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Open Strings and Extended Mirror Symmetry

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Abstract

The classical mirror theorems relate the Gromov-Witten theory of a Calabi-Yau manifold at genus 0 to the variation of Hodge structure of an associated mirror manifold. I review recent progress in extending these closed string results to the open string sector. Specifically, the open Gromov-Witten theory of a particular Lagrangian submanifold of the quintic hypersurface is related to the Abel-Jacobi map for a particular object in the derived category of coherent sheaves of the mirror quintic. I explain the relevance for the homological mirror symmetry program.

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1 Introduction

Mirror symmetry is one of the best-developed examples of the refreshed interaction between theoretical physics and certain areas of pure mathematics that has taken place over the past decades. From the string theory point of view, it is natural to expect that there exists a far reaching generalization of classical concepts of geometry that will incorporate the idea that the fundamental building blocks are extended objects with some degree of non-locality. There is at present no precise idea about the form such a theory should ultimately take. One has to be content that most results such as derived from mirror symmetry take the form of unusual looking statements about classical geometric objects. Mirror symmetry, in its simplest form, relates certain properties of two Calabi-Yau manifolds[1] X and Y of different topology. The

[1] For us here, a Calabi-Yau manifold is a Kähler manifold with trivial canonical bundle and whose Hodge numbers satisfy $h^{k,0} = 0$ for $0 < k < n$, where n denotes the complex

basic necessary condition for X and Y to form a *mirror pair* is that the Hodge numbers satisfy

$$h^{p,q}(Y) = h^{n-p,q}(X) \qquad \text{for } 0 \leq p, q \leq n \tag{1.1}$$

but this relation has to be filled with a lot more life before we call it mirror symmetry.

The tightest possible connection between mirror manifolds X and Y is the one offered by string theory: The physical theory associated with the string background defined by X is isomorphic to the physical theory associated with the string background defined by Y. The problem with this statement is that at present there is no complete definition of string theory, so this is best viewed either as a fond wish or as a constraint on a future definition. The statement of perturbative string theory, which is well-understood, is that the two-dimensional so-called sigma-models with X and Y as target spaces are isomorphic as $\mathcal{N} = 2$ superconformal field theories (SCFTs). This can also be expressed as saying that X and Y are identical when viewed in "string geometry". Although there are precise mathematical definitions of such superconformal field theories, the process of attaching an SCFT to a Calabi-Yau manifold is fairly complicated and mathematically out of reach at the moment.

Mirror symmetry celebrated its first success after a computation by Candelas, de la Ossa, Green and Parkes [1]. These authors used the above conjectural equivalence of $\mathcal{N} = 2$ SCFT (as well as some other information about the string theory interpretation of the Calabi-Yau background) to make a prediction about the number of rational curves on a generic quintic threefold. That prediction was expressed in terms of the periods of an assumed mirror manifold (called the mirror quintic) previously constructed by Greene and Plesser [2]. Thus mirror symmetry entered the world of mathematics as the challenge to understand and verify the predictions of Candelas et al.

The theory relevant to the computations on the mirror manifold ("B-model") was understood to be related to classical Hodge theory [3]. Based on the development of Gromov-Witten theory, the enumerative predictions

dimension of the manifold. We are interested in the case $n = 3$.

("A-model") of Candelas et al. were verified in subsequent years, culminating in the proof of the now classical "mirror theorems" [4–6], which amount to

$$\text{Gromov-Witten theory of } X \xleftrightarrow{\text{solved by}} \text{Hodge theory of } Y \tag{1.2}$$

We refer to [7, 8] for textbook treatments of these developments. What (1.2) does not provide, however, is an intrinsic "explanation" of why these theorems would hold, in other words, answer the question "What is mirror symmetry?"

There are two proposals that are in a sense midway between the classical mirror theorems and the full equivalence of the $\mathcal{N} = 2$ SCFTs. The first of those is known as the "homological mirror symmetry" program (or conjecture) of Kontsevich [9]. The basic idea is a kind of "categorification" of the enumerative predictions. To any Calabi-Yau manifold, one can attach, if not a full SCFT, at least two categories. The B-model category, $D^b(Y)$ is the bounded derived category of coherent sheaves, while the A-model category, $\mathrm{Fuk}(X)$, is a certain derived version of a symplectic category constructed by Fukaya (et al.) using (Lagrangian intersection) Floer theory. Homological mirror symmetry amounts to the statement that if X and Y are a mirror pair, the associated categories are equivalent up to interchange of A and B-model,

$$\mathrm{Fuk}(X) \cong D^b(Y) \tag{1.3}$$

(More precisely, the full symmetry includes the statement $D^b(X) \cong \mathrm{Fuk}(Y)$. One is normally interested only in (1.3), especially when studying manifolds that are not Calabi-Yau.)

The second approach to understanding mirror symmetry from the mathematical point of view was proposed by Strominger, Yau and Zaslow (SYZ) [10]. Roughly speaking, X and Y are a mirror pair if they can both be written as fibrations by (generically) special Lagrangian tori over a common base, in such a way that the tori for X are (generically) dual to the tori for Y.

$$\begin{array}{ccccc} T^n & \longrightarrow & X & \searrow & \\ & & & & B \\ (T^n)^\vee & \longrightarrow & Y & \nearrow & \end{array} \tag{1.4}$$

This SYZ conjecture thus gives a purely geometric way of understanding the relation between the two manifolds in a mirror pair.

The present mathematical challenges of mirror symmetry include making the statements (1.3), (1.4) more precise, proving them, and finding relations between either of them and (1.2). All these conjectures are strongly backed by physical ideas from string theory, centrally involving D-branes. D-branes are boundary conditions where open strings can end and are required when the $\mathcal{N}=2$ SCFTs are considered on two-dimensional spaces with boundary. The objects in the A- and B-model categories appearing in (1.3) are models of (topological) D-branes, and the equivalence of $\mathcal{N}=2$ SCFTs would imply the equivalence of D-brane categories. In a similar vein, the special Lagrangian tori appearing in (1.4) are a special type of D-branes, referred to as "BPS", and the original argument of [10] deduced the geometric picture from the equivalence of physical theories and the properties of BPS branes therein.

The purpose of the present contribution is to review recent progress in extending classical mirror symmetry (1.2) to the open string sector, in a way that takes it closer to the homological mirror symmetry conjecture (1.3). The underlying ideas have been worked out for the particular case of the real quintic and its mirror (definitions below) in the papers [11–13], and efforts are under way to adapt the technology to other situations.[2]

Remark. We concentrate in this review on the progress made on the compact Calabi-Yau quintic threefold. Tremendous progress has been made on open Gromov-Witten theory, and mirror symmetry, for non-compact (local) Calabi-Yau manifolds. The starting point of those developments is [14], see [15] for a review. Relations to the present work are suggestive, but have not been worked out in detail yet.

