Anomalies in QFT and Index Theory

Seminar of Professor Walcher, Summer term 2016

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1 Anomalies in QFT

The quantum field theory analogue of the Noether continuity equation is the Ward-Takahashi identity. It states that current conservation holds as an operator equation.

$$\int \mathcal{D}\phi\, e^{-S[\psi]} D_\alpha J^\alpha(x) = \langle D_\alpha J^\alpha(x) \rangle = 0 \tag{1}$$

The necessary precondition for this is the invariance of the path integral (PI) measure. If the measure is not invariant under a global or a gauge symmetry of the system we speak of an anomaly. From a physical point of view, global anomalies pose no problem but are rather good for testing QFT. The absence of gauge anomalies is a important consistency condition for a QFT. An anomalous gauge symmetry renders the usual Fadeev-Popov gauge fixing procedure inconsistent and thus leads to negative norm states and/or non-renormalizable theories. An additional motivation for us is that we need the $U(1)_A$ symmetry for the twisting of our 2d chiral theories but this could be anomalous in some models. Although our main interest will be the 2d sigma model, we will give a general overview on anomalies for gauge theories. This will seem odd a priori, since the 2d sigma model is no gauge theory but our mathematical examination will show how close they are related.

- Consider a matter theory $S[\psi]$ coupled to an external gauge field A^μ with Euclidean quantum effective action

$$e^{-\Gamma[A]} = \int \mathcal{D}\psi\, e^{-(S[\psi]-A_\mu \cdot J^\mu)}. \tag{2}$$

  The external A^μ is taken to be non-dynamical, i.e. $\mathcal{D}\psi$ as well as $S[\psi]$ include only the matter fields and $S[\psi] - A_\mu J^\mu$ is classicaly gauge invariant.

- A gauge anomaly arises if $\Gamma[A]$ is not invariant under a gauge transformation

$$\delta_\alpha A_\mu = D_\mu \alpha, \tag{3}$$

  with α an arbitrary local function.

- This occurs if the measure transforms as

$$\mathcal{D}\psi \to \mathcal{D}\psi\, e^{-2i\int dx\, \alpha(x)\mathcal{A}(x)} \tag{4}$$

  such that

$$\delta_\alpha(-\Gamma[A]) = 2i \int dx\, \alpha(x)\mathcal{A}(x) \tag{5}$$

  where we used that the anomaly term $\mathcal{A}(x)$ turns out to be independent of ψ.

- The gauge anomaly signals a non-conservation of J^μ at the operator level. First note that
$$-\frac{\delta\Gamma}{\delta A_\mu} = e^{\Gamma[A]} \int \mathcal{D}\psi\, e^{-(S[\psi]-A_\mu\cdot J^\mu)}\ (J^\mu)\ . \tag{6}$$
Therefore
$$\frac{\delta\Gamma}{\delta A_\mu} = -\langle J^\mu\rangle, \tag{7}$$
and it is easy to show that
$$\begin{aligned}\delta_\alpha\Gamma[A] &= \int \delta_\alpha A_\mu(x)\frac{\delta\Gamma}{\delta A_\mu}\\ &= \int D_\mu\alpha(x)\langle -J^\mu(x)\rangle\\ &= \langle\int \alpha(x)\, D_\mu J^\mu(x)\rangle.\end{aligned} \tag{8}$$

- A slight reformulation shows that an anomaly induces a $U(1)$ transformation of the functional determinant:
$$\begin{aligned}\delta_\alpha e^{-\Gamma[A]} &= \int \mathcal{D}\psi\, e^{-2i\int dx\,\alpha(x)\mathcal{A}(x)} e^{-(S-A\cdot J)}\\ &= \int \mathcal{D}\psi\, e^{-(S-A\cdot J)}\ e^{iG(A,\alpha)}\\ &= \det\, i\not{D}\ e^{iG(A,\alpha)},\end{aligned} \tag{9}$$
where we rephrased $iG(A,\alpha) = -2i\int dx\,\alpha(x)\mathcal{A}(x)$. So the absolute value of the functional determinant is always gauge invariant whereas the phase could receive contributions from an anomaly. Note that above identification of the functional integral with the determinant is **not well-defined** a priori. There are two problems:

  i) The Gaussian integration over the fermion field is divergent and needs to be regularized. This will not be of further importance for us as it can be shown that the anomaly is independent of the regularisation method. Nevertheless, the fact that we cannot find regulator which respects the classical symmetry is just what we call an anomaly.

  ii) Usually we consider only chiral theories. This is because a nonchiral theory is anomaly-free and furthermore, in the massless case, we can always seperate our Lagrangian in a positive and negative chirality part due to the reducibility of the spinor representation. But the Dirac operator maps spinors of definite chirality to the opposite chirality and therefore it has no well-defined eigenvalue problem. We cannot identify det $i\not{D} \neq \prod_i i\lambda_i$ where λ_i are the eigenvalues of the Dirac operator.

