Atiyah-Singer index theorem

Throughout M is a closed manifold of dimension m, E and F are complex vector bundles over M with fibre metrics $\langle,\rangle_E$ and $\langle,\rangle_F$, respectively.

1. Characteristic classes

1.1 Definition. The *total Chern class* of a complex vector bundle E with curvature 2-form F is given by

$$c(F) = \det\left(I + \frac{iF}{2\pi}\right) = 1 + c_1(F) + c_2(F) + ... \tag{1}$$

where $c_i(F) \in \Omega^{2i}(M)$ is called the ith Chern class.

1.2 Facts.

- These forms are gauge invariant, i.e. invariant under $F \to gFg^{-1}$ for $g \in G$.
- They are closed, i.e. $\mathrm{d}c_i(F) = 0$, due to the Bianchi identity $D_A F = \mathrm{d}F + \mathrm{ad}_A F = 0$. Thus, the Chern forms represent cohomology classes.
- The cohomology class of the Chern forms do not depend on the choice of the connection.
- If the Chern classes of two vector bundles do not agree, then the vector bundles cannot be isomorphic.
- Any characteristic class of a complex vector bundle is a polynomial in Chern classes.

1.3 Definition. Diagonalizing F as

$$g^{-1}\frac{i}{2\pi}Fg = \mathrm{diag}(x_1, ..., x_n) \tag{2}$$

where n is the fibre dimension of the bundle, the Chern class becomes

$$c(F) = \prod_{i=1}^{n}(1 + x_i). \tag{3}$$

We furthermore define the *Chern character*

$$\mathrm{ch}(F) := \mathrm{Tr}\exp\left(\frac{iF}{2\pi}\right) = \sum_{i=1}^{n}\exp(x_i), \tag{4}$$

the *Todd class*

$$\mathrm{Td}(F) := \prod_i \frac{x_i}{1 - e^{-x_i}} \tag{5}$$

and the $\hat{A}$ *(or Dirac) genus* by

$$\hat{A}(F) = \prod_{i=1}^{n} \frac{x_i/2}{\sinh(x_i/2)} \tag{6}$$

for any complex vector bundle E. The *Euler class* however is assigned to the tangent bundle TM of the manifold M:

$$e(TM) = \prod_{i=1}^{m/2} x_i \tag{7}$$

2. Atiyah-Singer index theorem

2.1 Definition. A *differential operator* D is a linear map of sections

$$D : \Gamma(M, E) \to \Gamma(M, F). \tag{8}$$

Its *adjoint* $D^\dagger : \Gamma(M, F) \to \Gamma(M, E)$ is defined by

$$\langle s', Ds \rangle_F \equiv \langle D^\dagger s', s \rangle_E \,. \tag{9}$$

We restrict ourselves to a certain class of operators called *elliptic* whose definition is given in the appendix.

2.2 Definition. Elliptic operators are called *Fredholm* if their kernel is finite:

$$\ker D = \{s \in \Gamma(M, E) | Ds = 0\}.$$

2.3 Fact. On compact manifolds every elliptic operator is Fredholm.

2.4 Definition. The *index* of an elliptic operator D is

$$\text{index} D := \dim \ker D - \dim \ker D^\dagger$$

2.5 Definition. A sequence

$$... \to \Gamma(M, E_{i-1}) \xrightarrow{D_{i-1}} \Gamma(M, E_i) \xrightarrow{D_i} \Gamma(M, E_{i+1}) \xrightarrow{D_{i+1}} ...$$

of vector bundles $\{E_i\}$ and elliptic operators D_i is called an elliptic complex if $D_i \circ D_{i-1} = 0 \ \forall i$. Such a complex gives rise to cohomology groups

$$H^i(E, D) := \ker D_i / \text{im} D_{i-1}$$

which we use to define the *index of the elliptic complex* (E, D):

$$\text{index}(E, D) := \sum_{i=0}^{m} (-1)^i \dim H^i(E, D)$$

2.6 Fact. The *Laplacian*

$$\Delta_i := D_{i-1}D_{i-1}^\dagger + D_i^\dagger D_i \tag{10}$$

gives rise to the Hodge decomposition of sections and

$$H^i(E, D) \cong \ker(\Delta_i) \tag{11}$$

The index of an elliptic complex is therefore truly analytical and can be shown to reduce to the index of an elliptic operator.

