Supersymmetry in Geometry and Quantum Physics

The Coleman-Mandula Theorem

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1 Introduction

The Coleman-Mandula Theorem is a no-go theorem stating, that given some physically reasonable assumptions, the symmetry group of the S-matrix is a direct product of the Poincaré group and an internal symmetry group. We will define these terms in the following section. The theorem was originally proved in Coleman and Mandula’s paper “All Possible Symmetries of the S Matrix” in 1967 (see [2]). We however, in section 5, will look at the proof by Weinberg in [5], which rearranges the original proof and directly considers symmetry (Lie) algebras and not the groups themselves, but the last step going back to groups is easy. We will look at the exact statement of the theorem in section 3.

In section 4 we will study Witten’s kinematic argument why the theorem should be true: Any additional symmetry beyond Poincaré in a relativistically covariant theory overconstrains the elastic scattering amplitudes and allows non-zero amplitudes only for discrete scattering angles. The assumption of analyticity of scattering angles therefore rules out such symmetries (see [6]).

Section 5 will be the rigorous proof and finally in section 6 we will shortly list possible loopholes of the theorem.

This is an introduction to a supersymmetry seminar, so you might wonder why we are looking at classical Lie algebras. The theorem tells us, that all generators of internal symmetries (gauge symmetries in the standard model) will commute with the generators of the Poincaré group (space-time symmetries) or as Coleman and Mandula put it: ”We prove a new theorem on the impossibility of combining space-time and internal symmetries in any but a trivial way.” (see [2])

This means that for a unified theory of Gravity and Gauge interactions, we are forced to look at a loophole of this theorem which is to consider superalgebras or general graded algebras. So as long as we do not want to give up field theory as a framework, we must consider supersymmetry. Because of this, the theorem was the starting point for physicist to try and implement SUSY into their theories and the general study of SUSY from a mathematical perspective.

In 1975 Haag, Lopuszanski and Sohnius generalised the theorem to constrain possible SUSYs in their paper “All Possible Generators of Supersymmetries of the S-Matrix” (see [3]).
2 Relativistic Field Theory and Scattering Theory

We will start with a short summary of important terms in field theory.

We will usually look at our symmetry acting on states. These states form the Hilbert space
\[ \mathcal{H} = \bigoplus_n \mathcal{H}^{(n)}, \quad (1) \]
where \( \mathcal{H}^{(n)} \) is the n-particle Hilbert space
\[ \mathcal{H}^{(n)} \simeq \bigotimes_n \mathcal{H}^{(1)}. \quad (2) \]

The S-matrix is a unitary operator on \( \mathcal{H} \) that maps “in” to “out” states
\[ |\alpha, \text{in}\rangle = S |\alpha, \text{out}\rangle, \quad (3) \]
where \( S_{\alpha\beta} \) is called the S-matrix element. The “in” and “out” states are elements of Hilbert spaces as above, which are non-interacting in the sense that the states are looked at at past and future infinity, where they are far apart from each other and can be thought of as asymptotically free. Any interaction in perturbative scattering theory is then captured by the isomorphism \( S \).

So the matrix elements \( S_{\alpha\beta} \) are what physically describes interaction.

This brings us to the definition of a symmetry transformation as a unitary operator \( U \), that fulfills
(1) \( U \) turns one-particle states into one-particle states
\[ U |1\text{-particle}\rangle = |1\text{-particle}'\rangle; \quad (4) \]
(2) \( U \) acts on many-particle states as if they were tensor products of one-particle states
\[ U |\alpha, \beta\rangle = (U \otimes 1 \otimes 1 \otimes U) |\alpha\rangle \otimes |\beta\rangle; \quad (5) \]
(3) \( U \) commutes with \( S \)
\[ [S, U] = 0, \quad (6) \]
such that ”the physics is unchanged”
\[ S_{\beta\alpha} = \langle \beta, \text{out} | \alpha, \text{in} \rangle = \langle \beta, \text{in} | SU^\dagger U |\alpha, \text{in}\rangle = \langle \beta, \text{in} | U^\dagger SU |\alpha, \text{in}\rangle = S_{\beta', \alpha'}, \quad (7) \]

Note that the associated element of the algebra (connected by the exponential map) is a Hermitian operator.

The S-matrix and therefore the theory is called Lorentz-invariant if it possesses a symmetry group that contains the Poincaré group as a subgroup (up to isomorphism). Then the states in \( \mathcal{H}^{(1)} \) are of the form \( |\alpha, p, n\rangle \), where \( p \) is four momentum, \( n \) is the spin and \( \alpha \) are all the other quantum numbers arising from the rest of the symmetry group, where we used the definition of the Hilbert space as the representation space of the symmetry group.

We call \( B \) an internal symmetry transformation if it commutes with the generators of the Poincaré group and therefore only acts on particle type indices \( \alpha \).

What is actually of interest is the connected S-matrix \( T \), which is the non-trivial part of \( S \)
\[ S = 1 - i(2\pi)^4 \delta(4)(P_\mu - P'_\mu)T. \quad (8) \]
Usually one does not care to write $T$ rather than $S$, but in some steps, where it is of importance we will make the distinction by actually writing $T$.

Lastly, we state the fact that generators $A$ of a symmetry are integral operators of the form

$$A |p, n\rangle = \sum_{n'} \int d^4 p' \left( \mathcal{A}(p', p) \right)_{n'n} |p', n'\rangle,$$

where the kernel $\mathcal{A}$ is a matrix-valued (we will justify below why the indices $n$ and $n'$ are discrete and finite) distribution in momentum space. (Note, if $[A, P] = 0$ then $\mathcal{A} = \delta^{(4)}(p - p') a(p)$ for some matrix-valued function $a(p)$.)