There are two solutions for the determinant problem, both redefinitions of the usual Dirac operator. We will discuss one now and the other one later.
Alvarez-Gaumé and Ginsparg proposed to define a new operator:

Definition

$$\hat{D} = \gamma^\mu \left(\partial_\mu + A_\mu P_+\right) = \partial_- + D_+ \ , \tag{10}$$

with $\partial_\pm = \not{\partial} P_\pm$.

Note that $\hat{D}$ is again a endomorphism and hence has a well-defined eigenvalue problem, so we can set the determinant to equal the partition function. But the gauge potential only couples to positive chirality spinors and since ∂_- does not have any non-trivial zero modes, $i\hat{D}$ has only positive chirality zero modes which are the same as the Weyl operator's. This is also necessary, since (**??**) leads to an effective perturbation theory and we want $\hat{D}$ to describe the same physical theory.
We use the following Euclidean convention

$$\begin{aligned} \gamma^{\mu\dagger} &= \gamma^\mu, \\ \gamma^5 &= i^n \prod_{\mu=1}^{2n} \gamma^\mu, \\ g^{\mu\nu} &= \delta^{\mu\nu}. \end{aligned} \tag{11}$$

Furthermore, we require $|\det\ (i\hat{D})|$ to be gauge invariant. The reason for this is in principle the same as above: the real part of the fermion effective action is always gauge invariant. But this is also satisfied for the new operator:

Lemma 1.1. *The determinant of $i\hat{D}$ is gauge invariant and proportional to the Dirac determinant,*

$$det\ (i\hat{D}) = e^{i\phi[A]} \cdot \sqrt{det(i\not{D})}. \tag{12}$$

2 Fibre bundles and Index

Before moving on to the calculation of the anomaly we have to introduce some notions of differential geometry and functional analysis.

2.1 Fibre bundles

Definition A fibre bundle (E, π, M, F, G) consists of the following elements:

i) A differentiable Manifold E called the **total space**.

ii) A differentiable Manifold M called the **base space**.

iii) A differentiable Manifold F called the **fibre**.

iv) A surjection $\pi : E \to M$ called the projection. The inverse image $\pi^{-1}(p) = F_p \cong F$ is called the fibre at p.

v) A Lie group G called the **structure group**, which acts on F on the left.

vi) A set of open covering $\{U_i\}$ of M with a diffeomorphism $\phi_i : U_i \times F \to \pi^{-1}(U_i)$ such that $\pi \circ \phi_i(p, f) = p$. The map ϕ_i is called the **local trivialization** since ϕ_i^{-1} maps $\pi^{-1}(U_i)$ onto the direct product $U_i \times F$

vii) Elements of G relate the local trivializations via **transition functions**: $t_{ij} : U_i \cap U_j \to G$.
$$\phi_j(p, f) = \phi(p, t_{ij}(p)f). \tag{13}$$
The transition functions describe the "twisting" of the bundle, i.e. how the fibres are glued together to form the bundle.

Examples of fibre bundles are

i) The **tangent bundle** $TM = \cup_{p \in M} T_pM$ over the base space M with $dim\, M = m$ which looks locally like $\mathbb{R}^m \times U_i$.

ii) The Möbius strip as well as the cylinder are both tangent bundles over S^1 with fibre $F = [-1, 1]$. They both differ only by the choice of local trivializations (or the transition functions), either with twist (Mbius strip) or without twist (cylinder).

We describe this here very shortly, only introducing the concepts absolutely necessary for us. For a (more) detailed treatment of this, see e.g. [2]. Fibre bundles are for special importance for physics because they are the geometrical setup for describing gauge theories.

