

FROM CLASSICAL TO NON-COMMUTATIVE IWASAWA THEORY AN INTRODUCTION TO THE GL_2 MAIN CONJECTURE

OTMAR VENJAKOB

This paper, which is an extended version of my talk ‘The GL_2 main conjecture for elliptic curves without complex multiplication’ given on the 4ECM, aims to give a survey on recent developments in non-commutative Iwasawa theory. It is written mainly for non-experts and does not contain neither proofs nor any new results, but hopefully serves as introduction to the original articles [4, 33]. Also, technical details are sometimes placed into footnotes in order to keep the main text as easily accessible as possible.

1. CLASSICAL IWASAWA THEORY

For the motivation of non-commutative Iwasawa theory it might be helpful to first go back to the origin of classical Iwasawa theory starting in some sense with the work of Kummer on cyclotomic fields.

The ideal class group $Cl(K)$ of a number field K measures the failure of unique factorisation into prime elements in its ring of integers O_K . It was Kummer who observed that the vanishing of $Cl(\mathbb{Q}(\zeta_p))$, where ζ_p denotes a primitive p th root of unity for a fixed odd prime p , implies that the famous equation

$$x^p + y^p = z^p$$

only has trivial solutions in \mathbb{Z} .¹ In fact, it is even sufficient that only the p -primary part $A_1 := Cl(\mathbb{Q}(\zeta_p))(p)$ of the ideal class group vanishes; in this case p is called *regular*, otherwise *irregular*. Thus, if all prime number would be regular the proof of Fermat’s last theorem would have been rather easy. Hence, it was important for Kummer to be able to tell the regular from the irregular primes and he found the following criterion which reveals a mysterious relationship between A_1 and certain special values of the complex Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}},$$

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¹Indeed, after adjoining ζ_p the right hand side decomposes into the product of pairwise prime elements $x + \zeta_p^i y$, $i = 0, \dots, p-1$, in $\mathbb{Z}[\zeta_p]$; the ideal class group being zero we can now consider the equation prime by prime showing that z must be the product of p pairwise prime numbers z_i which leads to a contradiction as is easily shown, see [35].

where the Euler product ranges over all prime numbers and converges for $\Re(s) > 1$.

Theorem 1.1 (Kummer). *A_1 is trivial if and only if p does not divide any of the numerators of the values $\zeta(-1), \zeta(-3), \dots, \zeta(4-p)$.*²

For example the first four irregular primes are 37, 59, 67, 103 and in contrast to the regular ones it is known that there exist infinitely many of them.

Now it was Iwasawa's idea to study more general the (p -primary) ideal class groups $A_n := Cl(k_n)(p)$ of the fields $k_n = \mathbb{Q}(\mu_{p^n})$ which arise by adjoining the p^n th root of unities μ_{p^n} to \mathbb{Q} , both in order to understand better Kummer's criterion and to see whether the order of A_n could be at least controlled in view of Fermat's last theorem. Studying the ideal class groups for the whole tower of number fields k_n simultaneously leads naturally to considering the infinite Galois extension $\mathbb{Q}_\infty := \mathbb{Q}(\mu_{p^\infty})$ of \mathbb{Q} whose Galois group \mathcal{G} is given explicitly by the cyclotomic character $\chi : \mathcal{G} \xrightarrow{\cong} \mathbb{Z}_p^\times$, i.e. $\zeta^g = \zeta^{\chi(g)}$ for all $g \in \mathcal{G}$ and $\zeta \in \mu_{p^\infty}$. We write \mathbb{Q}_{cyc} for the unique subextension whose Galois group is isomorphic to $\mathbb{Z}_p \cong 1 + p\mathbb{Z}_p \subseteq \mathbb{Z}_p^\times$ and denote the corresponding Galois group by Γ . We fix a topological generator γ , say $1+p$, of Γ . Also we set $\Delta := G(k_1/\mathbb{Q})$ and note that $\mathcal{G} \cong \Gamma \times \Delta$. Iwasawa was not only interested in the size of A_n but also in the finer structure of the ideal class group as Galois module. The natural Galois action on A_n for all n extends naturally to an action of the *Iwasawa algebra* $\Lambda := \Lambda(\mathcal{G})$, i.e. the completed group algebra

$$\mathbb{Z}_p[[\mathcal{G}]] := \varprojlim_n \mathbb{Z}_p[G(k_n/\mathbb{Q})],$$

on the projective limit

$$X := \varprojlim_n A_n.$$

In fact, Iwasawa showed that X is a finitely generated Λ -torsion module. Since $\Lambda \cong \mathbb{Z}_p[\Delta][[\Gamma]]$ decomposes into a finite product of rings³ each of which is isomorphic to the power series ring $\mathbb{Z}[[T]]$ in one variable $T = \gamma - 1$ there is a nice structure theory which assigns to any Λ -torsion module M a *characteristic polynomial* F_M (with coefficients in $\mathbb{Z}_p[\Delta]$), see 3.2 for more details. Neglecting for simplicity the \mathbb{Z}_p -torsion part of M , F_M can be interpreted as the characteristic polynomial of the endomorphism T acting on the free $\mathbb{Q}_p[\Delta](= \prod_{i=1}^{p-1} \mathbb{Q}_p)$ -module $M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. This F_X will be used to define the *algebraic p -adic zeta function* below.

On the other hand, Kummer had shown mysterious congruences between the values of the modified ζ -function

$$\zeta_{(p)}(s) := (1 - p^{-s})\zeta(s)$$

with the Euler factor at p eliminated, which turn out to be equivalent to the existence of a continuous function $\zeta_{p\text{-adic}} : \mathbb{Z}_p \setminus \{1\} \rightarrow \mathbb{Q}_p$ such that $\zeta_{p\text{-adic}}(1-n) = \zeta_{(p)}(1-n)$ for all $n > 1$. Remember that already Euler knew that for $n > 1$ the values $\zeta(1-n)$ are

²Note that ζ has trivial zeroes at the even negative integers, also the numerator of $\zeta(2-p)$ is never divisible by p . The proof of this theorem relies decisively on the analytic class number formula and a decomposition of A_1 and the zeta function of $\mathbb{Q}(\zeta_p)$ into eigenspaces and L -functions with respect to the powers ω^i of the Teichmüller character $\omega : G(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \rightarrow \mu_{p-1}$, respectively.

