^\alpha b_\alpha(p',q') = S(p',q';p,q)\ (c^\alpha b_\alpha(p,q))\ S^{-1}(p',q';p,q) = 0. \tag{30}
$$

This in turn tells us that

$$
c^\alpha \Big(b_\alpha(p')\Big)_{m'm}\ \delta_{n'n} = -c^\alpha \Big(b_\alpha(q')\Big)_{n'n}\ \delta_{m'm}, \tag{31}
$$

so we can read off that for such p' and q' $c^\alpha b_\alpha(p')$ and $c^\alpha b_\alpha(q')$ are proportional to the identity matrix.

If they were traceless they would therefore have to be zero, so let's make them traceless:

$$
\mathrm{Tr}\Big[b_\alpha(p',q')\Big] = \mathrm{Tr}\Big[Sb_\alpha(p,q)S^{-1}\Big] = \mathrm{Tr}\Big[b_\alpha(p,q)\Big]. \tag{32}
$$

$$
\begin{aligned}
\mathrm{Tr}\left[\Big(b_\alpha(p,q)\Big)_{m'n',mn}\right] &= \mathrm{Tr}\left[\Big(b_\alpha(p)\Big)_{m'm}\ \delta_{n'n} + \Big(b_\alpha(q)\Big)_{n'n}\ \delta_{m'm}\right] \\
&= \mathrm{tr}\left[\Big(b_\alpha(p)\Big)_{m'm}\right]\ \mathrm{tr}\Big[\delta_{n'n}\Big] + \mathrm{tr}\left[\Big(b_\alpha(q)\Big)_{n'n}\right]\ \mathrm{tr}\Big[\delta_{m'm}\Big] \\
&= N(\sqrt{-q_\mu q^\mu})\ \ \mathrm{tr}\ \ b_\alpha(p) + N(\sqrt{-p_\mu p^\mu})\ \ \mathrm{tr}\ \ b_\alpha(q),
\end{aligned}
\tag{33}
$$

where $N(m)$ is the multiplicity of particle types and spins with mass m (the trace goes over particle type and spin quantum numbers for any give momentum p).

So we have the following conditions for mass-shell momenta p, q, p', q'

$$\begin{aligned}\frac{\text{tr}\ b_\alpha(p)}{N(\sqrt{-p_\mu p^\mu})}+\frac{\text{tr}\ b_\alpha(q)}{N(\sqrt{-q_\mu q^\mu})}=&\frac{\text{tr}\ b_\alpha(p')}{N(\sqrt{-p'_\mu p'^\mu})}+\frac{\text{tr}\ b_\alpha(q')}{N(\sqrt{-q'_\mu q'^\mu})},\\ p+q=&p'+q',\end{aligned}\tag{34}$$

which have the solution

$$\frac{\text{tr}\ b_\alpha(p)}{N(\sqrt{-p_\mu p^\mu})}=a^\mu_\alpha p_\mu.\tag{35}$$

So the trace is just a linear function of p_μ. A possible constant term can be ruled out by looking at scattering with an unequel amount of states on each side.

With this at hand we can define new symmetry generators $B^\#_\alpha$ and their one-particle representation as

$$\begin{aligned}B^\#_\alpha&\equiv B_\alpha-a^\mu_\alpha P_\mu,\\ \Big(b^\#_\alpha(p)\Big)_{n'n}&=\Big(b^\alpha(p)\Big)_{n'n}-\frac{\text{tr}\ b_\alpha(p)}{N(\sqrt{-p_\mu p^\mu})}\delta_{n'n}.\end{aligned}\tag{36}$$

These still commute with P_μ and thus have the same algebra

$$\begin{aligned}\Big[B^\#_\alpha,B^\#_\beta\Big]&=iC^\gamma_{\alpha\beta}B_\gamma=iC^\gamma_{\alpha\beta}[B^\#_\gamma+a^\mu_\gamma P_\mu],\\ [b^\#_\alpha(p),b^\#_\beta(p)]&=iC^\gamma_{\alpha\beta}b_\gamma(p)=iC^\gamma_{\alpha\beta}[b^\#_\gamma(p)+a^\mu_\gamma p_\mu].\end{aligned}\tag{37}$$