2.1.1 A physicist introduction to fibre bundles

Consider a gauge theory with gauge group G and some matter field $\psi(x)$ in a k-dimensional representation R of G. At $x \in M$, the field takes values in k-dimensional representation space V_x which is of course isomorph to some "reference" k-dimensional vector space.

Distinguish the following cases:

- $M = \mathbb{R}^m$ let us define $\psi(x)$ globally well defined on M. Fields which lay in the same gauge orbit describe the same physics:

$$\psi'(x) = R\left(g(x)\right)\psi(x) \sim \psi(x) \tag{14}$$

  for some **globally defined** $g(x) \in G$.

- If M is a more general m-dimensional manifold with open covering $\{U_i\}$, we may have different $\psi^{(i)}(x)$ in each neighbourhood U_i, nevertheless only the orbit space of the fields is a non-redundant description.

$$R\left(g(x)\right)^{(i)}\psi^{(i)}(x) \sim \psi^{(i)}(x)(x) \tag{15}$$

  To glue this together to a global description of the theory on M, we require that for $x \in U_i \cap U_j$ there exists a gauge transformation $t_{ij}(x)$ such that

$$\boxed{\psi^{(i)}(x) = t_{ij}\psi^{(j)}(x).} \tag{16}$$

  They should describe the same physics and therefore consistency requires:

$$t_{ii} = 1 \quad , t_{ij} = t_{ji}^{-1}, \tag{17}$$

  and under gauge transformations

$$t_{ij} \to g^{(i)}(x)t_{ij}\left(g^{(j)}(x)\right)^{-1} \tag{18}$$

The structure we just described here is a **vector bundle**. The fibre is given by the representation space V_x and the fields describe (local) **sections** of the bundle, i.e. maps from the base space to the total space,

$$\psi : M \to E. \tag{19}$$

The t_{ij} are called transition functions and describe the topological non-triviality of the bundle. If there exists a choice of gauge transformations such that $t_{ij} = 1 \forall i, j$, the

setup is called topological trivial and the bundle is **globally** just the direct product $M \times V_R$.
We mention here that our physical theory is defined on a slightly different structure, namely the **gauge bundle**, which is the **associated vector bundle** to a principal G-bundle. For details see again [2]. In $\mathbb{R}^m$ or $\mathbb{R}^{1,m-1}$ the gauge bundle is always trivial. However, in presence of sources or defects one might have non-trivial topologies.

2.1.2 Connections

We need one more concept before we can move on. You already know the example of a connection on the tangent bundle of a manifold. There it defines the **directional derivative** of a vector field (a section s) along another vector field. The generalisation to arbitrary vector bundles is the directional derivative $\bigtriangledown_v(s)$ of a section $s : M \to E$ along a vector field $v \in TM$ such that $\bigtriangledown_v(s)$ is again a section of E. As in the first case, this induces **locally** a **connection 1-form**:

$$\bigtriangledown_\mu s = e_{iA} \left(\partial_\mu z_i^A + A_{i\,\mu B}^A z^B \right) \tag{20}$$

where $s = z_i^A e_{i\,A}$ is a local section in chart U_i. The greek indices denote components of TM whereas A, B label the bundle frame. These $A_{i\,\mu B}^A$ are therefore "generalised Christoffel symbols". But the requirement that $\bigtriangledown_v(s)$ is again a section fixes the transformation of $A_{i\,\mu B}^A$ under change of charts:

$$\boxed{A_{i\,\mu} = t_{ij} A_{j\,\mu} t_{ij}^{-1} + t_{ij} \partial_\mu t_{ij}^{-1}} \tag{21}$$

But this is exactly the gauge transformation of the Yang-Mills gauge potential. If we now utilize the fact that such local connection 1-forms induce also connections on a fibre bundle, we see that fibre bundles indeed are the natural geometric setup for a gauge theory. To be precise, our usual fermion fields live in the so called **twisted spinor bundle**, which is nothing else but $S_\pm \times E$ with $S_\pm$ denoting the spin bundle and E the vector bundle. Since we are not interested further in gravitational anomalies we ignore the spin bundle furthermore.

2.2 Index

Now we introduce the **index** of a differential operator.

