arXiv:math.NT/0507275 v1 13 Jul 2005

FROM THE BIRCH & SWINNERTON-DYER CONJECTURE OVER THE EQUIVARIANT TAMAGAWA NUMBER CONJECTURE TO NON-COMMUTATIVE IWASAWA THEORY - A SURVEY

AFTER BURNS/FLACH, FUKAYA/KATO, HUBER/KINGS, COATES, SUJATHA ...

OTMAR VENJAKOB

This paper aims to give a survey on Fukaya and Kato's article [19] which establishes the relation between the Equivariant Tamagawa Number Conjecture (ETNC) by Burns and Flach [7] and the noncommutative Iwasawa Main Conjecture (MC) (with p-adic L-function) as formulated by Coates, Fukaya, Kato, Sujatha and the author [11]. Moreover, we compare their approach with that of Huber and Kings [20] who formulate an Iwasawa Main Conjecture (without p-adic L-functions). We do not discuss these conjectures in full generality here, in fact we are mainly interested in the case of an abelian variety defined over $\mathbb{Q}$. Nevertheless we formulate the conjectures for general motives over $\mathbb{Q}$ as far as possible. We follow closely the approach of Fukaya and Kato but our notation is sometimes inspired by [7, 20]. In particular, this article does not contain any new result, but hopefully serves as introduction to the original articles. See [37] for a more down to earth introduction to the GL_2 Main Conjecture for an elliptic curve without complex multiplication. There we had pointed out that the Iwasawa main conjecture for an elliptic curve is morally the same as the (refined) Birch and Swinnerton Dyer (BSD) Conjecture for a whole tower of number fields. The work of Fukaya and Kato makes this statement precise as we are going to explain in these notes. For the convenience of the reader we have given some of the proofs here which had been left as an exercise in [19] whenever we had the feeling that the presentation of the material becomes more transparent thereby.

Since the whole paper bears an expository style we omit a lengthy introduction and just state briefly the content of the different sections:

In section 1 we recall the fundamental formalism of (non-commutative) determinants which were introduced first by Burns and Flach to formulate equivariant versions of the TNC. In section 2 we briefly discuss the setting of (realisations of) motives as they are used to formulate the conjectures concerning their L-functions, which are defined in section 3. There, also the absolute version of the TNC is discussed, which predicts the order of vanishing of the L-function at $s = 0$, the rationality and finally the precise value of the leading coefficient at $s = 0$ up to the period and regulator. In subsection 3.1 we sketch how one retrieves the BSD conjecture in its classical formulation if one

Date: September 22, 2005.

This survey was written during a stay at Centro de Investigacion en Matematicas (CIMAT), Mexico, as a Heisenberg fellow of the Deutsche Forschungsgemeinschaft (DFG) and I want to thank these institutions for their hospitality and financial support.

applies the TNC to the motive $h^1(A)(1)$ of an abelian variety $A/\mathbb{Q}$. Though well known to the experts this is not very explicit in the literature. In section 4 we consider a p-adic Lie extension of $\mathbb{Q}$ with Galois group G. In this context the TNC is extended to an equivariant version using the absolute version for all twists of the motive by certain representations of G. The compatibility of the ETNC with respect to Artin-Verdier/Poitou-Tate duality and the functional equation of the L-function is studied in section 5. A refinement leads to the formulation of the local ϵ-conjecture in subsection 5.1. In order to involve p-adic L-functions one has to introduce Selmer groups or better complexes. The necessary modifications of the L-function and the Galois cohomology - in a way that respects the functional equation - are described in section 6. From this the MC in the form of [11] is derived in subsection 6.2 after a short interlude concerning the new "localized K_1". In the Appendix we collect basic facts about Galois cohomology on the level of complexes.

Acknowledgements: I am very grateful to Takako Fukaya and Kazuya Kato for providing me with actual copies of their work and for answering many questions. I also would like to thank John Coates for his interest and Ramdorai Sujatha for reading parts of the manuscript and her valuable comments. Finally I am indebted to David Burns, Pedro Luis del Angel, Matthias Flach, Annette Huber, Adrian Iovita, Bruno Kahn and Guido Kings for helpful discussions.

1. Noncommutative determinants

The (absolute) TNC measures compares integral structures of Galois cohomology with values of complex L-functions. For this purpose the determinant is the adequate tool as is illustrated by the following basic

Example 1.1. Let T be a $\mathbb{Z}_p$-lattices in a finite dimensional $\mathbb{Q}_p$-vector space V and $f : T \to T$ a $\mathbb{Z}_p$-linear map which induces an automorphism of V. Then the cokernel of f is finite with cardinality $|\det(f)|_p^{-1}$ where $|-|_p$ denotes the p-adic valuation normalized as usual: $|p|_p = 1/p$.

Since the equivariant TNC involves the action of a possibly non-commutative ring R one needs a determinant formalism over an arbitrary (associative) ring R (with unit). This can be achieved by either using virtual objects a la Deligne as Burns and Flach [7, §2] do or by Fukaya and Kato's adhoc construction [19, 1.2], both approaches lead to an equivalent description.

Let $\mathrm{P}(R)$ denote the category of finitely generated projective R-modules and $(\mathrm{P}(R), is)$ its subcategory of isomorphisms, i.e. with the same objects, but whose morphisms are precisely the isomorphisms. Then there exists a category $\mathcal{C}_R$ and a functor

$$\mathbf{d}_R : (\mathrm{P}(R), is) \to \mathcal{C}_R$$

which satisfies the following properties:

a) $\mathcal{C}_R$ has an associative and commutative product structure $(M, N) \mapsto M \cdot N$ or written just MN with unit object $\mathbf{1}_R = \mathbf{d}_R(0)$ and inverses. All objects are of the form $\mathbf{d}_R(P)\mathbf{d}_R(Q)^{-1}$ for some $P, Q \in \mathrm{P}(R)$.
b) all morphisms of $\mathcal{C}_R$ are isomorphisms, $\mathbf{d}_R(P)$ and $\mathbf{d}_R(Q)$ are isomorphic if and only if their classes in $K_0(R)$ coincide. There is an identification of groups

$\mathrm{Aut}(\mathbf{1}_R) = K_1(R)$ and $\mathrm{Mor}(M,N)$ is either empty or an $K_1(R)$-torsor where $\alpha : \mathbf{1}_R \to \mathbf{1}_R \in K_1(R)$ acts on $\phi : M \to N$ as $\alpha\phi : M = \mathbf{1}_R \cdot M \overset{\alpha\cdot\phi}{\to} \mathbf{1}_R \cdot N = N$.

c) $\mathbf{d}_R$ preserves the "product" structures: $\mathbf{d}_R(P \oplus Q) = \mathbf{d}_R(P) \cdot \mathbf{d}_R(Q)$.

This functor can be naturally extended to complexes. Let $\mathrm{C}^p(R)$ be the category of bounded complexes in $\mathrm{P}(R)$ and $(\mathrm{C}^p(R), quasi)$ its subcategory of quasi-isomorphisms. For $C \in \mathrm{C}^p(R)$ we set $C^+ = \bigoplus_{i\ even} C^i$ and $C^- = \bigoplus_{i\ odd} C^i$ and define $\mathbf{d}_R(C) := \mathbf{d}_R(C^+)\mathbf{d}_R(C^-)^{-1}$ and thus we obtain a functor

$$\mathbf{d}_R : (\mathrm{C}^p(R), quasi) \to \mathcal{C}_R$$

with the following properties ($C, C', C'' \in \mathrm{C}^p(R)$)

d) If $0 \to C' \to C \to C'' \to 0$ is a short exact sequence of complexes, then there is a canonical isomorphism

$$\mathbf{d}_R(C) \cong \mathbf{d}_R(C')\mathbf{d}_R(C'')$$

which we take as an identification,

e) If C is acyclic, then the quasi-isomorphism $0 \to C$ induces a canonical isomorphism

$$\mathbf{1}_R \to \mathbf{d}_R(C).$$

f) $\mathbf{d}_R(C[r]) = \mathbf{d}_R(C)^{(-1)^r}$ where $C[r]$ denotes the r^{th} translate of C.

g) the functor $\mathbf{d}_R$ factorizes over the image of $\mathrm{C}^p(R)$ in $\mathrm{D}^p(R)$, the category of perfect complexes (as full triangulated subcategory of the derived category $\mathrm{D}^b(R)$ of the homotopy category of bounded complexes of R-modules), and extends to $(\mathrm{D}^p(R), is)$ (uniquely up to unique isomorphisms) [1].

h) If $C \in \mathrm{D}^p(R)$ has the property that all cohomology groups $\mathrm{H}^i(C)$ belong again to $\mathrm{D}^p(R)$, then there is a canonical isomorphism

$$\mathbf{d}_R(C) = \prod_i \mathbf{d}_R(\mathrm{H}^i(C))^{(-1)^i}.$$

Moreover, if R' is another ring, Y a finitely generated projective R'-module endowed with a structure as right R-module such that the actions of R and R' on Y commute, then the functor $Y \otimes_R - : \mathrm{P}(R) \to P(R')$ extends to a commutative diagram

$$\begin{array}{ccc} (\mathrm{D}^p(R), is) & \xrightarrow{\mathbf{d}_R} & \mathcal{C}_R \\ \downarrow{\scriptstyle Y\otimes^{\mathbb{L}}_R -} & & \downarrow{\scriptstyle Y\otimes_R -} \\ (\mathrm{D}^p(R'), is) & \xrightarrow{\mathbf{d}_{R'}} & \mathcal{C}_{R'} \end{array} .$$

In particular, if $R \to R'$ is a ring homomorphism and $C \in \mathrm{D}^p(R)$, we just write $\mathbf{d}_R(C)_{R'}$ for $R' \otimes_R \mathbf{d}_R(C)$.

[1] But property d) does not in general extend to arbitrary distinguished triangles, thus from a technical point of view all constructions involving complexes will have to be made carefully avoiding this problem. We will neglect this problem but see [7] for details.

Now let R° be the opposite ring of R. Then the functor $\mathrm{Hom}_R(-,R)$ induces an anti-equivalence between $\mathcal{C}_R$ and $\mathcal{C}_{R^\circ}$ with quasi-inverse induced by $\mathrm{Hom}_{R^\circ}(-,R^\circ)$; both functors will be denoted by $-^*$. This extends to a commutative diagram

$$\begin{array}{ccc} (\mathrm{D}^p(R), is) & \xrightarrow{\mathbf{d}_R} & \mathcal{C}_R \\ {\scriptstyle \mathrm{RHom}_R(-,R)}\downarrow & & \downarrow{\scriptstyle -^*} \\ (\mathrm{D}^p(R^\circ), is) & \xrightarrow{\mathbf{d}_{R^\circ}} & \mathcal{C}_{R^\circ} \end{array}$$

and similarly for $\mathrm{RHom}_{R^\circ}(-,R^\circ)$.

For the handling of the determinant functor in practice the following considerations are quite important:

Remark 1.2. (i) We have to distinguish at least two inverses of a map $\phi : \mathbf{d}_R(C) \to \mathbf{d}_R(D)$ with $C, D \in \mathrm{C}^p(R)$. The inverse with respect to composition will be denoted by $\overline{\phi} : \mathbf{d}_R(D) \to \mathbf{d}_R(C)$. But due to the product structure in $\mathcal{C}_R$ and the identification $\mathbf{d}_R(C) \cdot \mathbf{d}_R(C)^{-1} = \mathbf{1}_R$ the knowledge of ϕ is equivalent to that of

$$\mathbf{1}_R = \mathbf{d}_R(C) \cdot \mathbf{d}_R(C)^{-1} \xrightarrow{\phi \cdot \mathrm{id}_{\mathbf{d}_R(C)^{-1}}} \mathbf{d}_R(D) \cdot \mathbf{d}_R(C)^{-1}$$

or even

$$\phi^{-1} : \mathbf{d}_R(C)^{-1} \to \mathbf{d}_R(D)^{-1}$$

which is by definition $\overline{\mathrm{id}_{\mathbf{d}_R(D)^{-1}} \cdot \phi \cdot \mathrm{id}_{\mathbf{d}_R(C)^{-1}}}$ or in other words $\phi \cdot \phi^{-1} = \mathrm{id}_{\mathbf{1}_R}$. In particular, $\phi : \mathbf{d}_R(C) \to \mathbf{d}_R(C)$ corresponds uniquely to $\phi \cdot \mathrm{id}_{\mathbf{d}_R(C)^{-1}} : \mathbf{1}_R \to \mathbf{1}_R$. Thus it can and will be considered as an element in $K_1(R)$. Note that under this identification the elements in $K_1(R)$ assigned to each of ϕ^{-1} and $\overline{\phi}$ is the inverse to the element assigned to ϕ. Furthermore, the following relation between $\circ$ and $\cdot$ is easily verified: Let $A \xrightarrow{\phi} B$ and $B \xrightarrow{\psi} C$ be morphisms in $\mathcal{C}_R$. Then the composite $\psi \circ \phi$ is the same as the product $\psi \cdot \phi \cdot \mathrm{id}_{B^{-1}}$.

Convention: If $\phi : \mathbf{1} \to A$ is a morphism and B an object in $\mathcal{C}_R$, then we write $B \xrightarrow{\cdot\,\phi} B \cdot A$ for the morphism $\mathrm{id}_B \cdot \phi$. In particular, any morphism $B \xrightarrow{\phi} A$ can be written as $B \xrightarrow{\cdot\,(\mathrm{id}_{B^{-1}} \cdot\, \phi)} A$.

(ii) The determinant of the complex $C = [P_0 \overset{\phi}{\to} P_1]$ (in degree 0 and 1) with $P_0 = P_1 = P$ is by definition $\mathbf{d}_R(C) \overset{def}{=\!=} \mathbf{1}_R$ and is defined even if ϕ is not an isomorphism (in contrast to $\mathbf{d}_R(\phi)$). But if ϕ happens to be an isomorphism, i.e. if C is acyclic, then by e) there is also a canonical map $\mathbf{1}_R \overset{acyc}{\to} \mathbf{d}_R(C)$, which is in fact nothing else then

$$\mathbf{1}_R = \mathbf{d}_R(P_1)\mathbf{d}_R(P_1)^{-1} \xrightarrow{\mathbf{d}(\phi)^{-1} \cdot \mathrm{id}_{\mathbf{d}(P_1)^{-1}}} \mathbf{d}_R(P_0)\mathbf{d}_R(P_1)^{-1} = \mathbf{d}_R(C)$$

(and which depends in contrast to the first identification on ϕ). Hence, the composite $\mathbf{1}_R \overset{acyc}{\to} \mathbf{d}_R(C) \overset{def}{=\!=} \mathbf{1}_R$ corresponds to $\mathbf{d}_R(\phi)^{-1} \in K_1(R)$ according to the first remark. In order to distinguish the above identifications between $\mathbf{1}_R$ and $\mathbf{d}_R(C)$ we also say that C is *trivialized by the identity* when we refer to $\mathbf{d}_R(C) \overset{def}{=\!=} \mathbf{1}_R$ (or its inverse with respect to composition). For $\phi = \mathrm{id}_P$ both identifications agree obviously.

We end this section by considering the example where $R = K$ is a field and V a finite dimensional vector space over K. Then, according to [19, 1.2.4], $\mathbf{d}_K(V)$ can be identified with the highest exterior product $\bigwedge^{top} V$ of V and for an automorphism $\phi : V \to V$ the determinant $\mathbf{d}_K(\phi) \in K^\times = K_1(K)$ can be identified with the usual determinant $\det_K(\phi)$. In particular, we identify $\mathbf{d}_K = K$ with canonical basis 1. Then a map $\mathbf{1}_K \xrightarrow{\psi} \mathbf{1}_K$ corresponds uniquely to the value $\psi(1) \in K^\times$.

Remark 1.3. Note that every *finite* $\mathbb{Z}_p$-module A possesses a free resolution C as in Remark 1.2 (ii), i.e. $\mathbf{d}_{\mathbb{Z}_p}(A) \cong \mathbf{d}_{\mathbb{Z}_p}(C)^{-1} = \mathbf{1}_{\mathbb{Z}_p}$. Taking into account the above and Example 1.1 we see that modulo $\mathbb{Z}_p^\times$ the composite $\mathbf{1}_{\mathbb{Q}_p} \xrightarrow{acyc} \mathbf{d}_{\mathbb{Z}_p}(C)_{\mathbb{Q}_p} \overset{def}{=\!=} \mathbf{1}_{\mathbb{Q}_p}$ corresponds to the cardinality $|A| \in \mathbb{Q}_p^\times$.

2. K-Motives over $\mathbb{Q}$

In this survey we will be mainly interested in the Tamagawa Number Conjecture and Iwasawa theory for the motive $M = h^1(E)(1)$ of an elliptic curve E or the slightly more general $M = h^1(A)(1)$ of an abelian variety A defined over $\mathbb{Q}$. But as it will be important to consider certain twists of M we also recall basic facts on the Tate motive $\mathbb{Q}(1)$ and Artin motives. We shall simply view motives in the naive sense, as being defined by a collection of realizations satisfying certain axioms, together with their motivic cohomology groups. The archetypical motive is $h^i(X)$ for a smooth projective variety X over $\mathbb{Q}$ with its obvious étale cohomology $\mathrm{H}^i_{ét}(X \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_l)$, singular cohomology $\mathrm{H}^i(X(\mathbb{C}), \mathbb{Q})$ and de Rham cohomology $\mathrm{H}^i_{dR}(X/\mathbb{Q})$, their additional structures and comparison isomorphisms. More general, let K be a finite extension of $\mathbb{Q}$. A K-motive M over $\mathbb{Q}$, i.e. a motive over $\mathbb{Q}$ with an action of K, will be given by the following data, which for $M = h^n(X)_K$ arise by tensoring the above cohomology groups by K over $\mathbb{Q}$:

2.1. **The l-adic realization M_l of M (for every prime number l).** For a place λ of K lying above l and denote by K_λ the completion of K with respect to λ. Then M_λ is a continuous finite dimensional K_λ-linear representation of the absolute Galois group $G_{\mathbb{Q}}$ of $\mathbb{Q}$. We put $K_l := K \otimes_{\mathbb{Q}} \mathbb{Q}_l = \prod_{\lambda | l} K_\lambda$ and we denote by M_l the free K_l-module $\prod_{\lambda | l} M_\lambda$.

2.2. **The Betti realization M_B of M.** Attached to M is a finite dimensional K-vector space M_B which carries an action of complex conjugation ι and a $\mathbb{Q}$-Hodge structure $M_B \otimes_{\mathbb{Q}} \mathbb{C} \cong \bigoplus \mathcal{H}^{i,j}$ (over $\mathbb{R}$) with $\iota \mathcal{H}^{i,j} = \mathcal{H}^{j,i}$ where $\mathcal{H}^{i,j}$ are free $K_{\mathbb{C}} := K \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathbb{C}^{\Sigma_K}$-modules and where Σ_K denotes the set of all embeddings $K \to \mathbb{C}$. E.g. the motive $M = h^n(X)$ is pure of weight $w(M) = n$, i.e. $\mathcal{H}^{i,j} = 0$ if $i + j \neq n$.

2.3. **The de Rham realization M_{dR} of M.** M_{dR} is a finite dimensional K-vector space with a decreasing exhaustive filtration M^k_{dR}, $k \in \mathbb{Z}$. The quotient $t_M = M_{dR}/M^0_{dR}$ is called the *tangent space* of M.

2.4. **Comparison between M_B and M_l.** For each prime number l there is an isomorphism of K_l-modules

$$K_l \otimes_K M_B \xrightarrow[\cong]{g_l} M_l \tag{2.1}$$

which respects the action of complex conjugation, in particular it induces canonical isomorphisms

$$g_\lambda^+ : K_\lambda \otimes_K M_B^+ \cong M_\lambda^+ \text{ and } g_l^+ : K_l \otimes_K M_B^+ \cong M_l^+. \tag{2.2}$$

Here and in what follows, for any commutative ring R and $R[G(\mathbb{C}/\mathbb{R})]$-module X we denote by X^+ and X^- the R-submodule of X on which ι acts by $+1$ and -1, respectively.

2.5. **Comparison between M_B and M_{dR}.** There is a $G(\mathbb{C}/\mathbb{R})$-invariant isomorphism of $K_\mathbb{C}$-modules

$$\mathbb{C} \otimes_\mathbb{Q} M_B \xrightarrow[\cong]{g_\infty} \mathbb{C} \otimes_\mathbb{Q} M_{dR} \tag{2.3}$$

(on the left hand side ι acts diagonally while on the right hand side only on $\mathbb{C}$) such that for all $k \in \mathbb{Z}$

$$g_\infty\Big(\bigoplus_{i\geq k} \mathcal{H}^{i,j}(M)\Big) \cong \mathbb{C} \otimes_\mathbb{Q} M_{dR}^k.$$

This induces an isomorphism

$$(\mathbb{C} \otimes_\mathbb{Q} M_B)^+ \cong \mathbb{R} \otimes_\mathbb{Q} M_{dR} \tag{2.4}$$

and the period map

$$\mathbb{R} \otimes_\mathbb{Q} M_B^+ \xrightarrow{\alpha_M} \mathbb{R} \otimes_\mathbb{Q} t_M \tag{2.5}$$

We say that M is *critical* if this happens to be an isomorphism[2].

