Seminar on Supersymmetry in Geometry and Quantum Physics

Vacua of the $\mathbb{C}\mathrm{P}^{N-1}$ Nonlinear Sigma Model

Lukas Hahn

1. Introduction

The aim of these notes is to discuss aspects of the $\mathbb{C}\mathrm{P}^{N-1}$ nonlinear sigma model, in particular how it kind of magically arises as a certain limit of the gauged linear sigma model and the occurence of N massive vacua that are due to quantum corrections of the twisted superpotential and not at all expected from the classical point of view. These vacua will become important when investigating instanton effects in the $\mathbb{C}\mathrm{P}^{N-1}$ nonlinear sigma model and historically they are one of the main discoveries that led to mirror symmetry. Before adressing these topics a short introduction/revision of the gauged linear sigma model in general is given.
The talk and these notes are strongly oriented on [**1**] and [**2**].

2. Revision: Gauged linear sigma model

Consider a two dimensional supersymmetric linear sigma model with bosonic fields $\phi : \mathbb{R}^2 \to \mathbb{R}^N$ and $\mathscr{N} = (2, 2)$ supersymmetry. The kinetic term of the Lagrangian is given by

$$\mathcal{L}_{\mathrm{kin}} = \int d^4\theta\ \bar{\Phi}\Phi, \tag{2.1}$$

where Φ is a chiral superfield with component expression $\Phi = \Phi(\phi, \psi_\pm, \bar{\psi}_\pm)$ as usual. It enjoys a global U(1) symmetry

$$\Phi \mapsto e^{i\alpha}\Phi \qquad \alpha \in \mathbb{R}, \tag{2.2}$$

for which all Φ's are supposed to have "charge 1". In a local QFT, however, we have to demand that these transformations only depend on local coordinates for locality to be respected. For supersymmetric theories in superspace formalism this amounts to require

$$\alpha = \alpha(x^\mu, \theta^\pm, \bar{\theta}^\pm), \tag{2.3}$$

i.e. dependence on all superspace coordinates. By this redefinition invariance of the Lagrangian is destroyed, as

$$\bar{\Phi}\Phi \longmapsto \bar{\Phi}e^{-i\bar{\alpha}+i\alpha}\Phi. \tag{2.4}$$

This issue can be dealt with in a completely analogous way as it is usually done in standard QFT with gauge symmetries: Introducing a vector field (or connection) which will be denoted as $V(x^\mu, \theta^\pm, \bar\theta^\pm)$ and called *vector superfield.* By definition, it has to exhibit the transformation behaviour

$$V \longmapsto V + i(\bar\alpha - \alpha) \tag{2.5}$$

and the problem is fixed if the Lagrangian is modified to

$$\mathcal{L}_{\rm kin} = \int d^4\theta \; \bar\Phi \; e^V \Phi, \tag{2.6}$$

which is now invariant under the combined transformations (2.2) and (2.5). As for every superfield, V allows a finite expansion in terms of the Grassmann variables such that it depends on component fields

$$V = V(\lambda_\pm, \bar\lambda_\pm, D, \sigma, \bar\sigma, v_0, v_1), \tag{2.7}$$

where the λ's are Dirac spinors, the D is a real scalar, the σ's are complex scalars and the v's are vector (or 1-form) fields. There is a certain choice of gauge fixing which is usually used in this context called *Wess-Zumino gauge* in which this expression will take the form

$$\begin{aligned} V = \theta^-\bar\theta^-\left(v_0 - v_1\right) + \theta^+\bar\theta^+\left(v_0 + v_1\right) - \theta^-\bar\theta^+\sigma - \theta^+\bar\theta^-\bar\sigma \\ + i\theta^-\theta^+\left(\bar\theta^-\bar\lambda_- + \bar\theta^+\bar\lambda_+\right) + i\bar\theta^+\bar\theta^-\left(\theta^-\lambda_- + \theta^+\lambda_+\right) + \theta^-\theta^+\bar\theta^+\bar\theta^- D. \end{aligned} \tag{2.8}$$