### 3 The Coleman-Mandula Theorem

We now cite the original theorem from [2].

**Theorem:** Let $G$ be a connected symmetry group of the S-matrix, and let the following five conditions hold:

1. (Lorentz invariance.) $G$ contains a subgroup locally isomorphic to $P$.

2. (Particle-finiteness.) All particle types correspond to positive-energy representations of $P$. For any finite $M$, there are only a finite number of particle types with mass less than $M$.

3. (Weak elastic analyticity.) Elastic-scattering amplitudes are analytic functions of center-of-mass energy, $s$, and invariant momentum transfer, $t$, in some neighborhood of the physical region, except at normal thresholds.

4. (Occurrence of scattering.) Let $|p\rangle$ and $|p'\rangle$ be any two one-particle momentum eigenstates, and let $|p, p'\rangle$ be the two-particle state made from these. Then

$$T |p, p'\rangle \neq 0,$$

except perhaps for certain isolated values of $s$. Phrased briefly, at almost all energies, any two plane waves scatter.

5. (An ugly technical assumption.) The generators of $G$, considered as integral operators in momentum space, have distributions for their kernels. More precisely: There is a neighborhood of the identity in $G$ such that every element of $G$ in this neighborhood lies on some one-parameter group $g(t)$. Further if $x$ and $y$ are any two states in $D$, then

$$\frac{1}{i} \frac{d}{dt} (x, g(t)y) = (x, Ay),$$

exists at $t = 0$ and defines a continuous function of $x$ and $y$, linear in $y$ and antilinear in $x$.

*Then*, $G$ is locally isomorphic to the direct product of an internal symmetry group and the Poincaré group.

As already alluded to in the beginning, we will not work with a symmetry group, but rather its generators, i.e. its algebra and therefore already implementing assumption (5) of Coleman.
and Mandula to some extent. The statement of the theorem then reads that for any symmetry
generators $B$ not in the Poincaré algebra and given the assumptions stated above we must have
\[ [B, P] = 0, \] (12)
that is to say $B$ commutes with all generators of the Poincaré algebra $P$ (we will use the same
symbol for the group and the algebra).

4 Short, Non-Rigorous Kinematic Proof

The argument by Witten in the introduction, that any additional symmetry beyond Poincaré
overconstrains scattering amplitudes can be made quantitative as follows.
This calculation is taken from a lecture on supersymmetry (see [1]).
Assume there is a symmetry generator $Q_{\mu\nu}$, such that $[Q_{\mu\nu}, P_{\rho}] \neq 0$, $Q_{\mu\nu} \neq J_{\mu\nu} \in \text{so}(1,3)$
and $Q_{\mu\nu}$ symmetric and traceless.
From its symmetry and tracelessness we know that
\[ \langle p | Q_{\mu\nu} | p \rangle \propto p_{\mu}p_{\nu} - \frac{1}{4}\eta_{\mu\nu}p^2. \] (13)
It will act on one-particle states as a tensor product, so for orthonormal states $|p^1\rangle$ and $|p^2\rangle$ we have
\[ \langle p^1, p^2 | Q_{\mu\nu} | p^1, p^2 \rangle = \langle p^1 | Q_{\mu\nu} | p^1 \rangle + \langle p^2 | Q_{\mu\nu} | p^2 \rangle. \] (14)
Now, for an elastic scattering $p^1, p^2 \rightarrow q^1, q^2$ with $Q_{\mu\nu}$ conserved we have
\[ \langle p^1, p^2 | Q_{\mu\nu} | p^1, p^2 \rangle = \langle q^1, q^2 | Q_{\mu\nu} | q^1, q^2 \rangle \] (15)
and therefore using [13] and [14]
\[ p^1_\mu p^2_\nu + p^2_\mu p^1_\nu = q^1_\mu q^2_\nu + q^2_\mu q^1_\nu. \] (16)
Since for elastic scattering we know that $p^1_\mu + p^2_\mu = q^1_\mu + q^2_\mu$ we can see that making the ansatz
\[ q^1_\mu = p^1_\mu + a_\mu, q^2_\mu = p^2_\mu + b_\mu, \] (17)
that the momentum change has to be $a_\mu = -b_\mu$. Replacing the $q$’s above this finally gives
\[ a_\mu(p^1_\nu - p^2_\nu) + a_\nu(p^1_\mu - p^2_\mu) + 2a_\mu a_\nu = 0. \] (18)
If this is to hold for arbitrary $p$’s we see that we have $a_\mu = 0$ and therefore we can only have
trivial scattering.

5 Proof

We now want to prove the theorem rigorously on an infinitesimal level: If $\mathcal{G}$ is a symmetry
group of the S-matrix in a relativistic field theory, and the assumptions (2) to (4) hold, then
the generators of $\mathcal{G}$ consist only of the generators of the Poincaré group $P$ and the generators
of internal symmetries. Assumption (5) will only be needed to argue why we can work with ele-
ments of an algebra. The main steps of this proof are following [5] with some of the calculations
that Weinberg skipped carried out.
5.1 The subalgebra \( \mathcal{B} \)

Note again that we will use the same symbol for groups and their algebras. We begin with the subalgebra \( \mathcal{B} \) of \( G \) consisting of symmetry generators \( B_\alpha \) who commute with the four-momentum operator \( P_\mu \)

\[
[B_\alpha, P_\mu] = 0. \tag{19}
\]

These act on multiparticle states as on a tensor product of one-particle states, i.e.

\[
B_\alpha |p, m; q, n; \ldots\rangle = \sum_{m'} \left( b_\alpha(p) \right)_{m'm} |p, m'; q, n; \ldots\rangle + \sum_{n'} \left( b_\alpha(q) \right)_{n'n} |p, m'; q, n; \ldots\rangle + \ldots. \tag{20}
\]

