Supersymmetry and Geometry in Quantum Physics

The Chiral Ring

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In this talk, we want to investigate the so called *chiral ring* of physical operators. After some basic definitions, we will see how to use three-point correlation functions to calculate the chiral ring. Furthermore, we are going to determine the dependence on parameters and finally find correspondences between the chiral ring and vacuum states.
We mainly follow chapter 16 and 17 of [1].

1. Introduction

We want to focus only on B-twisted theories, i.e. $Q = Q_B = \overline{Q}_+ + \overline{Q}_-$ and restrict onto the case where $Z = \left\{\overline{Q}_+, \overline{Q}_-\right\} = 0$. Thus, we have that $Q^2 = 0$, so we can consider cohomology classes of operators. As usual, we demand for a physical operator $\mathcal{O}$ to commute with Q:

$$[Q, \mathcal{O}] = 0. \tag{1}$$

For a B-twisted theory, we call these operators *chiral operators*. As an example, consider a chiral superfield $\mathbf{\Phi}$. In the second talk, we have seen that we can expand these fields as

$$\mathbf{\Phi} = \phi + \Theta^\alpha \psi_\alpha + \Theta^+ \Theta^- F \tag{2}$$

As a $\mathbf{\Phi}$ satisfies $\overline{D}_\pm \mathbf{\Phi} = 0$ it follows that $\left[\overline{Q}_\pm, \phi\right] = 0$, i.e. ϕ is a chiral operator.

Properties of chiral operators

- *The worldsheet derivative is Q-exact.*
  To show this, we rewrite the derivatives by brackets with the generators and use the following relation

$$\{A, [B, C]\} = [\{A, B\}, C] - \{[A, C], B\} \tag{3}$$

which follows directly by expanding the (anti-)commutators. Let $\mathcal{O}$ be a chiral operator.

$$\begin{aligned}\frac{i}{2}\left(\frac{\partial}{\partial x^0}+\frac{\partial}{\partial x^1}\right)\mathcal{O} &= [(H+P),\mathcal{O}] = \left[\{Q_+,\overline{Q}_+\},\mathcal{O}\right] \\ &\stackrel{3}{=} \{[Q_+,\mathcal{O}],\overline{Q}_+\} + \{Q_+,[\overline{Q}_+,\mathcal{O}]\} \\ &\stackrel{1}{=} \{[Q_+,\mathcal{O}],\overline{Q}_+\} - \{Q_+,[\overline{Q}_+,\mathcal{O}]\} \\ &\stackrel{3}{=} \{[Q_+,\mathcal{O}],\overline{Q}_+\} - \underbrace{[\{Q_+,\overline{Q}_-\},\mathcal{O}]}_{=0} + \{\overline{Q}_-,[Q_+,\mathcal{O}]\} \\ &= \{Q,[Q_+,\mathcal{O}]\}\end{aligned} \tag{4}$$

Thus, we conclude that the cohomology class of $\mathcal{O}$ is independent of the worldsheet-position.

- *The product of two chiral operators is chiral*
  This follows directly from the product rule for commutators

$$[A, B\cdot C] = [A,B]\cdot C + B\cdot [A,C] \tag{5}$$

  if one sets $A = Q$ and B, C chiral operators.

From this we get a ring structure on the cohomology classes of chiral operators. This ring is called *chiral ring*.
Note: The same analysis can be done for A-twisted ($Q_A = \overline{Q}_+ + Q_-$) theories where we call the physical operators *twisted chiral operators*. This will lead to the so called *twisted chiral ring*.
In order to get a better intuition for the chiral ring, let $\{\phi_i\}$ be a basis of the Q-cohomology group of operators (in the sense that the physical operators also form a vector-space) and expand the product of two such basis elements

$$\phi_i\phi_j = \phi_k C_{ij}^k + [Q,\Lambda]. \tag{6}$$

We call the C_{ij}^k *structure constants* of the chiral ring (with respect to the basis $\{\phi_i\}$). They satisfy the following relations:

- The structure constants are antisymmetric in $i \leftrightarrow j$ if ϕ_i and ϕ_j are fermionic and symmetric in $i \leftrightarrow j$ else.
- From the associativity of the operator product $(\phi_i\phi_j)\phi_k = \phi_i(\phi_j\phi_k)$ it follows that

$$C_{il}^m C_{jk}^l = C_{lk}^m C_{ij}^l \tag{7}$$

- If we denote the identity operator in this basis by $1 = \phi_0$, we get

$$C_{i0}^k = C_{0i}^k = \delta_i^k \tag{8}$$

  as the product with other basis elements is trivial.