The v_μ's (note, that the μ denotes a worldsheet index) are left with a residual gauge symmetry

$$v_\mu(x) \longmapsto v_\mu(x) - \partial_\mu\beta(x) \tag{2.9}$$

in Wess-Zumino gauge and for some real function β, as can be read off from (2.8). Recall, that the covariant derivative for the supersymmetry is given by

$$\mathcal{D}_\pm = \frac{\partial}{\partial\theta^\pm} - i\bar\theta^\pm\partial_\pm \tag{2.10}$$

and accordingly for $\bar{\mathcal{D}}_\pm$. We define the *super-field strength* by

$$\Sigma = \bar{\mathcal{D}}_+\mathcal{D}_- V \tag{2.11}$$

which turns out to be the super-analog for the standard field strength in non-supersymmetric theories. As can be seen by direct computation, it is gauge invariant (in the above sense) and a twisted chiral superfield (i.e. $\bar{\mathcal{D}}_+\Sigma = \mathcal{D}_-\Sigma = 0$). It takes values in the adjoint of U(1) for some U(1) bundle over $2 \mid 4$ superspace.
Finally, taking into account all restrictions that are imposed by the requirement of supersymmetry as well as the properties of Grassmann integrals the *gauged linear sigma model* (GLSM) Lagrangian is found to be of the general form

$$\mathcal{L}_{\rm GLSM} = \int d^4\theta \left(\bar\Phi \; e^V \Phi - \frac{1}{2e^2}\bar\Sigma\Sigma\right) + \frac{1}{2}\left(\int d^2\widetilde\theta \; \widetilde{W}_{{\rm FI},\vartheta}(\Sigma) + c.c.\right) \tag{2.12}$$

where the factor in front of the gauge-kinetic term is due to fixing the dimension and the tilde over $\widetilde\theta$ means that the right components of the Grassmann coordinates are chosen such that $\widetilde{W}_{{\rm FI},\vartheta}$ is a twisted chiral superfield called *twisted superpotential.*

We assumed that there are no non-twisted superpotentials of any kind which is for now unjustified but will be automatically satisfied later on. In particular we are interested in the linear twisted superpotential

$$\widetilde{W}_{\mathrm{FI},\vartheta} = -t\Sigma \tag{2.13}$$

with a complex parameter

$$t = r - i\vartheta, \tag{2.14}$$

where r and ϑ are called *Fayet-Iliopolous parameter* and *theta-angle*, respectively (which explains the subscript). Both of them are dimensionless parameters of the theory.

In components, Wess-Zumino gauge and for this particular choice the GLSM Lagrangian is given by

$$\begin{aligned}\mathcal{L}_{\mathrm{GLSM}} = \sum_{j=1}^{N} \big[&- \mathcal{D}^{\mu}\bar{\phi}_j \mathcal{D}_{\mu}\phi_j + i\bar{\psi}_{j-}\left(\mathcal{D}_0 + \mathcal{D}_1\right)\psi_{j-} + i\bar{\psi}_{j+}\left(\mathcal{D}_0 - \mathcal{D}_1\right)\psi_{j+} \\ &-|\sigma|^2|\phi_j|^2 - \bar{\psi}_{j-}\sigma\psi_{j+} - \bar{\psi}_{j+}\bar{\sigma}\psi_{j-} - i\bar{\phi}_j\lambda_-\psi_{j+} + i\bar{\phi}_j\lambda_+\psi_{j-} + i\bar{\psi}_{j+}\bar{\lambda}_-\phi_j \\ &-i\bar{\psi}_{j-}\bar{\lambda}_+\phi_j\big] + \frac{1}{2e^2}\left(-\partial^{\mu}\bar{\sigma}\partial_{\mu}\sigma + i\bar{\lambda}_-(\partial_0+\partial_1)\lambda_- + i\bar{\lambda}_+(\partial_0-\partial_1)\lambda_+ + v_{01}^2\right) \\ &+\vartheta v_{10} - \frac{e^2}{2}\left(\sum_{i=1}^{N}|\phi_i|^2 - r\right)^2 .\end{aligned} \tag{2.15}$$

Here, the worldsheet covariant derivative is defined with respect to the residual gauge symmetry of v_μ in Wess-Zumino gauge, i.e. $\mathcal{D}_\mu = \partial_\mu + v_\mu$ and $v_{\mu\nu}$ is the related curvature

$$v_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu . \tag{2.16}$$

of the gauge transformation (2.9).