Using that the trace of a commutator of finite matrices is zero

$$\begin{aligned}0=\text{tr}[[b^\#_\alpha(p),b^\#_\beta(p)]]&=iC^\gamma_{\alpha\beta}\,\text{tr}[b^\#_\gamma(p)]+iC^\gamma_{\alpha\beta}a^\mu_\gamma p_\mu\,\text{tr}[\delta]\\&=iC^\gamma_{\alpha\beta}a^\mu_\gamma p_\mu N(\sqrt{-p_\mu p^\mu}),\end{aligned}\tag{38}$$

we find $C^\gamma_{\alpha\beta}a^\mu_\gamma=0$ for non-zero particle multiplicity. The $B^\#_\alpha$ are also symmetry generators of the S-matrix and therefore all considerations from before go through and we find

$$c^\alpha b^\#_\alpha(p,q)=0\Rightarrow c^\alpha b^\#_\alpha(p',q')=0\Rightarrow c^\alpha b^\#_\alpha(p')=c^\alpha b^\#_\alpha(q')=0,\tag{39}$$

for some coefficients c^α and for momenta p, q, p', q' as before, since now the tracelessness tells us that the proportionality constant between $c^\alpha b^\#_\alpha(p')$ and the identity matrix has to be zero.

Now we still need to argue why the last equation holds for any mass-shell momentum k given that it holds for only one momentum p. If

$$c^\alpha b^\#_\alpha(p,q)=c^\alpha b^\#_\alpha(p',q')=0,\tag{40}$$

then

$$c^\alpha b^\#_\alpha(p)=c^\alpha b^\#_\alpha(q)=c^\alpha b^\#_\alpha(p')=c^\alpha b^\#_\alpha(q')=0,\tag{41}$$

so also

$$0=c^\alpha b^\#_\alpha(p,q')=c^\alpha b^\#_\alpha(k,p+q'-k),\tag{42}$$

where the last equality is due to the similarity tranformation of 2-2-scattering $p,q'\longrightarrow k,(p+q'-k)$ that holds when k and $p+q'-k$ are mass-shell momenta. The momentum $p+q'-k$ was of course chosen like that to respect momentum conservation. This is the crucial step that Coleman and Mandula came up with to show that also $c^\alpha B^\#_\alpha=0$.

All we need to do is to reassure ourselves that indeed k can take any mass-shell value. What we start with are p and q, such that $m_p^2=p^2$ and $m_q^2=q^2$. We first look at the

scattering $p, q \longrightarrow p' = (p + q - q'), q'$. We need $m_p^2 = (p + q - q')^2$ and $m_q^2 = q'^2$ for the scattering to be elastic, which removes two degrees of freedom in our choice of q'. Next we consider $p, q' \longrightarrow k, (p + q' - k)$, for which we now need $m_p^2 = k^2$, which removes one degree of freedom from k, and $m_q^2 = (p + q' - k)^2$, which does not constrain k because we still have enough freedom in the choice of q' to fulfill this equation for any k. So we are free to choose $\boldsymbol{k}$, the three-vector components of k, to be anything we want.
So we have shown that if for some fixed mass-shell momenta p and q

$$c^\alpha b_\alpha^\#(p, q) = 0, \tag{43}$$

we have for almost all mass-shell momenta k

$$c^\alpha b_\alpha^\#(k) = 0. \tag{44}$$

Note: We got rid of momentum conservation!

But we now still need to show that this holds for *all* momenta k. By considering elastic scattering we have limited ourselves to momenta k which are on the same mass-shell as p. And even on this mass-shell we had to use that the scattering matrix is invertible only for *almost* all momenta. Luckily by our particle finiteness assumption we now that there are only fintely many possibilities for the zero component k_0 for a general three-momentum $\boldsymbol{k}$. So let's see what happens if, for some particular mass-shell momentum k^*, $c^\alpha b_\alpha^\#(k^*)$ does not vanish? Then a scattering process $k^*, k \longrightarrow k', k''$ would be forbidden by the symmetry generated by $c^\alpha B_\alpha$ for almost all k, k' and k'', since almost all $b_\alpha^\#(k', k'')$ are related to $b_\alpha^\#(p, q)$ by a similarity transformation, so if a scattering $k^*, k \longrightarrow k', k''$ would exist, also $c^\alpha b_\alpha^\#(k^*, k) \neq 0$ would be related to $c^\alpha b_\alpha^\#(p, q) = 0$. This cannot be. So for almost all k, k' and k'' this scattering amplitude must vanish, which is in contradiction to one of our assumptions, and therefore $c^\alpha b_\alpha^\#(k^*)$ must be zero.
In conclusion we find that