2.6. **Comparison between M_p and M_{dR}.** Let B_{dR} be the filtered field of de Rham periods with respect to $\overline{\mathbb{Q}_p}/\mathbb{Q}_p$, which is endowed with a continuous action of the absolute Galois group $G_{\mathbb{Q}_p}$ of $\mathbb{Q}_p$, and set as usual $D_{dR}(V) = (B_{dR} \otimes_{\mathbb{Q}_p} V)^{G_{\mathbb{Q}_p}}$ for a finite-dimensional $\mathbb{Q}_p$-vector space V endowed with a continuous action of $G_{\mathbb{Q}_p}$. The (decreasing) filtration B_{dR}^i of B_{dr} induces a filtration $D_{dR}^i = (B_{dR}^i \otimes_{\mathbb{Q}_p} V)^{G_{\mathbb{Q}_p}}$ of D_{dR}. Then there is an $G_{\mathbb{Q}_p}$-invariant isomorphism of filtered $K_p \otimes_{\mathbb{Q}_p} B_{dR}$-modules

$$B_{dR} \otimes_{\mathbb{Q}_p} M_p \xrightarrow[\cong]{g_{dR}} B_{dR} \otimes_\mathbb{Q} M_{dR} \tag{2.6}$$

which induces an isomorphism of filtered K_p-modules by taking $G_\mathbb{Q}$-invariants

$$D_{dR}(M_p) \xrightarrow[\cong]{g_{dR}} K_p \otimes_K M_{dR}, \tag{2.7}$$

[2]By [10, lem. 3] M is critical if and only if one of the following equivalent conditions holds: a) both infinite Euler factors $L_\infty(M,s)$ and $L_\infty(M^*(1),-s)$ (see section 5) are holomorphic at $s=0$, b) if $j<k$ and $\mathcal{H}^{j,k} \neq \{0\}$ then $j<0$ and $k\geq 0$, and, in addition, if $\mathcal{H}^{k,k} \neq \{0\}$, then ι acts on this space as $+1$ if $k<0$ and by -1 if $k \geq 0$. See also [12, lem. 2.3] for another criterion.

an isomorphism of K_p-modules

$$t(M_p) := D_{dR}(M_p)/D^0_{dR}(M_p) \xrightarrow[\cong]{g^t_{dR}} K_p \otimes_K t_M \tag{2.8}$$

and, for each place λ of K over p, an isomorphism of K_λ-vector spaces

$$t(M_\lambda) := D_{dR}(M_\lambda)/D^0_{dR}(M_\lambda) \xrightarrow[\cong]{g^t_{dR}} K_\lambda \otimes_K t_M. \tag{2.9}$$

The tensor product $M \otimes_K N$ of two K-motives is given by the data which arises from the tensor products of all realizations and their additional structures. Similar the dual M^* of the K-motive M is given by the duals of the corresponding realizations. In particular, we denote by $M(n)$, $n \in \mathbb{Z}$, the twist of M by the $|n|$-fold tensor product $\mathbb{Q}(n) = \mathbb{Q}(1)^{\otimes n}$ of the Tate motive if $n \geq 0$ and of its dual $\mathbb{Q}(-1) = \mathbb{Q}(1)^*$ if $n < 0$. For the motive $M = h^i(X)(j)$ where the dimension of X is d, Poincaré duality gives a perfect pairing

$$h^i(X)(j) \times h^{2d-i}(X)(d-j) \to h^{2d}(X)(d) \cong \mathbb{Q}$$

which identifies M^* with $h^{2d-i}(X)(d-j)$. Here $\mathbb{Q} = h^0(spec(\mathbb{Q}))(0)$ denotes the trivial $\mathbb{Q}$-motive.

Example 2.1. A) The Tate motive $\mathbb{Q}(1) = h^2(\mathbb{P}^1)^*$ should be thought of as $h_1(\mathbb{G}_m)$ even though the multiplicative group $\mathbb{G}_m$ is not proper. Its l-adic realisation is the usual Tate module $\mathbb{Q}_l(1)$ on which $G_\mathbb{Q}$ acts via the cyclotomic character $\chi_l : G_\mathbb{Q} \to \mathbb{Z}_l^\times$. The action of complex conjugation on $\mathbb{Q}(1)_B = \mathbb{Q}$ is by -1, its Hodge structure is pure of weight $w(M) = -2$ and given by $\mathcal{H}^{-1,-1}$. The filtration $\mathbb{Q}(1)^k_{dR}$ of $\mathbb{Q}(1)_{dR} = \mathbb{Q}$ is either $\mathbb{Q}$ or 0, according as $k \leq -1$ or $k > -1$, in particular we have $t_{\mathbb{Q}(1)} = \mathbb{Q}$. Finally, g_∞ sends $1 \otimes 1$ to $2\pi i \otimes 1$ while g_{dR} sends $1 \otimes 1$ to $t \otimes 1$, where $t =$ "$2\pi i$" is the p-adic period analogous to $2\pi i$.

B) For the $\mathbb{Q}$-motive $M = h^1(A)(1)$ of an abelian variety A over $\mathbb{Q}$ we have $M_l = H^1_{ét}(A_{\bar{\mathbb{Q}}}, \mathbb{Q}_l(1)) = \mathrm{Hom}_{\mathbb{Q}_l}(V_l A, \mathbb{Q}_l(1)) \cong V_l(A^\vee)$ via the Weil pairing. More generally, the Poincare bundle on $A \times A^\vee$ induces isomorphisms $M^*(1) \cong h^1(A^\vee)(1)$ and $M \cong h^1(A^\vee)^*$, while by fixing a (very) ample symmetric line bundle on A, whose existence is granted by [29, cor. 7.2], it is sometimes convenient to identify M with $h_1(A) := h^1(A)^*$ using the hard Lefschetz theorem ([34, 1.15,thm. 5.2 (iii)], see also [26]) (but in general better to work with the dual abelian variety $A^\vee$). Then M_l can be identified with $V_l(A)$, while $M_B = H^1(A(\mathbb{C}), \mathbb{Q})(1)$ can be identified with $H_1(A(\mathbb{C}), \mathbb{Q})$, the Hodge-decomposition (pure of weight -1) is given by $\mathcal{H}^{0,-1} = H^0(A(\mathbb{C}), \Omega^1_A)(\cong \mathrm{Hom}_\mathbb{C}(\mathrm{H}^1(A(\mathbb{C}), \Omega^0_A), \mathbb{C}))$ and $\mathcal{H}^{-1,0} = \mathrm{H}^1(A(\mathbb{C}), \Omega^0_A)(\cong \mathrm{Hom}_\mathbb{C}(\Omega^1(A), \mathbb{C}))$. Furthermore, we have $M^{-1}_{dR} = M_{dR}$, M^0_{dR} is the image of $\Omega^1_{A/\mathbb{Q}}(A)(\cong \mathrm{H}^1(A, \Omega^0_{A/\mathbb{Q}})^*)$ and $M^1_{dR} = 0$. In particular, $t_M = \mathrm{H}^1(A, \Omega^0_{A/\mathbb{Q}}) = Lie(A^\vee)$ (e.g. [27, thm. 5.11]) the Lie-algebra of $A^\vee$, can be identified with $t_{h_1(A)} = \mathrm{Hom}_\mathbb{Q}(\Omega^1_{A/\mathbb{Q}}(A), \mathbb{Q}) = Lie(A)$. The map α_M for the motive $M = h_1(A)$, which is in fact an isomorphism, is induced by sending a 1-cycle $\gamma \in \mathrm{H}_1(A(\mathbb{C}), \mathbb{Q})^+$ to $\int_\gamma \in \mathrm{Hom}_\mathbb{Q}(\Omega^1_{A/\mathbb{Q}}(A), \mathbb{R}) = Lie(A)_\mathbb{R}$ which sends a 1-form ω to $\int_\gamma \omega \in \mathbb{R}$.

C) Artin motives $[\rho]$ (with coefficients in a finite extension K of $\mathbb{Q}$) are direct summands of the K-motive $h^0(spec(F)) \otimes_\mathbb{Q} K$ but can also be identified with the category of finite-dimensional K-vector spaces V with an action by $G_{\mathbb{Q}}$, i.e. representations $\rho : G_{\mathbb{Q}} \to Aut_K(V)$ with finite image. We write $[\rho]$ for the corresponding motive and have $[\rho]_l = V \otimes_K K_l$ with $G_{\mathbb{Q}}$ acting just on V, $[\rho]_B = V$ with Hodge-Structure pure of Type $(0,0)$ and $[\rho]_{dR} = (V \otimes_{\mathbb{Q}} \bar{\mathbb{Q}})^{G_{\mathbb{Q}}}$, where $G_{\mathbb{Q}}$ acts diagonally. Since $[\rho]^k_{dR}$ is either $[\rho]_{dR}$ or 0 according as $k \leq 0$ of $k > 0$, we have $t_{[\rho]} = 0$. The inverse of g_∞ is induced by the natural inclusion $(V \otimes_{\mathbb{Q}} \bar{\mathbb{Q}})^{G_{\mathbb{Q}}} \subseteq V \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}$. E.g. if ψ denotes a Dirichlet character of conductor f considered via $(\mathbb{Z}/f\mathbb{Z})^* \cong G(\mathbb{Q}(\zeta_f)/\mathbb{Q})$ as character $G \to K^\times$ where $K = \mathbb{Q}(\zeta_{\varphi(f)})$ and φ denotes the Euler φ-function, then we obtain a basis of $[\psi]_{dR}$ over K by the Gauss sum

$$\sum_{1 \leq n < f, (n,f)=1} \psi(n) \otimes e^{-2\pi i n/f} \quad \in (K(\psi) \otimes_{\mathbb{Q}} \bar{\mathbb{Q}})^{G_{\mathbb{Q}}},$$

where $K(\psi)$ denotes the 1-dimensional K-vector space on which $G_{\mathbb{Q}}$ acts via ψ.

Of course, $h^0(spec(F)) \otimes_{\mathbb{Q}} K$ corresponds to the regular representation of $G(F/\mathbb{Q})$ on $K[G(F/\mathbb{Q})]$ considered as representation of $G_{\mathbb{Q}}$.

Other examples arise by taking symmetric products or tensor products of the above examples. In particular, we will be concerned with the motives

D) $[\rho] \otimes h^1(A)(1)$, where ρ runs through all Artin representations.
E) Finally, the motive $M(f)$ of a modular form is a prominent example, see [15, §7] and [33].

2.7. **Motivic cohomology.** The motivic cohomology K-vector spaces $\mathrm{H}^0_f(M) := \mathrm{H}^0(M)$ and $\mathrm{H}^1_f(M)$ may be defined by algebraic K-theory or motivic cohomology a la Voevodsky. They are conjectured to be finite dimensional. Instead of a general definition we just describe them in our standard examples.

Example 2.2. A) For the Tate motive we have $\mathrm{H}^0_f(\mathbb{Q}(1)) = \mathrm{H}^1_f(\mathbb{Q}(1)) = 0$ and for its Kummer dual $\mathrm{H}^0_f(\mathbb{Q}) = \mathbb{Q}$ while $\mathrm{H}^1_f(\mathbb{Q}) = 0$.

B) If $M = h^1(A)(1)$ for an abelian variety A over $\mathbb{Q}$ one has $\mathrm{H}^0_f(M) = 0$ and $\mathrm{H}^1_f(M) = A^\vee(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

C) For $M = h^0(spec(F))$ we have $\mathrm{H}^0_f(M) = \mathbb{Q}$ and $\mathrm{H}^1_f(M) = 0$ while for $M^*(1) = h^0(spec(F))(1)$ one has $\mathrm{H}^0_f(M^*(1)) = 0$ and $\mathrm{H}^1_f(M^*(1)) = \mathcal{O}_F^\times \otimes_{\mathbb{Z}} \mathbb{Q}$. More general, for an K-Artin motive $[\rho]$ one has $\mathrm{H}^0_f([\rho]) = K^n$, where n is the multiplicity with which $\mathbb{Q}$ occurs in $[\rho]$.

Unfortunately the functor H^i_f does not behave well with tensor products, i.e. in general one cannot derive $\mathrm{H}^*_f([\rho] \otimes h^1(A)(1))$ from $\mathrm{H}^*_1([\rho])$ and $\mathrm{H}^*_1(h^1(A)(1))$ (e.g. in form of a Künneth formula).

3. The Tamagawa Number Conjecture - absolute version

In [4] Bloch and Kato formulated a vast generalization of the analytic class number formula and the BSD-conjecture. While the conjecture of Deligne and Beilinson links the order of vanishing of the L-function attached to a motive M to its motivic cohomology and claims rationality of special L-values or more general leading coefficients (up to periods and regulators) the Tamagawa number conjecture by Bloch and Kato predicts the precise L-value in terms of Galois cohomology (assuming the conjecture of Deligne-Beilinson).

Later, Fontaine and Perrin-Riou [18] found an equivalent formulation using (commutative) determinants instead of (Tamagawa) measures[3]. In this section we follow closely their approach.

Let us first recall the definition of the complex L-function attached to a K-motive M. We fix a place λ of K lying over l and an embedding $K \to \mathbb{C}$. For every prime p take a prime $l \neq p$ and set

$$P_p(M_\lambda, X) = \det_{K_\lambda}(1 - \varphi_p X | (M_\lambda)^{I_p}) \in K_\lambda[X],$$

where φ_p denotes the geometric Frobenius automorphism of p in $G_{\mathbb{Q}_p}/I_p$ and I_p is the inertia subgroup of p in $G_{\mathbb{Q}_p} \subseteq G_{\mathbb{Q}}$. It is conjectured that $P_p(X)$ belongs to $K[X]$ and is independent of the choices of l and λ. For example this is known by the work of Deligne proving the Weil conjectures for $M = h^i(X)$ for places p where X has good reduction; by the compatibility of the system of l-adic realisations for abelian varieties [17, rem. 2.4.6(ii)] and Artin motives it is also clear for our examples A)-D). Then we have the L-function of M as Euler product

$$L_K(M, s) = \prod_p P_p(M_\lambda, p^{-s})^{-1},$$

defined and analytic for $\Re(s)$ large enough.

Example 3.1. A) The L-function $L_{\mathbb{Q}}(\mathbb{Q}(1), s-1)$ of the Tate motive is just the Riemann zeta function $\zeta(s)$. In general, one has $L_K(M(n), s) = L_K(M, s+n)$ for any K-motive M and any integer n.
B) If $M = h^1(A)(1)$ for an abelian variety A over $\mathbb{Q}$, then $L(M, s-1)$ is the classical Hasse-Weil L-function of $A^\vee$, which coincides with that for A because A and $A^\vee$ are isogenous.
C) $L_K([\rho], s)$ coincides with the usual Artin L-function of ρ, in particular we retrieve the Dedekind zeta-function $\zeta_F(s)$ as $L_{\mathbb{Q}}(h^0(spec(F), s)$.
D) The L-functions $L_K([\rho] \otimes h^1(A)(1), s)$ will play a crucial role for the interpolation property of the p-adic L-function.

Also, the meromorphic continuation to the whole plane $\mathbb{C}$ is part of the conjectural framework. The Taylor expansion

$$L_K(M, s) = L_K^*(M) s^{r(M)} + \dots$$

defines the leading coefficient $L_K^*(M) \in \mathbb{C}^\times$, which can be shown to belong to $\mathbb{R}^\times$ actually, and the order of vanishing $r(M) \in \mathbb{Z}$ of $L_K(M, s)$ at $s = 0$. The aim of the conjectures to be formulated now is to express $L^*(M)$ and $r(M)$ in terms of motivic and Galois cohomology.

Conjecture 3.2 (Order of Vanishing; Deligne-Beilinson)**.**

$$r(M) = \dim_K \mathrm{H}^1_f(M^*(1)) - \dim_K \mathrm{H}^0_f(M^*(1))$$

[3] The name comes from an analogy with the theory of algebraic groups, see [4].

According to the remark in [16] the duals of $\mathrm{H}^i_f(M^*(1))$ should be considered as "motivic cohomology with compact support $\mathrm{H}^{2-i}_c(M)$" and thus $r(M)$ is just their Euler characteristic. This explains why the Kummer duals $M^*(1)$ are involved here.

The link between the complex world, where the values $L^*(M)$ live, and the p-adic world, where the Galois cohomology lives, is formed by the *fundamental line* in $\mathcal{C}_K$ following the formulation of Fontaine and Perrin-Riou [18]:

$$\begin{aligned}\Delta_K(M) : \ &= \ \mathbf{d}_K(\mathrm{H}^0_f(M))^{-1}\mathbf{d}_K(\mathrm{H}^1_f(M))\mathbf{d}_K(\mathrm{H}^0_f(M^*(1))^*)\mathbf{d}_K(\mathrm{H}^1_f(M^*(1))^*)^{-1}\\ &\quad\ \mathbf{d}_K(M_B^+)\mathbf{d}_K(t_M)^{-1}.\end{aligned}$$

The relation of $\Delta_K(M)$ with the Betti and de Rham realization of M is given by the following

Conjecture 3.3 (Fontaine/Perrin-Riou)**.** *There exist an exact sequence of $K_\mathbb{R} := \mathbb{R}\otimes_\mathbb{Q} K$-modules*

$$0 \longrightarrow \mathrm{H}^0_f(M)_\mathbb{R} \xrightarrow{c} \ker(\alpha_M) \xrightarrow{r_B^*} (\mathrm{H}^1_f(M^*(1))_\mathbb{R})^*$$
$$\xrightarrow{h} \mathrm{H}^1_f(M)_\mathbb{R} \xrightarrow{r_B} \mathrm{coker}(\alpha_M) \xrightarrow{c^*} (\mathrm{H}^0_f(M^*(1))_\mathbb{R})^* \longrightarrow 0$$

where by $-_\mathbb{R}$ we denote the base change from $\mathbb{Q}$ to $\mathbb{R}$ (respectively from K to $K_\mathbb{R}$), c is the cycle class map into singular cohomology, r_B is the Beilinson regulator map and (if both $\mathrm{H}^1_f(M)$ and $\mathrm{H}^1_f(M^*(1))$ are nonzero so that M is of weight -1, then) h is a height pairing.

Example 3.4. A),C) For the motive $M = h^0(spec(F))$ the above exact sequence is just the $\mathbb{R}$-dual of the following

$$0 \longrightarrow \mathcal{O}_F^\times \otimes_\mathbb{Z} \mathbb{R} \xrightarrow{r} \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \xrightarrow{\Sigma} \mathbb{R} \longrightarrow 0\ ,$$

where r is the Dirichlet(=Borel) regulator map (see [5] for a comparison of the Beilinson and Borel regulator map) and r_1 and r_2 denote the number of real and complex places of F, respectively.
B) The Neron height pairing (see [3] and the references there)

$$<,>: A^\vee(\mathbb{Q}) \otimes_\mathbb{Z} \mathbb{R} \times A(\mathbb{Q}) \otimes_\mathbb{Z} \mathbb{R} \to \mathbb{R}$$

induces a homomorphism $A^\vee(\mathbb{Q}) \otimes_\mathbb{Z} \mathbb{R} \to \mathrm{Hom}_\mathbb{Z}(A(\mathbb{Q}), \mathbb{R})$, the inverse of which gives the exact sequence for the motive $M = h^1(A)(1)$.

We assume this conjecture. Using property e) and change of rings of the functor $\mathbf{d}$, it induces a canonical isomorphism (period-regulator map)

$$\vartheta_\infty : K_\mathbb{R} \otimes_K \Delta_K(M) \cong \mathbf{1}_{K_\mathbb{R}}. \tag{3.10}$$

Conjecture 3.5 (Rationality; Deligne-Beilinson)**.** *There is a unique isomorphism*

$$\zeta_K(M) : \mathbf{1}_K \to \Delta_K(M)$$

such that for every embedding $K \to \mathbb{C}$ we have

$$L^*_K(M) : \mathbf{1}_\mathbb{C} \xrightarrow{\zeta_K(M)_\mathbb{C}} \Delta_K(M)_\mathbb{C} \xrightarrow{(\vartheta_\infty)_\mathbb{C}} \mathbf{1}_\mathbb{C}$$

In other words, the preimage $\overline{\vartheta_\infty}(L_K^*(M))$ of $L_K^*(M)$ with respect to ϑ_∞ generates the K-vector space $\Delta_K(M)$ if the determinant functor is identified with the highest exterior product. Thus up to a period and a regulator (the determinant of ϑ_∞ with respect to a K-rational basis) the value $L_K^*(M)$ belongs to K.