³corresponding to the characters ω^i of Δ .

rational and thus can also be considered as elements of the local field \mathbb{Q}_p . Furthermore, Kubota and Leopold showed that $\zeta_{p\text{-adic}}$ can be expanded into a p -adic power series, thus being p -adic analytic. As Iwasawa observed $\zeta_{p\text{-adic}}$ can also be interpreted as an element of $Q(\mathcal{G})$, the total ring of fractions of Λ .⁴ First note that every continuous character $\psi : \mathcal{G} \rightarrow \mathbb{Z}_p^\times$ extends linearly to a ring homomorphism $\Lambda \rightarrow \mathbb{Z}_p$ which we also call ψ by abuse of language. Apart from some bad denominators this map extends also to $Q(\mathcal{G})$. In particular, elements $Z \in Q(\mathcal{G})$ can be considered as functions on certain subsets of the set of continuous characters of \mathcal{G} by setting $Z(\psi) := \psi(Z)$, if the latter is defined.

Theorem 1.2 (Iwasawa, Kubota, Leopoldt). *There exists a unique element $Z \in Q(\mathcal{G})$ such that*

$$Z(n) := Z(\chi^n) = \zeta_{(p)}(1 - n)$$

for all $k > 1$.

Note that $Z(n)$ is zero for all odd n due to the trivial zeroes of the Riemann zeta function. Also, by decomposing the cyclotomic character into the product of its projections onto $1 + p\mathbb{Z}_p$ and μ_{p-1} , respectively, one can extend Z to p -adic analytic functions $Z_{\omega^i}(s)$, $s \in \mathbb{Z}_p \setminus \{1\}$.⁵ Alternatively, Z is determined by an interpolation property with respect to Dirichlet characters instead of powers of the cyclotomic character. In this case the Dirichlet L -functions are involved.

In some sense generalizing the analytic class number formula Iwasawa detected a deep relationship between the " p -adic families" of ideal class groups A_n , namely X , on the algebraic side and of special values of ζ , namely Z on the (p -adic) analytic side, which he formulated in the following classical

Main Conjecture (Theorem of Mazur and Wiles) There is the following equality of ideals in Λ :

$$(F_{\mathbb{Z}_p(1)} \cdot Z) = (F_{X^+}).$$

Here $\mathbb{Z}_p(1) := \varprojlim_n \mu_{p^n}$ denotes the Tate-module and, due to the trivial zeroes of Z , one

only has to consider the $+1$ -eigenspace X^+ of X with respect to complex conjugation. In particular, the denominator of Z is controlled by $F_{\mathbb{Z}_p(1)}$. More heuristically, the main conjecture should be read as an identity in $Q(\mathcal{G})$

$$Z = \frac{F_{X^+}}{F_{\mathbb{Z}_p(1)}}$$

up to units in Λ .⁶

⁴ $Q(\mathcal{G})$ is isomorphic to the product $\prod_{i=1}^{p-1} Q(\mathbb{Z}_p[[T]])$ of fields of fractions of $\mathbb{Z}_p[[T]]$.

⁵More precisely, the Z_{ω^i} are the p -adic versions of the complex L -functions $L(\omega^i, s)$.

⁶This can be read as an alternating product of the characteristic polynomials of the action of γ on certain étale cohomology groups with coefficients in $\mathbb{Z}_p(1)$ which identify with the Λ -modules $\mathbb{Z}_p(1)$ and X^+ . This is analogous to the function field situation, namely the fact that the zeta function of a curve C over a finite field \mathbb{F} , $l \neq p$, can be expressed by means of the Lefschetz fix point formula using the action of the Frobenius endomorphism (instead of γ) on the étale cohomology of C . It was

While this classical theory concerned the multiplicative group \mathbb{G}_m - we adjoined the points of its p -primary torsion subgroup to \mathbb{Q} and considered the Galois modules $\mathbb{Z}_p(1)$ and X^+ which can be interpreted as Galois cohomology groups with coefficients in $\mathbb{Z}_p(1)$ - we will explain the corresponding theory for an elliptic curve E over \mathbb{Q} in the following sections.

References: [35, 22, 21, 8]

2. IWASAWA THEORY OF ELLIPTIC CURVES - THE PHILOSOPHY

2.1. Arithmetic of elliptic curves. In order to explain the Iwasawa theory of elliptic curves we first recall basic facts on (the arithmetic of) elliptic curves. To this end let E be an elliptic curve over \mathbb{Q} , i.e. a smooth projective curve of genus one with a distinguished \mathbb{Q} -rational point (the 0 of the underlying abelian group). Every such E can be realized in \mathbb{P}^2 by a (non-unique) Weierstrass equation of the form

$$W : y^2 = x^3 + Ax + B, \quad A, B \in \mathbb{Z}$$

the distinguished point being the point at infinity.

For every prime l this equation also defines a (not necessarily smooth) curve over the local field \mathbb{Q}_l and the finite field \mathbb{F}_l , respectively. One of the basic questions concerning the arithmetic of E is to determine the structure and in particular the size of the group

$$E(K) = ?$$

of K -rational points of E , i.e. the set of solutions of the equation (W) with coordinates in K , for K any number field, local field or finite field. Over a number field, e.g. $K = \mathbb{Q}$, there is the famous

Theorem 2.1 (Mordell-Weil). *The abelian group $E(\mathbb{Q})$ is finitely generated, i.e. it decomposes into*

$$E(\mathbb{Q}) = \mathbb{Z}^r \oplus E(\mathbb{Q})_{\text{tors}},$$

where $r = \text{rk}_{\mathbb{Z}} E(\mathbb{Q})$ is the rank of the Mordell-Weil group or the algebraic rank of E , while $E(\mathbb{Q})_{\text{tors}}$ is the finite torsion subgroup.