$$c^\alpha b_\alpha^\#(k) = 0 \; \forall k. \tag{45}$$

So now we know that the mapping of B_α into the $b_\alpha^\#(p, q)$ is an isomorphism, the above mentioned theorem is applicable and we can decompose $\mathcal{B}$ into a semi-simple compact Lie algebra and U(1)'s. What we now need to show for these summands of $\mathcal{B}$ individually is that their elements commute with the generators of Lorentz transformations, since then they commute with all elements of the Poincaré algebra and are therefore internal symmetries. At this point you might wonder what happened to the generator of translations P_μ, which by definition of $\mathcal{B}$ should be an element of it. But recall that we subtracted a linear function of P_μ from all generators to make them traceless. In this step we mapped P_μ to zero and therefore excluded it from further considerations. So rest assured P_μ itself is not an internal symmetry.

As a corollary, we can see that the number of independent matrices $b_\alpha^\#(p, q)$ cannot exceed $N(\sqrt{-p_\mu p^\mu})N(\sqrt{-q_\mu q^\mu})$ and therefore also the number of independent generators B_α must be finite, which we did not need to assume in the beginning.

5.1.1 The U(1)'s

Since the $B_\alpha^\#$ commute with P_μ and $[J, P_\mu]$, which is a linear combination of P_μ and where J is a generator of Lorentz transformations, the Jacobi identity is

$$0 = [P_\mu, [J, B_\alpha^\#]] + [J, [B_\alpha^\#, P_\mu]] + [B_\alpha^\#, [P_\mu, J]] = [P_\mu, [J, B_\alpha^\#]], \tag{46}$$

which tells us, that $[J, B^\#_\alpha]$ is a linear combination of $B^\#_\alpha$'s (tracelessness is clear because it is a commutator). Following the theorem decomposing the algebra $\mathcal{B}$ in a direct sum, we know that any U(1) generator $B^\#_i$ must commute with all other $B^\#_\alpha$'s, in particular

$$[B^\#_i, [J, B^\#_i]] \sim [B^\#_i, \sum_{\alpha \neq i} B^\#_\alpha + B^\#_i] = [B^\#_i, B^\#_i] = 0. \tag{47}$$

With some effort one can show by taking the expectation value of this commutator (using $J\,|p, m; q, n\rangle = \sigma(m, n)\,|p, m; q; n\rangle$), that

$$\begin{aligned} 0 &= \langle p, m; q, n|\,[B^\#_i, [J, B^\#_i]]\,|p, m; q, n\rangle \\ &= \langle p, m; q, n|\,(2B^\#_i J B^\#_i - J B^\#_i B^\#_i - B^\#_i B^\#_i J)\,|p, m; q, n\rangle \\ &= 2\,\langle p, m; q, n|\,(B^\#_i J B^\#_i - J B^\#_i B^\#_i)\,|p, m; q, n\rangle \\ &= 2 \sum_{m', n'} (\sigma(m', n') - \sigma(m, n)) \left| \left(b^\#_i(p, q) \right)_{m'n', mn} \right|^2, \end{aligned} \tag{48}$$

where from going from the second to the third line we used that $J^\dagger = J$ and in the fourth line acted with J once after and once before $B^\#_\alpha$ on the left state in the first and second term respectively. We also used the orthonormality of states. So we see that for any m, n, m' and n' for which $\sigma(m', n') \neq \sigma(m, n)$ $\left(b^\#_i(p, q) \right)_{m'n', mn}$ has to vanish. Since the $b^\#_i(p, q)$ are isomorphic to $B^\#_i$ this is just the statement that

$$[B^\#_i, J] = 0, \tag{49}$$

which proves the theorem for this subalgebra of the total symmetry algebra.

5.1.2 The semi-simple compact Lie algebra

We will only roughly sketch the idea for this part of the proof. For a full proof see section 24.1 in [5].

We look at $U(\Lambda) B_a U(\Lambda)^{-1}$, where B_a is a generator of the compact Lie algebra that resulted from the decomposition above and $U(\Lambda)$ is some representation of the Lorentz group. This is an element of the same algebra and therefore

$$U(\Lambda) B_a U(\Lambda)^{-1} = \sum_b D^b_a(\Lambda) B_b, \tag{50}$$

for some function $D(\Lambda)$ which turns out to also be a representation of the Lorentz group. From these $D(\Lambda)$ we can construct a *unitary finite-dimensional* representation of the *non-compact* $\mathrm{SO}(1, 3)$. This can therefore only be the trivial representation $D(\Lambda) = \mathbb{1}\ \forall \Lambda$. So the B_a also commute with the generators of the homogeneous Lorentz group.