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[2] During the Banff meeting, it was pointed out by G. Almkvist that results similar to those of [11] should hold for many other one-parameter Calabi-Yau manifolds. An extension of the corresponding Picard-Fuchs equations by an inhomogeneous term $\sim \sqrt{z}$ leads to an integral open string expansion similar to the quintic. It would be interesting to identify the correct normalization of these computations.

2 A-model

Let X be a generic quintic Calabi-Yau threefold

$$X = \{V(x_1, \ldots, x_5) = 0\} \subset \mathbb{P}^4 \tag{2.1}$$

where V is a homogeneous polynomial of degree 5 in 5 variables. In the A-model, one is interested in X as a symplectic manifold, so the precise choice of V, which corresponds to the choice of complex structure, does not matter. Namely, the *Fukaya category* $\mathrm{Fuk}(X)$ does not depend on V (at least as long as it is non-singular). It is nevertheless useful to remember V, for the following reason. A systematic way to define Lagrangian submanifolds of Calabi-Yau manifolds such as X is as the fixed point locus of an anti-holomorphic involution with respect to some complex structure. The generic quintic X in (2.1) admits an anti-holomorphic involution, given simply by complex conjugation on the homogeneous coordinates in $\mathbb{P}^4$, if V is real, and we can then consider the real locus

$$L = \{x_i = \bar{x}_i\} \subset X \tag{2.2}$$

This definition has the problem that while the A-model should not depend on V, the real locus L does. Both the topological type of L as well as even its homology class in $H_3(X;\mathbb{Z})$ change along certain singular loci in the moduli space of quintics. This problem does not affect the considerations in the present section, but for the application to mirror symmetry, it is necessary to fix a particular choice of V.

We will call by "the real quintic" the Lagrangian submanifold $L \subset X$ defined as the real locus of X in the complex structure at the Fermat point,

$$V_{\text{Fermat}} = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 \tag{2.3}$$

Topologically, the real quintic is isomorphic to $\mathbb{RP}^3$. This follows from the fact that the equation $V = 0$ over the reals has a unique solution for x_5 (say) in terms of $x_1, \ldots, x_4$.

In this section, we review the open Gromov-Witten theory of the pair (X, L), and describe its solution in terms of a certain differential equation that is an inhomogeneous version (or extension) of the ordinary hypergeometric differential equation governing closed string mirror symmetry for the quintic.

2.1 Open Gromov-Witten invariants

Classical Gromov-Witten theory is concerned with studying intersection theory on the moduli space of stable maps

$$\overline{\mathcal{M}}_{g,n}(X,\beta) \tag{2.4}$$

Here, the data are the genus g of the domain curve (Riemann surface) with some number n of marked points, a target manifold X, and a (non-zero) cohomology class $\beta \in H_2(X;\mathbb{Z})$. The target X is either as taken as a (projective) algebraic variety or a symplectic manifold. Cohomology classes on $\overline{\mathcal{M}}_{g,n}(X,\beta)$ are obtained by pulling back classes from X via evaluation at the marked points, or from the moduli space of stable curves via the projection $\overline{\mathcal{M}}_{g,n}(X,\beta) \to \overline{\mathcal{M}}_{g,n}$ (being propitious with that last statement).

One of the main motivations to develop Gromov-Witten theory was to verify the physicists' enumerative predictions about rational curves on the quintic. Calabi-Yau threefolds are special from the point of view of Gromov-Witten theory as the (virtual) dimension of $\overline{\mathcal{M}}_{g,0}(X,\beta)$ vanishes for any g and β. Since $H_2(X,\mathbb{Z}) \cong \mathbb{Z}$, we can encode the target class β in an integer degree d. The *Gromov-Witten invariants* are defined by the integral

$$\tilde{N}_d^{(g)} = \int_{[\overline{\mathcal{M}}_{g,0}(X,d)]^{\mathrm{vir}}} 1 \tag{2.5}$$

that formally computes the degree of the "virtual fundamental class". A nice feature of $g = 0$, any d, is that the Gromov-Witten theory of the quintic can be obtained from the Gromov-Witten theory of $\mathbb{P}^4$ by the "Euler class formula",

$$\tilde{N}_d \equiv \tilde{N}_d^{(0)} = \int_{\overline{\mathcal{M}}_{0,0}(\mathbb{P}^4,d)} \mathbf{e}(\mathcal{E}_d) \tag{2.6}$$

where

$$\mathcal{E}_d = \pi_* f^* \mathcal{O}_{\mathbb{P}^4}(5) \tag{2.7}$$

is the obstruction bundle whose fiber at $f \in \overline{\mathcal{M}}_{0,0}(\mathbb{P}^4,d)$ is the space of global section $H^0(\mathbb{P}^1, f^*\mathcal{O}(5))$ of the pull-back of the bundle of quintics on $\mathbb{P}^4$. The $\tilde{N}_d$ are in general rational numbers, but the Aspinwall-Morrison multi-cover formula,

$$\tilde{N}_d = \sum_{k|d} \frac{1}{k^3} N_{d/k} \tag{2.8}$$

allows the definition of “enumerative” invariants N_d, whose integrality was the main early reason for confidence in the Candelas et al. computation, but has been established rigorously only recently [16, 17].

It is suggested by the physics of D-branes or by applications to homological mirror symmetry that one should try to extend Gromov-Witten theory to the situation when the domain curve, Σ, is a Riemann surface with non-empty boundary, $\partial\Sigma \neq \emptyset$. The prototypical example is the unit disk, or the upper half-plane in $\mathbb{C}$. One should impose Dirichlet boundary conditions that require the boundary of Σ to map to a Lagrangian submanifold L of target space X (as a symplectic manifold). One then speaks of the Gromov-Witten theory of the pair (X, L).

Remark. The physics of orientifolds further suggests the definition of Gromov-Witten theory when Σ is not orientable (prototypical example being the real projective space $\mathbb{RP}^2$). One then requires the data of an anti-symplectic involution of the target space and the maps are equivariant maps from the orientation double cover of Σ to X. This has been studied in non-compact situations, *e.g.*, in [18–20].

The problems with defining Gromov-Witten theory when the domain curve has non-empty boundary or is non-orientable is that moduli spaces tend to be real manifolds. This means that one has to keep the symplectic side in the definitions, which is technically slightly more demanding. But more fundamentally, it results in the appearance of special (singular) loci in the moduli spaces in *real codimension one*. One needs to find suitable boundary conditions at these boundaries in moduli space in order to define an intersection theory. A general attitude towards this problem is that the physicists’ concept of a D-brane includes not only the boundary conditions on the maps, but also boundary conditions at these boundaries in moduli space. However, precise prescriptions have been given only in a limited number of cases so far, such as [21, 22].