3. $\mathbb{C}\mathrm{P}^{N-1}$ NLSM as limit of the GLSM

Consider again such a U(1) gauge theory with N chiral superfields obeying the GLSM Lagrangian. One can read off the potential for the ϕ's and σ's as

$$U(\phi) = \sum_{i=1}^{N}|\sigma|^2|\phi_i|^2 + \frac{e^2}{2}\left(\sum_{i=1}^{N}|\phi_i|^2 - r\right)^2 . \tag{3.1}$$

Whenever such a potential is present, those field configurations for which the potential vanishes is called, at least under the reasonable assumptions of non-emptiness and smoothness, *vacuum manifold* and denoted by M_{vac}. Quite generally, starting from the linear sigma model (i.e. a QFT in affine space) a theory on a submanifold of $\mathbb{R}^N$ can be obtained by turning on a potential whose zero-locus is given by this manifold. Further, theories on quotients of these submanifolds by the action of a continuous group are obtained in a similar way but for the related *gauged* linear sigma model. The $\mathbb{C}\mathrm{P}^{N-1}$ NLSM will reveal itself to be one of those cases.

For the potential at hand we will assume from now on that $r > 0$. Then the zero-locus is given by

$$\left\{ (\phi_1, ..., \phi_N) \mid \sum_{i=1}^{N} |\phi_i|^2 = r \right\} = S^{N-1}. \tag{3.2}$$

Note, that in this case we are forced to set $\sigma = 0$. Taking the gauge symmetry into account we end up with the vacuum manifold

$$M_{\text{vac}} = S^{N-1}/\text{U}(1) \cong \mathbb{C}\text{P}^{N-1} \tag{3.3}$$

as can be seen by thinking of S^{N-1} as embedded in $\mathbb{C}^N$ in the standard way (N is the complex dimension) and noting that the U(1) transformation acting on the ϕ's by

$$(\phi_1, ..., \phi_N) \longmapsto (e^{i\gamma}\phi_1, ..., e^{i\gamma}\phi_N) \tag{3.4}$$

acts on the sphere as a rotation along equatorial circles. Quotienting out the group action will identify all points on these. On the other hand all subspaces of $\mathbb{C}^N$ of complex dimension 1 that go through the origin intersect the sphere in exactly those equatorial circles and the identification above corresponds one-to-one to identifying complex 1 dimensional subspaces. This is exactly what the symbol $\mathbb{C}\text{P}^{N-1}$ is standing for.
The general outline of how the NLSM on $\mathbb{C}\text{P}^{N-1}$ arises is as follows: The only massless field excitations will turn out to be given by the ϕ's and the ψ's that are tangent to M_{vac}. All the rest aquires mass in some way and by making the masses very big they effectively "decouple" in a way that has to be made precise. Their equations of motion will pose algebraic constraints on the GLSM Lagrangian in such a way that the theory of massless fields on M_{vac} is precisely the $\mathbb{C}\text{P}^{N-1}$ NLSM.

- ϕ fields
  As can be read off from the component expression above, the masses of these fields are given by $|\sigma|$. As the potential is only minimized for $\sigma = 0$ the ϕ's that are tangent to M_{vac} have to be massless. The other configurations, i.e. the ones that are transverse to M_{vac} aquire mass in the usual way: The potential $U(\phi)$ admits a perturbative expansion $\widetilde{U}(\phi)$ around each point on the vacuum manifold whose Hessian $\partial_i\partial_j\widetilde{U}(\phi)$ (or rather its eigenvalues) determines the mass of these fields. By direct computations the mass is found to be $m = e\sqrt{2r}$.

- ψ fields
  One of the algebraic constraints that are determined below is given by

$$\sum_{i=1}^{N} \bar{\phi}_i\psi_{i\pm} = \sum_{i=1}^{N} \bar{\psi}_{i\pm}\phi_i = 0, \tag{3.5}$$

  for the tangent ϕ_i's. This is nothing but the vector $(\psi_\pm, \bar{\psi}_\pm)$ also being tangent to M_{vac} (remember: locally, the ψ's are just tangent vectors). Further, the masses of the remaining ψ's is in a similar way as above found to be again $m = e\sqrt{2r}$.