2. Three-point functions on $\Sigma = S^2$

As we have seen in the previous talk on *Topological Twisting* correlation functions of physical operators are independent of the choice of the metric. Also remember from 1 that the Q-cohomology class is independent of the worldsheet coordinate. Before we get to three-point functions, let us first consider two-point functions

$$\eta_{ij} = \left\langle \phi_i \phi_j \right\rangle . \tag{9}$$

For the theories we are interested in η_{ij} is an invertible matrix. We denote the inverse by η^{ij}, such that $\eta_{ij}\eta^{jk} = \delta_i^k$. The invertible two-point function can be considered as determining a metric on the parameter space. Hence, we call η_{ij} *topological metric*.

Now we come to three-point functions. We want to denote them by

$$C_{ijk} := \left\langle \phi_i \phi_j \phi_k \right\rangle . \tag{10}$$

Inserting the expansion given in 6, we get

$$\begin{aligned} C_{ijk} &= \left\langle \phi_i \left(\phi_l C^l_{jk} + [Q, \Lambda] \right) \right\rangle \\ &= \left\langle \phi_i \phi_l \right\rangle C^l_{jk} = \eta_{il} C^l_{jk}, \end{aligned} \tag{11}$$

where in the second step we used that the expectation value of a Q-exact term vanishes. As η was invertible we follow the interesting result

$$C^l_{jk} = \eta^{il} C_{ijk}. \tag{12}$$

We conclude that the chiral ring is fully determined by the calculation of three point function of chiral operators.

3. Dependence on Parameters

After we have seen how to calculate the chiral ring we now want to study how the chiral ring depends on the parameters of the theory. Remember that there were three term in the supersymmetric action, namely the D-term, the F-term and the twisted F-term (cf notes on *2d SUSY Introduction* section 1.2).

i) Variation of the D-term in the action
   This corresponds to an insertion of an operator in the path integral which is of the form
   $$\int \mathrm{d}^4\Theta \Delta K. \tag{13}$$
   Using the rules of fermionic integration, we find that the operator is pro-

portional to

$$\begin{aligned}
&\left\{\overline{Q}_+, \left[\overline{Q}_-, \int \mathrm{d}\Theta^+ \int \mathrm{d}\Theta^- \, \Delta \, K\Big|_{\overline{\Theta}^\pm}\right]\right\} \\
\overset{\overline{Q}_-^2=0}{=}&\left\{\overline{Q}_+ + \overline{Q}_-, \left[\overline{Q}_-, \int \mathrm{d}\Theta^+ \int \mathrm{d}\Theta^- \, \Delta \, K\Big|_{\overline{\Theta}^\pm}\right]\right\} \\
=&\left\{Q, \left[\overline{Q}_-, \int \mathrm{d}\Theta^+ \int \mathrm{d}\Theta^- \, \Delta \, K\Big|_{\overline{\Theta}^\pm}\right]\right\}.
\end{aligned} \tag{14}$$

Thus, we have seen that the inserted operator is Q-exact, which means that it's contribution to the correlation function vanishes.