The \( b_\alpha \) are, for any given momentum \( p \), finite (assumption (2)) Hermitian (symmetry generators have to be Hermitian, such that their exponentials are unitary operators) representation matrices, \( m, n, \) etc. are discrete indices labelling spin and particle type. The generators obey a Lie algebra (!) relation

\[
[B_\alpha, B_\beta] = i \sum_\gamma C_{\alpha\beta}^\gamma B_\gamma. \tag{21}
\]

We will, from now on, use sum convention for greek indices. From this we can calculate by acting on a state

\[
[B_\alpha, B_\beta] |p, m\rangle = \cdots = i \sum_{m'} C_{\alpha\beta}^\gamma \left( b_\gamma(p) \right)_{m'm} |p, m'\rangle
\]

\[
\Rightarrow [b_\alpha(p), b_\beta(p)] = iC_{\alpha\beta}^\gamma b_\gamma(p). \tag{22}
\]

So we can see that the \( b_\alpha(p) \) also form a Lie algebra.

We now want to use a theorem, proved in [4], chapter 15.2, that any Lie algebra of finite Hermitian matrices like the \( b_\alpha(p) \) must be a direct sum of a compact semi-simple Lie algebra and U(1) algebras. But this can only be applied to the algebra of interest, \( \mathcal{B} \), if there is an isomorphism between the \( B_\alpha \) and \( b_\alpha(p) \), which is not yet apparent (up to now it is only a homomorphism as can be seen in the shared commutation relations / algebra).

What do we need to look at? For a given momentum \( p \) we have the map

\[
B_\alpha \rightarrow b_\alpha(p). \tag{23}
\]

This could be degenerate in the sense that \( b_\alpha(p) = b_\beta(p) \) for \( \alpha \neq \beta \). To ensure that this is not the case we want to proof that if there exist coefficients \( c^\alpha \) such that the \( b_\alpha(p) \) are not linearly independent, then also the \( B_\alpha \) are not linearly independent, i.e.

\[
\{B_\alpha\} \simeq \{b_\alpha(p)\} \Leftrightarrow (c^\alpha b_\alpha(p) = 0 \Rightarrow c^\alpha B_\alpha = 0), \tag{24}
\]

for some momentum \( p \) and some coefficients \( c^\alpha \). To make it clearer: We want to show that the map is injective by showing that for any element of the image that is zero, already the preimage has to be zero, i.e. that the kernel of the map is zero.

Since \( B_\alpha \) is completely characterized by its representation on the Hilbert space, \( b_\alpha(k) \) for all momenta \( k \), above statement of linear dependence of \( B_\alpha \) is equivalent to \( c^\alpha b_\alpha(k) = 0 \forall k \).

At this point note, that by our particle finiteness assumption there will only be finitely many different masses in the spectrum of the theory and we only have to look at those momenta that are on mass-shell, i.e. those momenta for which \( k^2 = m^2 \) for some mass in the spectrum.
So let us try to prove this, by looking at two-particle states:

\[
B_\alpha |p, m; q, n\rangle = \sum_{m', n'} (b_\alpha(p, q))_{m', n', m m'} |p, m'; q, n'\rangle,
\]

(25)

\[
(b_\alpha(p, q))_{m', n', m m'} = \left( b_\alpha(p) \right)_{m m} \delta_{n'n} + \left( b_\alpha(q) \right)_{n' n} \delta_{m'm}.
\]

Since \(B_\alpha\) are symmetry generators we have for an elastic 2-2-scattering \((p, q \rightarrow p'q')\) with \(p + q = p' + q', p, q, p', q'\) all on mass-shell, \(p^2 = p'^2\) and \(q^2 = q'^2\),

\[
\langle p', m'; q', n' | [B_\alpha, S] | p, m; q, n \rangle = 0,
\]

(26)

which with quite a lot of work we can rewrite (using orthonormality of states of different particle type and spin) to

\[
b_\alpha(p', q')T(p', q'; p, q) = T(p', q'; p, q)b_\alpha(p, q).
\]

(27)

Here note that \(b_\alpha\) and \(T\) are both (finite) matrices:

\[
\sum_{m'n''n} \left( b_\alpha(p', q') \right)_{m'n', m'n''} \left( T(p', q'; p, q) \right)_{m'n'', mn} = \sum_{m'n''n} \left( T(p', q'; p, q) \right)_{m'n', m'n''} \left( b_\alpha(p, q) \right)_{m'n'', mn}.
\]

(28)

Assumptions (3) and (4) tell us that \(S\) (we will now go with the more common notation by using \(S\) rather than \(T\)) is non-singular and non-zero for almost all momenta (this "almost" will stick around for some time now, but we will only mention it again later), so it is invertible and above equation simply is a similarity transformation of matrices

\[
b_\alpha(p', q') = S(p', q'; p, q) b_\alpha(p, q) S^{-1}(p', q'; p, q).
\]

(29)

This means that if there are coefficients \(c^\alpha\) such that \(c^\alpha b_\alpha(p, q) = 0\) we have

\[
c^\alpha b_\alpha(p', q') = S(p', q'; p, q) (c^\alpha b_\alpha(p, q)) S^{-1}(p', q'; p, q) = 0.
\]

(30)

This in turn tells us that

\[
c^\alpha \left( b_\alpha(p') \right)_{m'm} \delta_{n'n} = -c^\alpha \left( b_\alpha(q') \right)_{n'n} \delta_{m'm},
\]

(31)

so we can read off that for such \(p'\) and \(q'\) \(c^\alpha b_\alpha(p')\) and \(c^\alpha b_\alpha(q')\) are proportional to the identity matrix.