- λ, σ, D and v fields
  First of all, the D is in any case only auxiliary and already integrated out in order to arrive at the component expression in Wess-Zumino gauge (2.8). For λ and σ the same argument as above holds and the mass is again determined to be

$m = e\sqrt{2r}$. For the v an exermination of the Higgs mechanism reveals the same mass in this case.

We formulate the large mass limit in the equivalent way of taking $e \longrightarrow \infty$ which, as we have seen, affects all the fields except the ϕ's and ψ's living on $\mathbb{C}\mathrm{P}^{N-1}$. In particular we get

$$-\frac{1}{2e^2}\int d^4\theta\ \bar{\Sigma}\Sigma \xrightarrow{e\longrightarrow\infty} 0 \tag{3.6}$$

which renders all the fields in the Σ multiplet non-dynamical. This justifies integrating them out by computing their equations of motion and plug them into the Lagrangian as the algebraic constraints advertised before. They are given as follows:

- <u>constraints by λ</u>
  The constraint equations are given by (3.5) and restrict the massless ψ's to M_{vac}.

- <u>constraints by v</u>
  Integrating out the equations of motion for v_μ yields

$$v_\mu = \frac{i}{2}\frac{\sum_{i=1}^{N}\left(\bar{\phi}_i\partial_\mu\phi_i - \partial_\mu\bar{\phi}_i\phi_i\right)}{\sum_{j=1}^{N}|\phi_j|^2}. \tag{3.7}$$

  In the NLSM the kinetic term for the ϕ's depends on the metric of the target geometry. In our case it is governed by the covariant derivative with respect to the gauge field v_μ. By plugging in this equation the metric in the above sense can be read off to be

$$ds^2 = \frac{r}{2\pi}g^{FS} \tag{3.8}$$

  with the *Fubini-Study* metric

$$g^{FS} = \frac{\sum_{i=1}^{N-1}|dz_i|^2}{1+\sum_{i=1}^{N-1}|z_i|^2} - \frac{\sum_{i=1}^{N-1}|\bar{z}_i dz_i|^2}{\left(1+\sum_{i=1}^{N-1}|z_i|^2\right)^2} \tag{3.9}$$

  in homogeneous coordinates $z_i = \frac{\phi_i}{\phi_N}$. This is also the natural choice of metric on $\mathbb{C}\mathrm{P}^{N-1}$ by considering the maps

$$\mathbb{C}^N - \{0\} \longrightarrow S^{N-1} \longrightarrow \mathbb{C}\mathrm{P}^{N-1}. \tag{3.10}$$

  and determining the respective induced metric.

- <u>constraints by σ</u>
  Here the equations of motion read

$$\sigma = -\frac{\sum_{i=1}^{N}\bar{\psi}_{i+}\psi_{i-}}{\sum_{j=1}^{N}|\phi_j|^2} \tag{3.11}$$

  and will provide us after a very tedious calculation with the typical term $\sim R_{ijkl}\psi_i\psi_j\bar{\psi}_k\bar{\psi}_l$ that includes the curvature into the NSLM Lagrangian.

Altogether in the large mass limit and by applying the logic explained above, the GLSM Lagrangian reduces to

$$\begin{aligned} \mathcal{L} = -g_{ij}^{FS} \partial^{\mu} \phi^i \partial_{\mu} \bar{\phi}^j + i g_{ij}^{FS} \bar{\psi}_-^j \left(\mathcal{D}_0 + \mathcal{D}_1 \right) \psi_-^i + i g_{ij}^{FS} \bar{\psi}_+^j \left(\mathcal{D}_0 - \mathcal{D}_1 \right) \psi_+^i \\ + R_{ijkl}^{FS} \psi_+^i \psi_-^j \bar{\psi}_+^k \bar{\psi}_-^l \end{aligned} \tag{3.12}$$

where R_{ijkl}^{FS} is the Riemann curvature tensor with respect to the Fubini-Study metric connection. This is exactly the $\mathbb{C}\mathrm{P}^{N-1}$ NLSM Lagrangian. However, this holds for now only *classically*. We will now take quantum effects into account.