ii) Dependence on (anti-)twisted chiral parameters
We only discuss the case of twisted chiral parameters as the anti-twisted case is a similar calculation. A deformation by a twisted chiral parameter corresponds to an insertion of an operator

$$\begin{aligned}
\int \mathrm{d}^2x \sqrt{h} \int \mathrm{d}\overline{\Theta}^- \int \mathrm{d}\Theta^+ \, \Delta\tilde{W}(\tilde{\mathbf{\Phi}}) \propto &\int \mathrm{d}^2x \sqrt{h}\{Q_+, [\overline{Q}_-, \Delta\tilde{W}(\tilde{\mathbf{\Phi}})]\} \\
=&\int \mathrm{d}^2x \sqrt{h}\{Q_+, [\overline{Q}_- + \overline{Q}_+, \Delta\tilde{W}(\tilde{\mathbf{\Phi}})]\} \\
=&-\int \mathrm{d}^2x \sqrt{h}\{Q, [Q_+, \Delta\tilde{W}(\tilde{\mathbf{\Phi}})]\} \\
&+ \text{total derivative},
\end{aligned} \tag{15}$$

where in the first step we used the rules of fermionic integration, in the second step the fact that $\tilde{W}$ is a twisted chiral operator, i.e. $[\overline{Q}_+, \tilde{W}] = 0$, and in the third step the relation 3. Thus, we again have seen that the inserted operator is Q-exact.

iii) Dependence on anti-chiral parameters
Such a deformation corresponds to an insertion of an operator

$$\begin{aligned}
\int \mathrm{d}^2x \sqrt{h} \int \mathrm{d}\overline{\Theta}^- \int \mathrm{d}\overline{\Theta}^+ \, \Delta\overline{W}(\overline{\mathbf{\Phi}}) \propto &\int \mathrm{d}^2x \sqrt{h}\{\overline{Q}_+, [\overline{Q}_-, \Delta\overline{W}(\overline{\mathbf{\Phi}})]\} \\
=&\int \mathrm{d}^2x \sqrt{h}\{\overline{Q}_+ + \overline{Q}_-, [\overline{Q}_-, \Delta\overline{W}(\overline{\mathbf{\Phi}})]\} \\
=&\int \mathrm{d}^2x \sqrt{h}\{Q, [\overline{Q}_-, \Delta\overline{W}(\overline{\mathbf{\Phi}})]\},
\end{aligned} \tag{16}$$

where in the second step we used that $\overline{Q}_-^2 = 0$. Once more we have an insertion of a Q-exact operator.

iv) Dependence on chiral operators
We recall that chiral operators are Q-closed, so we can use the descent relations to define the corresponding 2-form

$$\mathcal{O}^{(2)} = \mathrm{d}z \, \mathrm{d}\overline{z} \, \{Q_+, [Q_-, \mathcal{O}]\}. \tag{17}$$

A chiral deformation corresponds to an insertion of an operator

$$\begin{aligned} \int d^2x \sqrt{h} \int d\Theta^+ d\Theta^- \Delta W(\mathbf{\Phi}) &\propto \int d^2x \sqrt{h} \{Q_+, [Q_+, \Delta W(\mathbf{\Phi})]\} \\ &\propto \int \Delta W^{(2)}(\mathbf{\Phi}), \end{aligned} \tag{18}$$

where we have used the descent relation 17 for the chiral operator W. Thus, we have an expression that is not Q-exact.

We conclude that in a B-twisted theory correlation functions are only dependent on chiral parameters and the dependence is holomorphic. As we discussed in 2, the chiral ring is determined by calculation three-point functions of chiral operators. Hence, we have that the chiral ring is only holomorphic dependent on chiral parameters.

4. Chiral Ring and vacuum states

In the last section of this talk we want to investigate if there is a correspondence between the chiral ring and vacuum states.
Consider the space of ground states

$$V = \{|\alpha\rangle \in \mathcal{H} | Q |\alpha\rangle = Q^\dagger |\alpha\rangle = 0\}. \tag{19}$$