If they were traceless they would therefore have to be zero, so let's make them traceless:

\[
\text{Tr} \left[ b_\alpha(p', q') \right] = \text{Tr} \left[ S b_\alpha(p, q) S^{-1} \right] = \text{Tr} \left[ b_\alpha(p, q) \right].
\]

(32)

\[
\text{Tr} \left[ \left( b_\alpha(p, q) \right)_{m'n', mn} \right] = \text{Tr} \left[ \left( b_\alpha(p) \right)_{m'm} \delta_{n'n} + \left( b_\alpha(q) \right)_{n'n} \delta_{m'm} \right]
\]

\[
= \text{tr} \left[ \left( b_\alpha(p) \right)_{m'm} \right] \text{tr} \left[ \delta_{n'n} \right] + \text{tr} \left[ \left( b_\alpha(q) \right)_{n'n} \right] \text{tr} \left[ \delta_{m'm} \right]
\]

\[
= N(-\sqrt{-q_\mu q^\mu}) \text{ tr} \ b_\alpha(p) + N(\sqrt{-p_\mu p^\mu}) \text{ tr} \ b_\alpha(q),
\]

(33)

where \(N(m)\) is the multiplicity of particle types and spins with mass \(m\) (the trace goes over particle type and spin quantum numbers for any given momentum \(p\)).

6
Using the trace of a commutator of finite matrices is zero, we find:

\[
\frac{\text{tr } b_\alpha(p)}{N(\sqrt{-p_\mu p^\mu})} + \frac{\text{tr } b_\alpha(q)}{N(\sqrt{-q_\mu q^\mu})} = \frac{\text{tr } b_\alpha(p')}{N(\sqrt{-p'_\mu p'^\mu})} + \frac{\text{tr } b_\alpha(q')}{N(\sqrt{-q'_\mu q'^\mu})},
\]

\(p + q = p' + q',\)

which have the solution

\[
\frac{\text{tr } b_\alpha(p)}{N(\sqrt{-p_\mu p^\mu})} = c_\alpha^\mu p_\mu.
\]

So the trace is just a linear function of \(p_\mu\). A possible constant term can be ruled out by looking at scattering with an unequal amount of states on each side.

With this at hand we can define new symmetry generators \(B^\#_\alpha\) and their one-particle representation as

\[
B^\#_\alpha \equiv B_\alpha - a_\alpha^\mu P_\mu,
\]

\[
(b^\#_\alpha(p))_{n' n} = (b^n(p))_{n' n} - \frac{\text{tr } b_\alpha(p)}{N(\sqrt{-p_\mu p^\mu})} \delta_{n' n}.
\]

These still commute with \(P_\mu\) and thus have the same algebra

\[
\left[ B^\#_\alpha, B^\#_\beta \right] = i C^\gamma_{\alpha\beta} B_\gamma = i C^\gamma_{\alpha\beta} [B^\#_\gamma + a_\gamma^\mu P_\mu],
\]

\[
[b^\#_\alpha(p), b^\#_\beta(p)] = i C^\gamma_{\alpha\beta} b_\gamma(p) = i C^\gamma_{\alpha\beta} [b_\gamma(p) + a_\gamma^\mu P_\mu].
\]

Using that the trace of a commutator of finite matrices is zero

\[
0 = \text{tr} [(b^\#_\alpha(p), b^\#_\beta(p))] = i C^\gamma_{\alpha\beta} \text{tr} (b^\gamma_\alpha(p)) + i C^\gamma_{\alpha\beta} a_\gamma^\mu P_\mu \text{tr} [\delta]
\]

\[
= i C^\gamma_{\alpha\beta} a_\gamma^\mu P_\mu N(\sqrt{-p_\mu p^\mu}),
\]

we find \(C^\gamma_{\alpha\beta} a_\gamma^\mu = 0\) for non-zero particle multiplicity. The \(B^\#_\alpha\) are also symmetry generators of the S-matrix and therefore all considerations from before go through and we find

\[
c^\alpha b^\#_\alpha(p, q) = 0 \Rightarrow c^\alpha b^\#_\alpha(p', q') = 0 \Rightarrow c^\alpha b^\#_\alpha(p') = c^\alpha b^\#_\alpha(q') = 0,
\]

for some coefficients \(c^\alpha\) and for momenta \(p, q, p', q'\) as before, since now the tracelessness tells us that the proportionality constant between \(c^\alpha b^\#_\alpha(p')\) and the identity matrix has to be zero.

Now we still need to argue why the last equation holds for any mass-shell momentum \(k\) given that it holds for only one momentum \(p\). If

\[
c^\alpha b^\#_\alpha(p, q) = c^\alpha b^\#_\alpha(p', q') = 0,
\]

then

\[
c^\alpha b^\#_\alpha(p) = c^\alpha b^\#_\alpha(q) = c^\alpha b^\#_\alpha(p') = c^\alpha b^\#_\alpha(q') = 0,
\]

so also

\[
0 = c^\alpha b^\#_\alpha(p, q') = c^\alpha b^\#_\alpha(k, p + q' - k),
\]

where the last equality is due to the similarity tranformation of 2-2-scattering \(p, q' \rightarrow k, (p + q' - k)\) that holds when \(k\) and \(p + q' - k\) are mass-shell momenta. The momentum \(p + q' - k\) was of course chosen like that to respect momentum conservation. This is the crucial step that Coleman and Mandula came up with to show that also \(c^\alpha B^\#_\alpha = 0\).