While the possible structures of $E(\mathbb{Q})_{\text{tors}}$ were determined by Mazur, in particular, the order of this group is bounded by 16, it is not known whether the rank can be arbitrarily large when E/\mathbb{Q} varies. The properties of the Mordell-Weil group turn out to be, at least conjecturally, deeply related with L -functions, which we are going to recall now. For every prime l we denote by \tilde{E}_l the reduction of E modulo l , i.e. the curve which is given by the reduced equation

$$\tilde{W} : y^2 = x^3 + \tilde{A}x + \tilde{B}, \quad \tilde{A}, \tilde{B} \in \mathbb{F}_l.$$

this prototype which motivated Iwasawa to find a similar interpretation for the p -adic analytic zeta function. The non-trivial contribution of the étale cohomology comes from the jacobian of C , which is paralleled by the ideal class group in the number field case. Moreover, the extension from \mathbb{F} to its algebraic closure is achieved by adjoining roots of unity, which corresponds to taking the cyclotomic \mathbb{Z}_p -extension of \mathbb{Q} .

Here we assume that the Weierstrass equation (W) is a (global) minimal model of E over \mathbb{Z} , i.e. for all primes l the l -part of the integer $\Delta := -16(4A^3 + 27B^2)$ (*the discriminant*) is minimal with respect to all Weierstrass equations over the l -adic integers \mathbb{Z}_l which give rise to the same isomorphism class of elliptic curves. If \tilde{E}_l is again a smooth curve over \mathbb{F}_l , then E is said to have *good* (otherwise *bad*) reduction at l . In the previous case the integer a_l is defined by

$$\#\tilde{E}_l(\mathbb{F}_l) = 1 - a_l + l.$$

Otherwise \tilde{E}_l has either a node, i.e. *multiplicative* reduction, or a cusp, i.e. *additive* reduction. In the second case we set $a_l = 0$ while in the first case we set $a_l = 1$ if the multiplicative reduction is *split*, i.e. the tangent lines to the node on \tilde{E}_l have slopes defined over \mathbb{F}_l , and $a_l = -1$ if the reduction is *non-split*. Then the complex *Hasse-Weil L-function* of E is defined by the following Euler product

$$L(E/\mathbb{Q}, s) := \prod_l (1 - a_l l^{-s} + \epsilon(l) l^{1-2s})^{-1}, \quad s \in \mathbb{C}, \quad \Re(s) > \frac{3}{2},$$

where $\epsilon(l)$ equals by definition 1, if E has good reduction at l , and 0 otherwise. By the work of Wiles and Taylor-Wiles it is known that $L(E/\mathbb{Q}, s)$ has an analytic continuation to the entire complex plane. The following conjecture, which is a generalization of the analytic class number formula for number fields, predicts that the *analytic rank* of E , i.e. the vanishing order of $L(E/\mathbb{Q}, s)$ at $s = 1$, coincides with the *algebraic rank*. Moreover, the leading coefficient of the Taylor series expansion of the L -function at $s = 1$ can be expressed by the most important invariants of E : By $\text{III}(E/\mathbb{Q})$ we denote the *Tate-Shafarevich group* of E , which is conjectured to be finite though this is not known for a single elliptic curve. If \langle, \rangle denotes the height pairing of E and P_1, \dots, P_r form some set of generators of $E(\mathbb{Q})/E(\mathbb{Q})_{\text{tor}}$, then the *regulator* of E is defined to be the determinant of the matrix $(\langle P_i, P_j \rangle)_{i,j}$. Further, if we assume that (W) is a global minimal model of E over \mathbb{Z} , then the translation invariant holomorphic differential

$$\omega := \frac{dx}{2y}$$

is called the *Néron Differential* of E . The integration of it along a generator γ^+ of the real part $\pi_1(E(\mathbb{C}), 0)^+ := \pi_1(E(\mathbb{C}), 0)^{G(\mathbb{C}/\mathbb{R})}$ of the fundamental group of the complex manifold $E(\mathbb{C})$ defines the *real period*

$$\Omega_+ = \int_{\gamma^+} \omega$$

of E . Similarly, the period Ω_- is defined via integration along a generator γ_- of the -1 eigenspace of the fundamental group with respect to the action of complex conjugation. Finally, for any prime l we call *Tamagawa-number* at l the index $c_l = [E(\mathbb{Q}_l) : E^{ns}(\mathbb{Q}_l)]$ of the subgroup $E^{ns}(\mathbb{Q}_l)$ of the group of \mathbb{Q}_l -rational points $E(\mathbb{Q}_l)$ consisting of those points whose reduction modulo l is non-singular.

Conjecture 2.2 (Birch & Swinnerton-Dyer (BSD) Conjecture).

- I. $r := \text{ord}_{s=1} L(E/\mathbb{Q}, s) = \text{rk}_{\mathbb{Z}} E(\mathbb{Q})$
- II. $\lim_{s \rightarrow 1} (s-1)^r L(E/\mathbb{Q}, s) = \Omega_+ R_E \frac{\#\text{III}(E/\mathbb{Q})}{(\#E(\mathbb{Q})_{\text{tors}})^2} \prod_l c_l$

Note that the product over the Tamagawa numbers is actually finite as $C_l = 1$ whenever E has good reduction at l .

Thus this conjecture describes a mysterious relationship between the complex analytic L -function and the purely algebraically defined Mordell-Weil group. A similar conjecture can be formulated for elliptic curves over arbitrary number fields. The idea of Iwasawa theory is roughly speaking to study this deep connection between the values of (complex) L -functions and arithmetic invariants of E for a full tower of number fields simultaneously as we have already seen in section 1.