Solomon [22] has worked out a definition of open Gromov-Witten invariants when the worldsheet is the disk, and the target space data is a pair (X, L), where L is a Lagrangian submanifold that is the fixed point set of an anti-symplectic involution.[3] The essential idea is to eliminate (some of) the boundaries of moduli space by replacing any bubbling disk by its image under the anti-symplectic involution.

It is shown in [12] that for the real quintic pair (X, L) described above, and *odd degree*, $d \in 2\mathbb{Z}+1$,[4] the open Gromov-Witten invariants of [22] can moreover be computed from the open Gromov-Witten theory of the pair $(\mathbb{CP}^4, \mathbb{RP}^4)$, in a way similar to (2.6). The involution (complex conjugation on $\mathbb{CP}^4$) induces an involution on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^4, d)$. Over the real locus, $\overline{\mathcal{M}}^{\mathbb{R}}_{0,0}(\mathbb{P}^4, d)$, the involution induces a real structure on $\mathcal{E}_d$, and one has the Euler class formula,

$$\tilde{n}_d = \int_{\overline{\mathcal{M}}^{\mathbb{R}}_{0,0}(\mathbb{P}^4)} \mathbf{e}(\mathcal{E}_d^{\mathbb{R}}) \,. \tag{2.9}$$

The difficult parts in the proof and evaluation of this formula arise from subtle orientation issues, see [12] for details. Finally, we note that the analogue of (2.8) is

$$\tilde{n}_d = \sum_{k|d} \frac{1}{k^2} n_{d/k} \tag{2.10}$$

which is essentially the multi-cover formula for the disk first predicted by Ooguri and Vafa [23]. Given the results for the $\tilde{n}_d$ below, and the proof of integrality of the closed string invariants N_d [17], it should be possible to prove integrality of the n_d as well.

2.2 Schubert computation in degree 1

As an elementary application of the definition (2.9), let us compute the simplest case, $n_1 = \tilde{n}_1$. First we review the complex case, which can be done by Schubert calculus [24].

The space of lines in $\mathbb{P}^4$ is isomorphic to the space of 2-dimensional linear subspaces of $\mathbb{C}^5$, *i.e.*, the complex Grassmannian $G(2,5)$. This Grassmannian comes equipped with the tautological rank two bundle U, fitting into the sequence

$$U \to \mathbb{C}^5 \to V \tag{2.11}$$

[3] This involution might appear similar to, but should not be confused with, the involution appearing in the above remark on orientifolds.

[4] This degree is defined as the relative cohomology class in $H_2(X, L; \mathbb{Z}) \cong \mathbb{Z}$

where $\mathbb{C}^5$ stands for the trivial rank 5 bundle, and V is the universal quotient bundle. The cohomology ring of $G(2,5)$ is generated in degree 2 and 4 by the Chern classes of U, $c(U) = (1+a)(1+b) = 1 + u_1 + u_2$, where $a + b = u_1$, $a \cdot b = u_2$. The relations follow from (2.11), $c(U) \cdot c(V) = 1$. As is well-known, those relations can be integrated to a potential,

$$V(u_1, u_2) = \frac{a^6}{6} + \frac{b^6}{6} = \frac{u_1^6}{6} - u_2 u_1^4 + \frac{3}{2} u_2^2 u_1^2 - \frac{1}{3} u_2^3 \tag{2.12}$$

Namely, the intersection ring of $G(2,5)$ is $\mathbb{C}[u_1, u_2]/(\partial_1 V, \partial_2 V)$. A quintic on $\mathbb{P}^4$ induces a section of the bundle $\mathcal{E}_1 = \mathrm{Symm}^5(U))$ over $G(2,5)$. We have $c(\mathcal{E}_1) = \prod_{n=0}^{5}(1 + na + (5-n)b)$, from which

$$\begin{aligned} c_6(\mathcal{E}_1) &= 5a(4a+b)(3a+2b)(2a+3b)(a+4b)5b \\ &= 25u_2(9u_2^3 + 24u_2u_1^4 + 58u_2^2u_1^2) \end{aligned} \tag{2.13}$$

Thus,

$$N_1 = \int_{G(2,5)} c_6(\mathrm{Symm}^5 U^*) = 2875 \tag{2.14}$$

where we used the relations from (2.12), as well as $\int_{G(2,5)} u_2^3 = 1$. This last relation can be derived by expressing the Euler class of $G(2,5)$ in terms of u_1, u_2, and using $\chi(G(2,5)) = 10$.

To repeat the exercise over the reals, we use the Grassmannian of (oriented) real lines, $G^{\mathbb{R}}(2,5)$, which comes with its own "tautological sequence",

$$U_{\mathbb{R}} \to \mathbb{R}^5 \to V_{\mathbb{R}} \tag{2.15}$$

Actually, the sequence (2.15) gives easy access only to cohomology with $\mathbb{Z}_2$ coefficients, see [25]. For our purposes, it is more useful to think of $G^{\mathbb{R}}(2,5)$ as the Hermitian symmetric space

$$G^{\mathbb{R}}(2,5) \cong \frac{SO(5)}{SO(2) \times SO(3)} \tag{2.16}$$

which can also be represented as the quadric

$$G^{\mathbb{R}}(2,5) \cong \{z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 = 0\} \subset \mathbb{CP}^4 \tag{2.17}$$

The cohomology ring of $G^{\mathbb{R}}(2,5)$ has one generator x in degree 2 and one generator y in degree 3, and relations that can be integrated to the potential

$$V_{\mathbb{R}}(x,y) = \frac{1}{4}x^4 + xy^2 \tag{2.18}$$

From the presentation (2.17), we find $\chi(G^{\mathbb{R}}(2,5)) = 4 = \int 2x^3$. We also have $\mathbf{e}(U) = x$, and by setting $b = -a$ in (2.13), and taking a squareroot, we find

$$\mathbf{e}(\mathrm{Symm}^5(U_{\mathbb{R}})) = 15x^3 \tag{2.19}$$

which integrates to

$$n_1 = \int_{G^{\mathbb{R}}(2,5)} \mathbf{e}(\mathrm{Symm}^5(U)) = 30 \tag{2.20}$$

This is the result first obtained by Solomon [22]. The complete formula for all open Gromov-Witten invariants of the disk for the real quintic has been predicted in [11], and was proved in [12].

