THE HOMOTOPY FIBRE OF MAPS OF MOREL-VOEVODSKY SPACES

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1. The construction

Let $\Delta^{op}\,\mathrm{Shv}_{et}(\mathrm{Sm}\,k)_0$ denote the category of pointed connected simplicial étale sheaves on $\mathrm{Sm}(k)_{et}$ and $\mathcal{H}_0$ the homotopy category of pointed connected simplicial sets.
Given a morphism $f : (\mathcal{X}, x) \to (\mathcal{Y}, y)$ in $\Delta^{op}\,\mathrm{Shv}_{et}(\mathrm{Sm}\,k)_0$ we construct a pro-object $H_f \in \mathrm{pro}-\mathcal{H}_0$ such that we obtain a long exact homotopy sequence

$$\ldots \to \pi_{i+1}(\mathcal{Y}, y) \to \pi_i(H_f) \to \pi_i(\mathcal{X}, x) \to \pi_i(\mathcal{Y}, y) \to \ldots$$

For convenience we recall some facts about pro-objects. A pro-object of a category C is a functor $F : I \to C$ from a small left filtering category I to C. A map of pro-objects $F : I \to C$, $G : J \to C$ of C is an element of

$$\varprojlim_{j \in J} \varinjlim_{i \in I} \mathrm{Hom}_C(F(i), G(j)).$$

A strict map of pro-objects $F : I \to C$, $G : J \to C$ of C is a pair (α, ϕ) consisting of a functor $\alpha : J \to I$ and a natural transformation $\phi : F \circ \alpha \to G$. A strict map of pro-objects of course induces a map of pro-objects. We remark ([Fr1], ch. 4) that if $\alpha : J \to I$ is left final then $(\alpha, \mathrm{id} : F \circ \alpha \to F \circ \alpha)$ gives rise to an isomorphism from F to $F \circ \alpha$ (whose inverse need not be strict).

Definition 1.1. *Let $\pi Triv_0/\mathcal{X}$ denote the category whose objects are the pointed trivial local fibrations to $(\mathcal{X}, x)$ and whose morphisms are the obvious commutative triangles in $\pi\Delta^{op}\,\mathrm{Shv}_{et}(\mathrm{Sm}\,k)_0$. Let $\pi Triv_0(f)$ denote the category whose objects are commutative squares*

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[d] & V \arrow[d] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

in $\Delta^{op}\,\mathrm{Shv}_{et}(\mathrm{Sm}\,k)_0$ (which we denote again by g by abuse of notation) where the vertical maps are pointed trivial local fibrations. A map $g \to g'$ is a *commutative diagram*

$$\begin{tikzcd}
U \arrow[rrr, "g"] \arrow[dr] \arrow[dd] & & & V \arrow[dl] \arrow[dd] \\
& \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \arrow[dr] & \\
U' \arrow[rrr, "g'"] \arrow[ur] & & & V'
\end{tikzcd}$$

A morphism in $\pi Triv_0(f)$ is an equivalence class of maps $g \to g'$ where two maps $g \rightrightarrows g'$ are equivalent if there exists a pointed simplicial homotopy over f relating them.

Proposition 1.2. *The category $\pi Triv_0(f)$ is left filtering.*

Proof. This follows from the naturality of the construction of the homotopy left equalizer in $\pi Triv/\mathcal{X}$ as carried out in [Mo], Lemma 0.3. ☐

Recall ([GZ], VI 5.5.1) that to a map $S \to T$ of simplicial sets we can functorially associate a commutative triangle

$$\begin{tikzcd}
S \arrow[rr] \arrow[dr] & & T \\
& S^r \arrow[ur] &
\end{tikzcd}$$

such that $S \to S^r$ is a weak equivalence and $S^r \to T$ is a Kan fibration. We provide some functors between the above categories.

Definition 1.3. *Let*

$$\Pi_{\mathcal{X}} : \pi Triv_0/\mathcal{X} \to \mathcal{H}_0, \quad \Pi_{\mathcal{Y}} : \pi Triv_0/\mathcal{Y} \to \mathcal{H}_0$$

denote the connected component functor as in [S], §4. We define

$$H_f : \pi Triv_0(f) \to \mathcal{H}_0,\ g \mapsto (\Pi U)^r \times_{\Pi V} v$$

and

$$\Pi^r_f : \pi Triv_0(f) \to \mathcal{H}_0,\ g \mapsto (\Pi U)^r$$

where we make use of the factorization

$$\begin{tikzcd}
\Pi U \arrow[rr, "\Pi g"] \arrow[dr] & & \Pi V \\
& (\Pi U)^r \arrow[ur] &
\end{tikzcd}$$

of Πg into a weak equivalence followed by a Kan fibration and v denotes the basepoint of ΠV. We denote by

$$\mathrm{sr} : \pi Triv_0(f) \to \pi Triv_0/\mathcal{X},\ g \mapsto\ U \to \mathcal{X}$$

and

$$\mathrm{rg} : \pi Triv_0(f) \to \pi Triv_0/\mathcal{Y},\ g \mapsto\ V \to \mathcal{X}$$

the source and range functor, respectively.

Lemma 1.4. *The functors* sr *and* rg *are left final.*

Proof. This follows in the same way as in [Fr2], from Prop. 1.2. □

Proposition 1.5. *We have a long exact homotopy sequence*

$$\ldots \to \pi_{i+1}(\mathcal{Y}, y) \to \pi_i(H_f) \to \pi_i(\mathcal{X}, x) \to \pi_i(\mathcal{Y}, y) \to \ldots$$

Proof. We have obvious strict morphisms $H_f \to \Pi_f^r \to \Pi_{\mathcal{Y}} \circ \mathrm{rg}$ which by construction induce a long exact sequence of pro-groups

$$\ldots \to \pi_{i+1}(\Pi_{\mathcal{Y}} \circ \mathrm{rg}) \to \pi_i(H_f) \to \pi_i(\Pi_f^r) \to \pi_i(\Pi_{\mathcal{Y}} \circ \mathrm{rg}) \to \ldots$$

Since rg is left final, we obtain an isomorphism $\Pi_{\mathcal{Y}} \cong \Pi_{\mathcal{Y}} \circ \mathrm{rg}$ in pro-$\mathcal{H}_0$. The left finality of sr yields an isomorphism $\Pi_{\mathcal{X}} \cong \Pi_{\mathcal{X}} \circ \mathrm{sr}$, and by construction we have $\pi_i(\Pi_{\mathcal{X}} \circ \mathrm{sr}) \cong \pi_i(\Pi_f^r)$ for each i. This proves the claim. □

We remark that H_f coincides with $\mathrm{fib}(f_{ht})$ of [Fr1], ch. 10, if f is a map of schemes. We have a natural map $(\mathcal{X} \times_{\mathcal{Y}} y)_{et} \to H_f$ induced by

$$\pi Triv_0(f) \xrightarrow{\mathrm{sr}} \pi Triv_0/\mathcal{X} \to \pi Triv_0/\mathcal{X} \times_{\mathcal{Y}} y.$$

It would be interesting to know under which conditions this map is a weak equivalence.

References

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