References: [28, 29]

2.2. The Selmer group of E in towers of number fields. For technical reasons we make from now on the following

Assumption: $p \geq 5$ is a prime such that E has *good ordinary* reduction at p , i.e. the order of the group of p -division points $\tilde{E}_p(\overline{\mathbb{F}}_p)[p]$ equals p .

To study the Mordell-Weil group of E it is often more convenient to go over to the cohomologically defined (p -primary) *Selmer group* $\text{Sel}(E/K)$ for any finite extension K/\mathbb{Q} . Instead of giving the precise definition we just recall that induced by Kummer theory the Selmer group fits into the following short exact sequence, being the bridge between the (p -primary) Tate-Shafarevich group and the Mordell-Weil group:

$$0 \longrightarrow E(K) \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \text{Sel}(E/K) \longrightarrow \text{III}(E/K)(p) \longrightarrow 0 .$$

Assuming $\#\text{III}(E/K)(p) < \infty$, which can be checked - for fixed p - in many cases, it holds for the Pontryagin dual of the Selmer group

$$X(E/K) := \text{Sel}(E/K)^\vee := \text{Hom}(\text{Sel}(E/K), \mathbb{Q}_p/\mathbb{Z}_p),$$

that

$$\text{rk}_{\mathbb{Z}} E(K) = \text{rk}_{\mathbb{Z}_p} X(E/K).$$

Thus, indeed, the Selmer group (or its dual) bears significant arithmetic information of E .

Now we introduce a canonical tower of number fields associated with E . By $E[p^n] \cong \mathbb{Z}/p^n\mathbb{Z} \times \mathbb{Z}/p^n\mathbb{Z}$ we denote the p^n -division points of E over a fixed algebraic closure $\overline{\mathbb{Q}}$ of \mathbb{Q} . The action of the absolute Galois group $G_{\mathbb{Q}}$ on this group induces, after choosing a basis, the representation

$$\rho_{p^n} : G_{\mathbb{Q}} \longrightarrow \text{Aut}(E[p^n]) \cong GL_2(\mathbb{Z}/p^n\mathbb{Z}).$$

We define $K_n := \mathbb{Q}(E[p^n]), 0 \leq n < \infty$, to be the maximal subfield of $\overline{\mathbb{Q}}$ fixed under the kernel of ρ_{p^n} . Then $K_\infty := \bigcup_{n \geq 0} K_n$ is nothing else then the fixed field under the kernel of the representation

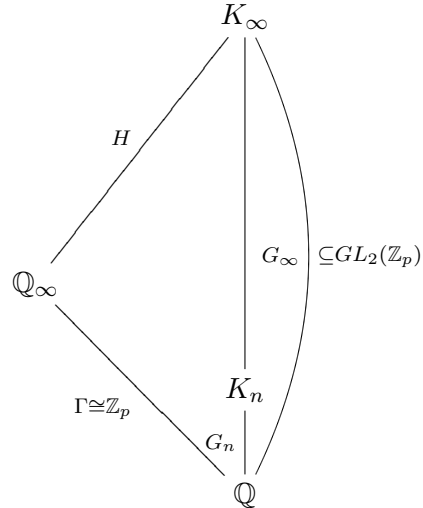
$$\rho_{p^\infty} : G_{\mathbb{Q}} \longrightarrow \text{Aut}_{\mathbb{Z}_p}(T_p E) \cong GL_2(\mathbb{Z}_p)$$

of $G_{\mathbb{Q}}$ on the Tate module

$$T_p E := \varprojlim_n E[p^n],$$

where the inverse limit is formed with respect to the multiplication by p maps. In particular, K_∞ is a Galois extension of \mathbb{Q} with Galois group $G := G(K_\infty/\mathbb{Q})$ isomorphic to a closed subgroup of $GL_2(\mathbb{Z}_p)$. Thus G is a p -adic Lie group. We want to stress that the L -function of E only depends on the Galois representation ρ_{p^∞} , thus the tower of number fields $\{K_n\}_n$ is most natural in order to study the arithmetic of E , in particular, to investigate properties of its L -function.

Note that due to the Weil pairing $\det \circ \rho$ is isomorphic to the cyclotomic character $\chi : G_{\mathbb{Q}} \longrightarrow \mathbb{Z}_p^\times$ which describes the action of $G_{\mathbb{Q}}$ on the p -power roots of unity $\mu_{p^\infty} : g\zeta = \zeta^{\chi(g)}$ for all $g \in G_{\mathbb{Q}}$ and $\zeta \in \mu_{p^\infty}$. Thus K_∞ contains the cyclotomic \mathbb{Z}_p -extension \mathbb{Q}_∞ of \mathbb{Q} . We write H for the Galois group $G(K_\infty/\mathbb{Q}_\infty)$ and obtain the diagram sideways.



As before the Iwasawa algebra of G

$$\Lambda(G) = \varprojlim_n \mathbb{Z}_p[G_n]$$

is the inverse limes of the group algebras $\mathbb{Z}_p[G_n]$ of G_n with coefficients in \mathbb{Z}_p . It is a compact, regular Noetherian ring. In contrast to the classical Iwasawa algebra $\Lambda(\Gamma)$ of Γ it is *not commutative in general*.

Now, for every $n \geq 1$, the Galois action makes $X(E/K_n) := \text{Sel}(E/K_n)^\vee$ into a compact $\mathbb{Z}_p[G_n]$ -module. To study, on the algebraic side, all these Selmer groups simultaneously for the whole tower of number fields means to go over to the inverse limit

$$X := X(E/K_\infty) := \varprojlim_n \text{Sel}(E/K_n)^\vee,$$

which turns out to be a finitely generated $\Lambda(G)$ -module, conjecturally even a *torsion* $\Lambda(G)$ -module. Roughly one should think of it as the family of all the Mordell-Weil groups $E(K_n)$ (and Tate-Shafarevich groups $\text{III}(E/K_n)(p)$). The analytic counterpart of this family will be discussed in the next subsection.