Definition The index of an Operator $T : E \to F$ is

$$\text{index } T = \text{dim ker } T - \text{dim coker } T \tag{22}$$

and with coker $T := F/\text{range } T \cong (\text{range } T)^{\perp} = \ker T^{\dagger}$

$$\text{index } t = \dim \ker T - \dim \ker T^{\dagger} \tag{23}$$

An example is given by the displacement operator on the space L^2 with complete orthonormal basis $\{\phi_n\}$:

$$T\phi_n = \begin{cases} \phi_{n-1} & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ \phi_n & \text{if } n < 0. \end{cases} \tag{24}$$

Then T is surjective, hence $\ker T^{\dagger} = (\text{range } T)^{\perp} = \emptyset$. The $\ker T$ is 1-dimensional, therefore the index is

$$\text{index } T = \dim \ker T - \dim \ker T^{\dagger} = 1 \tag{25}$$

Of special importance is the following theorem:

Theorem 2.1. *An elliptic differential operator over a compact boundaryless manifold has a well-defined index.*

This is because the Dirac operator is an elliptic differential operator.

Last we need the Atiyah-Singer index theorem which relates the **analytical index** of a differential operator with a topological invariant of the associated fibre bundle (from the mathematical point of view, these operators are maps of sections $D : \Gamma(M, E) \to \Gamma(M, F)$. We formulate the AS index theorem in the way we need it.

$$\text{index } D = \int_M \text{ch } (E) \tag{26}$$

3 Calculation of the Anomaly

Now we want to relate also the non-Abelian anomaly to an index of the Dirac operator. However, to do this in a precise mathematical derivation, we would need the family index theorem and K-theory, subjects unfamiliar to most physicists. Thus, we proceed with the derivation of [3] which is a more physical approach to this, avoiding these mathematical techniques. We define a new Dirac operator and relate the winding number of its phase with the anomaly. This winding number will be identified with the zero modes of an Dirac operator in 2n+2 as well as the first chern class of a line Bundle $\mathcal{L}$ over the gauge orbit space.

The relation between the 2n-dimensional non-abelian anomaly and a (2n + 2)-dimensional index theorem involves a specific two-parameter family of 2n-dimensional gauge field configurations. Consider Euclidian space $\mathbb{R}^{2n}$ compactified to a sphere S^{2n} and enlarge the domain of the gauge elements. The gauge group we take to be a simply connected semi-simple compact Lie group G. The gauge elements should now depend on an additional parameter,

$$g = g(x, \theta) \tag{27}$$

with $x \in S^{2n}$ and $\theta \in S^1$ with boundary condition

$$g(0, x_0) = g(2\pi, x_0) = 1. \tag{28}$$

Thus the new space of gauge potentials is just the smash product of these two spheres and is topological equivalent to S^{2n+1}.

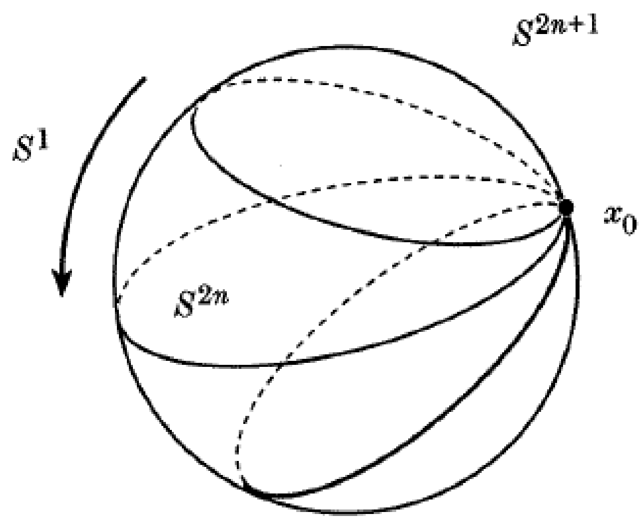

Figure 1: Visualizing the smash product, $S^1 \times S^{2n} \simeq S^{2n+1}$

We consider now a **one-parameter family of gauge potential transformations** on this new space.

$$A^g(x, \theta) = g^{-1}(x, \theta)[A(x) + d]g(x, \theta), \tag{29}$$

Due to the boundary conditions (28), this describes a circle in the affine space of gauge connections Sp $\mathscr{A}$. The fermion determinant $\det(i\hat{D}(A^\theta))$ can now be considered as a complex function of the gauge fields on S^1. We choose w.l.o.g. the reference gauge field to give a non-zero determinant and with lemma 1.1 we may write the Weyl determinant as

$$\begin{aligned} e^{-\Gamma[A^\theta]} &= \det\left(i\hat{D}(A^\theta)\right) \\ &= \sqrt{\det\left(i\not{D}(A)\right)} e^{i\omega(A,\theta)} \ , \end{aligned} \tag{30}$$