The rationality enables us to relate $L_K^*(M)$ to the p-adic world which we will describe now.

Let S be a finite set of places of $\mathbb{Q}$ containing p,∞ and the places of bad reduction of M, then $U := spec(\mathbb{Z}[\frac{1}{S}])$ is an open dense subset of $spec(\mathbb{Z})$. Then we have complexes $\mathrm{R}\Gamma_c(U,M_p)$, $\mathrm{R}\Gamma_f(\mathbb{Q},M_p)$ and $\mathrm{R}\Gamma_f(\mathbb{Q}_v,M_p)$ calculating the (global) cohomology $\mathrm{H}^i_c(U,M_p)$ with compact support, the finite part of global and local cohomology, $\mathrm{H}^i_f(\mathbb{Q},M_p)$ and $\mathrm{H}^i_f(\mathbb{Q}_v,M_p)$, respectively, see 7. These complexes fit into a distinguished triangle (see (7.60))

$$\mathrm{R}\Gamma_c(U,M_p) \longrightarrow \mathrm{R}\Gamma_f(\mathbb{Q},M_p) \longrightarrow \bigoplus_{v\in S} \mathrm{R}\Gamma_f(\mathbb{Q}_v,M_p) \longrightarrow \cdot \tag{3.11}$$

On the other hand motivic cohomology specializes to the finite parts of global Galois cohomology:

Conjecture 3.6. *There are natural isomorphisms* $\mathrm{H}^0_f(M)_{\mathbb{Q}_l} \cong \mathrm{H}^0_f(\mathbb{Q},M_l)$ *(cycle class maps) and* $\mathrm{H}^1_f(M)_{\mathbb{Q}_l} \cong \mathrm{H}^1_f(\mathbb{Q},M_l)$ *(Chern class maps).*

Hence, as there is a duality $\mathrm{H}^i_f(\mathbb{Q},M_l) \cong \mathrm{H}^{3-i}_f(\mathbb{Q},M_l^*(1))^*$ for all i, this conjecture determines all cohomology groups $\mathrm{H}^i_f(\mathbb{Q},M_l)$.

Using properties d), g) and change of rings of the determinant functor, the conjecture 3.6 for $l=p$, the canonical isomorphisms (see appendix 7.64)

$$\eta_p(M_p): \quad \mathbf{1}_{K_p} \to \mathbf{d}_{K_p}(\mathrm{R}\Gamma_f(\mathbb{Q}_p,M_p))\cdot \mathbf{d}_{K_p}(t(M_p)), \tag{3.12}$$

$$\eta_l(M_p): \quad \mathbf{1}_{K_p} \to \mathbf{d}_{K_p}(\mathrm{R}\Gamma_f(\mathbb{Q}_l,M_p)), \tag{3.13}$$

the comparison isomorphisms (2.2) and (2.8) as well as (3.11), we obtain a canonical isomorphism (p-adic period-regulator map)

$$\vartheta_p : \Delta_K(M)_{K_p} \cong \mathbf{d}_{K_p}(\mathrm{R}\Gamma_c(U,M_p))^{-1}, \tag{3.14}$$

which induces for any place λ above p

$$\vartheta_\lambda : \Delta_K(M)_{K_\lambda} \cong \mathbf{d}_{K_\lambda}(\mathrm{R}\Gamma_c(U,M_\lambda))^{-1}. \tag{3.15}$$

Now let T_λ be a Galois stable $\mathcal{O}_\lambda$-lattice of M_λ and $\mathrm{R}\Gamma_c(U,T_\lambda)$ its Galois cohomology with compact support, see section 7. Here, $\mathcal{O}_\lambda$ denotes the valuation ring of K_λ. Note that by Artin-Verdier/Poitou-Tate duality (see 7.61) the "cohomology" $\mathrm{R}\Gamma_c(U,T_\lambda)$ with compact support can also be replaced by the complex $\mathrm{R}\Gamma(U,T_\lambda^*(1))^* \oplus (T_\lambda^*(1))^+$ where $\mathrm{R}\Gamma(U,T_\lambda^*(1))$ calculates as usual the global Galois cohomology with restricted ramification.

The following conjecture, for every prime p, gives a precise description of the special L-value $L_K^*(M)\in\mathbb{R}^\times$ up to $\mathcal{O}_K^\times$, i.e. up to sign if $K=\mathbb{Q}$, where O_K denotes the ring of integers in K:

Conjecture 3.7 (Integrality; Bloch/Kato, Fontaine/Perrin-Riou)**.** *Assume conjecture 3.5. Then for every place λ above p there exist a (unique) isomorphism*

$$\zeta_{\mathcal{O}_\lambda}(T_\lambda) : \mathbf{1}_{\mathcal{O}_\lambda} \to \mathbf{d}_{\mathcal{O}_\lambda}(\mathrm{R}\Gamma_c(U, T_\lambda))^{-1}$$

which induces via $K_\lambda \otimes_{\mathcal{O}_\lambda} -$ the following map

$$\zeta_{\mathcal{O}_\lambda}(T_\lambda)_{K_\lambda} : \mathbf{1}_{K_\lambda} \xrightarrow{\zeta_K(M)_{K_\lambda}} \Delta_K(M)_{K_\lambda} \xrightarrow{\vartheta_\lambda} \mathbf{d}_{K_\lambda}(\mathrm{R}\Gamma_c(U, M_\lambda))^{-1}. \tag{3.16}$$

If we identify again the determinant functor with the highest exterior product, this conjecture can be rephrased as follows: $\vartheta_\lambda\overline{\vartheta_\infty}(L_K^*(M))$ generates the $\mathcal{O}_\lambda$-lattice $\mathbf{d}_{K_\lambda}(\mathrm{R}\Gamma_c(U, T_\lambda))^{-1}$ of $\mathbf{d}_{K_\lambda}(\mathrm{R}\Gamma_c(U, M_\lambda))^{-1}$. In other words, this generator is determined up to a unit in $\mathcal{O}_\lambda$.

It can be shown that this conjecture is independent of the choice of S and T_λ.

Example 3.8. (Analytic class number formula) For the motive $M = h^0(spec(F))$ we have that $r(M) = r_1 + r_2 - 1$ if $F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ and that $L^*(M) = \frac{-|Cl(\mathcal{O}_F)|R}{|\mu(F)|}$ for the unit regulator R. Thus Conjectures 3.2,3.5 and 3.7 are theorems in this case!

For other known cases of these conjectures we refer the reader to the excellent survey [16], where in particular the results of Burns-Greither [8] and Huber-Kings [21] are discussed.

Another example will be discussed in the following section.

3.1. **Equivalence to classical formulation of BSD.**

I am very grateful to Matthias Flach for some advice concerning this section, in which we assume $p \neq 2$ for simplicity. In order to see that the above conjectures for the motive $M = h^1(A)(1)$ of an abelian variety A are equivalent to the classical formulation involving all the arithmetic invariants of A one has to consider also integral structures for the finite parts of global and local Galois cohomology. For T_p we take the Tate-module $T_p(A^\vee)$ of $A^\vee$. In particular one can define perfect complexes of $\mathbb{Z}_p$-modules $\mathrm{R}\Gamma_f(\mathbb{Q}, T_p)$ and $\mathrm{R}\Gamma_f(\mathbb{Q}_v, T_p)$ such that the analogue of (3.11) holds, see [6, §1.5]. We just state some results concerning their cohomology groups H^i_f in the following

Proposition 3.9 ([6, (1.35)-(1.37)])**.** *(a)(global) If the Tate-Shafarevich group $\text{Ш}(A/\mathbb{Q})$ is finite, then one has*

$$\mathrm{H}^0_f(\mathbb{Q}, T_p) = 0 \qquad \mathrm{H}^3_f(\mathbb{Q}, T_p) \cong \mathrm{Hom}_{\mathbb{Z}}(A(\mathbb{Q})_{tors}, \mathbb{Q}_p/\mathbb{Z}_p) \tag{3.17}$$

$$\mathrm{H}^1_f(\mathbb{Q}, T_p) \cong A^\vee(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Z}_p \qquad \mathrm{H}^i_f(\mathbb{Q}, T_p) = 0 \textit{ for } i \neq 0,1,2,3 \tag{3.18}$$

and an exact sequence of $\mathbb{Z}_p$-modules

$$0 \to \text{Ш}(A/\mathbb{Q})(p) \longrightarrow \mathrm{H}^2_f(\mathbb{Q}, T_p) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(A(\mathbb{Q}), \mathbb{Z}_p) \to 0. \tag{3.19}$$

(b)(local) For all primes l one has

$$\mathrm{H}^0_f(\mathbb{Q}_l, T_p) = 0 \qquad \mathrm{H}^1_f(\mathbb{Q}_l, T_p) \cong A^\vee(\mathbb{Q}_l)^{\wedge p} \qquad \mathrm{H}^i_f(\mathbb{Q}_l, T_p) = 0 \textit{ for } i \neq 0,1, \tag{3.20}$$

where $A^\vee(\mathbb{Q}_l)^{\wedge p}$ denotes the p-completion of $A^\vee(\mathbb{Q}_l)$.

Note that one has $\mathrm{H}^i_f(-,T_p)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p \cong \mathrm{H}^i_f(-,M_p)$ for the local and global versions.

Recall from Example 2.2 B) that we have

(3.21)

$$\Delta_{\mathbb{Q}}(M) = \mathbf{d}_{\mathbb{Q}}(A^\vee(\mathbb{Q})\otimes_{\mathbb{Z}}\mathbb{Q})\mathbf{d}_{\mathbb{Q}}(\mathrm{Hom}_{\mathbb{Z}}(A(\mathbb{Q}),\mathbb{Q}))^{-1}\mathbf{d}_{\mathbb{Q}}(H_1(A^\vee(\mathbb{C}),\mathbb{Q})^+)\mathbf{d}_{\mathbb{Q}}(Lie(A^\vee))^{-1}$$

In order to define the period and regulator we have to choose bases: we first fix $P_1^\vee,\dots,P_r^\vee \in A^\vee(\mathbb{Q})$ (respectively $P_1,\dots,P_r \in A(\mathbb{Q})$), $r = \mathrm{rk}_{\mathbb{Z}}(A^\vee(\mathbb{Q})) = \mathrm{rk}_{\mathbb{Z}}(A(\mathbb{Q}))$, such that setting $T_{A^\vee} := \bigoplus \mathbb{Z}P_i^\vee$ (respectively $T_A^d := \bigoplus \mathbb{Z}P_i^d \subseteq \mathrm{Hom}_{\mathbb{Z}}(A(\mathbb{Q}),\mathbb{Z})$, where P_i^d denotes the obvious dual "basis") we obtain

$$A^\vee(\mathbb{Q}) \cong A^\vee(\mathbb{Q})_{tor} \oplus T_{A^\vee} \qquad\qquad \mathrm{Hom}_{\mathbb{Z}}(A(\mathbb{Q}),\mathbb{Z}) \cong T_A^d. \tag{3.22}$$

Similarly we fix a $\mathbb{Z}$-basis $\gamma^+ = (\gamma_1^+,\dots,\gamma_{d_+}^+)$ of $T_B^+ := H_1(A^\vee(\mathbb{C}),\mathbb{Z})^+$ and $\mathbb{Z}$-basis $\delta = (\delta_1,\dots,\delta_{d_+})$ of the $\mathbb{Z}$-lattice $Lie_{\mathbb{Z}}(A^\vee) := \mathrm{Hom}_{\mathbb{Z}}(\Omega^1_{\mathcal{B}/\mathbb{Z}}(\mathcal{B}),\mathbb{Z})$ of $Lie(A^\vee)$, respectively. Here $\mathcal{B}/\mathbb{Z}$ denotes the (smooth, but not proper) Neron model of $A^\vee$ over $\mathbb{Z}$. Thus we obtain an integral structure of $\Delta_{\mathbb{Q}}(M)$:

$$\Delta_{\mathbb{Z}}(M) := \mathbf{d}_{\mathbb{Z}}(T_{A^\vee})\mathbf{d}_{\mathbb{Z}}(T_A^d)^{-1}\mathbf{d}_{\mathbb{Z}}(T_B^+)\mathbf{d}_{\mathbb{Z}}(Lie_{\mathbb{Z}}(A^\vee))^{-1} \tag{3.23}$$

together with a canonical isomorphism

$$\mathbf{1}_{\mathbb{Z}} \xrightarrow{can_{\mathbb{Z}}} \Delta_{\mathbb{Z}}(M) \tag{3.24}$$

induced by the above choices of bases.[4]

Define the period $\Omega_\infty^+(A)$ and the regulator R_A of A to be the determinant of the maps α_M and h with respect to the bases chosen above, respectively. Then Conjecture 3.5 tells us that

$$\zeta_{\mathbb{Q}}(M) = \frac{L^*(M)}{\Omega_\infty^+(A)\cdot R_A}\cdot can_{\mathbb{Q}}, \tag{3.25}$$

where $can_{\mathbb{Q}} : \mathbf{1}_{\mathbb{Q}} \to \Delta_{\mathbb{Q}}(M)$ is induced from $can_{\mathbb{Z}}$ by base change. Indeed, we have $\Omega_\infty^+(A)R_A = (\vartheta_\infty)_{\mathbb{C}}\circ(can_{\mathbb{Q}})_{\mathbb{C}}$ and $L^*(M) = (\vartheta_\infty)_{\mathbb{C}}\circ(\zeta_{\mathbb{Q}}(M))_{\mathbb{C}}$ in $\mathrm{Aut}_{\mathcal{C}_{\mathbb{C}}}(\mathbf{1}_{\mathbb{C}}) = \mathbb{C}^\times$ and thus $\zeta_{\mathbb{Q}}(M)$ differs from $can_{\mathbb{Q}}$ by $\frac{L^*(M)}{\Omega_\infty^+(A)\cdot R_A}$.

On the other hand, using among others property h) of the determinant functor, Proposition 3.9 and the identification $T_B^+\otimes_{\mathbb{Z}}\mathbb{Z}_p \cong T_p^+$ (induced from (2.2)) one easily verifies that there is an isomorphism

$$\begin{aligned}\Delta_{\mathbb{Z}_p}(M) := \Delta_{\mathbb{Z}}(M)_{\mathbb{Z}_p} \cong\ & \mathbf{d}_{\mathbb{Z}_p}(\mathrm{R}\Gamma_f(\mathbb{Q},T_p))^{-1}\mathbf{d}_{\mathbb{Z}_p}(T_p^+)\mathbf{d}_{\mathbb{Z}_p}(Lie_{\mathbb{Z}_p}(A^\vee))^{-1}\\ &\cdot \mathbf{d}_{\mathbb{Z}_p}(Ш(A/\mathbb{Q})(p))\mathbf{d}_{\mathbb{Z}_p}(A(\mathbb{Q})(p))^{-1}\mathbf{d}_{\mathbb{Z}_p}(A^\vee(\mathbb{Q})(p))^{-1},\end{aligned} \tag{3.26}$$

where $Lie_{\mathbb{Z}_p}(A^\vee) := Lie_{\mathbb{Z}}(A^\vee)\otimes_{\mathbb{Z}}\mathbb{Z}_p$ is a $\mathbb{Z}_p$-lattice of $t(M_p) \cong \mathrm{H}^1(A,\mathcal{O}_A)\otimes_{\mathbb{Q}}\mathbb{Q}_p$.

In order to compare this with the integral structure $\mathrm{R}\Gamma_c(U,T_p)$ of $\mathrm{R}\Gamma_c(U,M_p)$ we have to introduce the *local Tamagawa numbers* $c_l(M_p)$ [18, I §4].

[4] The choice of the basis $P_i^\vee$ of $T_{A^\vee}$ induces a map $\mathbb{Z}^r \to T_{A^\vee}$ and, taking determinants, $can_{P^\vee} : \mathbf{d}_{\mathbb{Z}}(\mathbb{Z}^r) \to \mathbf{d}_{\mathbb{Z}}(T_{A^\vee})$. Similarly, we obtain canonical isomorphisms can_{P^d}, can_{γ^+} and can_δ for T_A^d, T_B^+ and $Lie_{\mathbb{Z}}(A^\vee)$, respectively. Set $can_{\mathbb{Z}} := can_{P^\vee}\cdot can_{P^d}^{-1}\cdot can_{\gamma^+}\cdot can_\delta^{-1}$.

We first assume $l \neq p$. Then there is an exact sequence (cf. [18, §4.2])

$$0 \longrightarrow T_p^{I_l} \xrightarrow{1-\phi_l} T_p^{I_l} \longrightarrow \mathrm{H}^1_f(\mathbb{Q}_l, T_p) \longrightarrow \mathrm{H}^1(I_l, T_p)_{tors}^{G_{\mathbb{Q}_l}} \longrightarrow 0 \tag{3.27}$$

which induces an isomorphism

$$\begin{aligned}\psi_l : \mathbf{1}_{\mathbb{Z}_p} \to \mathbf{d}_{\mathbb{Z}_p}([\, T_p^{I_l} \xrightarrow{1-\phi_l} T_p^{I_l}\,]) &\cong \mathbf{d}_{\mathbb{Z}_p}(\mathrm{R}\Gamma_f(\mathbb{Q}_l, T_p))\mathbf{d}_{\mathbb{Z}_p}(\mathrm{H}^1(I_l, T_p)_{tors}^{G_{\mathbb{Q}_l}})^{-1} \\ &\cong \mathbf{d}_{\mathbb{Z}_p}(\mathrm{R}\Gamma_f(\mathbb{Q}_l, T_p)).\end{aligned} \tag{3.28}$$

Here the first map arises as trivialization by the identity, the second comes from the above exact sequence (interpreted as short exact sequence of complexes) while the last comes again from trivializing by the identity according to Remark 1.3.

We define $c_l(M_p) := |\mathrm{H}^1(I_l, T_p)_{tors}^{G_{\mathbb{Q}_l}}|$ and remark that $(\psi_l)_{\mathbb{Q}_p}$ differs from $\eta_l(M_p)$ precisely by the map $\mathbf{1}_{\mathbb{Q}_p} \xrightarrow{acyc} \mathbf{d}_{\mathbb{Z}_p}(\mathbf{d}_{\mathbb{Z}_p}(\mathrm{H}^1(I_l, T_p)_{tors}^{G_{\mathbb{Q}_l}})_{\mathbb{Q}_p} \overset{def}{=\!=} \mathbf{1}_{\mathbb{Q}_p}$, which appealing to Remark 1.3 we also denote by $c_l(M_p)$. In other words we have

$$(\psi_l)_{\mathbb{Q}_p} = c_l(M_p) \cdot \eta_l(M_p). \tag{3.29}$$

Note also, that one has $|A^\vee(\mathbb{Q}_l) \otimes_{\mathbb{Z}} \mathbb{Z}_p| = |P_l(M_p, 1)|_p^{-1} \cdot c_l(M_p)$ and that $c_l(M_p) = 1$ whenever A has good reduction at l.

It can be shown [35, Exp. IX,(11.3.8)] that c_l is the order of the p-primary part of the group of $\mathbb{F}_l$-rational components $(\mathcal{E}/\mathcal{E}^0)(\mathbb{F}_l) \cong \mathcal{E}(\mathbb{F}_l)/\mathcal{E}^0(\mathbb{F}_l) \cong A(\mathbb{Q}_p)/A_0(\mathbb{Q}_p)$ of the special fibre $\mathcal{E} := \mathcal{A}_{\mathbb{F}_l}$ of the smooth (but not necessarily proper) Neron model $\mathcal{A}$ of A over $\mathbb{Z}$.[5]

Now let $l = p$. Similarly one defines maps (both depending on the choice of δ)

$$\psi_p : \mathbf{1}_{\mathbb{Z}_p} \to \mathbf{d}_{\mathbb{Z}_p}(\mathrm{R}\Gamma_f(\mathbb{Q}_p, T_p))\mathbf{d}_{\mathbb{Z}_p}(Lie_{\mathbb{Z}_p}(A^\vee)), \tag{3.30}$$

$$c_p(M_p) : \mathbf{1}_{\mathbb{Q}_p} \to \mathbf{1}_{\mathbb{Q}_p}, \tag{3.31}$$

such that

$$(\psi_p)_{\mathbb{Q}_p} = c_p(M_p) \cdot \eta_p(M_p) \tag{3.32}$$

holds.[6]

[5]The first isomorphism is a consequence of the theorem of Lang [28] that the map $x \mapsto \phi(x)x^{-1}$ on the $\overline{k}$-rational points of a connected algebraic group over a finite field k (with Frobenius ϕ) is surjective.