2.3 Result

Recall that classical mirror symmetry for the quintic is governed by the differential operator

$$\mathcal{L} = \theta^4 - 5z(5\theta+1)(5\theta+2)(5\theta+3)(5\theta+4)\,, \tag{2.21}$$

where $\theta = zd/dz$. Four linearly independent solutions, $\varpi_i(z)$, $i = 0,1,2,3$, of (2.21) can be obtained from the hypergeometric series

$$\mathfrak{F}(z,H) = \sum_{n=0}^{\infty} z^{H+n} \frac{\prod_{r=1}^{5n}(5H+r)}{\prod_{r=1}^{n}(H+r)^5} \tag{2.22}$$

as the coefficients of the Taylor series around $H = 0$

$$\mathfrak{F}(z,H) = \sum_{i=0}^{3} \varpi_i(z) H^i \bmod H^4 \tag{2.23}$$

The prediction of Candelas et al. [1] is that the generating function of genus 0 Gromov-Witten invariants on the quintic,

$$F(t) = \frac{5}{6}t^3 + \sum_{d=1}^{\infty} \tilde{N}_d q^d \tag{2.24}$$

is given by a certain "magical" combination of solutions of the differential equation

$$F(t(z)) = \frac{5}{2}\left(\frac{\varpi_1(z)}{\varpi_0(z)}\frac{\varpi_2(z)}{\varpi_0(z)} - \frac{\varpi_3(z)}{\varpi_0(z)}\right) \tag{2.25}$$

where the variables $q \equiv e^t$ are related to the variable z of (2.21) via the "mirror map",

$$q(z) = \exp\big(\varpi_1(z)/\varpi_0(z)\big)\,. \tag{2.26}$$

Turning to the real quintic, we form the generating function of open Gromov-Witten invariants,

$$T(t) = \sum_{\substack{d=1 \\ d \text{ odd}}}^{\infty} \tilde{n}_d q^{d/2} \tag{2.27}$$

and introduce a "mirror" function by

$$\tau(z) = 2 \sum_{d \text{ odd}} \frac{(5d)!!}{(d!!)^5}\, z^{d/2} \tag{2.28}$$

This function is a solution of the inhomogeneous Picard-Fuchs equation

$$\mathcal{L}\tau(z) = \frac{15}{8}\sqrt{z} \tag{2.29}$$

and is also related to the hypergeometric series (2.22) as

$$\tau(z) = 30\,\mathfrak{F}(z, 1/2) \tag{2.30}$$

Theorem A. *Let $T(t)$ be the generating function of open Gromov-Witten invariants of the pair (X, L), $q(z)$ be the mirror map, and $\tau(z)$ the solution of the inhomogeneous Picard-Fuchs equation* (2.29)*, all as defined above. We have*

$$T(t(z)) = \frac{\tau(z)}{\varpi_0(z)} \tag{2.31}$$

2.4 Proof by localization

Kontsevich has given a formula that computes the Gromov-Witten invariants $\tilde{N}_d$ from (2.6) via localization on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^4, d)$ with respect to the action of the five-dimensional torus

$$\mathbb{T}^5 \subset GL(5, \mathbb{C}) : \mathbb{CP}^4 \to \mathbb{CP}^4 \tag{2.32}$$

The fixed loci of the torus action are encoded combinatorially in certain decorated graphs Γ and can be described geometrically as moduli spaces of curves with marked points (and trivial target space). The integrals over the fixed loci are known explicitly, and allow the evaluation of the Atiyah-Bott equivariant localization formula

$$\tilde{N}_d = \sum_{\Gamma} \frac{1}{|\mathrm{Aut}\Gamma|} \int_{\overline{\mathcal{M}}_\Gamma} \frac{\mathbf{e}(\mathcal{E}_d)}{\mathbf{e}(\mathcal{N}_\Gamma)} \tag{2.33}$$

The essential idea to prove the analogous result for the open Gromov-Witten invariants is to use the presentation of $\tilde{n}_d$ as an integral over the real moduli space, see (2.9). Inspection shows that the involution σ (complex conjugation) commutes with a two-dimensional subtorus of (2.32)

$$\mathbb{T}^2 \subset GL(5,\mathbb{R}) : \mathbb{RP}^4 \to \mathbb{RP}^4 \tag{2.34}$$

The fixed loci of this $\mathbb{T}^2$ action on the real moduli space are identical to the fixed loci of the $\mathbb{T}^5$ action which are invariant under the involution. Thus, we need to evaluate a sum over the subset of graphs in (2.33) satisfying $\sigma(\Gamma) = \Gamma$. Inspection shows that these graphs all have a distinguished "central" edge that is invariant under the involution and is decorated by an odd integer degree, call it e. We cut the graphs through this distinguished edge into two pieces, γ and $\sigma(\gamma)$, each of which represents a disk of degree d. Finally, we recall that if $E = E_\mathbb{R} \otimes \mathbb{C}$ is a complex vector bundle with a real structure, we have $\mathbf{e}(E_\mathbb{R}) = \pm\sqrt{\mathbf{e}(E)}$, up to orientation. This allows the easy evaluation of the (equivariant) Euler class of the real bundles in (2.9). Since the moduli spaces of pointed curves all originate away from that distinguished edge, the integral over the fixed loci can be evaluated as before. We thus obtain the formula,

$$\tilde{n}_d = 2\sum_{\gamma} \frac{1}{|\mathrm{Aut}\gamma|} \int_{\overline{\mathcal{M}}_\gamma} \frac{\mathbf{e}(\mathcal{E}_d^\mathbb{R})}{\mathbf{e}(\mathcal{N}_\gamma^\mathbb{R})} = \sum_{\Gamma=\sigma(\Gamma)} \pm\sqrt{\frac{2}{e|\mathrm{Aut}\Gamma|}}\sqrt{\int_{\overline{\mathcal{M}}_\Gamma} \frac{\mathbf{e}(\mathcal{E}_d)}{\mathbf{e}(\mathcal{N}_\Gamma)}} \tag{2.35}$$

where the sign of the squareroot is fixed by careful consideration of the orientation of the various spaces involved.

To handle the combinatorics of (2.35), we view the graphs γ as graphs representing spheres of degree $(d-e)/2$ with one marked point at which is attached an "intersection disk". This observation allows to reduce the sum in (2.35) to the evaluation of a certain equivariant correlator in closed Gromov-Witten theory. Formulas for such correlators were proved in [5, 26], and the rest is straightforward, see [12]. □

3 B-model

Let W denote the one-parameter family of degree five polynomials in five variables

$$W = \frac{1}{5}\big(x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5\big) - \psi x_1 x_2 x_3 x_4 x_5 \tag{3.1}$$

Let μ_5 be the multiplicative group of 5th roots of unity. The family (3.1) is invariant under the action of

$$\Gamma = \operatorname{Ker}\big((\mu_5)^5 \xrightarrow{p} \mu_5\big)/\mu_5 \tag{3.2}$$

where p is the product of the five 5th roots and the identification is by the diagonal subgroup. The mirror quintic, Y, is constructed by blowing up the singular loci in the quotient of the one-parameter family of quintics by the group Γ

$$Y = \widetilde{\{W = 0\}}/\Gamma \tag{3.3}$$

We will usually confuse the manifold Y for fixed value of ψ with the family of manifolds $Y \xrightarrow{\pi} M$ varying over the base curve $M \ni \psi$.