All we need to do is to reassure ourselves that indeed \(k\) can take any mass-shell value. What we start with are \(p\) and \(q\), such that \(m_p^2 = p^2\) and \(m_q^2 = q^2\). We first look at the
scattering \( p, q \rightarrow p' = (p + q - q'), q' \). We need \( m_p^2 = (p + q - q')^2 \) and \( m_q^2 = q'^2 \) for the scattering to be elastic, which removes two degrees of freedom in our choice of \( q' \). Next we consider \( p, q' \rightarrow k, (p + q' - k) \), for which we now need \( m_p^2 = k^2 \), which removes one degree of freedom from \( k \), and \( m_q^2 = (p + q' - k)^2 \), which does not constrain \( k \) because we still have enough freedom in the choice of \( q' \) to fulfill this equation for any \( k \). So we are free to choose \( k \), the three-vector components of \( k \), to be anything we want.

So we have shown that if for some fixed mass-shell momenta \( p \) and \( q \)

\[
e^0 b^\#_\alpha(p, q) = 0, \tag{43}
\]

we have for almost all mass-shell momenta \( k \)

\[
e^0 b^\#_\alpha(k) = 0. \tag{44}
\]

Note: We got rid of momentum conservation!

But we now still need to show that this holds for all momenta \( k \). By considering elastic scattering we have limited ourselves to momenta \( k \) which are on the same mass-shell as \( p \). And even on this mass-shell we had to use that the scattering matrix is invertible only for almost all momenta. Luckily by our particle finiteness assumption we now have that there are only finitely many possibilities for the zero component \( k_0 \) for a general three-momentum \( k \). So let’s see what happens if, for some particular mass-shell momentum \( k^* \), \( e^0 b^\#_\alpha(k^*) \) does not vanish? Then a scattering process \( k^*, k \rightarrow k', k'' \) would be forbidden by the symmetry generated by \( e^0 B_\alpha \) for almost all \( k, k' \) and \( k'' \), since almost all \( b^\#_\alpha(k', k'') \) are related to \( b^\#_\alpha(p, q) \) by a similarity transformation, so if a scattering \( k^*, k \rightarrow k', k'' \) would exist, also \( e^0 b^\#_\alpha(k^*, k) \neq 0 \) would be related to \( e^0 b^\#_\alpha(p, q) = 0 \). This cannot be. So for almost all \( k, k' \) and \( k'' \) this scattering amplitude must vanish, which is in contradiction to one of our assumptions, and therefore \( e^0 b^\#_\alpha(k^*) \) must be zero.

In conclusion we find that

\[
e^0 b^\#_\alpha(k) = 0 \quad \forall k. \tag{45}
\]

So now we know that the mapping of \( B_\alpha \) into the \( b^\#_\alpha(p, q) \) is an isomorphism, the above mentioned theorem is applicable and we can decompose \( \mathcal{B} \) into a semi-simple compact Lie algebra and \( U(1)' \)’s. What we now need to show for these summands of \( \mathcal{B} \) individually is that their elements commute with the generators of Lorentz transformations, since then they commute with all elements of the Poincaré algebra and are therefore internal symmetries. At this point you might wonder what happened to the generator of translations \( P_\mu \), which by definition of \( \mathcal{B} \) should be an element of it. But recall that we subtracted a linear function of \( P_\mu \) from all generators to make them traceless. In this step we mapped \( P_\mu \) to zero and therefore excluded it from further considerations. So rest assured \( P_\mu \) itself is not an internal symmetry.

As a corollary, we can see that the number of independent matrices \( b^\#_\alpha(p, q) \) cannot exceed \( N(\sqrt{-p_\mu p^\mu})N(\sqrt{-q_\mu q^\mu}) \) and therefore also the number of independent generators \( B_\alpha \) must be finite, which we did not need to assume in the beginning.

### 5.1.1 The \( U(1)' \)’s

Since the \( B^\#_\alpha \) commute with \( P_\mu \) and \([J, P_\mu]\), which is a linear combination of \( P_\mu \) and where \( J \) is a generator of Lorentz transformations, the Jacobi identity is

\[
0 = [P_\mu, [J, B^\#_\alpha]] + [J, [B^\#_\alpha, P_\mu]] + [B^\#_\alpha, [P_\mu, J]] = [P_\mu, [J, B^\#_\alpha]], \tag{46}
\]
which tells us, that \([J, B^\#] \) is a linear combination of \(B^\#_\alpha\)'s (tracelessness is clear because it is a commutator). Following the theorem decomposing the algebra \(B\) in a direct sum, we know that any U(1) generator \(B^\#_i\) must commute with all other \(B^\#_\alpha\)'s, in particular

\[
[B^\#_i, [J, B^\#_i]] \sim [B^\#_i, \sum_{\alpha \neq i} B^\#_\alpha + B^\#_i] = [B^\#_i, B^\#_i] = 0. \tag{47}
\]

With some effort one can show by taking the expectation value of this commutator (using \(J |p, m; q, n\rangle = \sigma(m, n) |p, m; q, n\rangle\)), that

\[
0 = \langle p, m; q, n | [B^\#_i, [J, B^\#_i]] | p, m; q, n \rangle \\
= \langle p, m; q, n | (2B^\#_i JB^\#_i - J B^\#_i B^\#_i - B^\#_i B^\#_i J) | p, m; q, n \rangle \\
= 2 \langle p, m; q, n | (B^\#_i JB^\#_i - J B^\#_i B^\#_i) | p, m; q, n \rangle \\
= 2 \sum_{m', n'} (\sigma(m', n') - \sigma(m, n)) \left| (b^\#_i (p, q))_{m'n', mn} \right|^2,
\]

where from going from the second to the third line we used that \(J^\dagger = J\) and in the fourth line acted with \(J\) once after and once before \(B^\#_i\) on the left state in the first and second term respectively. We also used the orthonormality of states. So we see that for any \(m, n, m'\) and \(n'\) for which \(\sigma(m', n') \neq \sigma(m, n)\) \((b^\#_i (p, q))_{m'n', mn}\) has to vanish. Since the \(b^\#_i (p, q)\) are isomorphic to \(B^\#_i\) this is just the statement that

\[
[B^\#_i, J] = 0, \tag{49}\]

which proves the theorem for this subalgebra of the total symmetry algebra.