[6]Assume that A has dimension d and let $\widehat{B}$ be the formal group of $A^\vee$ over $\mathbb{Z}_p$, i.e. the formal completion of $\mathcal{B}$ along the zero-section in the fibre over p. Note that $Lie_{\mathbb{Z}_p}(A^\vee)$ can be identified with the tangent space $t_{\widehat{B}}(\mathbb{Z}_p)$ of $\widehat{B}$ with values in $\mathbb{Z}_p$ (a good reference for formal groups is [14]). Furthermore we write $\widehat{\mathbb{G}}_a$ for the formal additive group over $\mathbb{Z}_p$, $\mathcal{B}^\circ$ and $\widetilde{B}^\circ$ for the connected component of the identity of $\mathcal{B}$ and its fibre $\widetilde{B}$ over p, respectively, and Φ for the group of connected components of $\widetilde{B}$. Again by Lang's theorem we have $\Phi(\mathbb{F}_p) = \widetilde{B}(\mathbb{F}_p)/\widetilde{B}^\circ(\mathbb{F}_p)$. Moreover there are exact sequences

$$0 \longrightarrow \mathcal{B}^\circ(\mathbb{Z}_p) \longrightarrow \mathcal{B}(\mathbb{Z}_p) \longrightarrow \Phi(\mathbb{F}_p) \longrightarrow 0$$

and

$$0 \longrightarrow \widehat{B}(\mathbb{Z}_p) \longrightarrow \mathcal{B}^\circ(\mathbb{Z}_p) \longrightarrow \widetilde{B}^\circ(\mathbb{F}_p) \longrightarrow 0.$$

Using the integral version of (3.11), the maps ψ_l (analogously as η_l for ϑ_p (3.14))induce a canonical map

$$\kappa_p : \Delta_{\mathbb{Z}_p}(M) \to \mathbf{d}_{\mathbb{Z}_p}(\mathrm{R}\Gamma_c(\mathbb{Q}, T_p)) \tag{3.33}$$

where all the terms $\mathbf{d}_{\mathbb{Z}_p}(Ш(A/\mathbb{Q})(p))$, $\mathbf{d}_{\mathbb{Z}_p}(A(\mathbb{Q})(p))^{-1}$ and $\mathbf{d}_{\mathbb{Z}_p}(A^\vee(\mathbb{Q})(p))^{-1}$ are trivialized by the identity. Hence, using again Remark 1.3, we have

$$(\kappa_p)_{\mathbb{Q}_p} = \frac{|Ш(A/\mathbb{Q})|}{|A(\mathbb{Q})_{tors}||A^\vee(\mathbb{Q})_{tors}|} \prod c_l(M_p) \cdot \vartheta_p \tag{3.34}$$

modulo $\mathbb{Z}_p^\times$. Since $\zeta_{\mathbb{Z}_p}(T_p)$ equals $\kappa_p \circ can_{\mathbb{Z}_p}$ up to an element in $\mathbb{Z}_p^\times$, it follows immediately from (3.34),(3.25) and (3.16) that

$$\frac{L^*(M)}{\Omega_\infty^+(A) \cdot R_A} \sim \frac{|Ш(A/\mathbb{Q})|}{|A(\mathbb{Q})_{tors}||A^\vee(\mathbb{Q})_{tors}|} \prod c_l(M_p) \mod \mathbb{Z}_p^\times. \tag{3.35}$$

For all primes p this implies the classical statement of the BSD-Conjecture up to sign (and a power of 2 due to our restriction $p \neq 2$).

4. The TNC - equivariant version

The first equivariant version of the TNC with commutative coefficients (other than number fields) was given by Kato [24, 23] observing that classical Iwasawa theory is, roughly speaking, nothing else than the ETNC for a "big" coefficient ring. Inspired by Kato's work Burns and Flach formulated an ETNC where the coefficients of the motive are allowed to be (possibly non-commutative) finite-dimensional $\mathbb{Q}$-algebras, using for the first time the general determinant functor described in section 1 and relative K-groups. Their systematic approach recovers all previous versions of the TNC and more over all central conjectures of Galois module theory. It were Huber and Kings [20] who realized that the formulation of the ETNC by relative K-groups is equivalent to the perhaps more suggestive use of "generators," i.e. maps of the form $\mathbf{1}_R \to \mathbf{d}_R(?)$ in the category $\mathcal{C}_R$ for various rings R instead, see also Flach's survey [16, §6]. They used this approach to give - for motives of the form $M^*(1-k)$ with k big enough, i.e. with very negative weight - the first version of a ETNC over general p-adic Lie extensions, which they call Iwasawa Main Conjecture (while in this survey we reserve this name for versions involving p-adic L-functions). While Burns and Flach use "equivariant"

Now the logarithm map

$$Lie_{\mathbb{Z}_p}(A^\vee) \supseteq (p\mathbb{Z}_p)^d = \widehat{\mathbb{G}}_a(\mathbb{Z}_p)^d \xleftarrow[log]{\cong} \widehat{B}(\mathbb{Z}_p) \subseteq A^\vee(\mathbb{Q}_p)^{\wedge p} = \mathrm{H}^1_f(\mathbb{Q}_p, T_p)$$

induces the map ψ_p by trivializing all finite subquotients of the above line by the identity. Note that the first subquotient on the left has order p^d. Using [4, ex. 3.11], which says that the Bloch-Kato exponential map coincides, up to the identification induced by the Kummer map, with the usual exponential map of the corresponding formal group, it is easy to see that $c_p(M_p) := \eta_p^{-1} \cdot (\psi_p)_{\mathbb{Q}_p} = \bar{\eta_p} \circ (\psi_p)_{\mathbb{Q}_p}$ equals modulo $\mathbb{Z}_p^\times$

$$c_p(M_p) \quad = \quad p^{-d}|P(M_p,1)|_p^{-1} \# \widetilde{B}^\circ(\mathbb{F}_p)(p) \# \Phi(\mathbb{F}_p)(p) = \#\Phi(\mathbb{F}_p)(p),$$

where we used the relation $|P(M_p,1)|_p = |P(M_l,1)|_p = p^{-d} \# \widetilde{B}^\circ(\mathbb{F}_p)(p)$. For elliptic curves this is well known [36, appenndix §16], the general case is an exercise using the description of the reduction of abelian varieties in [35, Exp. IX].

motives and L-functions in their general formalism, Fukaya and Kato realized that, at least for the connection with Iwasawa theory which we have in mind, it is sufficient to use non-commutative coefficients only for the Galois cohomology, but to stick to number fields as coefficients for the involved motives. In this survey we closely follow their approach.

To be more precise, consider for any motive M the motive $h^0(spec(F)) \otimes M$ (both defined over $\mathbb{Q}$) for some finite Galois extension F of $\mathbb{Q}$ with Galois group $G = G(F/\mathbb{Q})$. This motive has a natural action by the group algebra $\mathbb{Z}[G]$ and thus will be of particular interest for Iwasawa theory where a whole tower of finite extensions F_n of $\mathbb{Q}$ is considered simultaneously. Since there is an isomorphism of K-motives (for K sufficiently big)

$$h^0(spec(F))_K \otimes M \cong \bigoplus_{\rho \in \widehat{G}} [\rho^*]^{n_\rho} \otimes M$$

where ρ runs through all absolute irreducible representations of G and n_ρ denotes the multiplicity with which it occurs in the regular representation of G on $K[G]$, it suffices - on the complex side - to consider the collection of K-motives $[\rho^*] \otimes M$ and their L-functions or more precisely the corresponding leading terms and vanishing orders. Indeed, the $\mathbb{C}$-algebra $\mathbb{C}[G]$ can be identified with $\prod_{\rho \in \widehat{G}} M_{n_\rho}(\mathbb{C})$ and thus its first K-group identifies with $\prod_{\rho \in \widehat{G}} \mathbb{C}^\times \cong \mathrm{center}(\mathbb{C}[G])$. In contrast, on the p-adic side, even more when integrality is concerned, such a decomposition for $\mathbb{Z}_p[G]$ is impossible in general.

This motivated Fukaya and Kato to choose the following form of the ETNC. In fact, in order to keep the presentation concise, we will only describe a small extract of their complex and much more general treatment.

Let F be a p-adic Lie extension of $\mathbb{Q}$ with Galois group $G = G(F/\mathbb{Q})$. By $\Lambda = \Lambda(G)$ we denote its Iwasawa algebra. For a $\mathbb{Q}$ -motive M over $\mathbb{Q}$ we fix a $G_{\mathbb{Q}}$-stable $\mathbb{Z}_p$ -lattice T_p of M_p and define a left Λ-module

$$\mathbb{T} := \Lambda \otimes_{\mathbb{Z}_p} T_p$$

on which Λ acts via multiplication on the left factor from the left while $G_{\mathbb{Q}}$ acts diagonally via $g(x \otimes y) = x\bar{g}^{-1} \otimes g(y)$, where $\bar{g}$ denotes the image of $g \in G_{\mathbb{Q}}$ in G. This is a "big Galois representation" in the sense of Nekovar [31]. Choose S as in the previous section and such that $\mathbb{T}$ is unramified outside S and denote, for any number field F', by $G_S(F')$ the Galois group of the maximal outside S unramified extension of F'. Then by Shapiro's Lemma the cohomology of $\mathrm{R}\Gamma(U, \mathbb{T})$ for example is nothing else than the perhaps more familiar $\Lambda(G)$-module $\mathrm{H}^i_{Iw}(F, T_p) := \varprojlim_{F'} \mathrm{H}^i(G_S(F'), T_p)$ where the limit is taken with respect to corestriction and F' runs over all finite subextensions of $F/\mathbb{Q}$.

Let K be a finite extension of $\mathbb{Q}$, λ a finite place of K, $\mathcal{O}_\lambda$ the ring of integers of the completion K_λ of K at λ and assume that $\rho : G \to GL_n(\mathcal{O}_\lambda)$ is a continuous representation of G which, for some suitable choice of a basis, is the λ-adic realization N_λ of a some K-motive N. We also write ρ for the induced ring homomorphism $\Lambda \to M_n(\mathcal{O}_\lambda)$ and we consider $\mathcal{O}_\lambda^n$ as a right Λ-module via action by ρ^t on the left, viewing $\mathcal{O}_\lambda^n$ as set of column vectors (contained in K_λ^n.) Note that, setting $M(\rho^*) := N^* \otimes M$, we obtain an isomorphism of Galois representations

$$\mathcal{O}_\lambda^n \otimes_\Lambda \mathbb{T} \cong T_\lambda(M(\rho^*)),$$

where $T_\lambda(M(\rho^*))$ is the $\mathcal{O}_\lambda$-lattice $\rho^* \otimes T_p$ of $M(\rho^*)_\lambda$ and ρ^* denotes the contragredient(=dual) representation of ρ.

Now the equivariant version of conjecture 3.7 reads as follows

Conjecture 4.1 (Equivariant Integrality; Fukaya/Kato)**.** *There exists a (unique)*[7] *isomorphism*

$$\zeta_\Lambda(M) := \zeta_\Lambda(\mathbb{T}) : \mathbf{1}_\Lambda \to \mathbf{d}_\Lambda(\mathrm{R}\Gamma_c(U, \mathbb{T}))^{-1}$$

with the following property:

For all K, λ and ρ as above the (generalized) base change $\mathcal{O}_\lambda^n \otimes_\Lambda -$ sends $\zeta_\Lambda(M)$ to $\zeta_{\mathcal{O}_\lambda}(T_\lambda(M(\rho^*)))$.

Note that this conjecture assumes conjecture 3.7 for all K-motives $M(\rho^*)$ with varying K. Furthermore, it is independent of the choice of S and of the lattices $T_p(M)$ and $T_\lambda(M(\rho^*))$.

One obtains a slight modification - to which we will refer as the *Artin-version* - of the above conjecture by restricting the representations ρ in question to the class of all Artin representations of G, (i.e. having finite image). If $F/\mathbb{Q}$ is finite, both versions coincide. Moreover, it is easy to see[8] that in this situation the conjecture is equivalent to (the p-part of) Burns and Flach's equivariant integrality conjecture [7, conj. 6] for the $\mathbb{Q}$-algebra $\mathbb{Q}[G]$ with $\mathbb{Z}$-order $\mathbb{Z}[G]$. Also, $\mathbb{T} = \mathbb{Z}_p[G] \otimes T_p(M)$ identifies with the induced representation $\mathrm{Ind}_{G_\mathbb{Q}}^{G_F} T_p(M)$.

Assume now that $F = \bigcup_n F_n$ is the union of finite extensions F_n of $\mathbb{Q}$ with Galois groups G_n. Putting $\zeta_{\mathbb{Q}_p[G_n]}(M) = \mathbb{Q}_p[G_n] \otimes_\Lambda \zeta_\Lambda(M)$ one recovers the "generator" $\delta_p(G_n, M, k)$[9] (for k big enough) in [20] as $\zeta_{\mathbb{Q}_p[G_n]}(M^*(1-k))$. Hence, up to shifting and Kummer duality, the Artin-version of conjecture 4.1 for F is (morally) equivalent to [20, Conj. 3.2.1]. Hence, using [20, lem. 6.0.2] we obtain the following

7 In fact, Fukaya and Kato assign such an isomorphism to each pair $(R, \mathbb{T})$ where R belongs to a certain class of rings containing the Iwasawa algebras for arbitrary p-adic Lie extensions of $\mathbb{Q}$ as well as the valuation rings of finite extensions of $\mathbb{Q}_p$ and where $\mathbb{T}$ is a projective R-module endowed with a continuous $G_\mathbb{Q}$-action. Then $\zeta_?(?)$ is supposed to behave well under arbitrary change of rings for such pairs. Moreover they require that the assignment $\mathbb{T} \mapsto \zeta_R(\mathbb{T})$ is multiplicative for short exact sequences. Only this full set of conditions leads to the uniqueness [19, §2.3.5], while e.g. for finite a group G the map $K_1(\mathbb{Z}_p[G]) \to K_1(\mathbb{Q}[G])$ need not be injective and thus $\zeta_{\mathbb{Z}_p[G]}(?)$ might not be unique if considered alone.

8 If we assume that K is big enough such that $K[G]$ decomposes completely into matrix algebras with coefficients in K, then the equivariant integrality statement (inducing (absolute) integrality for $M(\rho^*)$ for all Artin representations of G) amounts to an integrality statement for the generator $\zeta_{K_\lambda[G]}(M) := K_\lambda[G] \otimes_{\mathbb{Z}_p[G]} \zeta_{\mathbb{Z}_p[G]}(M)$ and thus Burns and Flach's version for the $\mathbb{Q}$-algebra $K[G]$ with order $\mathcal{O}_K[G]$. Using the functorialities of their construction [7, thm. 4.1], it is immediate that taking norms leads to the conjecture for the pair $(\mathbb{Q}[G], \mathbb{Z}[G])$.

9 To be precise, this is only morally true, since Huber and Kings take for the definition of their generators the leading coefficients of the modified L-function without the Euler factors in S. It is not clear to what extent this is compatible with our formulation above

Proposition 4.2 (Huber/Kings). *Assume Conjectures 3.2, 3.5 and 3.7 for all $M(\rho^*)$ where ρ varies over all absolutely irreducible Artin representations of G. Then the existence of $\zeta_{\Lambda(G)}(M)$ satisfying the Artin-version of Conjecture 4.1 is equivalent to the existence of $\zeta_{\Lambda(G_n)}(M)$ for all n.*

In general, as remarked in footnote 7, $\zeta_{\mathbb{Z}_p[G_n]}(M)$ might not be unique (if it is considered alone). But it is realistic to hope uniqueness for infinite G (cf. [25]) and then the previous zeta isomorphism would be unique by the requirement $\zeta_{\mathbb{Z}_p[G_n]}(M) = \mathbb{Z}_p[G_n] \otimes_{\Lambda(G)} \zeta_{\Lambda(G)}(M)$. Indeed, this is true at least if G is big enough, see [19, prop. 2.3.7]. Moreover, as Huber and Kings [20, §3.3] pointed out, by twist invariance (over trivializing extensions $F/\mathbb{Q}$ for a given motive) arbitrary zeta-isomorphism $\zeta_\Lambda(M)$ are reduced to those of the form $\zeta_{\mathbb{Z}_p[G(F/\mathbb{Q})]}(\mathbb{Q})$ for the trivial motive and where F runs through all finite extensions of $\mathbb{Q}$.

Question: Does the Artin-version imply the full version of Conjecture 4.1 ?

5. The functional equation and ϵ-isomorphisms

The L-function of a $\mathbb{Q}$-motive satisfies conjecturally a functional equation, which we want to state in the following way (to ease the notation we suppress the subscript $\mathbb{Q}$ in this section)

$$L(M,s) = \epsilon(M,s)\frac{L_\infty(M^*(1),-s)}{L_\infty(M,s)}L(M^*(1),-s)$$

where the factor L_∞ at infinity is built up by certain Γ-factors and certain powers of 2 and π depending on the Hodge structure of M_B. The ϵ-factor decomposes into local factors

$$\epsilon(M,s) = \prod_{v\in S} \epsilon_v(M,s),$$

where the definition for finite places is recalled in footnotes 11 and 14; $\epsilon_\infty(M,s)$ is a constant equal to a power of i.[10] We assume this conjecture. Then, taking leading coefficients induces

$$L^*(M) = (-1)^\eta \epsilon(M)\frac{L^*_\infty(M^*(1))}{L^*_\infty(M)}L^*(M^*(1))$$

where $\epsilon(M) = \prod \epsilon_v(M)$ with $\epsilon_\nu(M) = \epsilon(M,0)$ and η denotes the order of vanishing at $s=0$ of the completed L-function $L_\infty(M^*(1),s)L(M^*(1),s)$.

Example 5.1. For the motive $M = h^1(A)(1)$ of an abelian variety one has $L_\infty(M,s) = L_\infty(M^*(1),s) = 2(2\pi)^{-(s+1)}\Gamma(s+1)$, $L^*_\infty(M) = L^*_\infty(M^*(1)) = \pi^{-1}$, $\epsilon_\infty(M) = -1$ and $\eta = 0$.

It is in no way obvious that the ETNC is compatible with the functional equation and Artin-Verdier/Poitou-Tate duality. The following discussion is a combination and reformulation of [7, section 5] and [32, Appendix C]. In order to formulate the precise

[10] We fix once and for all the complex period $2\pi i$, i.e. a square root of -1, and, for every l, the l-adic period $t =$ "$2\pi i$", i.e. a generator of $\mathbb{Z}_l(1)$.

condition under which the compatibility holds we first return to the absolute case and define "difference" terms

$$L^*_{dif}(M) := L^*(M)L^*(M^*(1))^{-1} = (-1)^\eta \epsilon(M)\frac{L^*_\infty(M^*(1))}{L^*_\infty(M)}$$

and

$$\Delta_{dif}(M) := \mathbf{d}_{\mathbb{Q}}(M_B)\mathbf{d}_{\mathbb{Q}}(M_{dR})^{-1}.$$

We obtain an isomorphism

$$\vartheta^{PD} : \Delta_{\mathbb{Q}}(M)\cdot\Delta_{\mathbb{Q}}(M^*(1))^* \cong \Delta_{dif}(M)$$

which arises from the mutual cancellation of the terms arising from motivic cohomology, the following isomorphism

$$M_B^+ \oplus (M_B^*(1)^+)^* \cong M_B^+ \oplus M_B(-1)^+ \cong M_B,$$

where the last map is $(x,y)\mapsto x+2\pi i y$, and from the Poincare duality exact sequence

$$0 \longrightarrow (t_{M^*(1)})^* \longrightarrow M_{dR} \longrightarrow t_M \longrightarrow 0. \tag{5.36}$$

On the other side define an isomorphism

$$\vartheta^{dif}_\infty : \Delta_{dif}(M)_{\mathbb{R}} \cong \mathbf{1}_{\mathbb{R}}$$

applying the determinant to (2.4) and to the following isomorphism

$$(\mathbb{C}\otimes_{\mathbb{Q}} M_B)^+ = (\mathbb{R}\otimes_{\mathbb{Q}} M_B^+)\oplus(\mathbb{R}(2\pi i)^{-1}\otimes_{\mathbb{Q}} M_B^-) \cong (M_B)_{\mathbb{R}} \tag{5.37}$$

where the last map is induced by $\mathbb{R}(2\pi i)^{-1}\to\mathbb{R}, x\mapsto 2\pi i x$.

