NONPRINCIPAL REFLEXIVE LEFT IDEALS IN IWASAWA ALGEBRAS II

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In the following we use the same notation as in the appendix of [1]. There we constructed a principal ideal L of Λ which is nonprincipal. We will calculate the G- and H-homology of Λ/L.

Recall that L is generated by the elements

$$\begin{aligned} f = &\quad Z^2 - \frac{u\pi + \sigma^2(\pi)}{\sigma(\pi)} Z + u = Y^2 + (2 - \frac{u\pi + \sigma^2(\pi)}{\sigma(\pi)})Y + (u - \frac{u\pi + \sigma^2(\pi)}{\sigma(\pi)} + 1) \\ \pi h = &\quad \pi Z - \sigma(\pi) = \pi Y + (\pi - \sigma(\pi)) \end{aligned}$$

Lemma 1. *The following sequence is an exact sequence of Λ-modules:*

$$0 \to \Lambda \xrightarrow{\phi} \Lambda^2 \xrightarrow{\psi} \Lambda \to \Lambda/L \to 0.$$

The right map is the canonical projection and

$$\begin{aligned} \phi : &\quad e_1 \mapsto (M, N) := (\sigma(\pi), Z - u) = (\sigma(\pi), Y + (1-u)), \\ \psi : &\quad e_1 \mapsto f, \; e_2 \mapsto -\pi h. \end{aligned}$$

Proof. It suffices to show that $\operatorname{Im}\phi = \operatorname{Ker}\psi$. First we calculate

$$\begin{aligned} Mf &= \sigma(\pi) f = \sigma(\pi) Z^2 - (u\pi + \sigma^2(\pi))Z + \sigma(\pi)u = Z\pi Z - Z\sigma(\pi) - u\pi Z + u\sigma(\pi) \\ &= N\pi h \end{aligned}$$

which shows that $\operatorname{Im}\phi \subseteq \operatorname{Ker}\psi$. Let $(a, b) \in \operatorname{Ker}\psi$. We can write $b = cN + d$ with $c \in \Lambda, d \in R$. We obtain $af = (cN + d)\pi h$ and therefore $(a - cM)f = d\pi h = 0 \cdot f + d\pi h$. Since $\deg(d\pi h) = 1 < \deg f$ the uniqueness statement in the division theorem yields $a = cM$ and $d = 0$. Therefore $\operatorname{Ker}\psi \subseteq \operatorname{Im}\phi$. $\square$

Proposition 2. *It holds that*

$$\begin{aligned} H_0(G, \Lambda/L) &= \mathbb{Z}_p, \\ H_1(G, \Lambda/L) &= \mathbb{Z}_p \times \mathbb{Z}/p, \\ H_i(G, \Lambda/L) &= 0 \quad \text{for } i \geq 2. \end{aligned}$$

Moreover,

$$\begin{aligned} H_0(G, L) &= \mathbb{Z}_p \times \mathbb{Z}/p, \\ H_i(G, L) &= 0 \quad \text{for } i \geq 1. \end{aligned}$$

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Proof. If we denote the maps induced by taking G-coinvariants of the above sequence again with the same letters, then we obtain a sequence

$$\mathbb{Z}_p \xrightarrow{\phi} \mathbb{Z}_p^2 \xrightarrow{\psi} \mathbb{Z}_p$$

and $H_0(G, M) = \operatorname{Coker}\psi$, $H_1(G, M) = \operatorname{Ker}\psi/\operatorname{Im}\phi$, $H_2(G, M) = \operatorname{Ker}\phi$. In the proof of Lemma A.2 in [1] we showed that the absolute term of $(u\pi+\sigma^2(\pi))/\sigma(\pi)$ is $1+u$. The absolute term of $\pi - \sigma(\pi)$ is zero. Therefore $\psi : \mathbb{Z}_p^2 \to \mathbb{Z}_p$ is the zero map. Since the absolute term of $\sigma(\pi)$ is $-p$ and by Lemma A.2 of [1] we may write $u = 1 + \alpha p^l \mod (p, X)$ for some $l \geq 1$, ϕ is given by $e_1 \mapsto (-p, \alpha p^l)$. This implies the first claim. The second claim follows from the exact sequence

$$0 \to \Lambda \xrightarrow{\phi} \Lambda^2 \xrightarrow{\psi} L \to 0$$

by the same calculations. □

Corollary 3. *There does not exist a principal ideal $\tilde{L}$ in Λ with $\Lambda/L \cong \Lambda/\tilde{L}$.*

Proof. If there existed an $\tilde{L}$ with the above properties then we could write $\tilde{L} = \Lambda\tilde{f}$ with a distinguished polynomial $\tilde{f}$ and obtain a projective resolution

$$0 \to \Lambda \xrightarrow{\cdot\tilde{f}} \Lambda \to \Lambda/\Lambda\tilde{f} \to 0.$$

Then $H_1(G, \Lambda/\tilde{L}) = \operatorname{Ker}(\mathbb{Z}_p \xrightarrow{\cdot\tilde{f}_0} \mathbb{Z}_p)$ where $\tilde{f}_0$ denotes the absolute coefficient of $\tilde{f}$ and therefore $H_1(G, \Lambda/\tilde{L}) = \mathbb{Z}_p$ or 0 depending on whether this coefficient is zero or not. □

Lemma 4. *The following sequence is an exact sequence of Λ-modules:*

$$0 \to \Lambda/\Lambda N \xrightarrow{\chi} \Lambda/\Lambda f \to \Lambda/L \to 0.$$

The map χ is given by $\lambda + \Lambda N \mapsto \lambda\pi h + \Lambda f$ for $\lambda \in \Lambda$.

Proof. We have to determine the kernel of the surjection $\Lambda/\Lambda f \to \Lambda/L$. It is given by

$$(\Lambda\pi h + \Lambda f)/\Lambda f = \Lambda\pi h/(\Lambda f \cap \Lambda\pi h) = \Lambda\pi h/\Lambda N\pi h = \Lambda/\Lambda N$$

which gives the result. □

Proposition 5. *It holds that*

$$\begin{aligned} H_0(H, \Lambda/L) &= \mathbb{Z}_p \times \mathbb{Z}/p, \\ H_i(H, \Lambda/L) &= 0 \quad \text{for } i \geq 1. \end{aligned}$$

Proof. The exact sequence of the above lemma is a projective resolution for Λ/L as a $\Lambda(H)$-module. Taking H-coinvariants we obtain a map $\chi : \mathbb{Z}_p \to \mathbb{Z}_p^2$ with $H_0(H, \Lambda/L) = \operatorname{Coker}\chi$, $H_1(H, \Lambda/L) = \operatorname{Ker}\chi$. Since $\pi h = \pi Y + (\pi - \sigma(\pi))$ the map ψ is given by $e_1 \mapsto (-p, 0)$. This implies the result. □

The last proposition shows that Λ/L is not a free $\Lambda(H)$-module. However, it does not rule out the possibility that there might exist a principal reflexive left ideal $\tilde{L}$ of Λ (generated by a linear distinguished polynomial of Λ) for which there exists an injection $\Lambda/L \to \Lambda/\tilde{L} = \Lambda(H)$ of Λ-modules with pseudonull cokernel.

References

[1] O.Venjakob: *A non-commutative Weierstrass preparation theorem and its applications to Iwasawa theory. With an appendix by D. Vogel*, Journal für die reine und angewandte Mathematik 559(2003), 153-191

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