This does not vanish on $S^1 \subset$ Sp $\mathscr{A}$. The phase factor however may receive an anomalous gauge variation and defines an covering map

A gauge variation of the partition function only in θ gives us (by utilizing lemma 9:

$$\begin{aligned} -\delta_\alpha \Gamma[A^\theta] &= -i \int_{S^1} d\theta \, \alpha \mathscr{A} \\ &= i d\theta \frac{d}{d\theta} w(\theta, A) \end{aligned} \tag{31}$$

But this is just the local version of the winding number of the phase factor, which defines a covering map of $U(1)$:

$$f(\theta) := e^{i\omega(A,\theta)} : S^1 \to U(1) \simeq S^1, \tag{32}$$

which, as every covering map, is characterized by its winding number:

$$\begin{aligned} m &= \frac{1}{2\pi i} \int_0^{2\pi} d\theta \ f^{-1}(\theta) \frac{d}{d\theta} f(\theta) \\ &= \frac{1}{2\pi} \int_0^{2\pi} d\theta \, \frac{d}{d\theta} \left(\log f(\theta)\right). \end{aligned} \tag{33}$$

We will enlarge the one-parameter gauge family with a homotopy to construct a disk D^2 in Sp $\mathscr{A}$ by defining

$$A^{(t,\theta)} = t A^\theta \qquad t \in [0,1]. \tag{34}$$

Note here that this homotopy is **no** gauge transformation and thus the previous argument for $\det(i\hat{D}(A^{\theta,t})) \neq 0$ must be set aside for $t \neq 1$. Accordingly $i\hat{D}(A^{\theta,t})$ may contain zero-modes on the interior of the Disk. We do this for two reasons:

i) The projection of such a disk with interior zeroes onto the gauge orbit space gives exactly the non-contractible 2-spheres which characterize the complex line bundles over Sp $\mathscr{A}/G$ we mentioned at the beginning. Since the transition functions on such a line bundle are just elements of $U(1)$ we can therefore conclude again that the anomaly is given by the first chern class of the line bundle L:

$$-iG(\alpha, A) = \int_{S^1} c_1(L) \tag{35}$$

ii) The interior zero modes of $\det(i\hat{D}(A^{\theta,t}))$ on the disk equal the winding number on the boundary but the interior zeros can be related to an index of a specific 2n+2 dimensional Dirac operator, therefore we will now calculate the anomaly by relating this winding number to the index of an appropriate 2n+2 dimensional Dirac operator

- We begin with discussing the interior of the gauge disk D^2.

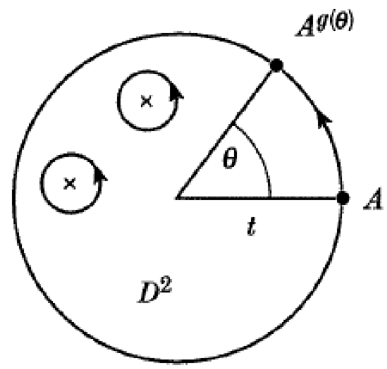

Figure 2: The disk D^2 with the interior zeros of $\det(i\hat{D}(A^{\theta,t}))$. The crosses mark the poles of the determinant phase

As said above, the determinant may receive zero-modes, but these occur at isolated points. By virtue of residue theorem we can now calculate

$$
\begin{aligned}
m &= \frac{1}{2\pi}\int_0^{2\pi} d\theta\ \frac{\partial}{\partial\theta}\omega(A,\theta) \\
&= \frac{1}{2\pi i}\oint_{S^1} \frac{\partial}{\partial\theta} i\omega(A,\theta) \\
&= \sum_{\text{zero modes}} \text{Res}_{\text{zero mode}}\left[\frac{\partial}{\partial\theta} i\omega(A,\theta)\right] \\
&= \sum_i \frac{1}{2\pi}\oint_{\partial U_i} \frac{\partial}{\partial\theta}\omega(A,\theta) \\
&= \sum_i m_i\ ,
\end{aligned}
\tag{36}
$$