Due to the autoduality of the exact sequence of Conjecture 3.3 (see [7, lem. 12]) we have a commutative diagram

$$\begin{array}{ccc}
\Delta_{\mathbb{Q}}(M)\cdot\Delta_{\mathbb{Q}}(M^*(1))^* & \xrightarrow{\vartheta^{PD}_{\mathbb{R}}} & \Delta_{dif}(M) \\
\Big\downarrow{\scriptstyle \vartheta_\infty(M)\cdot\overline{\vartheta_\infty(M^*(1))^*}} & & \Big\downarrow{\scriptstyle \vartheta^{dif}_\infty} \\
\mathbf{1}_{\mathbb{R}} & \xrightarrow{\mathrm{id}_{\mathbf{1}_{\mathbb{R}}}} & \mathbf{1}_{\mathbb{R}}
\end{array}$$

Thus we obtain the following

Proposition 5.2 (Rationality)**.** *Assume that Conjecture 3.5 is valid for the $\mathbb{Q}$-motive M. Then it is also valid for its Kummer dual $M^*(1)$ if and only there exists a (unique) isomorphism*

$$\zeta^{dif}(M) : \mathbf{1}_{\mathbb{Q}} \to \Delta_{dif}(M)$$

such that we have

$$L^*_{dif}(M) : \mathbf{1}_{\mathbb{C}} \xrightarrow{\zeta^{dif}(M)_{\mathbb{C}}} \Delta_{dif}(M)_{\mathbb{C}} \xrightarrow{(\vartheta^{dif}_\infty)_{\mathbb{C}}} \mathbf{1}_{\mathbb{C}}$$

Putting $t_H(M) := \sum_{r\in\mathbb{Z}} r h(r)$ with $h(r) := \dim_K gr^r(M_{dR}) (= \dim_{K_\lambda} gr^r(D_{dR}(M_\lambda)))$ for a K-motive M and noting that $t_H(M) = t_H(\det(M))$, we have in fact the following

Theorem 5.3 (Deligne [15, thm. 5.6], Burns-Flach [7, thm. 5.2])**.** *If the motive* $\det(M)$ *is of the form $\mathbb{Q}(-t_H(M))$ twisted by a Dirichlet character, then $\zeta^{dif}(M)$ exists.*

Deligne [15] conjectured that the condition of the theorem is satisfied for all motives. It is known to hold in all examples A)-E).

See (5.41) below for the rationality statement which is hidden in the formulation of this theorem. Now we have to check the compatibility with respect to the p-adic realizations. To this aim we define the isomorphism

$$\vartheta_p^{dif}:\Delta_{dif}(M)_{\mathbb{Q}_p}\cong \mathbf{d}_{\mathbb{Q}_p}(M_p)\cdot\prod_{S\backslash S_\infty}\mathbf{d}_{\mathbb{Q}_p}(\mathrm{R}\Gamma(\mathbb{Q}_l,M_p))$$

as follows: Apply the determinant to (2.2) and multiply the resulting isomorphism with

$$\mathrm{id}_{\mathbf{d}(M_{dR})^{-1}}\cdot\prod_{l\in S\backslash S_\infty}\Theta_l(M_p):\mathbf{d}(M_{dR})^{-1}\to\prod_{S\backslash S_\infty}\mathbf{d}_{\mathbb{Q}_p}(\mathrm{R}\Gamma(\mathbb{Q}_l,M_p))$$

where $\Theta_l(M_p)=\eta_l(M)\cdot\eta_l(M^*(1))$ is defined in the appendix 7.

On the other hand Artin-Verdier/Poitou-Tate Duality induces the following isomorphism

$$\begin{aligned}\mathbf{d}_{\mathbb{Q}_p}(\mathrm{R}\Gamma_c(U,M_p))^{-1}&\cong\mathbf{d}_{\mathbb{Q}_p}(\mathrm{R}\Gamma(U,M_p))^{-1}\mathbf{d}_{\mathbb{Q}_p}(\bigoplus_{v\in S}\mathrm{R}\Gamma(\mathbb{Q}_v,M_p))\\&\cong\mathbf{d}_{\mathbb{Q}_p}(\mathrm{R}\Gamma_c(U,M_p^*(1))^*)\mathbf{d}_{\mathbb{Q}_p}((M_p^*(1)^+)^*)\prod_{v\in S}\mathbf{d}_{\mathbb{Q}_p}(\mathrm{R}\Gamma(\mathbb{Q}_v,M_p))\\&\cong\mathbf{d}_{\mathbb{Q}_p}(\mathrm{R}\Gamma_c(U,M_p^*(1))^*)\mathbf{d}_{\mathbb{Q}_p}(M_p(-1)^+)\mathbf{d}_{\mathbb{Q}_p}(M_p^+)\prod_{l\in S\backslash S_\infty}\mathbf{d}_{\mathbb{Q}_p}(\mathrm{R}\Gamma(\mathbb{Q}_l,M_p)).\end{aligned}$$

Using the identification

$$M_p^+\oplus M_p(-1)^+=M_p^+\oplus M_p^-(-1)\cong M_p,\tag{5.38}$$

where the last map is induced by multiplication with the p-adic period $t=$ "$2\pi i$" : $M_p^-(-1)\to M_p^-$, we obtain

$$\vartheta_p^{AV}:\mathbf{d}_{\mathbb{Q}_p}(\mathrm{R}\Gamma_c(U,M_p))^{-1}\cdot\mathbf{d}_{\mathbb{Q}_p}(\mathrm{R}\Gamma_c(U,M_p^*(1))^*)^{-1}\cong\mathbf{d}_{\mathbb{Q}_p}(M_p)\cdot\prod_{S\backslash S_\infty}\mathbf{d}_{\mathbb{Q}_p}(\mathrm{R}\Gamma(\mathbb{Q}_l,M_p)).$$

Again one has to check the commutativity of the following diagram (cf. [7, lem. 12])

$$\begin{array}{ccc}\Delta_{\mathbb{Q}}(M)_{\mathbb{Q}_p}\cdot\Delta_{\mathbb{Q}}(M^*(1))^*_{\mathbb{Q}_p} & \xrightarrow{(\vartheta^{PD})_{\mathbb{Q}_p}} & \Delta^{dif}(M)_{\mathbb{Q}_p}\\ \Big\downarrow{\scriptstyle \vartheta_p(M)\cdot\overline{\vartheta_p(M^*(1))^*}} & & \Big\downarrow{\scriptstyle\vartheta_p^{dif}}\\ \mathbf{d}_{\mathbb{Q}_p}(\mathrm{R}\Gamma_c(U,M_p))^{-1}\cdot\mathbf{d}_{\mathbb{Q}_p}(\mathrm{R}\Gamma_c(U,M_p^*(1))^*)^{-1} & \xrightarrow{\vartheta_p^{AV}} & \mathbf{d}_{\mathbb{Q}_p}(M_p)\cdot\prod_{S\backslash S_\infty}\mathbf{d}_{\mathbb{Q}_p}(\mathrm{R}\Gamma(\mathbb{Q}_l,M_p)).\end{array}$$

Note that analogous maps exist and analogous properties hold also if we replace M_p by a Galois stable $\mathbb{Z}_p$-lattice T_p or even by the free Λ-module $\mathbb{T}$. Thus we obtain the following

Proposition 5.4 (Integrality). *Assume that Conjecture 3.7 is valid for the $\mathbb{Q}$-motive M. Then it is also valid for its Kummer dual $M^*(1)$ if and only there exists a (unique) isomorphism*

$$\zeta^{dif}_{\mathbb{Z}_p}(T_p) : \mathbf{1}_{\mathbb{Z}_p} \to \mathbf{d}_{\mathbb{Z}_p}(T_p) \cdot \prod_{S\setminus S_\infty} \mathbf{d}_{\mathbb{Z}_p}(\mathrm{R}\Gamma(\mathbb{Q}_l, T_p)).$$

which induces via $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} -$ the following map

$$\mathbf{1}_{\mathbb{Q}_p} \xrightarrow{\zeta^{dif}(M)_{\mathbb{Q}_p}} \Delta^{dif}(M)_{\mathbb{Q}_p} \xrightarrow{\vartheta_p^{dif}} \mathbf{d}_{\mathbb{Q}_p}(M_p) \cdot \textstyle\prod_{S\setminus S_\infty} \mathbf{d}_{\mathbb{Q}_p}(\mathrm{R}\Gamma(\mathbb{Q}_l, M_p)).$$

If this holds we have, using the above identifications, the functional equation

$$\zeta_{\mathbb{Z}_p}(T_p) = (\overline{\zeta_{\mathbb{Z}_p}(T_p^*(1))^*})^{-1} \cdot \zeta^{dif}_{\mathbb{Z}_p}(T_p)$$

Note that $(\overline{\zeta_{\mathbb{Z}_p}(T_p^*(1))^*})^{-1}$ is the same as $\zeta_{\mathbb{Z}_p}(T_p^*(1))^* \cdot \mathrm{id}_{\mathbf{d}_{\mathbb{Z}_p}(\mathrm{R}\Gamma_c(U,T_p^*(1))^*)}$ according to Remark 1.2(i). Needless to say that all the above has an analogous version for K-motives whose formulation we leave to the reader. Then it is clear how the equivariant version of this proposition looks like:

Proposition 5.5 (Equivariant Integrality). *Assume that Conjecture 4.1 is valid for the $\mathbb{Q}$-motive M. Then it is also valid for its Kummer dual $M^*(1)$ if and only if there exists a (unique) isomorphism*

$$\zeta^{dif}_{\Lambda}(M) : \mathbf{1}_{\Lambda} \to \mathbf{d}_{\Lambda}(\mathbb{T}) \cdot \prod_{S\setminus S_\infty} \mathbf{d}_{\Lambda}(\mathrm{R}\Gamma(\mathbb{Q}_l, \mathbb{T})).$$

with the following property $(*)$:

For all K, λ and ρ as before Conjecture 4.1 the (generalized) base change $\mathcal{O}_\lambda^n \otimes_\Lambda -$ sends $\zeta^{dif}_{\Lambda}(M)$ to $\zeta^{dif}_{\mathcal{O}_\lambda}(T_\lambda(M(\rho^*)))$, the analogue of $\zeta^{dif}_{\mathbb{Z}_p}(T_p)$ for the K-motive $M(\rho^*)$.

If this holds we have the functional equation

$$\zeta_{\Lambda}(M) = (\overline{\zeta_{\Lambda}(M^*(1))^*})^{-1} \cdot \zeta^{dif}_{\Lambda}(M)$$

Thus we formulate the

Conjecture 5.6 (Local Equivariant Tamagawa Number Conjecture). *The isomorphism $\zeta^{dif}_{\Lambda}(M)$ in the previous Proposition exists (uniquely).*

5.1. **ϵ-isomorphisms.** One obtains a refinement of the above functional equation if one looks more closely to which part of the Galois cohomology (and comparison isomorphisms) the factors occurring in $L^*_{dif}(M)$ belong precisely. We first recall from [32]

$$\frac{L^*_\infty(M^*(1))}{L^*_\infty(M)} = \pm 2^{d_-(M)-d_+(M)}(2\pi)^{-(d_-(M)+t_H(M))} \prod_{j\in\mathbb{Z}} \Gamma^*(-j)^{-h_j(M)} \tag{5.39}$$

where $\Gamma^*(-j)$ is defined to be $\Gamma(j) = (j-1)!$ if $j > 0$ and $\lim_{s\to j}(s-j)\Gamma(s) = (-1)^j((-j)!)^{-1}$ otherwise.

The factor $(2\pi)^{-(d_-(M)+t_H(M))}$ arises as follows. Assume for simplicity that $\det(M) = \mathbb{Q}(-t_H(M))$. Then fixing a $\mathbb{Q}$- basis $\gamma = (\gamma^+, \gamma^-)$ of M_B and $\omega = (\delta_M, \delta_{M^*(1)})$ of

M_{dR} which induce the canonical basis (cf. example A)) of $\det(M)_B$ and $\det(M)_{dR}$, respectively, gives rise to a map

$$\mathbf{1}_{\mathbb{Q}} \xrightarrow{can_{\gamma,\omega}} \mathbf{d}_{\mathbb{Q}}(M_B)\mathbf{d}_{\mathbb{Q}}(M_{dR})^{-1}. \tag{5.40}$$

Base and change and the comparison isomorphism (2.3) induce

$$\mathbf{1}_{\mathbb{C}} \xrightarrow{(can_{\gamma,\omega})_{\mathbb{C}}} \mathbf{d}_{\mathbb{Q}}(M_B)_{\mathbb{C}}\mathbf{d}_{\mathbb{Q}}(M_{dR})_{\mathbb{C}}^{-1} \xrightarrow{\mathbf{d}(g_\infty)} \mathbf{1}_{\mathbb{C}}$$

whose value Ω_∞^{dif} in $\mathbb{C}^\times$ is nothing else than the inverse of the determinant over $\mathbb{C}$ of the comparison isomorphism

$$\mathbb{C}\otimes_{\mathbb{Q}}\det(M)_B \to \mathbb{C}\otimes_{\mathbb{Q}}\det(M)_{dR}$$

and thus $\Omega_\infty^{dif} = (2\pi i)^{-t_H(M)}$. But note that due to the definition of (5.37) the above map differs from $(\vartheta_\infty^{dif})_{\mathbb{C}} \circ (can_{\gamma,\omega})_{\mathbb{C}}$ by the factor $(2\pi i)^{-d_-(M)}$. Thus we obtain an explanation of the factor $(2\pi i)^{-(t_H(M)+d_-(M))}$. Moreover, Theorem 5.3 and Proposition 5.2 tell us that

$$\frac{L^*_{dif}(M)}{(2\pi i)^{-d_-(M)}\Omega_\infty^{dif}} = \pm 2^{d_-(M)-d_+(M)} \frac{\epsilon_\infty(M)}{i^{-(t_H(M)+d_-(M))}} \prod_{l\in S\setminus S_\infty} \epsilon_l(M) \prod_{j\in\mathbb{Z}} \Gamma^*(-j)^{-h_j(M)} \tag{5.41}$$

is rational and that $\zeta_{dif}(M)$ is the map $can_{\gamma,\omega}$ multiplied by this rational number.

The factor $2^{d_-(M)-d_+(M)}$ arises as quotient of the Tamagawa factors of M and $M^*(1)$ at infinity. One can either cover it by defining Θ_∞ (see below) or changing the last map of the identification in (5.38) as follows: on the summand M_p^+ multiply with 2 and on M_p^- by $\frac{1}{2}$ (as Fukaya and Kato do).

The map (5.40) induces $\mathbf{1}_{\mathbb{Q}_p} \to \Delta_{dif}(M)_{\mathbb{Q}_p} \cong \mathbf{d}_{\mathbb{Q}_p}(M_p)\mathbf{d}_{\mathbb{Q}_p}(D_{dR}(M_p))^{-1}$ and furthermore

$$\mathbf{1}_{B_{dR}} \xrightarrow{(can_{\gamma,\omega})_{B_{dR}}} \Delta_{dif}(M)_{B_{dR}} \xrightarrow{\mathbf{d}_{\mathbb{Q}_p}(g_{dR})_{B_{dR}}} \mathbf{1}_{B_{dR}}$$

whose value Ω_p^{dif} in $B_{dR}^\times$ is nothing else than the inverse of the determinant over B_{dR} of the comparison isomorphism

$$B_{dR}\otimes_{\mathbb{Q}_p} D_{dR}(\det(M)_p) \to B_{dR}\otimes_{\mathbb{Q}_p}\det(M)_p$$

and thus $\Omega_p^{dif} = (2\pi i)^{-t_H(M)}$, where we consider the p-adic period $t = 2\pi i$ as an element of B_{dR}. Note that $(B_{dR})^{I_p} = \widehat{\mathbb{Q}_p^{nr}}$, the completion of the maximal unramified extension $\mathbb{Q}_p^{nr}$ of $\mathbb{Q}_p$. We need the following

Lemma 5.7 ([19, prop. 3.3.5],[32, C.2.8])**.** *The map $\epsilon_p(M)\cdot\Omega_p^{dif}\cdot(can)_{B_{dR}}$ comes from a map*

$$\epsilon_{dR}(M_p) : \mathbf{1}_{\widehat{\mathbb{Q}_p^{nr}}} \to \mathbf{d}_{\widehat{\mathbb{Q}_p^{nr}}}(M_p)\mathbf{d}_{\widehat{\mathbb{Q}_p^{nr}}}(D_{dR}(M_p))^{-1}.$$

Moreover, let L be any finite extension of $\mathbb{Q}_p$. Then a similar statement holds for any finite dimensional L-vector space V with continuous $G_{\mathbb{Q}_p}$-action instead of M_p[11]*. We*

[11] $\epsilon_p(V) = \epsilon(D_{pst}(V))$ where $D_{pst}(V)$ is endowed with the linearized action of the Weil-group and thereby considered as a representation of the Weil-Deligne group, see [19, §3.2], [18] or [32, appendix C].

write $\epsilon_{dR}(V)$ *for the corresponding map, which is defined over* $\widetilde{L} := \widehat{\mathbb{Q}_p^{nr}} \otimes_{\mathbb{Q}_p} L$ *(see* [19, prop. 3.3.5] *for details).*

For any V as in the lemma we define an isomorphism

$$\epsilon_{p,L}(V) : \mathbf{1}_{\widetilde{L}} \to \big(\mathbf{d}_L(\mathrm{R}\Gamma(\mathbb{Q}_p, V))\mathbf{d}_L(V)\big)_{\widetilde{L}}$$

as product of $\Gamma_L(V) := \prod_{\mathbb{Z}} \Gamma^*(j)^{-h(-j)}$, $\Theta_p(V)$ (see appendix 7.65) and $\epsilon_{dR}(V)$, where $h(j) = \dim_L gr^j D_{dR}(V)$.

Now let T be a Galois stable $\mathcal{O} := \mathcal{O}_L$-lattice of V and set $\widetilde{\mathcal{O}} := W(\overline{\mathbb{F}_p}) \otimes_{\mathbb{Z}_p} \mathcal{O}$, where $W(\overline{\mathbb{F}_p})$ denotes the Wittring of $\overline{\mathbb{F}_p}$. The following conjecture is a local integrality statement

Conjecture 5.8 (Absolute ϵ-isomorphism)**.** *There exists a (unique) isomorphism*

$$\epsilon_{p,\mathcal{O}}(T) : \mathbf{1}_{\widetilde{\mathcal{O}}} \to \big(\mathbf{d}_{\mathcal{O}}(\mathrm{R}\Gamma(\mathbb{Q}_p, T))\mathbf{d}_{\mathcal{O}}(T)\big)_{\widetilde{\mathcal{O}}}$$

with induces $\epsilon_{p,L}(V)$ *by base change* $L \otimes_{\mathcal{O}} -$.

This conjecture, which is equivalent to conjecture $C_{EP}(V)$ in [18, III 4.5.4], or more precisely its equivariant version below is closely related to the conjecture $\delta_{\mathbb{Z}_p}(V)$ [32] via the explicit reciprocity law $R\acute{e}c(V)$, which was conjectured by Perrin-Riou and proven independently by Benois [1], Colmez [13], and Kurihara/Kato/Saito [22]. In particular, the above conjecture is known for ordinary cristalline p-adic representations [32, 1.28,C.2.10] and for certain semi-stable representations, see [2].

To formulate an equivariant version, define

$$\widetilde{\Lambda} := \widehat{\mathbb{Z}_p^{nr}}[\![G]\!] = \varprojlim_n \big(W(\overline{\mathbb{F}_p}) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[G/G_n]\big),$$

where $\widehat{\mathbb{Z}_p^{nr}} = W(\overline{\mathbb{F}_p})$ denotes the ring of integers of $\widehat{\mathbb{Q}_p^{nr}}$. We assume $L = \mathbb{Q}_p$ and set as before $\mathbb{T} := \Lambda \otimes_{\mathbb{Z}_p} T$ (but later T might differ from our global T_p). We write $T(\rho^*)$ for the $\mathcal{O}$-lattice $\rho^* \otimes T$ of $\rho^* \otimes V$, which we assume de Rham.

Conjecture 5.9 (Equivariant ϵ-isomorphism)**.** *There exists a (unique)*[12] *isomorphism*

$$\epsilon_{p,\Lambda}(\mathbb{T}) : \mathbf{1}_{\widetilde{\Lambda}} \to \big(\mathbf{d}_{\Lambda}(\mathrm{R}\Gamma(\mathbb{Q}_p, \mathbb{T}))\mathbf{d}_{\Lambda}(\mathbb{T})\big)_{\widetilde{\Lambda}}$$

such that for all $\rho : G \to GL_n(\mathcal{O}) \subseteq GL_n(L)$, L *a finite extension of* $\mathbb{Q}_p$ *with valuation ring* $\mathcal{O}$, *we have*

$$\mathcal{O}^n \otimes_{\Lambda} \epsilon_{p,\Lambda}(\mathbb{T}) = \epsilon_{p,\mathcal{O}}(T(\rho^*)).$$

Furthermore, we suppress the dependence of the choice of a Haar measure and of $t = 2\pi i$ in the notation. The choice of $t = (t_n) \in \mathbb{Z}_p(1)$ determines a homomorphism $\psi_p : \mathbb{Q}_p \to \bar{\mathbb{Q}}_p^{\times}$ with $\ker(\psi_p) = \mathbb{Z}_p$ sending $\frac{1}{p^n}$ to $t_n \in \mu_{p^n}$.