5.1.2 The semi-simple compact Lie algebra

We will only roughly sketch the idea for this part of the proof. For a full proof see section 24.1 in \([5]\).

We look at \(U(\Lambda) B_\alpha U(\Lambda)^{-1}\), where \(B_\alpha\) is a generator of the compact Lie algebra that resulted from the decomposition above and \(U(\Lambda)\) is some representation of the Lorentz group. This is an element of the same algebra and therefore

\[
U(\Lambda) B_\alpha U(\Lambda)^{-1} = \sum_b D^b_\alpha(\Lambda) B_b, \tag{50}\]

for some function \(D(\Lambda)\) which turns out to also be a representation of the Lorentz group. From these \(D(\Lambda)\) we can construct a unitary finite-dimensional representation of the non-compact SO(1, 3). This can therefore only be the trivial representation \(D(\Lambda) = \mathbb{1} \forall \Lambda\). So the \(B_\alpha\) also commute with the generators of the homogeneous Lorentz group.

This concludes the discussion of \(B\): All generators \(B_\alpha \in B\) commute with both \(P_\mu\) (by definition of \(B\)) and \(J\) (as shown for the U(1) and the compact part separately) and therefore with all generators of the Poincaré group. They are internal symmetries!

5.2 The subset \(A\)

We still have all the generators left, that do not commute with \(P_\mu\). These we will call \(A_\alpha\). We want to show that this set \(A\) of \(A_\alpha\)'s can only consist of the generators of Lorentz transformations for which we already know that

\[
[J, P_\mu] \sim P_\mu \neq 0. \tag{51}\]
The action of a general symmetry generator $A_\alpha$ on a one-particle state is

$$A_\alpha |p, n\rangle = \sum_{n'} \int d^4p' \left( \mathcal{A}_\alpha(p', p) \right)_{n'n'} |p', n'\rangle,$$

where again $n$ and $n'$ label discrete spin and particle type indices. The kernel $\mathcal{A}_\alpha(p', p)$ must vanish unless both $p$ and $p'$ are on the mass-shell.

In the following we will first show that $\mathcal{A}_\alpha(p', p) = 0$ for $p \neq p'$. If $A_\alpha$ is a symmetry generator, then so is

$$A_\alpha^I \equiv \int d^4x \exp(iP \cdot x) \left( \mathcal{A}_\alpha(p', p) \right)_{n'n'} \exp(-iP \cdot x)f(x),$$

for any function $f(x)$. Acting on one-particle states we can evaluate the momentum operators $P_\mu$ and carry out the Fourier transformation, which gives

$$A_\alpha^I |p, n\rangle = \sum_{n'} \int d^4p' \tilde{f}(p' - p) \left( \mathcal{A}_\alpha(p', p) \right)_{n'n'} |p, n'\rangle,$$

where we used the Fourier transform $\tilde{f}(k)$ of $f$.

Now suppose there is a scattering process $p, q \rightarrow p', q'$ for mass-shell momenta $p, q, p', q'$. Suppose also there is $\Delta \neq 0$ such that $p + \Delta$ is still on the mass-shell and $\mathcal{A}_\alpha(p, p + \Delta) \neq 0$. For generic values of $q, p'$ and $q'$ the momenta $q + \Delta, p' + \Delta$ and $q' + \Delta$ will be off-shell. Choose $\tilde{f}$ to vanish outside of a small region around $k = \Delta$. We then have

$$A_\alpha^I |p, n\rangle = \sum_{n'} f(\Delta) \left( \mathcal{A}_\alpha(p + \Delta, p) \right)_{n'n'} |p + \Delta, n'\rangle \neq 0,$$

$$A_\alpha^I |q, n\rangle = A_\alpha^I |p', n\rangle = A_\alpha^I |q', n\rangle = 0,$$

where the last line holds because the kernel $\mathcal{A}_\alpha(q + \Delta, q)$ has to vanish since $q + \Delta$ is not on mass-shell and a symmetry cannot map a state in the Hilbert space to an unphysical state outside the Hilbert space, which of course is also true for $p'$ and $q'$. So we find that this symmetry forbids the scattering $p, q \rightarrow p', q'$ for generic $q, p'$ and $q'$ which is in contradiction to our assumption that scattering occurs at almost all energies except for isolated energies!

This leaves us with the options that either $\mathcal{A}_\alpha(p, p')$ commutes with $P_\mu$, which would leave us with a generator $B_\alpha$ that we already discussed, or that the kernel is proportional to $\delta^{(4)}(p' - p)$ and its derivatives, in which case $A_\alpha^I$ would only be a symmetry if $\tilde{f}$ has support in $k = 0$. The latter being of course the option we have to further investigate.

So we consider

$$\left( \mathcal{A}_\alpha(p', p) \right)_{n'n'} = \sum_{i=1}^{D_\alpha} \left( a_\alpha^{(i)}(p', p) \right)_{n'n'} \mu_1^{i} \ldots \mu_i \frac{\partial^i}{\partial p_{\mu_1} \ldots \partial p_{\mu_i}} \delta^{(4)}(p' - p),$$

with the coefficients $a_\alpha^{(i)}$ of each term being matrix valued functions of momentum. The "ugly technical assumption" of Coleman and Mandula now is needed to argue that there is a finite
number $D_\alpha$ of derivatives in the distributions $\mathcal{A}_\alpha$.