In this section, we are concerned with the "B-model" on Y. This means that we are interested in Y as an algebraic variety and in the variation of various data attached with Y as the parameter ψ is varied. The precise way in which we do the resolution (3.3) is part of Kähler data. It affects the A-model on Y, but not any of the considerations here.

3.1 Variation of Hodge structure

The third cohomology group of Y is $H^3(Y;\mathbb{C}) \cong \mathbb{C}^4$, and has the *Hodge decomposition*

$$H^3(Y;\mathbb{C}) = H^{3,0}(Y) \oplus H^{2,1}(Y) \oplus H^{1,2}(Y) \oplus H^{0,3}(Y) \tag{3.4}$$

As Y varies with ψ, the decomposition of $H^3(Y;\mathbb{C}) \cong H^3(Y;\mathbb{Z})\otimes\mathbb{C}$ (as a topological invariant) into the Dolbeault cohomology group $H^{p,q}(Y) \cong H^q_{\bar\partial}(\Omega^p_Y)$ changes. This gives rise to a so-called *variation of Hodge structure*, see *e.g.*, [27]. The first key fact in that theory is that the Hodge decomposition (3.4) does not vary holomorphically over $M \ni \psi$. However, the *Hodge filtration*

$$F^pH^3(Y) = \bigoplus_{p'\geq p} H^{p',3-p'}(Y) \tag{3.5}$$

does vary holomorphically. The next key element in the theory of variation of Hodge structure is the flat ("Gauss-Manin") connection ∇ which originates in the local triviality of $H^3(Y;\mathbb{Z})$ over M, and satisfies Griffiths transversality

$$\nabla F^pH^3(Y) \subset F^{p-1}H^3(Y) \otimes \Omega_M \tag{3.6}$$

In this context, the Picard-Fuchs equation $\mathcal{L}\varpi(z) = 0$ that we have met (in the A-model) in the previous section, can be understood as follows. Since Y is Calabi-Yau, there exists a unique-up-to-scale section of the canonical bundle, in other words, a holomorphic $(3,0)$-form, Ω. As ψ varies, we are dealing with a section, also denoted Ω, of the holomorphic line bundle $F^3H^3(Y)$ over M.

Now given any topological three-cycle $\Gamma \in H_3(Y;\mathbb{Z})$, we can consider the *period integral*

$$\varpi = \int_\Gamma \Omega \tag{3.7}$$

The Picard-Fuchs equation is a differential equation satisfied by any period integral, as a function of the complex structure parameters. For the one-parameter family of quintics (3.1), and a particular section, $\hat\Omega$, of $F^3H^3(Y)$, the Picard-Fuchs equation takes the form $\mathcal{L}\varpi(z) = 0$, with

$$\mathcal{L} = \theta^4 - 5z(5\theta+1)(5\theta+2)(5\theta+3)(5\theta+4)\,, \tag{3.8}$$

as in (2.21). Here, $z = (5\psi)^{-5}$ and $\theta = zd/dz$. This equation is the same before or after quotient by the group $\Gamma = (\mu_5)^3$.

The structure above is not special to any particular dimension, we have just written it out in dimension 3 for convenience. The central character in closed string mirror whose properties are special to dimension 3 is the so-called *Yukawa coupling*. It arises from the third iterate of the differential period mapping, $H^1(TY) \to \bigoplus \mathrm{Hom}\big(H^{p,q}(Y), H^{p-1,q+1}(Y)\big)$ by contraction with your choice of section of F^3H^3, and is usually written as

$$\kappa = \int_Y \Omega \wedge \nabla^3 \Omega \tag{3.9}$$

Classical closed string mirror symmetry provides an identification between the generating function of genus 0 Gromov-Witten invariants of some Calabi-Yau X and the Yukawa coupling of the mirror manifold Y. There are two key steps. The first is to provide the *mirror map* between the "formal" (Kähler) parameters of X and the complex structure parameters of Y. The second is a normalization of the Yukawa coupling, *i.e.*, the identification of a particular distinguished choice of holomorphic three-form. Both steps can be formalized by studying the degeneration of the variation of Hodge structure at certain special points in moduli space known under the name "maximal unipotent monodromy". This is explained for the quintic in [3], for full details, see *e.g.*, [7].

We now turn to the part of Hodge theory on Y that provides the B-model counterpart to the generating function for the open Gromov-Witten invariants of the real quintic studied in the previous section.

3.2 Normal functions

Let $\big(H^3(Y;\mathbb{Z}), F^*H^3(Y)\big)$ be an integral variation of Hodge structure of weight 3 as we had in the previous subsection. The *Griffiths intermediate Jacobian fibration* is the fibration $J^3(Y) \to M$ of complex tori

$$J^3(Y) = \frac{H^3(Y)}{F^2H^3(Y) \oplus H^3(Y;\mathbb{Z})} \tag{3.10}$$

An equivalent definition of J^3 is obtained by noting that by Poincaré duality we have an isomorphism $H^3(Y)/F^2H^3(Y) \cong (F^2H^3(Y))^*$, under which $H^3(Y;\mathbb{Z}) \cong H_3(Y;\mathbb{Z})$. Thus,

$$J^3(Y) = (F^2H^3(Y))^*/H_3(Y;\mathbb{Z}) \tag{3.11}$$

Next, a *Poincaré normal function* of the variation of Hodge structure is a holomorphic section ν of $J^3(Y)$ satisfying *Griffiths transversality for normal functions*

$$\nabla\tilde{\nu} \in F^1H^3(Y) \otimes \Omega_M \tag{3.12}$$

where $\tilde{\nu}$ is an arbitrary lift of ν from $J^3(Y)$ to $H^3(Y)$ (the condition (3.12) does not depend on the lift).

An important source of normal functions are *homologically trivial algebraic cycles*, where in our situation, we are interested in cycles of codimension 2. Let C be such a homologically trivial algebraic cycle (formal integral combination of holomorphic curves). We are here assuming that C varies in a nice family with Y over M, again confusing some notation. C being homologically trivial means that there exists a three-chain Γ, varying over M and unique modulo $H_3(Y;\mathbb{Z})$ such that for all $m \in M$, $\partial\Gamma_m = C_m$.

To define a normal function ν_C attached to C, we use the definition (3.11). Any class $[\omega] \in F^2H^3(Y)$ can be represented locally on M by a three-form $\omega \in F^2\mathcal{A}^3$ pointing in the fiber direction with at least two holomorphic indices. One can show that the definition

$$\nu_C([\omega])_m := \int_{\Gamma_m} \omega_m \tag{3.13}$$

satisfies all conditions from the above definition of a normal function (holomorphicity and transversality). This association defines a map from algebraic cycles modulo rational equivalence to the intermediate Jacobian $J^3(Y)$, known as the *Abel-Jacobi map*.