[12] Again, Fukaya and Kato assign such an isomorphism to each triple $(R, \mathbb{T}, t)$, where R is as before, $\mathbb{T}$ is a projective R-module endowed with a continuous $G_{\mathbb{Q}_p}$-action and t is a generator of $\mathbb{Z}_p(1)$. Then $\epsilon_{p,?}(?)$ is supposed to behave well under arbitrary change of rings for such pairs. Moreover they require that the assignment $\mathbb{T} \mapsto \epsilon_{p,R}(\mathbb{T})$ is multiplicative for short exact sequences, that it satisfies a duality relation when replacing $\mathbb{T}$ by $\mathbb{T}^*(1)$, that the group $G_{\mathbb{Q}_p}^{ab}$ acts on a predetermined way (modifying t) compatible in a certain sense with the Frobenius ring homomorphism on $\widetilde{\Lambda}$ induced from the absolute Frobenius of $\overline{\mathbb{F}_p}$. Only this full set of conditions may lead to the uniqueness in general.

If $T = T_p \subseteq M_p$ is fixed we also write $\epsilon_{p,\Lambda}(M)$ for $\epsilon_{p,\Lambda}(\mathbb{T})$.

Similarly we proceed in the case $l \neq p$, formulating just one

Conjecture 5.10. *There exists a (unique)*[13] *isomorphism*

$$\epsilon_{l,\Lambda}(\mathbb{T}) : \mathbf{1}_{\widetilde{\Lambda}} \to \mathbf{d}_\Lambda(\mathrm{R}\Gamma(\mathbb{Q}_l, \mathbb{T}))_{\widetilde{\Lambda}}$$

such that for all $\rho : G \to GL_n(L)$, L a finite extension of $\mathbb{Q}_p$, we have

$$\mathcal{O}^n \otimes_\Lambda \epsilon_{l,\Lambda}(\mathbb{T}) = \epsilon_{l,\mathcal{O}}(T(\rho^*)).$$

Here $\epsilon_{l,\mathcal{O}}(T(\rho^*))$ is the analogue of the above with respect to $\mathcal{O}$ instead of Λ and required to induce

$$L \otimes_\mathcal{O} \epsilon_{l,\mathcal{O}}(T(\rho^*))) = \Theta_l(V) \cdot \epsilon_l(V(\rho^*))$$

and its existence is part of the conjecture[14].

For commutative Λ this conjecture was proved by S. Yasuda [38] and it seems that he can extend his methods to cover the non-commutative case, too.

If $T = T_p \subseteq M_p$ we also write $\epsilon_{l,\Lambda}(M)$ for $\epsilon_{l,\Lambda}(\mathbb{T})$.

Finally we set $\epsilon_{\infty,\Lambda}(M) = \pm 2^{d_-(M) - d_+(M)} \frac{\epsilon_\infty(M)}{i^{-(t_H(M) + d_-(M))}}$ where the sign is that which makes (5.39) correct. The following result is now immediate.

Theorem 5.11 (cf. [19, conj. 3.5.5])**.** *Assume Conjectures 5.8, 5.9 and 5.10. Then Conjecture 5.6 holds, $\zeta_\Lambda^{dif}(M) = \prod_{v \in S} \epsilon_{v,\Lambda}(M)$ and we have the functional equation*

$$\zeta_\Lambda(M) = (\overline{\zeta_\Lambda(M^*(1))^*})^{-1} \cdot \prod_{v \in S} \epsilon_{v,\Lambda}(M)$$

6. p-adic L-functions and the Iwasawa main conjecture

A p-adic L-function attached to a $\mathbb{Q}$-motive M should be considered as a map on certain class of representations of G which interpolates the L-values of the twists $M(\rho^*)$ at 0. The experience from those cases where such p-adic L-functions exist, shows that one has to modify the complex L-values by certain factors before one can hope to obtain a p-adic interpolation (cf. [9, 10] or [32]). The reason for this becomes clearer if one considers the Galois cohomology involved together with the functional equation; in fact, that was the main motivation of the previous section.

In order to evaluate e.g. the ζ-isomorphism or a modification of it at a representation ρ over a finite extension L of $\mathbb{Q}_p$, one needs that the complex $\rho \otimes_\Lambda C$, where C is (a modification of) $\mathrm{R}\Gamma_c(U, \mathbb{T})$, becomes acyclic: then the induced map $\mathbf{1}_L \to \mathbf{d}_L(\rho \otimes_\Lambda C) \to \mathbf{1}_L$ can be considered as value in $K_1(L) = L^\times$ at ρ.

[13] A similar comment as in the previous statement applies here.

[14] $\epsilon_l(V) = \epsilon(V)$ where V is considered as representation of the Weil-Deligne group of $\mathbb{Q}_l$ and where we suppress the dependence of the choice of a Haar measure and of $t = 2\pi i$ in the notation. The choice of $t = (t_n) \in \mathbb{Z}_l(1)$ determines a homomorphism $\psi_l : \mathbb{Q}_l \to \bar{\mathbb{Q}}_l^\times$ with $Ker(\psi_l) = \mathbb{Z}_l$ sending $\frac{1}{l^n}$ to $t_n \in \mu_{l^n}$. The formulation of this conjecture is equivalent to [19, conj. 3.5.2] where the constants ϵ_0 are used instead of ϵ and where θ_l does not occur. More precisely, our $\epsilon_{l,\Lambda}(\mathbb{T})$ equals $\epsilon_{0,\Lambda}(\mathbb{Q}_l, T, \xi) \cdot s_l(T)$ in [19, 3.5.2,3.5.4]

In general, $R\Gamma_c(U,\mathbb{T})$ does not behave good enough and will have to be replaced by some Selmer complex, which we well achieve in two steps. This modification corresponds to a shifting of certain Euler- and ϵ-factors from one side of the functional equation to the other such that both sides are balanced.

Though the following part of the theory holds in much greater generality (e.g. in the *ordinary* good reduction case, but not in the *supersingular* good reduction case) we just discuss the case of abelian varieties in order to keep the situation as concise as possible. Thus let A be an abelian variety over $\mathbb{Q}$ with good ordinary reduction at a fixed prime $p\neq 2$ and set $M=h^1(A)(1)$ as before. Let F_∞ be an infinite p-adic Lie extension of $\mathbb{Q}$ with Galois group G. For simplicity we assume also that G has no element of order p, hence its Iwasawa algebra $\Lambda=\Lambda(G)$ is a regular ring.

Due to our assumption on the reduction type of A, we have the following fact: There is a unique $\mathbb{Q}_p$-subspace $\hat{V}$ of $V=M_p$ which is stable under the action of $G_{\mathbb{Q}_p}$ and such that

$$D_{dR}(\hat{V})\cong D_{dR}(V)/D^0_{dR}(V). \tag{6.42}$$

More precisely, $\hat{V}=V_p(\widehat{A^\vee})$ where $\widehat{A^\vee}$ denotes the formal group of the dual abelian variety $A^\vee$, i.e. the formal completion of the Neron model $\mathcal{A}/\mathbb{Z}_p$ of $A^\vee$ along the zero section of the special fibre $\widetilde{\mathcal{A}}$. Then (6.42) arises from the *unit root splitting* $D^0_{dR}(V_p(\widetilde{\mathcal{A}}))\cong D^0_{dR}(V)\subseteq D_{dR}(V)$ (see [30, 1.31]) which is induced from applying $D^0_{dR}(-)$ to the exact sequence of $G_{\mathbb{Q}_p}$-modules

$$0\longrightarrow V_p(\widehat{A^\vee})\longrightarrow V_p(A^\vee)\longrightarrow V_p(\widetilde{\mathcal{A}})\longrightarrow 0.$$

Let T be the $G_{\mathbb{Q}}$-stable $\mathbb{Z}_p$-lattice $T_p(A^\vee)$ of V and set

$$\hat{T}:=T\cap\hat{V},$$

a $G_{\mathbb{Q}_p}$-stable $\mathbb{Z}_p$-lattice of $\hat{V}$. As before let $\mathbb{T}$ denote the big Galois representation $\Lambda\otimes_{\mathbb{Z}_p}T$ and put $\hat{\mathbb{T}}:=\Lambda\otimes_{\mathbb{Z}_p}\hat{T}$ similarly. Then $\hat{\mathbb{T}}$ is $G_{\mathbb{Q}_p}$-stable sub-Λ-module of $\mathbb{T}$. In fact, it is a direct summand of $\mathbb{T}$ and we have an isomorphism of Λ-modules

$$\beta:\mathbf{d}_\Lambda(\mathbb{T}^+)\cong\mathbf{d}_\Lambda(\hat{\mathbb{T}}).\text{[15]} \tag{6.46}$$

[15] which arises as follows: Choose a basis $\gamma^+=(\gamma_1^+,\dots,\gamma_r^+)$ of $\mathrm{H}_1(A^\vee(\mathbb{C}),\mathbb{Q})^+$ and $\gamma^-=(\gamma_1^-,\dots,\gamma_r^-)$ of $\mathrm{H}_1(A^\vee(\mathbb{C}),\mathbb{Q})^-$, which gives rise to a $\mathbb{Z}_p$-basis of $T^+\cong(\mathrm{H}_1(A^\vee(\mathbb{C}),\mathbb{Z})\otimes_\mathbb{Z}\mathbb{Z}_p)^+$ and $T^-\cong(\mathrm{H}_1(A^\vee(\mathbb{C}),\mathbb{Z})\otimes_\mathbb{Z}\mathbb{Z}_p)^-$ respectively, where $r=d_+(M)=d_-(M)$. Then we obtain an isomorphisms

$$\Lambda^r\cong\mathbb{T}^+\quad\text{and}\quad\phi:\mathbf{d}_\Lambda(\Lambda^r)\to\mathbf{d}_\Lambda(T^+) \tag{6.44}$$

using the Λ-basis $\frac{1+\iota}{2}\otimes\gamma_j^+ + \frac{1-\iota}{2}\otimes\gamma_j^-$, $1\leq j\leq r$, of $\mathbb{T}^+$. On the other hand one can choose a $\mathbb{Q}$-basis $\delta=(\delta_1,\dots,\delta_r)$ of $Lie(A^\vee)$ (e.g. a $\mathbb{Z}$-basis of $Lie_\mathbb{Z}(A^\vee)$ as in section 3.1) such that the isomorphism

$$\mathbf{d}_{\mathbb{Q}_p}(\mathbb{Q}_p^r)_{\widehat{\mathbb{Q}_p^{nr}}}\cong\mathbf{d}_{\mathbb{Q}_p}(\mathbb{Q}_p\otimes_\mathbb{Q} t_M)_{\widehat{\mathbb{Q}_p^{nr}}}\cong\mathbf{d}_{\mathbb{Q}_p}(D_{dR}(\hat{V}))_{\widehat{\mathbb{Q}_p^{nr}}}\cong\mathbf{d}_{\mathbb{Q}_p}(\hat{V})_{\widehat{\mathbb{Q}_p^{nr}}}, \tag{6.45}$$

which is induced by δ, (2.8), (6.42) and $\epsilon_{dR}(\hat{V})$) (according to Lemma 5.7, but note that $\epsilon_p(\hat{V})=1$ due to the good reduction), comes from an isomorphism

$$\mathbf{d}_{\mathbb{Z}_p}(\mathbb{Z}_p^r)_{\widehat{\mathbb{Z}_p^{nr}}}\cong\mathbf{d}_{\mathbb{Z}_p}(\hat{T})_{\widehat{\mathbb{Z}_p^{nr}}},$$

where $\widehat{\mathbb{Z}_p^{nr}}:=W(\overline{\mathbb{F}_p})$. Then base change $\widetilde{\Lambda}\otimes_{\widehat{\mathbb{Z}_p^{nr}}}-$ induces an isomorphism

$$\psi:\mathbf{d}_\Lambda(\Lambda^r)_{\widetilde{\Lambda}}\cong\mathbf{d}_\Lambda(\hat{\mathbb{T}})_{\widetilde{\Lambda}}. \tag{6.46}$$

In this good ordinary case one can now first replace $\mathrm{R}\Gamma_c(U,\mathbb{T})$ by the Selmer complex $SC_U := SC_U(\hat{\mathbb{T}},\mathbb{T})$ (see (7.66)) which fits into the following distinguished triangle

$$\mathrm{R}\Gamma_c(U,\mathbb{T}) \longrightarrow SC_U \longrightarrow \mathbb{T}^+ \oplus \mathrm{R}\Gamma(\mathbb{Q}_p,\hat{\mathbb{T}}) \longrightarrow \tag{6.47}$$

and thus induces an isomorphism

$$\mathbf{d}_\Lambda(\mathrm{R}\Gamma_c(U,\mathbb{T}))^{-1} \cong \mathbf{d}_\Lambda(SC_U)^{-1}\mathbf{d}_\Lambda(\mathbb{T}^+)\mathbf{d}_\Lambda(\mathrm{R}\Gamma(\mathbb{Q}_p,\hat{\mathbb{T}})).$$

The p-adic L-function $\mathcal{L}_U = \mathcal{L}_{U,\beta}(M, F_\infty/\mathbb{Q})$ arises from the zeta-isomorphism $\zeta_\Lambda(M)$ by a suitable cancellation of the two last terms, compatible with the functional equation. This is achieved by putting

$$\mathcal{L}_U := (\beta \cdot \mathrm{id}_{\mathbf{d}_\Lambda(SC_U)^{-1}\mathbf{d}_\Lambda(\hat{\mathbb{T}})^{-1}}) \circ (\epsilon_{p,\Lambda}(\hat{\mathbb{T}})^{-1} \cdot \zeta_\Lambda(M)) : \mathbf{1}_{\widetilde{\Lambda}} \to \mathbf{d}_\Lambda(SC_U)^{-1}_{\widetilde{\Lambda}}. \tag{6.48}$$

In order to arrive at a p-adic L-function which is independent of U one has to replace SC_U by another Selmer complex $SC := SC(\hat{\mathbb{T}},\mathbb{T})$ ((7.67)), which fits into a distinguished triangle

$$SC_U \longrightarrow SC \longrightarrow \bigoplus\nolimits_{l\in S\setminus\{p,\infty\}} \mathrm{R}\Gamma_f(\mathbb{Q}_l,\hat{\mathbb{T}}) \longrightarrow, \tag{6.49}$$

where for $l \neq p$

$$\mathrm{R}\Gamma_f(\mathbb{Q}_l,\hat{\mathbb{T}}) \cong [\ \hat{\mathbb{T}}^{I_l} \xrightarrow{1-\varphi_l} \hat{\mathbb{T}}^{I_l}] \tag{6.50}$$

in the derived category. Replacing $\hat{\mathbb{T}}^{I_l}$ by a projective resolution if necessary and using the identity isomorphism of it we obtain isomorphisms

$$\zeta_l(M) = \zeta_l(M, F_\infty/\mathbb{Q}) : \mathbf{1}_\Lambda \to \mathbf{d}_\Lambda(\mathrm{R}\Gamma_f(\mathbb{Q}_l,\hat{\mathbb{T}}))^{-1} \tag{6.51}$$

and we define the p-adic L-function

$$\mathcal{L} = \mathcal{L}(M) = \mathcal{L}_U \cdot \prod_{l\in S\setminus\{p,\infty\}} \zeta_l(M) : \mathbf{1}_{\widetilde{\Lambda}} \to \mathbf{d}_\Lambda(SC)^{-1}_{\widetilde{\Lambda}}. \tag{6.52}$$

Let Υ be the set of all $l \neq p$ such that the ramification index of l in $F_\infty/\mathbb{Q}$ is infinite. Note that Υ is empty if G has a commutative open subgroup.

Lemma 6.1. [19, prop. 4.2.14(3)] $\hat{\mathbb{T}}^{I_l} = 0$ *and thus* $\zeta_l(M) = 1$ *in* $K_1(\Lambda)$ *for all* l *in* Υ.

Let us derive the *interpolation property* of $\mathcal{L}_U$ and $\mathcal{L}$. Whenever $L^n \otimes^{\mathbb{L}}_\Lambda SC_U$ is acyclic for a continuous representation $\rho : G \to GL_n(\mathcal{O}_L)$, L a finite extension of $\mathbb{Q}_p$, we obtain an element $\mathcal{L}_U(\rho) \in \widehat{L^{nr}}^\times$ from the isomorphism

$$\mathbf{1}_{\tilde{L}} \xrightarrow{L^n\otimes_\Lambda \mathcal{L}_U} \mathbf{d}_{\tilde{L}}(L^n \otimes^{\mathbb{L}}_\Lambda SC_U)^{-1} \xrightarrow{\textit{acyclic}} \mathbf{1}_{\tilde{L}}$$

which via $K_1(\tilde{L}) \to K_1(\widehat{L^{nr}})$ can be considered as element of $\widehat{L^{nr}}^\times$.

Let K be a finite extension of $\mathbb{Q}$, $\rho : G \to GL_n(\mathcal{O}_K)$ an Artin representation, $[\rho^*]$ the Artin motive corresponding to ρ^*. Fix a place λ of K above p, put $L := K_\lambda$ and consider the L-linear representation of $G_\mathbb{Q}$ or its restriction to $G_{\mathbb{Q}_l}$

$$W := M(\rho^*)_\lambda = [\rho^*]_\lambda \otimes_{\mathbb{Q}_p} M_p$$

Now $\beta = \beta_{\gamma,\delta}$ is $\psi \circ \overline{\phi}$.

and the $G_{\mathbb{Q}_p}$-representation

$$\hat{W} := [\rho^*]_\lambda \otimes_{\mathbb{Q}_p} \hat{V}.$$

For a $G_{\mathbb{Q}_p}$-representation V define $P_{L,l}(V,u) := \det_L(1-\varphi_l u|V^{I_l}) \in L[u]$ if $l \neq p$ and $P_{L,p}(V,u) := \det_L(1-\varphi_p u|D_{cris}(V)) \in L[u]$ otherwise.

Some conditions for acyclicity are summarized in the next

Proposition 6.2 ([19, 4.2.21, 4.1.6-8])**.** *Assume the following conditions:*

(i) $H^j_f(\mathbb{Q},W) = H^j_f(\mathbb{Q},W^*(1)) = 0$ *for* $j = 0,1$,

(ii) $P_{L,l}(W,1) \neq 0$ *for any* $l \in \Upsilon$ *(respectively for any* $l \in S\setminus\{p,\infty\}$*).*

(iii) $\{P_{L,p}(W,u)P_{L,p}(\hat{W},u)^{-1}\}_{u=1} \neq 0$ *and* $P_{L,p}(\hat{W}^*(1),1) \neq 0$.

Then the following complexes are acyclic: $L^n \otimes^{\mathbb{L}}_{\Lambda,\rho} SC$ *(respectively* $L^n \otimes^{\mathbb{L}}_{\Lambda,\rho} SC_U$*),* $R\Gamma_f(\mathbb{Q}_l,W) = L^n \otimes^{\mathbb{L}}_{\Lambda,\rho} R\Gamma_f(\mathbb{Q}_l,\mathbb{T})$*, for any* $l \in \Upsilon$ *(respectively for any* $l \in S\setminus\{p\}$*). Furthermore, there is a quasi-isomorphism*

$$R\Gamma(\mathbb{Q}_p,\hat{W}) \to R\Gamma_f(\mathbb{Q}_p,W).$$

Finally, assuming Conjectures 3.2 and 3.6, $L_K(M(\rho^*),s)$ *has neither zero or pole at* $s=0$.

Henceforth we assume the conditions (i)-(iii).