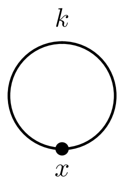
Quantum corrections

We consider the corresponding QFT with GLSM Lagrangian. The large mass limit is realised in this context by specifying an energy scale $\mu \ll e\sqrt{2r} = m$. Even if classically the massless fields vanish in the large mass limit and will have no impact on any physical effect, there is the possibility for those fields to have a nonzero vacuum expectation value that enters the potential energy terms in the Lagrangian and originates in processes beyond our characteristic energy scale. Furthermore, in order to arrive at the effective theory we are also forced to integrate out massless field excitations characterised by momentum k in the range $\mu \leq |k| \leq \Lambda_{\mathrm{UV}}$ for some UV-cutoff Λ_{UV}. This effect will only be important for the potential terms proportional to $|\phi|^2$, i.e. (3.1). Applying our logic we have to replace

$$|\phi|^2 \longrightarrow \langle |\phi|^2 \rangle = \langle \phi^{\dagger}(x) \phi(x) \rangle. \tag{3.13}$$

This correlation function is essentially given by the isolated 1-loop diagram of the scalar field

(3.14)



$$= \int_{\mu}^{\Lambda_{\mathrm{UV}}} \frac{d^2 k}{(2\pi)^2} \frac{2\pi}{k^2 + |\sigma|^2},$$

where the propagator can as always be determined from the Lagrangian but is not too surprisingly just the standard propagator for massive, scalar fields (the additional factor of 2π is just a matter of convention). The result is given by

$$\log \left(\frac{\Lambda_{\mathrm{UV}}}{\mu} \right) + \log \left(\frac{\mu}{|\sigma|} \right), \tag{3.15}$$

where the second term is finite but the first one suffers from an UV-divergence in the limit $\Lambda_{\mathrm{UV}} \to \infty$. Note that for the massless excitations the loop integral is

$$\int_{\mu}^{\Lambda_{\mathrm{UV}}} \frac{d^2 k}{(2\pi)^2} \frac{2\pi}{k^2} \tag{3.16}$$

and leads to the same divergence but the finite term is zero. In order to cope with the divergence we note that there is a dimensionless parameter, namely the Fayet-Iliopolous parameter, that is still a bare input. I.e. we can use the standard methods of renormalisation to cancel the divergence in terms of a redefinition of r that is characteristic for the energy scale μ. To be precise consider

$$r_0 = r + \log\left(\frac{\Lambda_{\mathrm{UV}}}{\mu}\right), \tag{3.17}$$

where r_0 and r are the bare and renormalised FI parameters, respectively. We have to demand a scale dependence of $r = r(\mu)$ in such a way that

$$r(\mu) = N\ \log\left(\frac{\mu}{\Lambda}\right) \tag{3.18}$$

exactly cancels the divergences in all loop diagrams of the N scalar fields (hence the additional factor of N). For this to happen, Λ has to be chosen accordingly but is fixed for given r_0 and cutoff Λ_{UV}. In any case, however, it is a finite parameter of mass dimension in contrast to the dimensionless bare FI parameter in the full theory. This phenomenon, sometimes called *dimensional transmutation*, is common in renormalised QFT's: A mass scale is dynamically generated by processes beyond our characteristic energy scale in a former scale invariant theory. Λ is called *renormalisation scale* and determines the renormalisation group flow equations that in this case govern the scale dependence of the parameter $r(\mu)$. We refer for a more detailed discussion on these topics to the standard literature or [**5**].

The final observation is the matching behaviour of the RG-flow: As can be seen directly from the definition (3.18) the FI parameter r' at a smaller scale μ' is in a simple relationship with r

$$r = r' + N\ \log\left(\frac{\mu}{\mu'}\right). \tag{3.19}$$

Remember, that from the NLSM perspective the running parameter is the "radius" of $\mathbb{CP}^{N-1}$, i.e. the factor r in the Fubini-Study metric. It is a general feature of this metric that it is in principle determined by the Ricci tensor:

$$R^{FS}_{ij} = N\ g^{FS}_{ij} \tag{3.20}$$

Following chapter 14 of [**1**] the scale dependence of the metric in the NLSM is given by