3.3 Extended Picard-Fuchs equation

After these general definitions, we consider again the one-parameter family of quintics (3.1). It contains for any ψ a twin family of algebraic curves of degree 2 given by

$$C_\pm = \{x_1 + x_2 = 0, x_3 + x_4 = 0, x_5^2 \pm \sqrt{5\psi}x_1x_3 = 0\} \subset \{W = 0\} \tag{3.14}$$

Since the second Betti number of the quintic is equal to 1, those two curves must lie in the same homology class, and we can consider the normal function attached to $C = C_+ - C_-$. More precisely, we are here interested in the mirror quintic, so we should really be discussing this after taking the quotient by μ_5^3. However, this requires some explicit choices for the resolution of the singularities, which is somewhat cumbersome. We refer to [13] for details.

Given the normal function attached to $C = C_+ - C_-$, we can consider its "truncation", namely, the restriction of the integral to $[\omega] \in F^3H^3$,[5]

$$\mathcal{T}_B = \mathcal{T}_B(z) = \int_\Gamma \hat{\Omega} \tag{3.15}$$

where $\hat{\Omega}$ is a particular choice of holomorphic three-form on Y, defined as Poincaré residue by the formula

$$\hat{\Omega} = \left(\frac{5}{2\pi i}\right)^3 \psi \, \mathrm{Res}_{W=0} \frac{\alpha}{W} \tag{3.16}$$

where α is the four-form on projective space

$$\alpha = \sum_i (-1)^{i-1} x_i dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_5 \tag{3.17}$$

The choice of holomorphic three-form in (3.16) is precisely the one for which the Picard-Fuchs equation of the mirror quintic takes the form (3.8).

Theorem B. *Let $\mathcal{T}_B$ be the truncated normal function of the algebraic cycle $C = C_+ - C_-$, for the choice of holomorphic three-form given in* (3.16)*. Let $\mathcal{L}$ be the Picard-Fuchs operator in* (3.8)*. Then*

$$\mathcal{L}\mathcal{T}_B(z) = \frac{15}{16\pi^2}\sqrt{z} \tag{3.18}$$

The proof [13] is an explicit computation that uses the algorithm of Griffiths-Dwork to derive the Picard-Fuchs equation of hypersurfaces, and keeps careful track of all boundary terms originating from the fact that $\partial\Gamma = C$. One begins with the definition of $\hat{\Omega}$ as a residue to write the integral $\int_\Gamma \hat{\Omega}$ as an integral over a four-chain that is a little tube over Γ around the hypersurface $W = 0$. Application of $\mathcal{L}$ reduces this to an integral of a certain meromorphic three-form over a tube around the algebraic cycle C. The latter can be evaluated based on the following fact. The curves C_+, C_- lie entirely in the plane $P = \{x_1 + x_2 = 0, x_3 + x_4 = 0\}$. If one could restrict the tube to lie entirely in P as well, the integral of any meromorphic three-form over it would vanish. The reason one cannot restrict the computation to P is that the intersection of P with the hypersurface is the reducible degree five curve $x_5^5 - 5\psi x_5 x_1^2 x_3^2$.

[5] Recall, $z = (5\psi)^{-5}$.

Any tube in P around say C_+ will intersect one of the other components, and therefore not encircle $W = 0$ as it should. For the same reason, one can reduce the computation to the intersection points of the components of $P \cap \{W = 0\}$, which is a straightforward computation in an affine patch. □

4 Mirror Symmetry

Let's summarize what we have learned so far. By comparing (2.29) with (3.18) we learn that the generating function for open Gromov-Witten invariants of the real quintic satisfies, up to an overall normalization, the same inhomogeneous Picard-Fuchs equation as the truncated normal function associated with the algebraic cycle $C = C_+ - C_-$ on the mirror quintic. In this section, we put this result in the context of mirror symmetry.

We begin by presenting explicitly two objects in the category of B-branes on Y that we conjecture are equivalent under mirror symmetry to the two objects in the Fukaya category of X that can be defined from the real quintic. We will then explain the relation between those objects and the normal function ν_C. Finally, we will complete the generating function of open Gromov-Witten invariants to a full identification of the normal function in the A-model.

We also include two somewhat speculative considerations. The first is a formal statement relating real sections of the quintic with objects in the category of B-branes for the mirror. The second is another application of this proposed correspondence and concerns the A_∞ structure or Floer homology of the real quintic. A better understanding of these issues could be a possible starting point for establishing homological mirror symmetry for the quintic (and other hypersurfaces).

4.1 Matrix factorizations

Let $V \in \mathbb{C}[x_1, x_2, \ldots, x_5]$ be a polynomial. A *matrix factorization* of V is a $\mathbb{Z}_2$-graded free $\mathbb{C}[x_1, \ldots, x_5]$ module M equipped with an odd endomorphism $Q : M \to M$ of square V,

$$Q^2 = V \cdot \mathrm{id}_M \tag{4.1}$$

The category $\mathrm{MF}(V)$ is the triangulated category of matrix factorizations with morphisms given by Q-closed morphisms of free modules, modulo Q-exact morphisms. Matrix factorizations are well-known objects since the mid '80's, see in particular [28], and it was proposed by Kontsevich that $\mathrm{MF}(V)$ should be a good description of B-type D-branes in a Landau-Ginzburg model based on the worldsheet superpotential V [29–31]. To apply this to the case of interest, we need a little bit of extra structure.

When V is of degree 5, the so-called homological Calabi-Yau/Landau-Ginzburg correspondence [32–35] states that the derived category of coherent sheaves of the projective hypersurface $X = \{V = 0\} \subset \mathbb{P}^4$ is equivalent to the graded, equivariant category of matrix factorizations of the corresponding Landau-Ginzburg superpotential,

$$D^b(X) \cong \mathrm{MF}(V/\mu_5) \tag{4.2}$$

where μ_5 is the group of 5-th roots of unity acting diagonally on $x_1, \ldots, x_5$. The analogue statement for the mirror quintic (specialize V to W from (3.1)) is

$$D^b(Y) \cong \mathrm{MF}(W/(\mu_5)^4) \tag{4.3}$$

where $(\mu_5)^4$ is the subgroup of phase symmetries of W whose product is equal to 1 (*i.e.*, $\mathrm{Ker}(p)$ in the notation of (3.2)).