This concludes the discussion of $\mathcal{B}$: All generators $B_\alpha \in \mathcal{B}$ commute with both P_μ (by definition of $\mathcal{B}$) and J (as shown for the U(1) and the compact part separately) and therefore with all generators of the Poincaré group. They are internal symmetries!

5.2 The subset $\mathcal{A}$

We still have all the generators left, that do not commute with P_μ. These we will call A_α. We want to show that this set $\mathcal{A}$ of A_α's can only consist of the generators of Lorentz transformations for which we already know that

$$[J, P_\mu] \sim P_\mu \neq 0. \tag{51}$$

The action of a general symmetry generator A_α on a one-particle state is

$$A_\alpha |p,n\rangle = \sum_{n'} \int d^4p' \Big(\mathscr{A}_\alpha(p',p)\Big)_{n'n} |p',n'\rangle \,, \tag{52}$$

where again n and n' label discrete spin and particle type indices. The kernel $\mathscr{A}_\alpha(p',p)$ must vanish unless both p and p' are on the mass-shell.

In the following we will first show that $\mathscr{A}_\alpha(p',p) = 0$ for $p \neq p'$.
If A_α is a symmetry generator, then so is

$$A_\alpha^f \equiv \int d^4x \; exp(iP\cdot x)\Big(\mathscr{A}_\alpha(p',p)\Big)_{n'n} exp(-iP\cdot x) f(x), \tag{53}$$

for any function $f(x)$. Acting on one-particle states we can evaluate the momentum operators P_μ and carry out the Fourier transformation, which gives

$$A_\alpha^f |p,n\rangle = \sum_{n'} \int d^4p' \; \tilde{f}(p'-p)\Big(\mathscr{A}_\alpha(p',p)\Big)_{n'n} |p,n'\rangle \,, \tag{54}$$

where we used the Fourier transform $\tilde{f}(k)$ of f.

Now suppose there is a scattering process $p,q \longrightarrow p',q'$ for mass-shell momenta p, q, p' and q'. Suppose also there is $\Delta \neq 0$ such that $p+\Delta$ is still on the mass-shell and $\mathscr{A}_\alpha(p,p+\Delta) \neq 0$. For generic values of q, p' and q' the momenta $q+\Delta$, $p'+\Delta$ and $q'+\Delta$ will be off-shell. Choose $\tilde{f}$ to vanish outside of a small region around $k = \Delta$. We then have

$$\begin{aligned} A_\alpha^f |p,n\rangle &= \sum_{n'} f(\Delta)\Big(\mathscr{A}_\alpha(p+\Delta,p)\Big)_{n'n} |p+\Delta,n'\rangle \neq 0, \\ A_\alpha^f |q,n\rangle &= A_\alpha^f |p',n\rangle = A_\alpha^f |q',n\rangle = 0, \end{aligned} \tag{55}$$

where the last line holds because the kernel $\mathscr{A}_\alpha(q+\Delta,q)$ has to vanish since $q+\Delta$ is not on mass-shell and a symmetry cannot map a state in the Hilbert space to an unphysical state outside the Hilbert space, which of course is also true for p' and q'.
So we find that this symmetry forbids the scattering $p,q \longrightarrow p',q'$ for generic q, p' and q' which is in contradiction to our assumption that scattering occurs at almost all energies except for isolated energies!

This leaves us with the options that either $\mathscr{A}_\alpha(p,p')$ commutes with P_μ, which would leave us with a generator B_α that we already discussed, or that the kernel is proportional to $\delta^{(4)}(p'-p)$ and its derivatives, in which case A_α^f would only be a symmetry if $\tilde{f}$ has support in $k=0$. The latter being of course the option we have to further investigate.