References: [2, 5, 23, 24, 13]

2.3. Twisted L -functions. For every $n \geq 0$, let $\text{Irr}(G_n)$ denote the set of isomorphism classes of (absolutely) irreducible representations of G_n , realized over an appropriate number field embedded into \mathbb{C} or over a local field contained in $\overline{\mathbb{Q}_l}$ (depending on n). Via the canonical projection $G \rightarrow G_n$ they are also considered as representations of G , to which we shall refer as *Artin representation*. Let R be the finite set of primes of \mathbb{Q} containing p and all primes l at which E has bad reduction.

On the analytic side one is searching for a function \mathcal{L}_E , the *p -adic analytic L -function of E* , on the set $\bigcup_n \text{Irr}(G_n)$ which assigns to ρ the value at $s = 1$ of the complex L -function $L(E, \rho, s)$ of E twisted by ρ or rather its modified version $L_R(E, \rho, s)$ with the Euler factors at primes in R eliminated.^{7 8}

Heuristically, summarizing the (generalized) BSD conjecture over all the fields K_n leads directly to the Iwasawa Main Conjecture of E ⁹. Since the (modified) L -function $L_R(E/K_n, s)$ of E over K_n (similarly defined as over \mathbb{Q} and without the Euler factors in R) decomposes into the product of twisted L -functions (with multiplicities), the idea is that on the analytic side of the picture the family of special values at $s = 1$ of $L_R(E, \rho, s)$ can be interpolated p -adically, which should lead to the *p -adic analytic L -function*. On the other hand on the algebraic side there should be some procedure to assign to the $\Lambda(G)$ -module $X = X(E/K_\infty)$ (as for any torsion $\Lambda(G)$ -module) some *characteristic element* F_X bearing hopefully many arithmetic information of E . The (heuristic) comparison of the algebraic and analytic aspect when going over to towers of number fields are illustrated in the following diagram

⁷we restrict to those primes not lying in R , because the corresponding factors at primes in R usually do not behave well p -adically and thus have to be eliminated from the usual definition of the L -function in order to expect a p -adic L -function in whatever sense.

⁸ For the interested reader we recall the definition of $L_R(E, \rho, s)$. Again it is defined as an Euler product, which converges only for $\Re(s) > \frac{3}{2}$,

$$L_R(E, \rho, s) := \prod_{q \notin R} P_q(E, \rho, q^{-s})^{-1}, \quad s \in \mathbb{C},$$

where the $P_q(E, \rho, T)$ are polynomials to be defined below. The only thing known about its analytic continuation at present is that it has a meromorphic continuation when ρ factors through a soluble extension of \mathbb{Q} . We will assume the analytic continuation of $L(E, \rho, s)$ to $s = 1$ for all Artin characters ρ of G in what follows. If q is any prime number we write Frob_q for the Frobenius automorphism of q in $G(\overline{\mathbb{Q}_q}/\mathbb{Q}_q)/I_q$, where, as usual, I_q denotes the inertia subgroup. Assume now that $\rho \in \text{Irr}(G_n)$ is realized on a vector space V_ρ over a number field K of dimension n_ρ . For a fixed place λ of K lying above $l \neq q$ we denote by K_λ the completion of K with respect to λ and we set

$$V_{\rho, \lambda} = V_\rho \otimes_K K_\lambda.$$

Also we consider the l -adic Tate module $V_l E := H_1(E(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}_l \cong T_l E \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ and set

$$H_l^1(E) := \text{Hom}(V_l E, \mathbb{Q}_l).$$

Finally we put for any prime l different from q

$$P_q(E, \rho, T) := \det(1 - \text{Frob}_q^{-1} \cdot T | (H_l^1(E) \otimes_{\mathbb{Q}_l} V_{\rho, \lambda})^{I_q}).$$

It can be shown that for ρ the trivial representation the local L -function $P_q(E, p^{-s}) := P_q(E, \rho, p^{-s})$ coincides with the Euler factor at q of the Hasse-Weil L -function of E . In particular, the integers a_q are just the traces of Frob_q acting on the maximal unramified quotient $(V_l E)_{I_q}$ of the Tate module.

⁹In fact this can be made precise in the context of the Equivariant Tamagawa Number Conjecture (ETNC), a natural generalisation of the BSD conjecture, see [11], also [1, 16].

algebraic	analytic
$X(E/K_n)$ as G_n -module	$L_R(E/K_n) = \prod_{\text{Irr}(G_n)} L_R(E, \rho, s)^{n_\rho}$
<i>p</i> -adic families	
$X(E/K_\infty)$	$(L_R(E, \rho, 1))_{\rho \in \text{Irr}(G_n), n < \infty}$
<i>p</i> -adic <i>L</i> -functions	
$F_E := F_X$ characteristic element	\mathcal{L}_E analytic <i>p</i> -adic <i>L</i> -function

This comparison culminates in the

Main Conjecture: $F_E \equiv \mathcal{L}_E,$

which says that the characteristic element of E and the *p*-adic analytic *L*-functions (if they exist at all) are essentially the same, in a sense we have to make precise later. If such a relation should hold it must be a very deep relationship since it connects two totally different aspects of E living in completely different worlds and in some sense "explaining" the mysterious BSD-conjecture.