where i indicates the zero modes in the interior of D^2 and U_i is an open subset of the respective zero mode.
In [4], it is shown by an 'adiabatic approximation' [1] that these interior zeros of the determinant correspond exactly to the zero-modes of a specific Dirac operator $i\not{D}_{2n+2}$ in $(2n+2)$ dimensions, whereas the local winding numbers turn out to be $m_i = \pm 1$, thus representing the chirality of these zero modes and therefore we can identify the total winding number of the phase (36) with the sum over the chirality zero-modes, which is just the index of the $(2n+2)$-dimensional Weyl operator:

$$
m = n_+ - n_- = \text{index } i\not{D}_{2n+2}\ . \tag{37}
$$

Therefore we can relate the local winding of the phase to the index density:

$$
m = \frac{1}{2\pi}\int_{S^1} \delta\omega(A,\theta) = \text{index } iD_{+,\ 2n+2} =: \int_{S^1} \text{Index } iD_{+,\ 2n+2} \tag{38}
$$

Thus, by calculating the index of this Weyl operator we can determine the anomaly.

- For the explicit construction of this Weyl operator see [1, p.441]. We sketch the procedure here: The base of our bundle is

$$
M = S^2 \times S^{2n}. \tag{39}
$$

[1] The idea is to deform the operator in such a way that the contribution of the 2 extra dimension in the Operator get small when the size of our extra dimensions get large compared to the 2n dimensional space

where S^2 is a sphere in Sp $\mathscr{A}/G$ and S^{2n} is our compactified spacetime [2].

Then with the AS index theorem, we get

$$\boxed{\begin{aligned} \text{anomaly } &= m = \text{index } i\not{D}_{2n+2} \\ &= \int_{S^2\times S^{2n}} \text{ch } (E), \end{aligned}} \tag{40}$$

where E denotes our gauge bundle over the base space $S^2 \times S^{2n}$.

A Possible subtletie in regard to our application of this proof to non-linear sigma models is the assumption of a compactified base space.

Finally, we want to point out the topological aspects of the anomaly which this derivation exposed. With (**??**) it is easy to see that we need a non-vanishing winding number for an anomaly. This is because the variation of the phase around the boundary of the gauge potential disc D^2 prevents the effective action from being a single-valued functional on the physical space of gauge potentials, the gauge orbit space $Sp\mathscr{A}/\mathcal{G}$. Alternatively, one can show that the winding number is proportional to the integer characterizing the map $g(\theta, x)$, thus an element of the homotopy group $\Pi_{2n+1}(G)$. This is done by calculating the index in another way as above, see [1, p.444] or [4]. Consequently, for a non-vanishing winding number we need a non-trivial class for a gauge transformation on the disk. Moreover, it is easy to see that the fundamental group of the space of all pointed gauge transformations,

$$\mathcal{G} := \{g(x) : S^{2n} \to G | g(x_0) = 1\}, \tag{41}$$

with x_0 corresponding to ∞ in the uncompactified theory, agrees with the $2n+1$ homotopy group of G:

$$\pi_{2n+1}(G) \sim \pi_1(\mathcal{G}) \ . \tag{42}$$

But the topology of the gauge orbit space $Sp\mathscr{A}/\mathcal{G}$ is determined only by that of $\mathcal{G}$, since $Sp\mathscr{A}$ is as an affine space topologically trivial. Now look again on the disc D^2 in this affine space under the aspect of topology. As the boundary of this disc is gauge equivalent, it will be identified when projecting the disc onto the physical space, hence giving us a 2-sphere in $Sp\mathscr{A}/\mathcal{G}$. This projection is part of the quantization process, as we need the gauge orbit space for a well-defined PI. But as mentioned

[2] We have to consider the sphere in the gauge orbit space, otherwise our 2n+2 dimensional Dirac operator would not have a well-defined index.

before the topology of the gauge orbit space is determined by $\mathcal{G}$ and so this 2-sphere will be contractible if and only if the loop in $\mathcal{G}$ determined by the boundary of D^2 is contractible. Therefore, we get

$$\pi_2(Sp\mathscr{A}/\mathcal{G}) \sim \pi_1(\mathcal{G}) \sim \pi_{2n+1}(G) \tag{43}$$

Hence, such noncontractible 2-spheres in the gauge orbit space correspond to a non-vanishing winding number m and thus to an anomaly.