So we consider

$$\Big(\mathscr{A}_\alpha(p',p)\Big)_{n'n} = \sum_{i=1}^{D_\alpha} \Big(a_\alpha^{(i)}(p',p)\Big)_{n'n}^{\mu_1\ldots\mu_i} \frac{\partial^i}{\partial p'^{\mu_1}\ldots\partial p'^{\mu_i}} \delta^{(4)}(p'-p), \tag{56}$$

with the coefficients $a_\alpha^{(i)}$ of each term being matrix valued functions of momentum. The "ugly technical assumption" of Coleman and Mandula now is needed to argue that there is a finite number D_α of derivatives in the *distributions* $\mathscr{A}_\alpha$.

We now look at the D_α-fold commutator of A_α with P_μ's

$$B_\alpha^{\mu_1\cdots\mu_{D_\alpha}} \equiv [P^{\mu_1},[P^{\mu_2},[\ldots[P^{\mu_{D_\alpha}},A_\alpha]\ldots]]. \tag{57}$$

To illustrate what happens let's look at $D_\alpha = 1$ and act on a one-particle state

$$\begin{aligned}
B_\alpha^\mu\,|p,n\rangle &= [P^\mu,A_\alpha]\,|p,n\rangle = \sum_{n'}[P^\mu,\Big(a^1(p)\Big)^\nu_{n'n}\frac{\partial}{\partial p^\nu}]\,|p,n\rangle \\
&= -\sum_{n'}\frac{\partial}{\partial p^\nu}P^\mu\Big(a^1(p)\Big)^\nu_{n'n}\,|p,n\rangle \\
&= -\sum_{n'}\delta^\mu_\nu\Big(a^1(p)\Big)^\nu_{n'n}\,|p,n\rangle\,,
\end{aligned} \tag{58}$$

where in the first line we used that the commutator with a function a^0 vanishes and evaluated P^μ before carrying out the derivative.
Further we find for the matrix elements

$$\begin{aligned}
&\langle p'|[B_\alpha^\mu,P^\nu]\,|p\rangle = \langle p'|\,[[P^\mu,A_\alpha],P^\nu]\,|p\rangle \\
&\quad= \langle p'|\,P^\mu A_\alpha P^\nu - A_\alpha P^\mu P^\nu - P^\nu P^\mu A_\alpha + P^\nu A_\alpha P^\mu\,|p\rangle \\
&\quad= \langle p'|\,p'^\mu A_\alpha p^\nu - A_\alpha p^\mu p^\nu - p'^\nu p'^\mu A_\alpha + p'^\nu A_\alpha p^\mu\,|p\rangle \\
&\quad= -(p'-p)^\mu(p'-p)^\nu\,\langle p'|\,A_\alpha\,|p\rangle \\
&\quad= -(p'-p)^\mu(p'-p)^\nu(a'^{(0)}_\alpha(p)+a^{(1)}_\alpha(p)^\nu\frac{\partial}{\partial p^\nu})\,\langle p'\,|p\rangle\,,
\end{aligned} \tag{59}$$

where $a'^{(0)}_\alpha(p) \equiv a^{(0)}_\alpha(p) + \frac{\partial}{\partial p^\nu}a^{(1)}_\alpha(p)$ and we used

$$\begin{aligned}
&\langle p'|\,A_\alpha\,|p\rangle = \int d^4\tilde p\,\langle p'|\,\big(a^{(0)}_\alpha(\tilde p,p)\delta^{(4)}(\tilde p-p) + a^{(1)}_\alpha(\tilde p,p)^\nu\frac{\partial}{\partial\tilde p^\nu}\delta^{(4)}(\tilde p-p)\big)\,|\tilde p\rangle \\
&= \int d^4\tilde p\;\delta^{(4)}(\tilde p-p)\big(a^{(0)}_\alpha(\tilde p,p) - \frac{\partial}{\partial\tilde p^\nu}(a^{(1)}_\alpha(\tilde p,p)^\nu) - a^{(1)}_\alpha(\tilde p,p)^\nu\frac{\partial}{\partial\tilde p^\nu}\big)\,\langle p'\,|\tilde p\rangle \\
&= (a'^{(0)}_\alpha(p) + a^{(1)}_\alpha(p)^\nu\frac{\partial}{\partial p^\nu})\,\langle p'\,|p\rangle\,,
\end{aligned} \tag{60}$$

where we partially integrated from the first to the second line and performed the integral using the delta function. Using the usual normalisation of states $\langle p'\,|p\rangle \propto \delta^{(3)}(\boldsymbol{p}-\boldsymbol{p'})$ and noting that p and p' must be on the same mass-shell we get

$$\langle p'|\,[B_\alpha^\mu,P^\nu]\,|p\rangle \propto -(p'-p)^\mu(p'-p)^\nu(a'^{(0)}_\alpha(p)+a^{(1)}_\alpha(p)^\nu\frac{\partial}{\partial p^\nu})\delta^{(4)}(p'-p). \tag{61}$$