\mathbb{G}_m	E
$\zeta(s)$	$L(E, s)$
$\mathbb{Q}(\mu_{p^\infty})$	$\mathbb{Q}(E[p^\infty])$
$Cl(k_n)$	$E(K_n)$
X^+	$X(E/K_\infty)$
$\zeta_{p\text{-adic}}$	\mathcal{L}_E

We conclude this section by illustrating the analogy between the \mathbb{G}_m - and the E -case:

3. IWASAWA THEORY OF ELLIPTIC CURVES - RECENT DEVELOPMENTS

3.1. What is new? Before we try to describe different approaches to make the philosophy explained above precise we would like to mention that we have to distinguish two totally different cases. Consider the following explicit elliptic curves

$$E_1 : y^2 = x^3 - x$$

and

$$E_2 : y^2 + y = x^3 - x^2.$$

At a first glance who would expect an essentially difference between them? But while the first one has a "big" ring of endomorphisms - one can show that $\text{End}(E_1) \cong \mathbb{Z}[i] \neq \mathbb{Z}$, i.e. E admits complex multiplication (CM) - the second one only has the endomorphisms

arising from multiplication with integers: $\text{End}(E_2) \cong \mathbb{Z}$, i.e. E does *not* admit complex multiplication.

Now it follows that in the *CM-case*, the group G has the form

$$G \cong \mathbb{Z}_p^2 \times \text{finite abelian group},$$

in particular it is *abelian*. This commutative theory is rather well known, the *2-variable main conjecture*¹⁰ is a Theorem of Rubin [25] in many cases, see also at the end of 3.5.

Thus we want to concentrate on the second, the *GL₂-case*. By a deep result of Serre [27] now G is of the form

$$G \subseteq_o GL_2(\mathbb{Z}_p) \quad \text{open subgroup},$$

in particular it is *not abelian*.

In this case it was not even known how to formulate a *GL₂ main conjecture* and it is this case where the substantial progress we want to describe in these notes has been achieved recently. This development concerns unfortunately only the algebraic side of the picture drawn above, in particular, the *existence of characteristic elements* has been established, while the *p-adic analytic part* will be purely conjectural.

We also should mention that there is a well-developed (commutative) Iwasawa theory of elliptic curves over the \mathbb{Q}_∞ , [19, 12, 20] We refer to the corresponding main conjecture as *1-variable main conjecture*. For CM-elliptic curves this is consequence of the 2-variable main conjecture. In the non-CM case there are partial results by Kato [17] and recent results by Urban and Skinner [?], which together prove the latter main conjecture in several cases.

3.2. Structure Theory. In this section let G be any compact p -adic Lie group without element of order p (G can always be realized as an closed subgroup of $GL_n(\mathbb{Z}_p)$ for some n).

In order to define the characteristic element of a $\Lambda(G)$ -module it is tempting to imitate the approach of classical Iwasawa theory, i.e. the case where

$$G \cong \mathbb{Z}_p^n$$

and thus there is an isomorphism $\Lambda = \Lambda(G) \cong \mathbb{Z}_p[[X_1, \dots, X_n]]$ for any choice of a minimal system of topological generators $\gamma_1, \dots, \gamma_n$ of G by the identities $X_i = \gamma_i - 1$. In particular, $\Lambda(G)$ is a complete, regular local ring of dimension $n + 1$. In the case $n = 1$ it was Iwasawa himself - and more generally for integrally closed (commutative) domains Serre - who established a structure theorem, similar to that concerning modules over principal ideal domains: Every finitely generated $\Lambda(G)$ -torsion module M is up to pseudo-null modules a direct product of cyclic modules

$$M \sim \prod_i \Lambda / \Lambda f_i^{n_i},$$

¹⁰Since G has dimension 2 as p -adic Lie group the p -adic L -function is a power series in two variables.

where n_i are uniquely defined integers and f_i are irreducible elements of $\Lambda(G)$, unique up to units. The pseudo-null¹¹ modules have to be considered as small - in fact, for $n = 1$ they are precisely the class of all finite modules - and the idea is that they do not contribute essential information in the arithmetic applications. Now one defines the characteristic element using the above invariants attached to M

$$F_M := \prod f_i^{n_i}.$$

Returning to the general case, i.e. to a not necessarily commutative group G , the concept of *pseudo-null* modules was developed in the authors thesis [30, 31, 34] by cohomological methods establishing the fundamental fact that $\Lambda(G)$ is an Auslander regular ring, for details see (loc. cit.). Then Coates, Schneider and Sujatha went on establishing a structure theorem in this non-commutative setting almost totally parallel to the above mentioned theory.

Theorem 3.1 (Coates, Schneider, Sujatha). *For every torsion Λ -module M there exist left ideals L_1, \dots, L_r such that, up to pseudo-null modules, M decomposes into a product of cyclic modules*

$$M \sim \prod_{i=1}^r \Lambda/L_i.$$

For details and the mild technical further assumption on G needed in this theorem, see [7] and [3]. Unfortunately, it turned out that the G -Euler characteristic, an important arithmetic invariant if e.g. applied to the Selmer group and which will be defined later, is not invariant under pseudo-isomorphisms¹². Moreover, the ideals L_i occurring above need not be principal in general (see [32, appendix] for a counterexample). Thus, at moment, the theorem cannot be used to attach an characteristic element to M and it is still not clear which role this astonishing structure result will play in non-commutative Iwasawa theory. In order to circumvent this dilemma we are going to apply techniques from algebraic K -theory and localisation of (possibly non-commutative) rings.

References: [7, 6, 22, 15, 14, 32, 26]

3.3. Localisation of Iwasawa algebras and characteristic elements. The following theory stems from joint work of Coates, Fukaya, Kato and Sujatha with the author [4] and makes heavily use of the following

Assumption: There exists a normal closed subgroup $H \trianglelefteq G$ such that the quotient $\Gamma := G/H$ is isomorphic to \mathbb{Z}_p .

Recall that it is satisfied in our application because K_∞ contains the cyclotomic \mathbb{Z}_p -extension \mathbb{Q}_{cyc} of \mathbb{Q} .