We now look at the $D_\alpha$-fold commutator of $A_\alpha$ with $P_\mu$’s

$$B^{\mu_1 \cdots \mu_{D_\alpha}}_\alpha \equiv [P^{\mu_1}, [P^{\mu_2}, \ldots [P^{\mu_{D_\alpha}}, A_\alpha] \ldots]].$$  \hfill (57)

To illustrate what happens let’s look at $D_\alpha = 1$ and act on a one-particle state

$$B^\mu_\alpha |p, n\rangle = |P^\mu, A_\alpha \rangle |p, n\rangle = \sum_{n'} [P^\mu, \left( a^\dagger(p) \right)^\nu]_{n'n} |p, n\rangle$$

$$= -\sum_{n'} \frac{\partial}{\partial p^\nu} P^\mu\left( a^\dagger(p) \right)^\nu_{n'n} |p, n\rangle$$

$$= -\sum_{n'} \delta_\nu\left( a^\dagger(p) \right)^\nu_{n'n} |p, n\rangle,$$  \hfill (58)

where in the first line we used that the commutator with a function $a^0$ vanishes and evaluated $P^\mu$ before carrying out the derivative.

Further we find for the matrix elements

$$\langle p'| [B^\mu_\alpha, P^\nu] |p\rangle = \langle p'| [P^\mu, A_\alpha, P^\nu] |p\rangle$$

$$= \langle p'| P^\mu A_\alpha P^\nu - A_\alpha P^\mu P^\nu - P^\mu P^\nu A_\alpha + P^\nu A_\alpha P^\mu |p\rangle$$

$$= \langle p'| p^\mu A_\alpha p^\nu - A_\alpha p^\mu p^\nu - p^\mu p^\nu A_\alpha + p^\nu A_\alpha p^\mu |p\rangle$$

$$= -(p' - p)^\mu(p' - p)^\nu \left( a^{(0)}_\alpha(p) + a^{(1)}_\alpha(p) \frac{\partial}{\partial p^\nu} \right) \langle p'| p\rangle,$$  \hfill (59)

where $a^{(0)}_\alpha(p) \equiv a^{(0)}_\alpha(p) + \frac{\partial}{\partial p^\nu} a^{(1)}_\alpha(p)$ and we used

$$\langle p'| A_\alpha |p\rangle = \int d^4 \tilde{p} \langle p'| \left( a^{(0)}_\alpha(\tilde{p}, p) \delta^{(4)}(\tilde{p} - p) + a^{(1)}_\alpha(\tilde{p}, p) \nu \frac{\partial}{\partial \tilde{p}^\nu} \delta^{(4)}(\tilde{p} - p) \right) |p\rangle$$

$$= \int d^4 \tilde{p} \delta^{(4)}(\tilde{p} - p) \left( a^{(0)}_\alpha(\tilde{p}, p) - \frac{\partial}{\partial p^\nu} \left( a^{(1)}_\alpha(\tilde{p}, p) \nu \right) - a^{(1)}_\alpha(\tilde{p}, p) \nu \frac{\partial}{\partial \tilde{p}^\nu} \right) \langle p'| p\rangle$$

$$= \left( a^{(0)}_\alpha(p) + a^{(1)}_\alpha(p) \nu \frac{\partial}{\partial p^\nu} \right) \langle p'| p\rangle,$$  \hfill (60)

where we partially integrated from the first to the second line and performed the integral using the delta function. Using the usual normalisation of states $\langle p'| p \rangle \propto \delta^{(3)}(p - p')$ and noting that $p$ and $p'$ must be on the same mass-shell we get

$$\langle p'| [B^\mu_\alpha, P^\nu] |p\rangle \propto -(p' - p)^\mu(p' - p)^\nu \left( a^{(0)}_\alpha(p) + a^{(1)}_\alpha(p) \nu \frac{\partial}{\partial p^\nu} \right) \delta^{(4)}(p' - p).$$  \hfill (61)

Generally we will find that the matrix elements $\langle p'| [B^{\mu_1 \cdots \mu_{D_\alpha}}_\alpha, P^\nu] |p\rangle$ will be proportional to $D_\alpha + 1$ factors of $p' - p$ and polynomials of order $D_\alpha$ in derivatives acting on $\delta^{(4)}(p' - p)$. The delta functions set $p'$ to $p$ such that the prefactors $p' - p$ vanish, which means that these commutators vanish. As the notation already suggested we see that the $B^{\mu_1 \cdots \mu_{D_\alpha}}_\alpha$ are in the subalgebra $\mathcal{B}$.

As proved above the $B^{\mu_1 \cdots \mu_{D_\alpha}}_\alpha$ act on one-particle states as

$$\left( b^{\mu_1 \cdots \mu_{D_\alpha}}_\alpha(p) \right) \equiv \left( b^{\mu_1 \cdots \mu_{D_\alpha}}_\alpha(p) \right)_{n'n} + a^{\mu_1 \cdots \mu_{D_\alpha}}_\alpha p_\mu \delta_{n'n},$$  \hfill (62)
where the matrices $b^{\mu_1 \cdots \mu D_\alpha}_\alpha$ are momentum-independent, traceless representations of generators of internal symmetries and the $a^{\mu_1 \cdots \mu D_\alpha}_\alpha$ are momentum-independent constants.