Generally we will find that the matrix elements $\langle p'|\,[B_\alpha^{\mu_1\cdots\mu_{D_\alpha}},P^\nu]\,|p\rangle$ will be proportional to $D_\alpha{+}1$ factors of $p'{-}p$ and polynomials of order D_α in derivatives acting on $\delta^{(4)}(p'{-}p)$. The delta functions set p' to p such that the prefactors $p'-p$ vanish, which means that these commutators vanish. As the notation already suggested we see that the $B_\alpha^{\mu_1\cdots\mu_{D_\alpha}}$ are in the subalgebra $\mathcal{B}$.

As proved above the $B_\alpha^{\mu_1\cdots\mu_{D_\alpha}}$ act on one-particle states as

$$\Big(b_\alpha^{\mu_1\cdots\mu_{D_\alpha}}(p)\Big)_{n'n} = \Big(b_\alpha^{\#\mu_1\cdots\mu_{D_\alpha}}\Big)_{n'n} + a_\alpha^{\mu\mu_1\cdots\mu_{D_\alpha}}p_\mu\,\mathbb{1}_{n'n}, \tag{62}$$

where the matrices $b_\alpha^{\#\mu_1\ldots\mu_{D_\alpha}}$ are momentum-independent, traceless representations of generators of internal symmetries and the $a_\alpha^{\mu\mu_1\ldots\mu_{D_\alpha}}$ are momentum-independent constants.
We know by assumption (2) that the mass operator $-P_\mu P^\mu$ has discrete eigenvalues, so since A_α cannot take one-particle states off the mass-shell we have $[-P_\mu P^\mu, A_\alpha] = 0$ which for $D_\alpha \geq 1$ gives

$$0 = [P^{\mu_1}P_{\mu_1}, [P^{\mu_2}, [\ldots[P^{\mu_{D_\alpha}}, A_\alpha]\ldots]]] = 2P_{\mu_1}B_\alpha^{\mu_1\ldots\mu_{D_\alpha}}. \tag{63}$$

We can directly see that the momentum-independent term of $b_\alpha^{\mu_1\ldots\mu_{D_\alpha}}(p)$ must vanish since above equation suggests

$$b_\alpha^{\#\mu_1\ldots\mu_{D_\alpha}}p_{\mu_1} + a_\alpha^{\mu\mu_1\ldots\mu_{D_\alpha}}p_{\mu_1}p_\mu = 0, \tag{64}$$

which must be satisfied for any timelike momentum p, as long as there are massive particles in the theory, which tells us that the coefficients in each order of p have to vanish individually. The last equation therefore also tells us in order p^2 that

$$a_\alpha^{\mu\mu_1\mu_2\ldots\mu_{D_\alpha}} = -a_\alpha^{\mu_1\mu\mu_2\ldots\mu_{D_\alpha}}. \tag{65}$$

For these coefficients $a_\alpha^{\mu\mu_1\ldots\mu_{D_\alpha}}$ we now have to distinguish between $D_\alpha = 0, 1$ and $D_\alpha \geq 2$. Note that $b_\alpha^{\mu_1\ldots\mu_{D_\alpha}}$ is symmetric in all its indices by definition of $B_\alpha^{\mu_1\ldots\mu_{D_\alpha}}$ via the commutator and therefore $a_\alpha^{\mu\mu_1\ldots\mu_{D_\alpha}}$ is symmetric in the indices $\mu_1, \ldots, \mu_{D_\alpha}$.