In this situation we are able to define a multiplicatively closed subset \mathcal{T}^{13} consisting of non-zerodivisors of $\Lambda := \Lambda(G)$ hoping that one can localize Λ with respect to it.

¹¹A finitely generated $\Lambda(G)$ -module M is *pseudo-null* if its support has codimension at least 2 in the spectrum of $\Lambda(G)$.

¹²A homomorphism of modules whose kernel and cokernel are pseudo-null

¹³First define $\mathcal{T}' := \{\lambda \in \Lambda \mid \Lambda/\Lambda\lambda \text{ finitely generated over } \Lambda(H)\}$ and then saturate it with the powers of p , i.e. $\mathcal{T} := \bigcup_{i \geq 0} p^i \mathcal{T}' \subseteq \Lambda$.

While this is always possible for commutative rings this is a quite subtle issue for *non-commutative* rings: one has to check that \mathcal{T} satisfies the Ore-condition, which means roughly speaking that every right fraction with denominator in \mathcal{T} can also be written as left fraction, and vice-versa. If the localisation with respect to \mathcal{T} exists, it should be related - by construction - to the following subcategory of the category of Λ -torsion modules:

$\mathfrak{M}_H(G)$ category of Λ -modules M such that modulo \mathbb{Z}_p -torsion M is finitely generated over $\Lambda(H) \subseteq \Lambda(G)$.

Thus, from a technical point of view the following theorem is the key result of our construction:

Theorem 3.2. *The localization $\Lambda_{\mathcal{T}}$ of Λ with respect to \mathcal{T} exists and there is a surjective map arising from K -theory¹⁴*

$$\partial : (\Lambda_{\mathcal{T}})^{\times} \rightarrow K_0(\mathfrak{M}_H(G))$$

from the group of units $(\Lambda_{\mathcal{T}})^{\times}$ of $\Lambda_{\mathcal{T}}$ to the Grothendieck group $K_0(\mathfrak{M}_H(G))$ of $\mathfrak{M}_H(G)$.

This leads directly to the following

Definition 3.3. Any $F_M \in (\Lambda_{\mathcal{T}})^{\times}$ with $\partial[F_M] = [M]$ is called *characteristic element* of $M \in \mathfrak{M}_H(G)$.

In order to show that this is not just a sophisticated but useless definition we state some basic properties of our construction. In particular, F_M behaves well with Euler characteristics.

Properties

- (i) Any $f \in (\Lambda_{\mathcal{T}})^{\times}$ can be interpreted as a map on the isomorphism classes of (continuous) representations $\rho : G \rightarrow Gl_n(\mathcal{O}_K)$, where \mathcal{O}_K runs through the ring of integers of finite extensions K of \mathbb{Q}_p :

$$\rho \mapsto f(\rho) \in K \cup \{\infty\} \subseteq \overline{\mathbb{Q}_p} \cup \{\infty\}.$$

- (ii) The evaluation of F_M at ρ gives the generalized G -Euler characteristic¹⁵ $\chi(G, M(\rho))$

$$|F_M(\rho)|_p^{-[K:\mathbb{Q}_p]} = \chi(G, M(\rho))$$

¹⁴There is an exact localization sequence

$$\begin{array}{ccccccc} \Lambda^{\times} & & (\Lambda_{\mathcal{T}})^{\times} & & & & \\ \downarrow & & \downarrow & & & & \\ K_1(\Lambda) & \longrightarrow & K_1(\Lambda_{\mathcal{T}}) & \xrightarrow{\partial} & K_0(\mathfrak{M}_H(G)) & \longrightarrow & 0, \end{array}$$

where the surjectivity claims need little arguments, see [4, §4].

¹⁵Note that with M also every twist $M(\rho) := M \otimes_{\mathbb{Z}_p} \mathcal{O}_K^{\times}$ (with diagonal G -action, via ρ on the right factor) belongs to $\mathfrak{M}_H(G)$. By definition, $\chi(G, M(\rho)) := \prod_{i \geq 0} (\#H_i(G, M(\hat{\rho})))^{(-1)^i}$, if all groups are finite and where $\hat{\rho}$ denotes the contragredient representation of ρ .

if the Euler-characteristic is defined. Here the p -adic valuation is normalized as usual by $|p|_p = \frac{1}{p}$.

For more details and proofs, see [4, § 3],[33, §8].

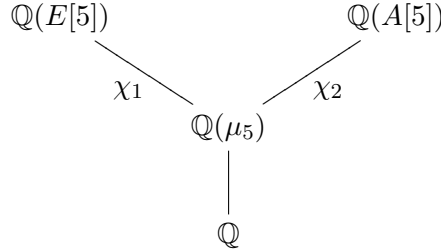
3.4. Numerical Example. Consider the two elliptic curves

$$\begin{aligned} E = X_1(11) & : y^2 + y = x^3 - x^2, \\ A & : y^2 + y = x^3 - x^2 - 7820x - 263580 \end{aligned}$$

and let $p = 5$. One can show that $X \in \mathfrak{M}_H(G)$, i.e. that F_X exists. Now $G = G(\mathbb{Q}(E(5))/\mathbb{Q})$ has 2 irreducible Artin Representations of degree 4 :

$$\rho_i = \text{Ind}\chi_i : G \rightarrow GL_4(\mathbb{Z}_5),$$

which are in fact induced by the characters χ_i , $i = 1, 2$, corresponding to the cyclic extensions of degree 5 as indicated in the following diagram



Calculations show that $\chi(G, X(\rho_i))$ equals 5^3 and 5 for $i = 1$ and $i = 2$, respectively. Thus

$$F_X(\rho_1) \sim 5^3, \quad F_X(\rho_2) \sim 5$$

up to \mathbb{Z}_5^\times .