4 Non-linear sigma model

Now all of this may seem not useful for non-linear sigma models, since a priori we have no gauge symmetry in these models. But we will show now how we can utilize this proof nevertheless.
A non-linear sigma model is a field theory in which the (bosonic) dynamical variables φ take their values in a Riemannian manifold M (typical a complex Kähler manifold), i.e. $\varphi \in \mathscr{G} = \{\varphi : X \to M\}$. The manifold M is called the **target space**, X is the d-dimensional base manifold (WS). The bosonic action is given by

$$S_b = \int_X \langle d\varphi, d\varphi \rangle = \int_X g_{ab}(\varphi(x))\partial_\mu \varphi^a \partial^\mu \varphi^b d^d x \,. \tag{44}$$

We could couple spinors to the WS, for example via supersymmetry $\delta\varphi = \bar{\epsilon}\psi$, where ϵ is a spinor on the WS. Then $\psi(x)$ is a section of the w.l.o.g. positive spinor bundle as well as the pullback bundle of the tangent space of M, namely $\varphi^*(TM)$. Therefore our total field configuration is specified by $\varphi \in \mathscr{G}$ and $\psi, \bar{\psi} \in \mathscr{H}^{\pm}$, where

$$\mathscr{H}^{\pm} = \left\{\text{sections of } S^{\pm} \otimes \varphi^*(TM)\right\} \tag{45}$$

We will call this tensor bundle $E^{\pm}_{\varphi} = S^{\pm} \otimes \varphi^*(TM)$. This suggests the following fermionic action:

$$S_f = \int_X \langle \bar{\psi}, \not{D}_\varphi \psi \rangle = \int_X h_{ij}(\varphi)\bar{\psi}^i \left(\delta^j_k \not{\partial} + \Theta^j_{ka}(\varphi)\partial\varphi^a\right) \psi^k. \tag{46}$$

The fibre metric is here denoted by h_{ij} and the pulled back local connection 1-form is given by $\Theta^j_{ka}(\varphi)\partial\varphi^a$. For brevity we have dropped the spin connection as well as possible quartic interaction terms in the fermions and will continue to do so. This is because although they alter possible anomalous terms, they cannot remove them and therefore are not of interest in the question of their existence.

Thus there is no problem in defining a classical theory. In the process of quantising, however there could be the same problems we encountered before for the fermion determinant (the generating functional) of our gauge theories: the theory needs to be regularised and the fermion determinant is no well-defined object due to chirality change.

One might try to rectify the chirality-flip by considering another Dirac operator (similarly to the one proposed by Alvarez-Gaumé and Ginsparg) $\hat{D}_\varphi = \not{D}_{\varphi_0}^{-1}\not{D}_\varphi$ but this makes no more sense than $\not{D}_\varphi$ since $\not{D}_\varphi : \mathscr{H}_\varphi^\pm \to \mathscr{H}_\varphi^\mp$ while $\not{D}_{\varphi_0}^{-1} : \mathscr{H}_{\varphi_0}^\mp \to \mathscr{H}_{\varphi_0}^\pm$. But if we choose isomorphisms

$$T^\pm(\varphi, \varphi_0) : \mathscr{H}_{\varphi_0}^\pm \to \mathscr{H}_\varphi^\pm, \tag{47}$$

we can take

$$e^{-\Gamma[\varphi]} = \det\left[T^+(\varphi,\varphi_0)\not{D}_{\varphi_0}^{-1}T^-(\varphi_0,\varphi)\not{D}_\varphi\right] \tag{48}$$

We have ignored one important fact: the Hilbert spaces $\mathscr{H}_\varphi^\pm$ for different φ are not naturally isomorphic, i.e. we can only define the isomorphisms in neighbourhood of the reference φ_0. Therefore, we have to cover $\mathscr{G}$ by patches $\{\mathscr{P}_\alpha\}$ with reference configurations φ_α and define the effective action patchwise $e^{-\Gamma[\varphi_\alpha]}$. The different action must be related in the following way,

$$e^{-\Gamma[\varphi_\alpha]} = g_{\alpha\beta}e^{-\Gamma[\varphi_\beta]}, \tag{49}$$

with

$$g_{\alpha\beta} = \det\left[\not{D}_{\varphi_\alpha}^{-1}T_\alpha^+(\varphi_\alpha,\varphi)T_\beta^-(\varphi,\varphi_\beta)\not{D}_{\varphi_\beta}T_\beta^+(\varphi_\beta,\varphi)T_\alpha^+(\varphi,\varphi_\alpha)\right]. \tag{50}$$