We define $\Omega_\infty(M(\rho^*)) \in \mathbb{C}^\times$ to be the determinant of the period map $\mathbb{C}\otimes_{\mathbb{R}}\alpha_{M(\rho^*)}$ with respect to the K-basis which arise from γ (respectively δ) and the basis given by ρ. It is easy to see that we have

$$\Omega_\infty(M(\rho^*)) = \Omega^+_\infty(M)^{d_+(\rho)}\Omega^-_\infty(M)^{d_-(\rho)}, \tag{6.53}$$

where $d_\pm(\rho) = d_\pm([\rho])$ and $\Omega^\pm_\infty(M)$ is the determinant of $\mathbb{C}\otimes_{\mathbb{Q}} M^\pm_B \cong \mathbb{C}\otimes_{\mathbb{Q}} t_M$ with respect to the basis $\gamma^\pm$ and δ. Assuming Conjecture 3.5 we have

$$\frac{L_K(M(\rho^*),0)}{\Omega_\infty(M(\rho^*))} \in K^\times.$$

We claim that, using Proposition 7.2, the isomorphism $\mathcal{L}_U(\rho)$

$$\mathbf{1} \xrightarrow[\vartheta_\lambda\circ\zeta(M(\rho^*))]{\zeta_\Lambda(M)(\rho)=} \mathbf{d}(R\Gamma_c(U,W))^{-1} \xrightarrow[=\cdot\,\epsilon_{p,L}(\hat{W})^{-1}]{\cdot\,\epsilon_{p,\Lambda}(\hat{\mathbb{T}})^{-1}(\rho)} \mathbf{d}(SC_U(\hat{W},W))^{-1}\mathbf{d}(\hat{W})^{-1}\mathbf{d}(W^+) \longrightarrow$$

$$\xrightarrow{\mathrm{id}\cdot\beta(\rho)} \mathbf{d}(SC_U(\hat{W},W))^{-1} \xrightarrow{acyc} \mathbf{1}$$

(we suppress for ease of notation the subscripts and remind the reader of our convention in Remark 1.2) is the product of the following automorphisms of $\mathbf{1}$:

(1) $L_K(M(\rho^*),0)\Omega_\infty(M(\rho^*))^{-1}$,

(2) $\Gamma_L(\hat{W})^{-1} = \Gamma_{\mathbb{Q}_p}(\hat{V})^{-1}$,

(3) $\Omega_p(M(\rho^*))$ which is, by definition, the composite

$$\mathbf{d}(\hat{W}) \xrightarrow{\cdot\,\epsilon_{dR}(\hat{W})^{-1}} \mathbf{d}(D_{dR}(\hat{W})) \xrightarrow{\mathbf{d}(g^t_{dR})} \mathbf{d}(t_{M(\rho^*)}) \xrightarrow{\cdot\,can_{\gamma,\delta}} \mathbf{d}\big((M(\rho^*)^+_B)_L\big) \xrightarrow{\mathbf{d}(g^+_\lambda)} \mathbf{d}(W^+) \xrightarrow{\beta(\rho)} \mathbf{d}(\hat{W}) \tag{6.54}$$

where we apply Remark 1.2 to obtain an automorphism of $\mathbf{1}$,[16]

(4) $\prod_{S\setminus\{p,\infty\}} P_{L,l}(W,1): \ \mathbf{1} \xrightarrow{\prod \eta_l(W)} \prod \mathbf{d}(\mathrm{R}\Gamma_f(\mathbb{Q}_l, W)) \xrightarrow{acyc} \mathbf{1}$ where the first map comes from the trivialization by the identity and the second from the acyclicity,

(5) $\{P_{L,p}(W,u)P_{L,p}(\hat{W},u)^{-1}\}_{u=1} \ : \ \mathbf{1} \xrightarrow{\eta_W \cdot \eta_{\hat{W}}^{-1}} \mathbf{d}(\mathrm{R}\Gamma_f(\mathbb{Q}_p, W))\mathbf{d}(\mathrm{R}\Gamma(\mathbb{Q}_p, \hat{W}))^{-1} \xrightarrow{quasi} \mathbf{1}$,
where we use that $t(W) = D_{dR}(\hat{W}) = t(\hat{W})$ and the quasi-isomorphism mentioned in the above Proposition, and

(6) $P_{L,p}(\hat{W}^*(1),1) \ : \ \mathbf{1} \xrightarrow{\overline{(\eta_{\hat{W}^*(1)})^*}} \mathbf{d}(\mathrm{R}\Gamma_f(\mathbb{Q}_p, \hat{W}^*(1))) \xrightarrow{acyc} \mathbf{1}$, where we use that $t(\hat{W}^*(1)) = D^0_{dR}(\hat{W}) = 0$.

For $\mathcal{L}$ we need beneath Lemma 6.1 another

Lemma 6.3 ([19, lem. 4.2.23]). *Let $l \neq p$ be not in Υ. Then $L^n \otimes^{\mathbb{L}}_{\Lambda,\rho} \mathrm{R}\Gamma_f(\mathbb{Q}_l, \hat{\mathbb{T}})$ is acyclic if and only if $P_{L,l}(W,1) \neq 0$. If this holds then we have $\zeta_l(M)(\rho) = P_{L,l}(W,1)^{-1}$.*

Thus we obtain the following

Theorem 6.4 ([19, thm. 4.2.26]). *Under the conditions (i)-(iii) from Proposition 6.2 and assuming Conjecture 4.1 for M and Conjecture 5.9 for $\hat{\mathbb{T}}$ the value $\mathcal{L}(\rho)$ (respectively $\mathcal{L}_U(\rho)$) is*

$$\frac{L_K(M(\rho^*),0)}{\Omega_\infty(M(\rho^*))} \cdot \Omega_p(M(\rho^*)) \cdot \Gamma_{\mathbb{Q}_p}(\hat{V})^{-1} \cdot$$
$$\cdot \{P_{L,p}(W,u)P_{L,p}(\hat{W},u)^{-1}\}_{u=1} \cdot P_{L,p}(\hat{W}^*(1),1) \cdot \prod_B P_{L,l}(W,1),$$

where $B = \Upsilon \subseteq S \setminus \{p,\infty\}$ (respectively $B = S \setminus \{p,\infty\}$).

Remark 6.5. Note that conditions (ii) and (iii) are satisfied in the case of an abelian variety with good ordinary reduction and p. Furthermore, the quotient $\Omega_p(M(\rho^*))/\Omega_\infty(M(\rho^*)$ is independent of the choice of basis γ and δ. Also, it is easy to see[17] that for some suitable choice we have $\Omega_p(M(\rho^*)) = \epsilon_p(\hat{W})^{-1}$ which, according to standard properties of ϵ-constants (cf. [19, §3.2]) using that $\hat{V}$ is unramified as module under the Weil-group, in turn is equal to $\epsilon_p(\rho^*)^{-k} \cdot \nu^{-f_p(\rho)}$ where $k = \dim_{\mathbb{Q}}(t_M) = \dim_{\mathbb{Q}_p}(\hat{V})$, $\nu = \det_{\mathbb{Q}_p}(\varphi_p|D_{cris}(\hat{V}))$ and where $f_p(\rho)$ is the p-adic order of the Artin-conductor of

[16] Using Remark 1.2(i) it is easy to see that this amounts to taking the product of the following isomorphisms and identifying the target with $\mathbf{1}$ afterwards

$$\mathbf{1} \xrightarrow{can_{\gamma,\delta}} \mathbf{d}\big((M(\rho^*)^+_B)_L\big)\mathbf{d}(t_{M(\rho^*)})^{-1}, \qquad \mathbf{1} \xrightarrow{\mathrm{id}_- \cdot \mathbf{d}(g^+_\lambda)} \mathbf{d}(W^+)\mathbf{d}\big((M(\rho^*)^+_B)_L\big)^{-1},$$
$$\mathbf{1} \xrightarrow{\mathrm{id}_- \cdot \mathbf{d}(g^t_{dR})} \mathbf{d}(D_{dR}(\hat{W}))^{-1}\mathbf{d}(t_{M(\rho^*)}), \qquad \mathbf{1} \xrightarrow{\epsilon_{dR}(\hat{W})^{-1}} \mathbf{d}(\hat{W})^{-1}\mathbf{d}(D_{dR}(\hat{W})),$$
$$\mathbf{1} \xrightarrow{\mathrm{id}_- \cdot \beta(\rho)} \mathbf{d}(\hat{W})\mathbf{d}(W^+)^{-1},$$

where the identity maps are those of $\mathbf{d}\big((M(\rho^*)^+_B)_L\big)^{-1}$, $\mathbf{d}(D_{dR}(\hat{W}))^{-1}$ and $\mathbf{d}(W^+)^{-1}$, respectively.

[17] Note that in the definition of β and thus in $\beta(\rho)$ the epsilon factor $\epsilon_p(\hat{V})$ in $\epsilon_{dR}(\hat{V})$ equals 1 and thus $\Omega_p(M(\rho^*)) = \beta(\rho) \circ (\epsilon_p(\hat{W})^{-1} \cdot \overline{\beta(\rho)})$.

ρ. Due to the compatibility conjecture C_{WD} in [17, 2.4.3], which is known for abelian varieties (loc.cit., rem 2.4.6(ii)) and for Artin motives, one obtains the ϵ- and Euler-factors either from $D_{pst}(W)$ or from the corresponding l-adic realisations with $l \neq p$. Furthermore, we have $P_{L,p}(W,1) \neq 0$ and $P_{L,p}(\hat{W},1) \neq 0$ for weight reasons. Thus, noting that for abelian varieties $\Gamma(\hat{V}) = 1$, the above formula becomes

$$\text{(Int)} \qquad \frac{L_{K,\Upsilon'}(M(\rho^*),0)}{\Omega_\infty(M(\rho^*))} \cdot \epsilon_p(\rho^*)^{-k} \cdot \nu^{-f_p(\rho)} \cdot \frac{P_{L,p}(\hat{W}^*(1),1)}{P_{L,p}(\hat{W},1)},$$

where $L_{K,\Upsilon'}$ denotes the modified L-function without the Euler-factors in $\Upsilon' := \Upsilon \cup \{p\}$.[18]

Proof. We consider the case $\mathcal{L}_U$. First observe that due to the vanishing of the motivic cohomology the map

$$\zeta_K(M(\rho^*)) : \mathbf{1}_K \to \mathbf{d}(\Delta(M(\rho^*))) = \mathbf{d}(M(\rho^*)^+_B)\mathbf{d}(t_{M(\rho^*)})^{-1}$$

is just the map $can_{\gamma,\delta} : \mathbf{1} \cong \mathbf{d}(M(\rho^*)^+_B)\mathbf{d}(t_{M(\rho^*)})^{-1}$, induced by the bases arising from γ and δ, multiplied with $L_K(M(\rho^*),0)\Omega_\infty(M(\rho^*))^{-1}$. Secondly, since $\mathbf{d}(\mathrm{R}\Gamma_f(\mathbb{Q},W)) = \mathbf{1}$, the isomorphism $\vartheta_\lambda(M(\rho^*)) : \mathbf{d}(\Delta(M(\rho^*)))_L \cong \mathbf{d}(\mathrm{R}\Gamma_c(U,W))^{-1}$ corresponds up to the identification $\mathbf{d}(M(\rho^*)^+_B)_L \cong \mathbf{d}(W^+)$ to the product of

$$\mathbf{d}(t_{M(\rho^*)})_L^{-1} \xrightarrow{\overline{\mathbf{d}(g^t_{dR})}^{-1}} \mathbf{d}(t(W))^{-1} \xrightarrow{\cdot\eta_\lambda(W)} \mathbf{d}(\mathrm{R}\Gamma_f(\mathbb{Q}_p,W))$$

with $\prod_B P_{L,l}(W,1)$. Thirdly, the contribution from $\epsilon_{p,L}(\hat{W})$ is $\eta_p(\hat{W}) \cdot \overline{\eta_p(\hat{W}^*(1))^*} \cdot \Gamma_L(\hat{W}) \cdot \epsilon_{dR}(\hat{W})$ up to the canonical local duality isomorphism. Together with $\beta(\rho)$ we thus obtain all the factors (1)-(6) above. To finish the proof in the case $\mathcal{L}$ use Lemmata 6.1, 6.3 and (7.69). □

6.1. **Interlude - Localized K_1.** The following construction of a localized K_1 is one of the differences to the approach of Huber and Kings [20][19]. For a moment let Λ be an arbitrary ring with unit and let Σ be a full subcategory of $C^p(\Lambda)$ satisfying (i) if C is quasi-isomorphic to an object in Σ then it belongs to Σ, too, (ii) Σ contains the trivial complex, (iii) all translations of objects in Σ belong again to Σ and (iv) any extension C in $C^p(\Lambda)$ (by an exact sequence of complexes) of $C', C'' \in \Sigma$ is again in Σ. Then Fukaya and Kato construct a group $K_1(\Lambda,\Sigma)$ whose objects are all of the form $[C,a]$ with $C \in \Sigma$ and an isomorphism $a : \mathbf{1}_\Lambda \to \mathbf{d}_\Lambda(C)$ (in particular, $[C] = 0$ in $K_0(\Lambda)$) satisfying certain relations, see [19, 1.3]. This group fits into an exact sequence

$$(6.55) \qquad K_1(\Lambda) \longrightarrow K_1(\Lambda,\Sigma) \xrightarrow{\partial} K_0(\Sigma) \longrightarrow K_0(\Lambda),$$

[18] In order to compare this formula with (107) in [11] we remark that, with the notation of (loc. cit.), $u = \det(\phi_l|\widehat{V}(-1)) = p\nu = p\omega^{-1}$. Then by [19, rem. 4.2.27] one has $\epsilon(\rho^*)^{-d}\nu^{-f_p(\rho)} = \epsilon(\rho)^d u^{-f_p(\rho)}$ (strictly speaking one has to replace the period t by $-t$ in the second epsilon factor). !!!!! But it seems that one has to interchange ρ and $\widehat{\rho}$ on the right hand side of (107)!!!!!!!!!!!!!!!!!

[19] Instead of the localised K_1 they work with K_1 of the ring $\varprojlim_n \mathbb{Q}_p[G/G_n]$, which occurs in the context of distributions, see [13].

where $K_0(\Sigma)$ is the abelian group generated by $[[C]]$, $C \in \Sigma$ and satisfying certain relations. Here the first map is given by sending the class of an automorphism $\Lambda^r \to \Lambda^r$ to $[[\Lambda^r \to \Lambda^r], can]$, where can denotes the trivialization of the complex $[\Lambda^r \to \Lambda^r]$ by the identity according to Remark 1.2, ∂ maps $[C, a]$ to $[[C]]$ while the last map is given by $[[C]] \mapsto [C]$. If S is an left denominator set of Λ, $\Lambda_S := S^{-1}\Lambda$ the corresponding localization and Σ_S the full subcategory of $C^p(\Lambda)$ consisting of all complexes C such that $\Lambda_S \otimes_\Lambda C$ is acyclic, then $K_1(\Lambda, \Sigma_S)$ and $K_0(\Sigma_S)$ can be identified with $K_1(\Lambda_S)$ and $K_0(S\text{-tor}^{pd})$, respectively. Here $S\text{-tor}^{pd}$ denotes the category of S-torsion Λ-modules with finite projective dimension.

6.2. **Iwasawa main conjecture I.** Let $\mathcal{O}$ be the ring of integers of the completion at any place λ above p of the maximal abelian outside p unramified extension $F_\infty^{ab,p}$ of $\mathbb{Q}$ inside F_∞. Note that the latter extension is finite because every non-finite abelian p-adic Lie extension of $\mathbb{Q}$ contains the cyclotomic $\mathbb{Z}_p$-extension, which is ramified at p. Then by [19, thm. 4.2.26(2)] $\epsilon_{p,\Lambda}(\hat{\mathbb{T}})$ and thus $\mathcal{L}$ is already defined over $\Lambda_\mathcal{O} := \mathcal{O} \otimes_{\mathbb{Z}_p} \Lambda(G)$ instead of $\widetilde{\Lambda}$.

Now let $\Sigma = \Sigma_{SC}$ be the smallest full subcategory of $C^p(\Lambda_\mathcal{O})$ containing SC and satisfying the conditions (i)-(iv) above. Then the evaluation of $\mathcal{L}$ factorizes over its class in $K_1(\Lambda_\mathcal{O}, \Sigma)$ which we still denote by $\mathcal{L}$. By the construction of $\mathcal{L}$ we have

Theorem 6.6 ([19, thm. 4.2.22])**.** *Assume Conjectures 4.1 for (M, Λ) and 5.9 for $(\hat{\mathbb{T}}, \Lambda)$. Then the following holds:*

(i) $\partial(\mathcal{L}) = [[SC]]$

(ii) $\mathcal{L}$ satisfies the interpolation property (Int).

Question: If one knows the existence of $\mathcal{L} \in K_1(\Lambda_\mathcal{O}, \Sigma)$ with the above properties, what is missing to obtain the zeta-isomorphism?

6.3. **Canonical Ore set.** Now assume that the cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}_{cyc}$ is contained in F_∞ and set $H := G(F_\infty/\mathbb{Q}_{cyc})$. In this situation there exist a canonical left and right denominator set of $\Lambda_\mathcal{O}$

$$S^* = \bigcup_{i \geq 0} p^i S$$

with

$$S = \{\lambda \in \Lambda_\mathcal{O} | \Lambda_\mathcal{O}/\Lambda_\mathcal{O}\lambda \text{ is a finitely generated } \Lambda_\mathcal{O}(H)\text{-module}\}$$

as was shown in [11].

In this case we write $\mathfrak{M}_H(G)$ for the category of S^*-torsion modules and identify $K_0(\mathfrak{M}_H(G))$ with $K_0(\Sigma_{S^*})$ recalling that $\Lambda_\mathcal{O}$ is regular.

We write

$$X = Sel(A/F_\infty)^\vee$$

for the Pontryagin dual of the classical Selmer group of A over F_∞, see [11].

Conjecture 6.7 ([11, conj. 5.1])**.** $X \in \mathfrak{M}_H(G)$.

It is shown in [19, prop. 4.3.7] that the conjecture is equivalent to SC belonging to Σ_{S^*}. We assume the conjecture. Observe that then $\Sigma \subseteq \Sigma_{S^*}$, which induces a commutative diagram

$$\begin{array}{ccccccc} K_1(\Lambda_{\mathcal{O}}) & \longrightarrow & K_1(\Lambda_{\mathcal{O}},\Sigma) & \xrightarrow{\partial} & K_0(\Sigma) & \longrightarrow & K_0(\Lambda_{\mathcal{O}}) \\ \| & & \downarrow & & \downarrow & & \| \\ K_1(\Lambda_{\mathcal{O}}) & \longrightarrow & K_1((\Lambda_{\mathcal{O}})_{S^*}) & \xrightarrow{\partial} & K_0(\mathfrak{M}_H(G)) & \longrightarrow & K_0(\Lambda_{\mathcal{O}}) \end{array}$$

Question: Determine the (co)kernel of the middle vertical maps.

In [11, §3] it is explained how to evaluate elements of $K_1((\Lambda_{\mathcal{O}})_{S^*})$ at representations. By [19, lem. 4.3.10] this is compatible with the evaluation of elements in $K_1(\Lambda_{\mathcal{O}},\Sigma)$. The following version of a Main Conjecture was formulated in (loc. cit.).

Conjecture 6.8 (Noncommutative Iwasawa Main Conjecture)**.** *There exist a (unique) element $\mathcal{L}$ in $K_1((\Lambda_{\mathcal{O}})_{S^*})$ such that*

(i) $\partial\mathcal{L} = [X_{\mathcal{O}}]$ in $K_0(\mathfrak{M}_H(G))$ and

(ii) $\mathcal{L}$ satisfies the interpolation property (Int).

The connection with the previous version is given by the following

Proposition 6.9. *Let F_∞ be e.g. $\mathbb{Q}(A(p))$ or $\mathbb{Q}(\mu(p), \sqrt[p^\infty]{\alpha})$ for some $\alpha \in \mathbb{Q}^\times \setminus \mu$ (false Tate curve)*[20]*. Then*

$$[[X]] = [[SC]]$$

in $K_0(\Sigma_{S^*})$. In particular, Conjecture 6.8 is a consequence of Conjecture 4.1 for (M,Λ) and Conjecture 5.9 for $(\hat{\mathbb{T}},\Lambda)$.

The advantage of the localisation $(\Lambda_{\mathcal{O}})_{S^*}$ relies on the fact that one has an explicit description of its first K-group since the natural map $(\Lambda_{\mathcal{O}})^\times_{S^*} \to K_1((\Lambda_{\mathcal{O}})_{S^*})$ induces quite often an isomorphism of the maximal abelian quotient of $(\Lambda_{\mathcal{O}})^\times_{S^*}$ onto $K_1((\Lambda_{\mathcal{O}})_{S^*})$, see [11, thm. 4.4]. On the other hand, the localized $K_1(\Lambda_{\mathcal{O}},\Sigma)$ exists without the assumption that G maps surjectively onto $\mathbb{Z}_p$, e.g. if $G = SL_n(\mathbb{Z}_p)$. Also, if G has p-torsion elements, i.e. if $\Lambda_{\mathcal{O}}(G)$ is *not* regular, one can still formulate the Main Conjecture using the complex SC instead of the classical Selmer group X (which could have infinite projective dimension).

Question: To which extend does a p-adic L-function $\mathcal{L}$ together with Conjecture 6.8 determine the ζ-isomorphism in Conjecture 4.1? In other words, does the Main conjecture imply the ETNC?