3.5. Analytic p -adic L -function and the GL_2 -main conjecture. In contrast to the algebraic theory above the following analytic part is purely conjectural. First of all we have Delignes [9]

Period - Conjecture:
$$\frac{L_R(E, \rho, 1)}{\Omega(E, \rho)} \in \bar{\mathbb{Q}}$$

for a suitable period $\Omega(E, \rho) \in \mathbb{C}$, which permits to consider $\frac{L_R(E, \rho, 1)}{\Omega(E, \rho)}$ as value in $\bar{\mathbb{Q}}_p$, i.e. in the same target where the elements of $(\Lambda(G)_{\mathcal{T}})^\times$ interpreted as functions take their values. In analogy with classical Iwasawa theory we call such an element which interpolates these values p -adic analytic L -function though one could criticize that there is no p -adic analysis involved at present.

Conjecture 3.4 (Existence of analytic p -adic L -function). *Let $p \geq 5$ and assume that E has good ordinary reduction at p . Then there exists*

$$\mathcal{L}_E \in (\Lambda(G)_{\mathcal{T}})^\times,$$

such that for all Artin representations ρ of G one has $\mathcal{L}_E(\rho) \neq \infty$ and

$$\mathcal{L}_E(\rho) \sim \frac{L_R(E, \rho, 1)}{\Omega(E, \rho)}$$

up to some modifications of the Euler factor at p .

The precise formula¹⁶ describing the interpolation property can be deduced from Fukaya and Kato's version¹⁷ of the Equivariant Tamagawa Number Conjecture (ETNC) together with their ϵ -conjecture and thus follows a precise recipe whose explanation is unfortunately out of the scope of this article. In particular, the following version of a non-commutative Iwasawa main conjecture is compatible with the ETNC corresponding to our tower K_n of number fields:

Conjecture 3.5 (Main Conjecture). *Assume that $p \geq 5$, E has good ordinary reduction at p , and $X(E/K_\infty)$ belongs to $\mathfrak{M}_H(G)$. Granted the existence of the p -adic L -function, \mathcal{L}_E is a characteristic element of $X(E/K_\infty)$:*

$$\partial[\mathcal{L}_E] = [X(E/K_\infty)].$$

Before we discuss evidence for this conjectures we would like to comment on some of its implications. Assuming the existence of \mathcal{L}_E , one can show

- (i) that the GL_2 -main conjecture implies the 1-variable main conjecture (over \mathbb{Q}_{cyc}).
- (ii) that, assuming also the GL_2 -main conjecture, it holds

$$\chi(G, X(\rho)) \text{ finite} \Leftrightarrow L_R(E, \rho, 1) \neq 0.$$

In this case one has with $m_\rho := [K : \mathbb{Q}_p]$:

$$\chi(G, X(\rho)) = |\mathcal{L}_E(\rho)|_p^{-m_\rho}.$$

- (iii) that, if $L(E, 1) \neq 0$, by results of Kolyvagin the groups $E(\mathbb{Q})$ and $\text{III}(E/\mathbb{Q})$ are finite and the p -part of the BSD conjecture (II.) holds:

$$\frac{L(E/\mathbb{Q}, 1)}{\Omega_+} \sim_p \frac{\#\text{III}(E/\mathbb{Q})}{(\#E(\mathbb{Q}))^2} \prod_l c_l$$

up to \mathbb{Z}_p^\times .

¹⁶Since E is ordinary at p , we have $P_p(E, 1, T) = 1 - a_p T + pT^2 = (1 - uT)(1 - wT)$ with $u \in \mathbb{Z}_p^\times$. We put $p^{f_\rho} = p$ -part of conductor of ρ , and denote by $e_p(\rho)$ the local ϵ -factor of ρ at p . Finally we set $P_p(\rho, T) := \det(1 - \text{Frob}_q^{-1} \cdot T | V_\rho^{I_p})$. Then the interpolation formula is

$$\mathcal{L}_E(\rho) = \frac{L_R(E, \rho, 1)}{\Omega(E, \rho)} \cdot e_p(\rho) \cdot \frac{P_p(\hat{\rho}, u^{-1})}{P_p(\rho, w^{-1})} \cdot u^{-f_\rho},$$

where $\Omega(E, \rho) = \Omega_+(E)^{d^+(\rho)} \Omega_-(E)^{d^-(\rho)}$ while $d^+(\rho)$ and $d^-(\rho)$ denote the dimension of the subspace of V_ρ on which complex conjugation acts by $+1$ and -1 , respectively (see [4, 5.7]).

¹⁷The original ETNC was formulated by Burns and Flach [1] inspired by [18]. A different version of an Iwasawa main conjecture (without p -adic L -functions) was discussed by Huber and Kings [16]

We conclude this survey by giving some evidence for Main Conjecture:

In the *CM-case* the existence of \mathcal{L}_E follows from the existence of the 2-variable p -adic L -function (Manin-Vishik [], Katz [?], Yager [36]). If $X \in \mathfrak{M}_H(G)$, then the main conjecture follows from the 2-variable main conjecture (Rubin, Yager).

In the *GL₂-case* almost nothing is known! There is only weak numerical evidence by calculations of T. and V. Dokchitser [10]. Let $E = X_1(11)$, $p = 5$, and ρ_i , $i = 1, 2$, be the two unique irreducible Artin representations of degree 4 as before. Then they verify that the relation

$$\chi(G, X(\rho_i)) = |\mathcal{L}_E(\rho_i)|_p^{-1}, \quad i = 1, 2$$

holds as is predicted by the main conjecture, see above. Here $\mathcal{L}_E(\rho_i)$ denotes the term describing the interpolation property of \mathcal{L}_E if the p -adic L -function should exist.

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UNIVERSITÄT HEIDELBERG, MATHEMATISCHES INSTITUT, IM NEUENHEIMER FELD 288, 69120 HEIDELBERG, GERMANY.

E-mail address: otmar@mathi.uni-heidelberg.de

URL: <http://www.mathi.uni-heidelberg.de/~otmar/>