These are just the transition functions of a complex line bundle L over $\mathscr{G}$ with $\det\left[T^+(\varphi,\varphi_0)\not{D}_{\varphi_0}^{-1}T^-(\varphi_0,\varphi)\not{D}_\varphi\right]$ interpreted as a section of this bundle. Only if L is trivial we can choose ismorphisms **globally** and the effective action can be regarded as a well-defined functional.

This bundle over $\mathscr{G}$ is also characterized completely by its restriction to two-cells Y (see [2]). By analogy to magnetic monopole theory, we state here that the twist of this bundle (therefore the problem of defining the functional integral as global function) is characterised by the winding number of its transition functions (note that the structure group of such a complex line bundle L is given by $U(1)$):

$$m = \frac{1}{2\pi i}\int_S^1 g_{\mathrm{NS}}^{-1}d(g_{\mathrm{NS}}) = \frac{1}{2\pi i}\int_S^1 d(\log g_{\mathrm{NS}} \tag{51}$$

The S^1 is here the equator of our S^2, the intersection of our two patches which we choose as the two hemispheres. But this is precisely the first chern class or chern character of the bundle:

$$m = \int_Y \text{ch } L. \tag{52}$$

Therefore we see that again the twist of a complex line bundle over configuration space gives us the obstruction to define a quantum theory. Plugin our transition functions 50 in, we get after some algebra

$$m = \frac{1}{2\pi i} \int_{S^1} \text{Tr}_{\text{f}} \left(\left[dT_N^{(+)} \left(T_N^{(+)} \right)^{-1} - dT_S^{(+)} \left(T_S^{(+)} \right)^{-1} \right] - \left[dT_N^{(-)} \left(T_N^{(-)} \right)^{-1} - dT_S^{(-)} \left(T_S^{(-)} \right)^{-1} \right] \right). \tag{53}$$

With a specific choice of regulator, we see that the trace becomes finite dimensional and the forms $dT_{N,S}^{(\pm)} \left(T_{N,S}^{(\pm)} \right)^{-1}$ become the connections of the finite-dimensional subbundles $\mathscr{H}_{\text{low}}^{\pm}$, thus

$$m = \int_Y \text{ch}_1 \mathscr{H}_{\text{low}}^{+} - \text{ch}_1 \mathscr{H}_{\text{low}}^{-}. \tag{54}$$

To relate this now to the topology of our target space, we make the following identifications to utilize our calculation via index theorem:

- The **gauge potential** is induced by the pullback of the connection on TM, $\varphi^*(\Theta)$. The local connection 1-form is given by $\Theta_{ka}^{j}(\varphi)\partial^a \varphi$.
- The **gauge transformations** are given by the transition functions of the pullback bundle
- The **gauge orbit space** Sp $\mathscr{A}/G$ is identified with the space of all maps, $\mathscr{G} = \{\varphi : X \to M\}$. This space is as well characterized completely by its restriction to two-cells.
- The **anomaly** is again the non-triviality of the complex line bundle over configuration space $\mathscr{G}$, to be precise its first chern class.

Via the proof we gave we can now relate:

$$\begin{aligned} \text{anomaly} &= c_1(L) \\ &= \text{index } D_{2n+2} = \int_{Y \times X} \text{ch}_{n+1} \left(\varphi^*(B) \right) \end{aligned} \tag{55}$$

With the naturality of the Chern class, $\text{ch}_{n+1}(\varphi^*(TM)) = \varphi^* \text{ch}_{n+1}(TM)$, we see that the theory is anomaly free iff the $n+1$-th chern character of the tangent space of M is vanishing.

For $n = 1$, we can exploit this further:

$$\mathrm{ch}_2(TM) = \frac{1}{2} c_1(X)\, c_1(M) \tag{56}$$

Therefore a consistent 2d model requires either generic base spaces and vanishing first chern class of the target manifold or we have to restrict to trivial base spaces. This gives us the famous result that the non-linear sigma model is anomaly free if and only if the first chern class of the target manifold vanishes, i.e. if the target manifold is a Calabi-Yau manifold !

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