To describe an object mirror to the real quintic, we begin with finding a matrix factorization of the one-parameter family of polynomials (3.1). If $S \cong \mathbb{C}^5$ is a 5-dimensional vector space, we can associate to its exterior algebra a $\mathbb{C}[x_1, \ldots, x_5]$-module $M = \wedge^* S \otimes \mathbb{C}[x_1, \ldots, x_5]$. It naturally comes with the decomposition

$$M = M_0 + M_1 + M_2 + M_3 + M_4 + M_5\,, \quad \text{where } M_s = \wedge^s S \otimes \mathbb{C}[x_1, \ldots, x_5], \tag{4.4}$$

and the $\mathbb{Z}_2$-grading $(-1)^i$. Let η_i be a basis of S and $\bar\eta_i$ the dual basis of S^*, both embedded in $\mathrm{End}(M)$. We then define two families of matrix factorizations $(M, Q_\pm)$ of W by

$$Q_\pm = \frac{1}{\sqrt{5}} \sum_{i=1}^{5} (x_i^2 \eta_i + x_i^3 \bar\eta_i) \pm \sqrt{\psi} \prod_{i=1}^{5} (\eta_i - x_i \bar\eta_i) \tag{4.5}$$

To check that $Q_\pm^2 = W \cdot \mathrm{id}_M$, one uses that η_i, $\bar\eta_i$ satisfy the Clifford algebra

$$\{\eta_i, \bar\eta_j\} = \delta_{ij} \tag{4.6}$$

as well as the ensuing relations

$$\{(x_i^2\eta_i + x_i^3\bar\eta_i), (\eta_i - x_i\bar\eta_i)\} = 0 \qquad \text{and} \qquad (\eta_i - x_i\bar\eta_i)^2 = -x_i \tag{4.7}$$

The matrix factorization (4.5) is quasi-homogeneous ($\mathbb{C}^*$-gradable), but we will not need this data explicitly.

Now to specify objects in $\mathrm{MF}(W/\Gamma)$, where $\Gamma = \mu_5$ or $(\mu_5)^4$ for the quintic and mirror quintic, respectively, we have to equip M with a representation of Γ such that $Q_\pm$ is equivariant with respect to the action of Γ on the x_i. Since $Q_\pm$ is irreducible, this representation of Γ on M is determined up to a character of Γ by a representation on S, *i.e.*, an action on the η_i. For $\gamma \in \Gamma$, we have $\gamma(x_i) = \gamma_i x_i$ for some fifth root of unity γ_i. We then set $\gamma(\eta_i) = \gamma_i^{-2}\eta_i$, making $Q_\pm$ equivariant. As noted, this representation is unique up to an action on M_0, *i.e.*, a character of Γ.

For the mirror quintic, $\Gamma = \mathrm{Ker}((\mu_5)^5 \to \mu_5)$, so $\Gamma^* = (\mu_5)^5/\mu_5$, and we label the characters of Γ as $[\chi]$. The corresponding objects of $\mathrm{MF}(W/\Gamma)$ constructed out of $Q_\pm$ (4.5) are classified as $Q_\pm^{[\chi]} = (M, Q_\pm, \rho_{[\chi]})$, where $\rho_{[\chi]}$ is the representation on M we just described.

4.2 A homological mirror symmetry conjecture

In section 2, we have introduced the real quintic L as the Lagrangian defined as the real points of the Fermat quintic. To get closer to homological mirror symmetry, we need to make two small amendments to this definition.

First, to define an object in the Fukaya category, $\mathrm{Fuk}(X)$, we need to specify not only a Lagrangian submanifold, but also a flat $U(1)$ local system on L. Since $H_1\big(L \cong \mathbb{RP}^3; \mathbb{Z}\big) = \mathbb{Z}_2$, there are two possible choices, leading to two objects in $\mathrm{Fuk}(X)$ which we denote by $L_\pm$.

Second, we note that the Fermat quintic is invariant under more than one anti-holomorphic involution. If as before μ_5 denotes the multiplicative group of fifth roots of unity, we define for $\chi = (\chi_1, \ldots, \chi_5) \in (\mu_5)^5$ an anti-holomorphic involution σ_χ of $\mathbb{P}^4$ by its action on homogeneous coordinates

$$\sigma_\chi : x_i \to \chi_i \bar x_i \tag{4.8}$$

The Fermat quintic is invariant under any σ_χ. The involution and the fixed point locus only depend on the class of χ in $(\mu_5)^5/\mu_5 \cong (\mu_5)^4$, and we obtain in this way $5^4 = 625$ pairs of objects $L_\pm^{[\chi]}$ in $\mathrm{Fuk}(X)$.

We emphasize again that although we have defined the Lagrangians $L_\pm^{[\chi]}$ as fixed point sets of anti-holomorphic involutions of the Fermat quintic, we can think of the corresponding objects of $\mathrm{Fuk}(X)$ without reference to the complex structure.

Conjecture 1. *There is an equivalence of categories* $\mathrm{Fuk}(X) \cong \mathrm{MF}(W/(\mu_5)^4)$ *which identifies the 625 pairs of objects* $L_\pm^{[\chi]}$ *with the 625 pairs of equivariant matrix factorizations* $Q_\pm^{[\chi]}$*.*

Remark. One can formulate a similar conjecture for any hypersurface in weighted projective space which has a Fermat point in its complex structure moduli space.

There are at present two pieces of evidence for the above conjecture. The first is a matching of the intersection indices, defined in the A-model by the geometric (transversal) intersection number of Lagrangian submanifolds, and in the B-model as the Euler character on the morphism spaces. That evidence was first noted in [36], and is reviewed in detail also in [13]. One of the central consequences of that computation is that the image of $L_\pm^{[\chi]}$ in homology actually generate $H_3(X;\mathbb{Z})$. This follows from the fact that the rank of the 625×625-dimensional intersection matrix is equal to 204, which is the rank of $H_3(X;\mathbb{Z})$.

The second piece of evidence comes from the matching between open Gromov-Witten theory of the real quintic and the normal function ν_C. Before filling in the connection, we note that this structure does not depend on the group theoretic data, so we can return to focus on the ordinary real quintic $L_\pm = L_\pm^{[\chi=1]}$ and its mirror object $Q_\pm = Q_\pm^{[\chi=1]}$.

4.3 Assembly

Recall that the theory of algebraic Chern classes provides a map

$$c_i^{\mathrm{alg}} : D^b(Y) \to \mathrm{CH}^i(Y) \tag{4.9}$$

from the derived category of coherent sheaves to the Chow groups of algebraic cycles modulo rational equivalence. The Chern classes factor through algebraic K-theory, and in particular, split any exact triangle in $D^b(Y)$. The image of c_i^{alg} in rational cohomology coincides with the topological Chern class $c_i \equiv c_i^{\mathrm{top}}$. Combining this with the Abel-Jacobi map described in section 3, we obtain the basic construction that associates a normal function to any object in $D^b(Y)$ of trivial topological Chern class. Since $\mathrm{MF}(W/(\mu_5)^4) \cong D^b(Y)$, we can extend this definition to the category of matrix factorizations.