For $D_\alpha = 0$ we trivially have $A_\alpha = B_\alpha$ since then the kernel is simply a delta function and the A_α therefore commute with P_μ.
For $D_\alpha = 1$ we get the condition

$$a_\alpha^{\mu\mu_1} = -a_\alpha^{\mu_1\mu}. \tag{66}$$

For $D_\alpha \geq 2$ we have this anti-symmetry in μ, μ_1 as in the case of $D_\alpha = 1$, but we also get it in all other indices because of the symmetry in the second to $(D_\alpha + 1)$th indices, which can only have the solution

$$a_\alpha^{\mu\mu_1\ldots\mu_{D_\alpha}} = 0. \tag{67}$$

To reassure ourselves we can switch indices to see that

$$\begin{aligned} a_\alpha^{\mu\mu_1\mu_2\ldots\mu_{D_\alpha}} &= -a_\alpha^{\mu_1\mu\mu_2\ldots\mu_{D_\alpha}} \\ &= -a_\alpha^{\mu_1\mu_2\mu\ldots\mu_{D_\alpha}} \\ &= a_\alpha^{\mu_2\mu_1\mu\ldots\mu_{D_\alpha}} \\ &= a_\alpha^{\mu_2\mu\mu_1\ldots\mu_{D_\alpha}} \\ &= -a_\alpha^{\mu\mu_2\mu_1\ldots\mu_{D_\alpha}} \\ &= -a_\alpha^{\mu\mu_1\mu_2\ldots\mu_{D_\alpha}}. \end{aligned} \tag{68}$$

So far we have reduced $\mathcal{A}$ to generators for which

$$[P^\mu, A_\alpha] = a_\alpha^{\mu\nu}P_\nu, \tag{69}$$

where $a_\alpha^{\mu\nu}$ is some constant which is anti-symmetric in its indices.
From the Poincaré algebra relation

$$[P^\mu, J^{\rho\sigma}] = -i\eta^{\nu\rho}P^\sigma + i\eta^{\nu\sigma}P^\rho \tag{70}$$

We can now see that above commutator $[P^\mu, A_\alpha]$ is simply that of

$$[P^\rho, A_\alpha] = [P^\rho, -\frac{i}{2}a_\alpha^{\mu\nu}J_{\mu\nu} + B_\alpha], \tag{71}$$

where B_α is some generator that commutes with P^μ, which is why we can conclude that

$$A_\alpha = -\frac{i}{2} a_\alpha^{\mu\nu} J_{\mu\nu} + B_\alpha. \tag{72}$$

We have found that A_α is just the sum of some internal symmetry generator (A_α and $J^{\mu\nu}$ are symmetry generators, so B_α also has to be one) and the generator of Lorentz transformations.

Altogether we have shown that the only possible Lie algebra of symmetry generators consists of the generators of the Poincaré group P_μ and $J_{\mu\nu}$ and the generators of some internal symmetries B_α. This is just the infinitesimal version of the statement

$$\mathcal{G} = \mathcal{P} \oplus \mathcal{B}. \tag{73}$$

□

6 Loopholes

Finally we will look at loopholes of the theorem.

Continuing in the proof we can see that for purely massless theories ($p^2 = 0$) above argument of anti-symmetry of $a_\alpha^{\mu\mu_1\mu_2\ldots\mu_{D_\alpha}}$ does not work. In such theories we can have additional generators of the conformal group, which indeed do mix with those of the Poincaré group.

There are also symmetries that are not captured by the theorem at all. These symmetries are for example discrete symmetries and spontaneously broken symmetries. The latter being symmetries that do not act on the S-matrix level and therefore evading the requirement to commute with S. Rather, these symmetries are symmetries of the action of the theory and are spontaneously broken before any calculation.

Another important loophole are 1+1 dimensional theories. In such theories we can only have forward and backward scattering and therefore scattering cannot satisfy assumption 3 about the analyticity of scattering angles.

Lastly, as was already mentionend in the introduction, this proof heavily depended on the use of commutation relations. So obviously we can simply replace commutators by other brackets, so to evade the theorem we can look at graded Lie algebras or Lie superalgebras.

References

[1] P.C. Argyres, *An Introduction to Global Supersymmetry*, lecture at Cornwell University, p.2.

[2] S. Coleman and J. Mandula, *All Possible Symmetries of the S Matrix*, Phys. Rev. 159 (1967), 1251.

[3] R. Haag, J.T. Lopuszanski, and M. Sohnius, *All Possible Generators of Supersymmetries of the S Matrix*, Nucl. Phys. B88 (1975) 257.

[4] S. Weinberg, *The quantum theory of fields. Vol. 2*, Cambridge University Press, 1995.

[5] S. Weinberg, *The quantum theory of fields. Vol. 3*, Cambridge University Press, 2000.

[6] E. Witten, in: *The unity of fundamental interactions. Proceedings, International School of Subnuclear Physics, Erice 1981*, ed. A. Zichichi, Plenum Press, 1983, p.305.