$$g^{FS}_{ij} = g'^{FS}_{ij} + \frac{1}{2\pi}\log\left(\frac{\mu}{\mu'}\right) R^{FS}_{ij}, \tag{3.21}$$

where some of the aspects of NSLM RG-flow can be found in [**6**]. By using the identification for the Fubini-Study metric (3.20) this can be rewritten as

$$g'^{FS}_{ij} = \frac{1}{2\pi}\left(r - N\ \log\left(\frac{\mu}{\mu'}\right)\right) g^{FS}_{ij}, \tag{3.22}$$

which is exactly the metric obtained from the GLSM reduction at the lower energy scale μ' upon using (3.19). The remarkable observation is: The RG-flow equations that are obtained for $r(\mu)$ in the GLSM precisely match the ones that are determined in the pure $\mathbb{CP}^{N-1}$ NLSM where the (initially dimensionless) parameter $r(\mu)$ is just the "radius" of $\mathbb{CP}^{N-1}$, i.e. the parameter in the Fubini-Study metric.
With this in mind the final conclusion is that not only classically the GLSM reduces to the $\mathbb{CP}^{N-1}$ NLSM but also in the related QFT: Quantum corrections will

not pose any problems with our previous analysis in the classical world, but rather render the theory scale dependent in precisely the way that we hoped for.
As a final remark, note that this seems to only hold at the 1-loop level: There may be higher order loop corrections to the VEV calculated above. For higher loops, however, the finite terms in (3.15) receive factors $\sim 1/|\sigma|$ which will be killed in the large mass limit. Similarly the renormalisation of the UV-divergence by utilising the FI parameter will go through in each order in a BPHZ-like way [**2**].

4. Massive Vacua

We are now ready to discuss the main statement of this talk: The existence of N massive vacua in the $\mathbb{C}\mathrm{P}^{N-1}$ NLSM. First of all note, that this is not expected classically as we introduced no potential terms in the Lagrangian that would lead to the related minima. Therefore the occurence of these vacua is exclusively due to quantum effects that can be determined in two different ways:
Either from starting with the $\mathbb{C}\mathrm{P}^{N-1}$ NLSM and discussing the effective quantum theory in the large radius limit or from thinking of this NLSM as a certain regime of the GLSM (as we have explained above) and determining a special effective theory in this picture. We carry out the second plan which amounts to considering the large mass limit in the GLSM

$$|\sigma| = m = e\sqrt{2r} \longrightarrow \infty \tag{4.1}$$

and a comparatively low energy scale $\mu \ll |\sigma|$. I.e. we are looking at an effective GLSM for this energy scale with Lagrangian that is generally of the form

$$S_{\text{eff}}(\Sigma) = \int d^4\theta \left(-K_{\text{eff}}(\Sigma, \bar{\Sigma})\right) + \frac{1}{2}\left(\int d^2\widetilde{\theta}\ \widetilde{W}_{\text{eff}}(\Sigma) + c.c.\right), \tag{4.2}$$

as classically we can set the massive matter chiral superfields to zero $\Phi \to 0$ in the large mass limit. Here we reinstated the possibility of non-chiral superpotentials that are in general given by an effective Kähler potential as above.
Until now there was no mention about R-symmetries whatsoever, but note that as $\mathbb{C}\mathrm{P}^{N-1}$ is *not* Calabi-Yau, $c_1(\mathbb{C}\mathrm{P}^{N-1}) = N \neq 0$, the B-twisted NSLM is in this case not even well defined as it suffers from sigma model anomalies [**3**]. So in order for all of this to make sense we have to demand A-twisted R-symmetries throughout: In this case it suffices to have a Kähler target, which is of course the case here. It was shown in one of the talks [**4**] that in A-twisted theories there can be no impact of non-chiral superpotentials by which we justify setting the effective Kähler potential to zero

$$K_{\text{eff}}(\Sigma, \bar{\Sigma}) = 0 \qquad \text{in the A-twisted theory.} \tag{4.3}$$