The following result derives from [13].

Proposition C. *Let $Q_\pm$ be the pair of matrix factorizations defined in* (4.5). *Let $C_\pm \subset Y$ be the pair of algebraic curves studied in section 3. We have*

$$c_2^{\mathrm{alg}}(Q_+) - c_2^{\mathrm{alg}}(Q_-) = [C_+ - C_-] \in \mathrm{CH}^2(Y) \tag{4.10}$$

The essential idea that explains the coincidence of Gromov-Witten theory of the real quintic with the normal function ν_C associated with $C = C_+ - C_-$ should now be clear. Following mirror symmetry, the A-model has been equipped with a variation of Hodge structure isomorphic to that defined by the B-model on the mirror quintic [37]. For homological mirror symmetry to make sense, there ought to exist an A-model version of the Abel-Jacobi map into the intermediate Jacobian of the A-model variation of Hodge structure.

Based on considerations in string theory, it was argued in [11] that for the object $L_+ - L_-$, the truncated normal function in the A-model takes the form

$$\mathcal{T}_A(t) = \frac{t}{2} + \Big(\frac{1}{4} + \frac{1}{2\pi^2} \sum_{d \text{ odd}} \tilde{n}_d q^{d/2}\Big) \tag{4.11}$$

where $q = e^{2\pi i t}$, and $\tilde{n}_d$ are the open Gromov-Witten invariants of the real quintic, as we had it before (with slightly different conventions on t).

Recalling eq. (2.29), we see that

$$\mathcal{L}\big(\varpi_0(z)\mathcal{T}_A(t(z))\big) = \frac{15}{16\pi^2}\sqrt{z} \tag{4.12}$$

which is the same equation satisfied by $\mathcal{T}_B(z)$ from (3.18). The boundary conditions on $\mathcal{T}_B(z)$ are most conveniently fixed around $z^{-1} = 0$, namely [13]

$$\mathcal{T}_B(z) = -\frac{4}{3} \sum_{m=0}^{\infty} \frac{\Gamma(-3/2 - 5m)}{\Gamma(-3/2)} \frac{\Gamma(1/2)^5}{\Gamma(1/2 - m)^5} z^{-(m+1/2)} \tag{4.13}$$

Note that this definition implicitly assumes a particular lift of the quotient by $H_3(Y;\mathbb{Z})$ in the definition of the intermediate Jacobian.

The following result is proved in [11].

Proposition D. *Let $\mathcal{T}_B(z)$ be the truncated normal function of the cycle $C_+ - C_-$. Let $\mathcal{T}_A(t)$ be given by the series in (4.11), extended to the z-plane by the mirror map and analytic continuation. We have*

$$\varpi_0(z)\mathcal{T}_A((t(z)) = \mathcal{T}_B(z)\,, \tag{4.14}$$

up to an integral period, $\int_{\Gamma^c}\hat{\Omega}(z)$, for some $\Gamma^c \in H_3(Y;\mathbb{Z})$

4.4 Floer Homology of the real quintic

There is an interesting feature of the one-parameter family of matrix factorization $Q_\pm$ that we have so far ignored, although it was really the initial motivation to investigate open mirror symmetry for the real quintic.

Consider the endomorphism algebra $\mathrm{Hom}^*(Q,Q)$ of the matrix factorization $Q = Q_+$, as objects in $\mathrm{MF}(W/(\mu_5)^4)$. This algebra is $\mathbb{Z}$-graded thanks to the homogeneity of W [32]. We also have $\mathrm{Hom}^0(Q,Q) \cong \mathbb{C}$ since Q is irreducible, and this implies $\mathrm{Hom}^3(Q,Q) \cong \mathbb{C}$ by Serre duality. Finally, it is shown in [38, 39] that

$$\mathrm{Hom}^1(Q,Q) = \mathrm{Hom}^2(Q,Q) = \begin{cases} 0 & \psi \neq 0 \\ \mathbb{C} & \psi = 0 \end{cases} \tag{4.15}$$

The appearance of an additional cohomology element in $\mathrm{Hom}^1(Q,Q)$ is signaled by the fact that $\mathcal{T}_B(z)$ in (4.13) vanishes at $\psi = 0$ (recall $z = (5\psi)^{-1}$).

To interpret (4.15) in the A-model, we recall that the morphism algebra of objects in the Fukaya category is defined using Lagrangian intersection Floer homology [40]. For the endomorphism algebra of a single Lagrangian, Floer homology is essentially a deformation of ordinary Morse homology by holomorphic disks.

For example, consider the real quintic $L \cong \mathbb{RP}^3$. Think of $\mathbb{RP}^3$ as $S^3/\mathbb{Z}_2$, and embed the S^3 in $\mathbb{R}^4 \ni (y_0,\dots,y_3)$ as $y_0^2 + y_1^2 + y_2^2 + y_3^2 = 1$. A standard Morse function for $\mathbb{RP}^3$ in this presentation is given by $f = y_1^2 + 2y_2^2 + 3y_3^2$

restricted to the S^3. This Morse function is self-indexing and has one critical point in each degree $i = 0, 1, 2, 3$. The Morse complex takes the form

$$C^0 \xrightarrow{0} C^1 \xrightarrow{\delta} C^2 \xrightarrow{0} C^3 \tag{4.16}$$

Working with integer coefficients, $C^i \cong \mathbb{Z}$ for all i, we have $\delta = 2$, and the complex (4.16) computes the well-known integral cohomology of $\mathbb{RP}^3$.

To compute Floer homology of the real quintic, we have to deform (4.16) by holomorphic disks, *i.e.*, $\delta = 2 + \mathcal{O}(e^{-t/2})$. In the standard treatments, such as [40], but also the more recent works such as [41], this requires taking coefficients from a certain formal (Novikov) ring with uncertain convergence properties. In other words, Floer homology is at present only defined in an infinitesimal neighborhood of the large volume point in moduli space (which leads to the often heard remark that $\mathrm{HF}^*(L, L)$ is isomorphic to $H^*(L)$). However, to make contact with (4.15), we have to understand $\mathrm{HF}^*(L, L)$ at the opposite end of moduli space, $z^{-1} = 0$. The results about the enumeration of disks ending on the real quintic that we have reviewed in this paper make one optimistic that this difficulty can be overcome, and support the conjecture made in [39]

Conjecture 2. *The Floer homology* $\mathrm{HF}^*(L, L)$ *of the real quintic can be defined in an open neighborhood of large volume. The complex* (4.16)*, with complex coefficients, can be analytically continued over the entire* z*-plane. The differential* δ *has a single zero at the point* $z^{-1} = 0$*.*

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