2.7 Theorem (Atiyah-Singer index theorem)**.** The index of an elliptic complex (E, D) over an m-dimensional closed manifold M is given by

$$\text{index}(E, D) = (-1)^{m(m+1)/2} \int_M \text{ch}\left(\oplus_r (-1)^r E_r\right) \frac{\text{Td}(TM^{\mathbb{C}})}{e(TM)}\bigg|_{\text{vol}}$$

3. The de Rham complex

The index of the complexified de Rham complex

$$\ldots \xrightarrow{\text{d}} \Omega^{r-1}(M)^{\mathbb{C}} \xrightarrow{\text{d}} \Omega^r(M)^{\mathbb{C}} \xrightarrow{\text{d}} \Omega^{r+1}(M)^{\mathbb{C}} \xrightarrow{\text{d}} \ldots \tag{12}$$

for $\Omega^r(M)^{\mathbb{C}} = \Gamma(M, \wedge^r T^* M \otimes \mathbb{C})$ is given by the Euler characteristic of M:

$$\begin{aligned} \text{index(d)} &= \sum_r (-1)^r \dim_{\mathbb{C}} H^r(M; \mathbb{C}) \\ &= \sum_r (-1)^r \dim_{\mathbb{R}} H^r(M; \mathbb{R}) \\ &= \chi(M) \end{aligned} \tag{13}$$

For even-dimensional M, the Chern characteristic can be calculated as (see appendix)

$$\text{ch}\left(\oplus_{r=0}^m (-1)^r \wedge^r T^* M^{\mathbb{C}}\right) = \prod_{i=1}^m (1 - e^{-x_i})(TM^{\mathbb{C}}). \tag{14}$$

Together with the Todd genus and the Euler class as given before, we arrive at

$$\chi(M) = \text{index(d)} = \int_M e(TM) \tag{15}$$

4. Dirac complex

Consider a spin bundle $S(M)$ over an even-dimensional manifold M^m. Sections $\psi \in \Delta(M) := \Gamma(M, S(M))$, i.e. Dirac spinors, are irreducible representations of the Clifford algebra which we can take to be of the form

$$\gamma^\beta = \begin{pmatrix} 0 & i\alpha_\beta \\ -i\bar{\alpha}_\beta & 0 \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \tag{16}$$

Irreducible representations of Spin(m) are obtained by seperating $\Delta(M)$ according to the eigenvalues of Γ:

$$\Delta(M) = \Delta^+(M) \oplus \Delta^-(M) \tag{17}$$

Introducing an orthonormal frame ($e^\alpha{}_\mu \mathrm{d}x^\mu$), i.e. $g_{\mu\nu} = e^\alpha{}_\mu e^\beta{}_\mu \eta_{\alpha\beta}$, on (M, g) the Dirac operator is given by

$$i\nabla\!\!\!/\psi = i\gamma^\mu(\partial_\mu + \omega_\mu)\psi \tag{18}$$

for the spin connection $\omega_\mu = \frac{i}{2}(\omega_\mu)^{\alpha\beta}\Sigma_{\alpha\beta}$ and $\gamma^\mu = \gamma^\alpha e_\alpha{}^\mu$. With the above form of the gamma matrices it becomes

$$i\nabla\!\!\!/ = \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix} \tag{19}$$

with

$$D = \bar{\alpha}^\beta e_\beta{}^\mu(\partial_\mu + \omega_\mu) \quad \text{and} \quad D^\dagger = -\alpha^\beta e_\beta{}^\mu(\partial_\mu + \omega_\mu). \tag{20}$$

We arrive at the two-term *spin complex*

$$\Delta^+(M) \underset{D^\dagger}{\overset{D}{\rightleftarrows}} \Delta^-(M) \tag{21}$$

with index

$$\mathrm{index}(D) = \dim\ker D - \dim\ker D^\dagger \tag{22}$$

which is the number of zero-energy modes of positive chirality minus the number of zero-energy modes of negative chirality. Using [1]

$$\mathrm{ch}(\Delta^+(M) - \Delta^-(M)) = (-1)^{m/2} \prod_{i=1}^{m/2} \left(e^{x_i/2} - e^{-x_i/2}\right) \tag{23}$$

the topological side becomes

$$\mathrm{index}(D) = \int_M \hat{A}(TM)\Big|_{\mathrm{vol}} \tag{24}$$

5. Supersymmetric proof for Dirac operator

A more thorough treatment can be found in [2]. In the excellent original papers [3] and [4] all classical complexes are 'proven' via supersymmetry.

Consider the (0+1)-dimensional supersymmetric $\sigma - model$ with target space M:

$$L = \frac{1}{2} g_{ij}\dot{\phi}^i\dot{\phi}^j + \frac{i}{2} g_{ij}\psi^i(\mathrm{d}\psi^j/\mathrm{d}t + \Gamma^j_{kl}\dot{\phi}^k\psi^l) \tag{25}$$