Further, we have to again take quantum corrections into account. These are in a quite similar manner as above obtained by (path-) integrating out all massive Φ modes and all Σ modes in the range $\mu \leq |k| \leq \Lambda_{\text{UV}}$, i.e. beyond the characteristic energy scale. At the 1-loop level the correction may be expressed as

$$e^{S_{\text{eff}}^{(\text{1-loop})}(\Sigma)} = \int \mathscr{D}\Phi\mathscr{D}\Sigma\ e^{iS(\Phi,\Sigma)}. \tag{4.4}$$

Here we denote the operation explained above by symbolically writing these path integrals. In principle this calculation can be carried out explicitly after mode expanding the fields in the measure: A quite detailed survey is given in the mirror symmetry book [**1**]. As it is rather lengthy, it will not be written down in these

notes. We will only state the two main discoveries:
First, the integration over $\mathscr{D}\Sigma$ will only yield corrections to the effective Kähler potential, $K_{\text{eff}}(\Sigma, \bar{\Sigma})$, that is killed in the A-twist anyway so it suffices to carry out the integral over the matter fields $\mathscr{D}\Phi$. It turns out that these corrections have an effect on the twisted superpotential and are 1-loop exact: The explicit calculation is quite similar to the loop integral above and the one loop exactness is for the same reason as discussed beforehand in this context. What we end up with is the (quantum) corrected effective twisted superpotential

$$\widetilde{W}_{\text{eff}}(\Sigma) = -t\Sigma \underbrace{-N\Sigma\left(\log\left(\frac{\Sigma}{\Lambda\text{'}}\right) - 1\right)}_{\text{1-loop corrections}}, \tag{4.5}$$

where Λ' is a slightly modified renormalisation scale whose details will not bother us. Note, that this regime of very large masses of the matter fields and comparatively low energy scale exactly matches the low energy regime of the $\mathbb{C}\text{P}^{N-1}$ NLSM by the previous discussion. To summarise, by considering the broader picture of the GLSM it is possible to compute quantum corrections to the twisted superpotential that will enter the Lagrangian of the $\mathbb{C}\text{P}^{N-1}$ NLSM as an additional potential term, namely the second term in (4.5). The minima of this new potential that has a purely quantum nature, are the vacua we are searching for.
We are interested in the critical points with respect to the mass σ, i.e. those (still very large) masses that minimise the potential

$$\partial_\sigma \widetilde{W}^{\text{1-loop}}_{\text{eff}}(\Sigma) = N\,\log\left(\frac{\sigma}{\Lambda\text{'}}\right) \stackrel{!}{=} 0. \tag{4.6}$$

The solutions to this equation are

$$\sigma = \Lambda\text{'} e^{\frac{2\pi i k}{N}} \;\; ; \quad k = 0, ..., N-1, \tag{4.7}$$

the N massive vacua of the $\mathbb{C}\text{P}^{N-1}$ NLSM that were advertised from the very beginning. Note, however, that as we only considered the critical points with respect to σ it is not clear if those are really all vacua of the twisted superpotential correction. It turns out to be true by using methods related to mirror symmetry but there is at least a hint for this from our present understanding: Using standard methods the Euler characteristic of $\mathbb{C}\text{P}^{N-1}$ is given by

$$\chi(\mathbb{C}\text{P}^{N-1}) = \text{Tr}\,(-1)^F = N. \tag{4.8}$$

So the number of massive vacua determined above exactly matches Witten's index that counts the number of supersymmetric ground states. That is no proof, as we are still in a particular limit, but a strong confirmation.

References

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[2] P. Deligne, P. Etingof, D.S. Freed, L.C. Jeffrey, D. Kazhdan, J.W. Morgan, D.R. Morrison, E. Witten; *Quantum Fields and Strings: A Course for Mathematicians*. American Mathematical Society, 1999.

[3] Moritz' Notes: *Anomalies in QFT and Index Theory*. `https://mathi.uni-heidelberg.de/~walcher/teaching/sose16/geo_phys/Anomalies.pdf`

[4] Fabio's Notes: *The Chiral Ring*. Soon on the Seminar Homepage.

[5] Timo Weigand's lecture notes: *Quantum Field Theory I+II*. `http://www.thphys.uni-heidelberg.de/~weigand/QFT2-14/SkriptQFT2.pdf`

[6] Lukas' Notes: *Sigma Model β-Function*. Soon on the Seminar Homepage.