As the ground states can depend on the parameters of the theory we denote $V = V(m)$ with $m \in \mathcal{M}$ the parameter space. However, it has a fixed dimension as the number of ground states stays constant. Let ϕ be a chiral field. It can be viewed as an operator by acting on a ground state $|\alpha\rangle \in V(m)$, as $\phi |\alpha\rangle \in \mathcal{H}$ defines a state in the Hilbert space of the theory. Consider the projection of $\phi |\alpha\rangle$ onto $V(m)$. We observe that the projection is only dependent on the Q-cohomology class, as we can transform $\phi \to \phi + [Q, \rho]$ without changing the result by the definition of the ground states. Hence, we can use chiral fields to relate different ground states. Furthermore, it can be shown that in fact all ground states can be obtained by acting with chiral fields on a canonical ground state.
We now want to define this canonical ground state. Consider the path-integral on the hemisphere with $\mathcal{H}$ based on the boundary. We can interpret the path-integral (PI) as a state in the Hilbert space as it defines a map

$$\text{PI} : \{\text{boundary field configurations}\} \mapsto \text{numbers}. \tag{20}$$

The projection of a boundary state onto $V(m)$ can be obtained in the following way. We glue a flat cylinder of length T to the S^1-boundary of the hemisphere. This corresponds to a evolution of boundary states with the operator e^{-TH}

$$|\psi\rangle \to e^{-TH} |\psi\rangle\,, \tag{21}$$

where H is the Hamiltonian. Taking the limit $T \to \infty$ is then equivalent to the projection onto a ground state, as contributions of states with non-zero eigenvalue of H will vanish. Remember that the twisted and untwisted theory do

not differ on a flat space, so the construction is well defined. Also note that the state we obtain is independent of the metric, as varying the metric yields to an insertion of a Q-trivial operator that gets annihilated by the evolution operator e^{-TH}. Hence, we get that the path-integral picks up a distinguished element of the Hilbert space $\mathcal{H}$ which we denote by $|0\rangle$.
Similar, by inserting a chiral field ϕ_i into the path-integral, we get a ground state we denote by $|i\rangle$. Remember that changing the position of ϕ_i does not modify the state, so we can "move" ϕ_i to the boundary of the hemisphere. Doing this, we get the relation

$$|i\rangle = \phi_i |0\rangle \,. \tag{22}$$

From this, we get

$$\phi_i |j\rangle = \phi_i \phi_j |0\rangle = C_{ij}^k \phi_k |0\rangle = C_{ij}^k |k\rangle \,, \tag{23}$$

so we see that the ground states provide a realization of the chiral ring. However, we do not always get a one-to-one correspondence (e.g. topological Landau-Ginzburg models with A-twist), but for many theories it is. Especially for the Sigma model with A-twist, which is the theory we are interested in most, there is a one-to-one correspondence. But note that this does not imply that both are equal. If we stick to the Sigma model, we have that $Q = d$ the exterior derivative, hence the chiral ring is given by the De-Rham cohomology classes but the Hamiltonian H is proportional to the Laplacian Δ, so $V(m)$ is given by the harmonic forms.

One can do even more: In $\mathcal{N} = (2.2)$ theories the number of ground states does not change if we change the superpotential terms in the action. As we have discussed in 3, the chiral ring does depend on chiral parameters. As it turns out, there is further information about the structure of the vacuum states encoded in the chiral ring.
As $V(m)$ has fixed dimension $\forall m \in \mathcal{M}$ and the states, etc. are continuous functions of m, $\{V(m)\}_{m\in\mathcal{M}}$ forms a bundle over $\mathcal{M}$, which is called *vacuum bundle*. It is obvious that $\mathcal{H}$ itself does not change with m. Thus it forms a trivial bundle over $\mathcal{M}$ with the vacuum bundle as a sub bundle. On a trivial bundle, we can always define a connection and via projection we get an induced connection on $V(m)$.
If we use similar constructions with the path integral as described above, it is possible to show that the connection on the vacuum bundle is compatible with the complex structure on the parameter space. Furthermore, one can show the so called tt^*-equations which are equivalent to the existence of an improved connection which is flat. This improved connection can be constructed with the given connection and the action of chiral fields on vacuum states (cf [1] chapter 17).

References

[1] Kentaro Hori, Sheldon Katz, Albrecht Klemm, Rahul Pandharipande, Richard Thomas, Cumrun Vafa, Ravia Vakil, and Eric Zaslow. *Mirror symmetry*. Clay mathematics monographs. AMS, Providence, RI, 2003.