It turns out that the supercharge is the Dirac operator $Q \sim \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix}$, the Hamiltonian $H \sim Q^2 = \begin{pmatrix} D^\dagger D & 0 \\ 0 & DD^\dagger \end{pmatrix}$ and $(-1)^F = \Gamma = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$. For $\beta > 0$ arbitrary and eigensections $\{\phi_n\}$ and $\{\psi_n\}$ we have

$$\begin{aligned}
\mathrm{Tr}(-1)^F e^{-\beta H} &= \mathrm{Tr} e^{-\beta D^\dagger D} - \mathrm{Tr} e^{-\beta D D^\dagger} \\
&= \sum_{\lambda_n \neq 0} \langle \phi_n | e^{-\beta D^\dagger D} | \phi_n \rangle - \sum_{\lambda_n \neq 0} \langle \psi_n | e^{-\beta D D^\dagger} | \psi_n \rangle \\
&\quad + \sum_i \langle \phi_i^0 | \phi_i^0 \rangle - \sum_j \langle \psi_j^0 | \psi_j^0 \rangle \\
&= \sum_{\lambda_n \neq 0} e^{-\beta \lambda_n} (\langle \phi_n | \phi_n \rangle - \langle \psi_n | \psi_n \rangle) + \sum_i 1 - \sum_j 1 \\
&= \dim \ker D - \dim \ker D^\dagger \\
&= \mathrm{index}(D)
\end{aligned} \tag{26}$$

Evaluation of the path integral

$$\mathrm{index}(D) = \mathrm{Tr}\Gamma e^{-\beta H} = \int_{\mathrm{PBC}} Dx D\psi e^{-\int_0^\beta \mathrm{d}t L}. \tag{27}$$

where $(-1)^F$ turned the anti-periodic boundary conditions of the fermions into periodic ones [3] is rather tedious and gives the Dirac genus [2].

Appendices

This talk was mainly based on [2] and [1].

A. Characteristic classes

A.1 Definition. A characteristic class associates to any vector bundle E over M a cohomology class $c(E) \in H^*(M; G)$ for some group G such that, if $f : N \to M$ is a continuous map, then $c(f^*P) = f^*c(P)$.

Therefore, $c(E_1) = c(E_2)$ for isomorphic vector bundles $E_1 \cong E_2$.

B. Atiyah-Singer index theorem

Using the following multi-index notation in local coordinates x^μ,

$$\begin{aligned}
N &:= (\mu_1, ..., \mu_m), \qquad \mu_i \in \mathbb{Z}_{\geq 0} \\
|N| &:= \mu_1 + ... + \mu_m \\
D_N &:= \frac{\partial^{|N|}}{\partial x^N} = \frac{\partial^{\mu_1 + ... + \mu_m}}{\partial (x^1)^{\mu_1} ... \partial (x^n)^{\mu_m}}
\end{aligned}$$

the most general form of a differential operator D is

$$[Ds(x)]^f = \sum_{\substack{|N|\leq O \\ 1\leq e\leq \dim E}} (A^N)^f_e(x) D_N s^e(x) \qquad 1 \leq f \leq \dim F$$

Here, s is a section and A^N a $\dim E \times \dim F$ matrix. The positive integer O is called the *order* of D.

B.1 Definition. D is called *elliptic* if its symbol

$$\sigma(D,\xi) := \sum_{|N|=O} A^N(x)\xi_N \tag{28}$$

(a $\dim E \times \dim F$ matrix) is invertible for all $x \in M$ and $\xi = (\xi_1, ..., \xi_m) \in \mathbb{R}^m - \{0\}$.

C. The de Rham complex

$$\begin{aligned}
\mathrm{ch}\left(\oplus_{r=0}^m (-1)^r \wedge^r T^*M^{\mathbb{C}}\right) &= \sum_{r=0}^m (-1)^r \mathrm{ch}\left(\wedge^r T^*M^{\mathbb{C}}\right) \\
&= 1 - \mathrm{ch}(T^*M^{\mathbb{C}}) + \mathrm{ch}\left(\wedge^2 T^*M^{\mathbb{C}}\right) - ... + (-1)^m \mathrm{ch}\left(\wedge^m T^*M^{\mathbb{C}}\right) \\
&= 1 - \sum_{i=1}^m e^{-x_i}(TM^{\mathbb{C}}) + \sum_{i<j}^m e^{-x_i}e^{-x_j}(TM^{\mathbb{C}}) \\
&\quad + ... + e^{-x_1}e^{-x_2}...e^{-x_m}(TM^{\mathbb{C}}) \\
&= \prod_{i=1}^m (1 - e^{-x_i})(TM^{\mathbb{C}}).
\end{aligned} \tag{29}$$

using the splitting principle, $x_i(T^*M^{\mathbb{C}}) = -x_i(TM^{\mathbb{C}})$ and $\mathrm{ch}(L_i) = \exp(x_i)$ where x_i is the first Chern class of the line bundle L_i.

The top Chern class obeys

$$x_1 x_2 ... x_m = c_m(TM^{\mathbb{C}}) = (-1)^{m/2} e(TM \oplus TM) = (-1)^{m/2} e^2(TM) \tag{30}$$

which is used in equation 15.

References

[1] P. Shanahan, *The Atiyah-Singer Index Theorem.* 1978.

[2] M. Nakahara, *Geometry, topology and physics.* 2003.

[3] L. Alvarez-Gaume, *Supersymmetry and the Atiyah-Singer Index Theorem*, *Commun. Math. Phys.* **90** (1983) 161.

[4] L. Alvarez-Gaume, *A Note on the Atiyah-singer Index Theorem*, *J. Phys.* **A16** (1983) 4177.