We know by assumption (2) that the mass operator $-P_\mu P^\mu$ has discrete eigenvalues, so since $A_\alpha$ cannot take one-particle states off the mass-shell we have $[-P_\mu P^\mu, A_\alpha] = 0$ which for $D_\alpha \geq 1$ gives

$$0 = [P^\mu P_\mu_1, [P^\mu_2, \ldots [P^\mu D_\alpha, A_\alpha] \ldots]] = 2P_\mu_1 B^{\mu_1 \cdots \mu D_\alpha}_\alpha. \quad (63)$$

We can directly see that the momentum-independent term of $b^{\mu_1 \cdots \mu D_\alpha}_\alpha(p)$ must vanish since above equation suggests

$$b^{\mu_1 \cdots \mu D_\alpha}_\alpha \mu_1 + a^{\mu_1 \cdots \mu D_\alpha}_\alpha \mu_1 \mu_\alpha = 0,$$  \quad (64)

which must be satisfied for any timelike momentum $p$, as long as there are massive particles in the theory, which tells us that the coefficients in each order of $p$ have to vanish individually.

The last equation therefore also tells us in order $p^2$ that

$$a^{\mu_1 \mu_2 \cdots \mu D_\alpha}_\alpha = -a^{\mu_1 \mu \mu_2 \cdots \mu D_\alpha}_\alpha. \quad (65)$$

For these coefficients $a^{\mu_1 \cdots \mu D_\alpha}_\alpha$ we now have to distinguish between $D_\alpha = 0, 1$ and $D_\alpha \geq 2$. Note that $b^{\mu_1 \cdots \mu D_\alpha}_\alpha$ is symmetric in all its indices by definition of $B^{\mu_1 \cdots \mu D_\alpha}_\alpha$ via the commutator and therefore $a^{\mu_1 \cdots \mu D_\alpha}_\alpha$ is symmetric in the indices $\mu_1, \ldots, \mu D_\alpha$.

For $D_\alpha = 0$ we trivially have $A_\alpha = B_\alpha$ since then the kernel is simply a delta function and the $A_\alpha$ therefore commute with $P_\mu$.

For $D_\alpha = 1$ we get the condition

$$a^{\mu_1}_\alpha = -a^{\mu_1}_\alpha. \quad (66)$$

For $D_\alpha \geq 2$ we have this anti-symmetry in $\mu, \mu_1$ as in the case of $D_\alpha = 1$, but we also get it in all other indices because of the symmetry in the second to $(D_\alpha + 1)$th indices, which can only have the solution

$$a^{\mu_1 \cdots \mu D_\alpha}_\alpha = 0. \quad (67)$$

To reassure ourselves we can switch indices to see that

$$a^{\mu_1 \mu_2 \cdots \mu D_\alpha}_\alpha = -a^{\mu_1 \mu_2 \cdots \mu D_\alpha}_\alpha = a^{\mu_2 \mu_1 \cdots \mu D_\alpha}_\alpha = a^{\mu_2 \mu_1 \cdots \mu D_\alpha}_\alpha = -a^{\mu_2 \mu_1 \cdots \mu D_\alpha}_\alpha = -a^{\mu_2 \mu_1 \cdots \mu D_\alpha}_\alpha = a^{\mu_1 \mu_2 \cdots \mu D_\alpha}_\alpha = a^{\mu_1 \mu_2 \cdots \mu D_\alpha}_\alpha. \quad (68)$$

So far we have reduced $\mathcal{A}$ to generators for which

$$[P^\mu, A_\alpha] = a^{\mu \nu}_\alpha P^\nu, \quad (69)$$

where $a^{\mu \nu}_\alpha$ is some constant which is anti-symmetric in its indices.

From the Poincaré algebra relation

$$[P^\mu, J^{\rho \sigma}] = -i\eta^{\rho \sigma} P^\mu + i\eta^{\mu \sigma} P^\rho \quad (70)$$

We can now see that above commutator $[P^\mu, A_\alpha]$ is simply that of

$$[P^\rho, A_\alpha] = [P^\rho, -\frac{i}{2} a^{\mu \nu}_\alpha J_{\mu \nu} + B_\alpha], \quad (71)$$

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where $B_\alpha$ is some generator that commutes with $P^\mu$, which is why we can conclude that

$$A_\alpha = -\frac{i}{2}a_\alpha^{\mu\nu}J_{\mu\nu} + B_\alpha. \quad (72)$$

We have found that $A_\alpha$ is just the sum of some internal symmetry generator ($A_\alpha$ and $J_{\mu\nu}$ are symmetry generators, so $B_\alpha$ also has to be one) and the generator of Lorentz transformations.

Altogether we have shown that the only possible Lie algebra of symmetry generators consists of the generators of the Poincaré group $P_\mu$ and $J_{\mu\nu}$ and the generators of some internal symmetries $B_\alpha$. This is just the infinitesimal version of the statement

$$\mathcal{G} = \mathcal{P} \oplus \mathcal{B}. \quad (73)$$

6 Loopholes

Finally we will look at loopholes of the theorem.

Continuing in the proof we can see that for purely massless theories ($p^2 = 0$) above argument of anti-symmetry of $a_\alpha^{\mu_1\mu_2...\mu_D\alpha}$ does not work. In such theories we can have additional generators of the conformal group, which indeed do mix with those of the Poincaré group.

There are also symmetries that are not captured by the theorem at all. These symmetries are for example discrete symmetries and spontaneously broken symmetries. The latter being symmetries that do not act on the S-matrix level and therefore evading the requirement to commute with $S$. Rather, these symmetries are symmetries of the action of the theory and are spontaneously broken before any calculation.

Another important loophole are 1+1 dimensional theories. In such theories we can only have forward and backward scattering and therefore scattering cannot satisfy assumption 3 about the analyticity of scattering angles.

Lastly, as was already mentionend in the introduction, this proof heavily depended on the use of commutation relations. So obviously we can simply replace commutators by other brackets, so to evade the theorem we can look at graded Lie algebras or Lie superalgebras.

References