7. Appendix: Galois cohomology

The main reference for this appendix is [19, §1.6], but see also [7, 6]. For simplicity we assume $p \neq 2$ throughout this section. Let $U = spec(\mathbb{Z}[\frac{1}{S}])$ be a dense open subset of $spec(\mathbb{Z})$ where S contains $S_p := \{p\}$ and $S := \{\infty\}$ (by abuse of notation). We write G_S for the Galois group of the maximal outside S unramified extension of $\mathbb{Q}$. Let X be

[20]See [19, prop. 4.3.15-17] for a more general statement.

a topological abelian group with a continuous action of G_S. Examples we have in mind are $X = T_p, M_p, \mathbb{T}$, etc. Using continuous cochains one defines a complex $\mathrm{R}\Gamma(U,X)$[21] whose cohomology is $\mathrm{H}^n(G_S,X)$. Then $\mathrm{R}\Gamma_c(U,X)$ is defined by the exact triangle

$$\mathrm{R}\Gamma_c(U,X) \longrightarrow \mathrm{R}\Gamma(U,X) \longrightarrow \bigoplus_{v\in S} \mathrm{R}\Gamma(\mathbb{Q}_v,X) \longrightarrow \tag{7.56}$$

where the $\mathrm{R}\Gamma(\mathbb{Q}_l,X)$ and $\mathrm{R}\Gamma(\mathbb{R},X)$ denote the continuous cochain complexes calculating the local Galois groups $\mathrm{H}^n(\mathbb{Q}_l,X)$ and $\mathrm{H}^n(\mathbb{R},X)$. Its cohomology is concentrated in degrees $0,1,2,3$.

Let L be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$. Now we define the local and global *"finite parts"* for a finite dimensional L-vector space V with continuous $G_{\mathbb{Q}_v}$- and $G_{\mathbb{Q}}$-action, respectively. For $\mathbb{Q}_v = \mathbb{R}$ we set

$$\mathrm{R}\Gamma_f(\mathbb{R},V) := \mathrm{R}\Gamma(\mathbb{R},V)$$

while for a finite place $\mathrm{R}\Gamma_f(\mathbb{Q}_l,V)$ is defined as a certain subcomplex of $\mathrm{R}\Gamma(\mathbb{Q}_l,V)$, concentrated in degree 0 and 1, whose image in the derived category is isomorphic to

$$\mathrm{R}\Gamma_f(\mathbb{Q}_l,V) \cong \begin{cases} [\ V^{I_l} \xrightarrow{1-\varphi_l} V^{I_l}\ ] & \text{if } l \neq p, \\ [\ D_{cris}(V) \xrightarrow{(1-\varphi_p,1)} D_{cris}(V) \oplus D_{dR}(V)/D^0_{dR}(V)\ ] & \text{if } l = p. \end{cases} \tag{7.57}$$

Here φ_l denotes the geometric Frobenius (inverse of the arithmetic) and the induced map $D_{dR}(V)/D^0_{dR}(V) \to H^1_f(\mathbb{Q}_p,V)$ is called exponential map $exp_{BK}(V)$ of Bloch-Kato, where we write $\mathrm{H}^n_f(\mathbb{Q}_l,V)$ for the cohomology of $\mathrm{R}\Gamma_f(\mathbb{Q}_l,V)$.

Defining $\mathrm{R}\Gamma_{/f}(\mathbb{Q}_l,V)$ as mapping cone

$$\mathrm{R}\Gamma_f(\mathbb{Q}_l,V) \longrightarrow \mathrm{R}\Gamma(\mathbb{Q}_l,V) \longrightarrow \mathrm{R}\Gamma_{/f}(\mathbb{Q}_l,V) \longrightarrow \tag{7.58}$$

we finally define $\mathrm{R}\Gamma_f(\mathbb{Q},V)$, whose cohomology is concentrated in degrees $0,1,2,3$, as mapping fibre

$$\mathrm{R}\Gamma_f(\mathbb{Q},V) \longrightarrow \mathrm{R}\Gamma(U,V) \longrightarrow \bigoplus_{S\setminus S_\infty} \mathrm{R}\Gamma_{/f}(\mathbb{Q}_l,V) \longrightarrow . \tag{7.59}$$

This is independent of the choice of U. The octahedral axiom induces an exact triangle

$$\mathrm{R}\Gamma_c(U,V) \longrightarrow \mathrm{R}\Gamma_f(\mathbb{Q},V) \longrightarrow \bigoplus_{S} \mathrm{R}\Gamma_f(\mathbb{Q}_v,V) \longrightarrow . \tag{7.60}$$

7.1. **Duality.** Let G, $\Lambda = \Lambda(G)$, $\mathbb{T}$ as in section 4. By abuse of notation we write $-^*$ for both (derived) functors $\mathbf{R}\mathrm{Hom}_\Lambda(-,\Lambda)$ and $\mathbf{R}\mathrm{Hom}_{\Lambda^\circ}(-,\Lambda^\circ)$. Then Artin-Verdier/Poitou-Tate duality induces the existence of the following distinguished triangle in the derived category of Λ-modules

$$\mathrm{R}\Gamma_c(U,\mathbb{T}) \longrightarrow \mathrm{R}\Gamma(U,\mathbb{T}^*(1))^*[-3] \longrightarrow \mathbb{T}^+ \longrightarrow \tag{7.61}$$

[21]For ease of notation we do not distinguish between complexes and their image in the derived category, though this is sometimes necessary in view of the correct use of the determinant functor and exact sequences of complexes.

and similarly for T (a Galois stable $\mathcal{O}$-lattice of V) and V as coefficients (with Λ replaced by $\mathcal{O}$ and L, respectively).[22]

For the finite parts one obtains from Artin-Verdier/Poitou-Tate and local Tate-duality the following isomorphisms

$$\mathrm{R}\Gamma_f(\mathbb{Q}_l, V) \cong (\mathrm{R}\Gamma(\mathbb{Q}_l, V^*(1))/\mathrm{R}\Gamma_f(\mathbb{Q}_l, V^*(1)))^*[-2], \tag{7.62}$$

$$\mathrm{R}\Gamma_f(\mathbb{Q}, V) \cong \mathrm{R}\Gamma_f(\mathbb{Q}, V^*(1))^*[-3]. \tag{7.63}$$

Set $t(V) := D_{dR}(V)/D^0_{dR}(V)$ if $l = p$ and $t(V) = 0$ otherwise. Trivializing V^{I_l} and $D_{cris}(V)$, respectively, in (7.57) by the identity induces, for each l, an isomorphism

$$\eta_l(V): \quad \mathbf{1}_L \quad \to \mathbf{d}_L(\mathrm{R}\Gamma_f(\mathbb{Q}_l, V))\mathbf{d}_L(t(V)). \tag{7.64}$$

Then, setting $D(V) = D_{dR}(V)$ if $l = p$ and $D(V) = 0$ otherwise, the isomorphism

$$\Theta_l(V): \mathbf{1}_L \to \mathbf{d}_L(\mathrm{R}\Gamma(\mathbb{Q}_l, V)) \cdot \mathbf{d}_L(D(V)) \tag{7.65}$$

is by definition induced from $\eta_l(V) \cdot \overline{(\eta_l(V^*(1))^*)}$ followed by an isomorphism induced by local duality (7.62) and using the analogue $D^0_{dR}(V) = t(V^*(1))^*$ of (5.36) if $l = p$.[23]

7.2. Selmer complexes.

For $l \neq p$ we define $\mathrm{R}\Gamma_f(\mathbb{Q}_l, \mathbb{T})$ as in (7.57) and $\mathrm{R}\Gamma_{/f}(\mathbb{Q}_l, \mathbb{T})$ as in (7.58) with V replaced by $\mathbb{T}$, see also (6.50). We do *not* define $\mathrm{R}\Gamma_f(\mathbb{Q}_p, \mathbb{T})$ since there is in general no integral version of $D_{cris}(V)$.

The Selmer complex $SC_U(\hat{\mathbb{T}}, \mathbb{T})$ is by definition the mapping fibre
(7.66)

$$SC_U(\hat{\mathbb{T}}, \mathbb{T}) \longrightarrow \mathrm{R}\Gamma(U, \mathbb{T}) \longrightarrow \mathrm{R}\Gamma(\mathbb{Q}_p, \mathbb{T}/\hat{\mathbb{T}}) \oplus \bigoplus_{S \setminus (S_p \cup S_\infty)} \mathrm{R}\Gamma(\mathbb{Q}_l, \mathbb{T}) \longrightarrow$$

while $SC(\hat{\mathbb{T}}, \mathbb{T})$ is the mapping fibre
(7.67)

$$SC(\hat{\mathbb{T}}, \mathbb{T}) \longrightarrow \mathrm{R}\Gamma(U, \mathbb{T}) \longrightarrow \mathrm{R}\Gamma(\mathbb{Q}_p, \mathbb{T}/\hat{\mathbb{T}}) \oplus \bigoplus_{S \setminus (S_p \cup S_\infty)} \mathrm{R}\Gamma_{/f}(\mathbb{Q}_l, \mathbb{T}) \longrightarrow \ .$$

---

[22] A more precise form to state the duality is the following. Let $\mathrm{R}\Gamma_{(c)}(U, \mathbb{T})$ be defined like $\mathrm{R}\Gamma_c(U, \mathbb{T})$ but using Tate cohomology $\widehat{\mathrm{R}\Gamma}(\mathbb{R}, \mathbb{T})$ instead of the usual group cohomology $\mathrm{R}\Gamma(\mathbb{R}, \mathbb{T})$. Then one has isomorphisms

$$\mathrm{R}\Gamma(U, \mathbb{T}^*(1))^* \cong \mathrm{R}\Gamma_{(c)}(U, \mathbb{T})[3] \cong \mathrm{R}\Gamma(U, \mathbb{T}^\vee(1))^\vee$$

where $-^\vee = \mathrm{Hom}_{cont}(-\mathbb{Q}_p/\mathbb{Z}_p)$ denotes the Pontryagin dual.

[23] More explicitly, $\theta_p(V)$ is obtained from applying the determinant functor to the following exact sequence

$$0 \longrightarrow \mathrm{H}^0(\mathbb{Q}_p, V) \longrightarrow D_{cris}(V) \longrightarrow D_{cris}(V) \oplus t(V) \xrightarrow{exp_{BK}(V)} \mathrm{H}^1(\mathbb{Q}_p, V) \longrightarrow$$

$$\xrightarrow{exp_{BK}(V^*(1))^*} D_{cris}(V^*(1))^* \oplus t(V^*(1))^* \longrightarrow D_{cris}(V^*(1))^* \longrightarrow \mathrm{H}^2(\mathbb{Q}_p, V) \longrightarrow 0$$

which arises from joining the defining sequences of $exp_{BK}(V)$ with the dual sequence for $exp_{BK}(V*(1))$ by local duality 7.62.

Thus by the octahedral axiom one obtains the distinguished triangles (6.47), (6.49) and, using Artin-Verdier/Poitou-Tate duality,

(7.68)
$$\mathrm{R}\Gamma(U,\mathbb{T}^\vee(1))^\vee \longrightarrow SC(\hat{\mathbb{T}},\mathbb{T}) \longrightarrow \mathrm{R}\Gamma(\mathbb{Q}_p,\hat{\mathbb{T}}) \oplus \bigoplus_{S\setminus(S_p\cup S_\infty)} \mathrm{R}\Gamma_f(\mathbb{Q}_l,\mathbb{T}) \longrightarrow .$$

With the notation of section 6 the Selmer complexes $SC_U(\hat{W},W)$ and $SC(\hat{W},W)$ are defined analogously and satisfy analogous properties.

The following properties [19, (4.2),propositions 1.6.5, 2.1.3 and 4.2.15] are necessary conditions for the existence of the zeta-isomorphism $\zeta_\Lambda(\mathbb{T})$ in Conjecture 4.1 and the p-adic L-functions $\mathcal{L}_U$ (6.52) and $\mathcal{L}$ (6.48).

Proposition 7.1. *The complexes* $\mathrm{R}\Gamma_c(U,\mathbb{T})$ *and* $SC_U(\hat{\mathbb{T}},\mathbb{T})$ *are perfect*[24] *and in* $K_0(\Lambda)$ *we have*
$$[\mathrm{R}\Gamma_c(U,\mathbb{T})] = [SC_U(\hat{\mathbb{T}},\mathbb{T})] = 0.$$
If G *does not have* p*-torsion, also* $SC(\hat{\mathbb{T}},\mathbb{T})$ *is perfect and we have* $[SC(\hat{\mathbb{T}},\mathbb{T})] = 0$.

7.3. **Descent properties.** For the evaluation at representations one needs good descent properties of the complexes involved.

Proposition 7.2. [19, prop. 1.6.5] *With the notation as in section 6 we have canonical isomorphisms (for all* l*)*
$$L^n \otimes_{\Lambda,\rho} \mathrm{R}\Gamma(U,\mathbb{T}) \cong \mathrm{R}\Gamma(U,W), \qquad L^n \otimes_{\Lambda,\rho} \mathrm{R}\Gamma_c(U,\mathbb{T}) \cong \mathrm{R}\Gamma_c(U,W),$$
$$L^n \otimes_{\Lambda,\rho} \mathrm{R}\Gamma_{(c)}(U,\mathbb{T}) \cong \mathrm{R}\Gamma_{(c)}(U,W), \qquad L^n \otimes_{\Lambda,\rho} \mathrm{R}\Gamma(\mathbb{Q}_l,\mathbb{T}) \cong \mathrm{R}\Gamma(\mathbb{Q}_l,W),$$
$$L^n \otimes_{\Lambda,\rho} SC_U(\hat{\mathbb{T}},\mathbb{T}) \cong SC_U(\hat{W},W).$$
For $l \notin \Upsilon \cup S_p$ *we also have:* $L^n \otimes_{\Lambda,\rho} \mathrm{R}\Gamma_f(\mathbb{Q}_l,\mathbb{T}) \cong \mathrm{R}\Gamma_f(\mathbb{Q}_l,W)$.

But note that the complex $\mathrm{R}\Gamma_f(\mathbb{Q}_l,\hat{\mathbb{T}})$ for $l \in \Upsilon$ and thus $SC(\hat{\mathbb{T}},\mathbb{T})$ does *not* descent like this in general. Instead, according to [19, prop. 4.2.17] one has a distinguished triangle

(7.69)
$$L^n \otimes_{\Lambda,\rho} SC(\hat{\mathbb{T}},\mathbb{T}) \longrightarrow SC(\hat{W},W) \longrightarrow \bigoplus_{l\in\Upsilon} \mathrm{R}\Gamma_f(\mathbb{Q}_l,W) \longrightarrow .$$

References

1. D. Benois, *On Iwasawa theory of crystalline representations*, Duke Math. J. **104** (2000), no. 2, 211–267.
2. L. Berger, *Tamagawa numbers of some crystalline representations*, arXiv:math.NT/0209233.
3. S. Bloch, *A note on height pairings, Tamagawa numbers, and the Birch and Swinnerton-Dyer conjecture*, Invent. Math. **58** (1980), no. 1, 65–76.
4. S. Bloch and K. Kato, *L-functions and Tamagawa numbers of motives*, The Grothendieck Festschrift, Vol. I, Progr. Math., vol. 86, Birkhäuser Boston, Boston, MA, 1990, pp. 333–400.
5. J. I. Burgos Gil, *The regulators of Beilinson and Borel*, CRM Monograph Series, vol. 15, American Mathematical Society, Providence, RI, 2002.
6. D. Burns and M. Flach, *Motivic L-functions and Galois module structures*, Math. Ann. **305** (1996), no. 1, 65–102.

[24] $\mathrm{R}\Gamma_c(U,\mathbb{T})$ is even perfect for $p=2$, this is the reason that it is better for the formulation of the ETNC than $\mathrm{R}\Gamma(U,\mathbb{T})$.

7. ——, *Tamagawa numbers for motives with (non-commutative) coefficients*, Doc. Math. **6** (2001), 501–570.
8. D. Burns and C. Greither, *On the equivariant Tamagawa number conjecture for Tate motives*, Invent. Math. **153** (2003), no. 2, 303–359.
9. J. Coates, *On p-adic L-functions*, Astérisque **177-178** (1989), no. Exp. No. 701, 33–59.
10. ——, *Motivic p-adic L-functions*, L-functions and arithmetic (Durham, 1989), London Math. Soc. Lecture Note Ser., vol. 153, Cambridge Univ. Press, Cambridge, 1991, pp. 141–172.
11. J. Coates, T. Fukaya, K. Kato, R. Sujatha, and O. Venjakob, *The GL_2 main conjecture for elliptic curves without complex multiplication*, Publ. Math. IHES. (2005).
12. J. Coates and B. Perrin-Riou, *On p-adic L-functions attached to motives over* **Q**, Algebraic number theory, Adv. Stud. Pure Math., vol. 17, Academic Press, Boston, MA, 1989, pp. 23–54.
13. P. Colmez, *Théorie d'Iwasawa des représentations de de Rham d'un corps local*, Ann. of Math. (2) **148** (1998), no. 2, 485–571.
14. B. Conrad and M. Lieblich, *Galois representations arising from p-divisible groups*, http://www.math.lsa.umich.edu/˜ bdconrad/papers/pdivbook.pdf.
15. P. Deligne, *Valeurs de fonctions L et périodes d'intégrales*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 313–346.
16. M. Flach, *The equivariant Tamagawa number conjecture: a survey*, Stark's conjectures: recent work and new directions, Contemp. Math., vol. 358, Amer. Math. Soc., Providence, RI, 2004, With an appendix by C. Greither, pp. 79–125.
17. J.-M. Fontaine, *Représentations p-adiques semi-stables*, Astérisque (1994), no. 223, 113–184.
18. J.-M. Fontaine and B. Perrin-Riou, *Autour des conjectures de Bloch et Kato: cohomologie galoisienne et valeurs de fonctions L*, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, pp. 599–706.
19. T. Fukaya and K. Kato, *A formulation of conjectures on p-adic zeta functions in non-commutative Iwasawa theory*, preprint (2003).
20. A. Huber and G. Kings, *Equivariant Bloch-Kato conjecture and non-abelian Iwasawa main conjecture*, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002) (Beijing), Higher Ed. Press, 2002, pp. 149–162.
21. ——, *Bloch Kato Conjecture and Main Conjecture of Iwasawa Theory for Dirichlet Characters*, Duke Math. Journal **119** (2003), no. 3, 393–464.
22. M. Kurihara K. Kato and T. Tsuji, *Local iwasawa theory of perrin-riou and syntomic complexes*, 1996.
23. K. Kato, *Iwasawa theory and p-adic Hodge theory*, Kodai Math. J. **16** (1993), no. 1, 1–31.
24. ——, *Lectures on the approach to Iwasawa theory for Hasse-Weil L-functions via B_{dR}. I*, Arithmetic algebraic geometry (Trento, 1991), Lecture Notes in Math., vol. 1553, Springer, Berlin, 1993, pp. 50–163.
25. ——, *K_1 of some non-commutative completed group rings*, preprint (2004).
26. S. L. Kleiman, *Algebraic cycles and the Weil conjectures*, Dix esposés sur la cohomologie des schémas, North-Holland, Amsterdam, 1968, pp. 359–386.
27. ——, *The picard scheme*, arXiv:math.AG/0504020v1 (2005).
28. S. Lang, *Algebraic groups over finite fields*, Amer. J. Math. **78** (1956), 555–563.
29. J. S. Milne, *Abelian varieties*, Arithmetic geometry (Storrs, Conn., 1984), Springer, New York, 1986, pp. 103–150.
30. J. Nekovář, *On p-adic height pairings*, Séminaire de Théorie des Nombres, Paris, 1990–91, Progr. Math., vol. 108, Birkhäuser Boston, Boston, MA, 1993, pp. 127–202.
31. ——, *Selmer complexes*, preprint (2003).
32. B. Perrin-Riou, *p-adic L-functions and p-adic representations*, SMF/AMS Texts and Monographs, vol. 3, American Mathematical Society, Providence, RI, 2000.
33. A. J. Scholl, *Motives for modular forms*, Invent. Math. **100** (1990), no. 2, 419–430.
34. ——, *Classical motives*, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, pp. 163–187.

35. SGA7, *Groupes de monodromie en géométrie algébrique. I*, Springer-Verlag, Berlin, 1972, Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 I), Dirigé par A. Grothendieck. Avec la collaboration de M. Raynaud et D. S. Rim, Lecture Notes in Mathematics, Vol. 288.
36. J. H. Silverman, *The arithmetic of elliptic curves*, Graduate Texts in Mathematics, vol. 106, Springer-Verlag, New York, 199?
37. O. Venjakob, *From classical to non-commutative Iwasawa theory - an introduction to the GL_2 main conjecture*, 4ECM Stockholm 2004, EMS, 2005.
38. S. Yasuda, *Local constants in torsion rings*, preprint (2005).

Universität Heidelberg, Mathematisches Institut, Im Neuenheimer Feld 288, 69120 Heidelberg, Germany.

E-mail address: otmar@mathi.uni-heidelberg.de

URL: http://www.mathi.uni-heidelberg.de/~otmar/