HERR-COMPLEXES IN THE LUBIN-TATE SETTING

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ABSTRACT. In this article we extend work of Herr from the case of cyclotomic (φ, Γ)-modules to the general case of Lubin-Tate (φ, Γ)-modules. In particular, we define generalized φ- and ψ-Herr complexes, which calculate Galois cohomology, when applied to the étale (φ, Γ)-modules attached to the coefficients.

1. INTRODUCTION

Fontaine's theory [14] of (cyclotomic) (φ, Γ)-modules plays a central role both in the p-adic local Langlands programme, more specifically in Colmez' celebrated work [13], as well as in (local) Iwasawa, where, for example, it contributes to proofs of reciprocity formulas or the construction of regulator maps and big exponential maps [2] à la Perrin Riou. One reason for this impact stems from the possibility to explicitly calculate Iwasawa and Galois cohomology of a Galois representation V in terms of the associated (φ, Γ)-module $\mathcal{D}(V)$. While the description of Iwasawa cohomology was given by Fontaine himself, it was his disciple Herr [18, 19] who described Galois cohomology as the cohomology of the following complex, now named after him. To this end we fix an odd prime p and consider a $\mathbb{Z}_p$-representation V of the absolute Galois group $G_{\mathbb{Q}_p}$ of the p-adic numbers $\mathbb{Q}_p$. Then the complex $\mathcal{C}_\varphi^\bullet(\Gamma, \mathcal{D}(V))$

$$0 \longrightarrow \mathcal{D}(V) \xrightarrow{(\varphi-1, \gamma-1)} \mathcal{D}(V) \oplus \mathcal{D}(V) \xrightarrow{(\gamma-1)\mathrm{pr}_1 - (\varphi-1)\mathrm{pr}_2} \mathcal{D}(V) \longrightarrow 0$$

computes the group cohomology of $G_{\mathbb{Q}_p}$ with values in V, where φ denotes the Frobenius endomorphism while γ is a topological generator of $\Gamma = G(\mathbb{Q}_p(\mu(p))/\mathbb{Q}_p)$. (cf. e.g. [12, Theorem 5.2.2., p. 93–94] and [12, Theorem 5.3.15, p. 103–104]). Upon replacing φ by its left-inverse ψ we obtain the complex $\mathcal{C}_\psi^\bullet(\Gamma, \mathcal{D}(V))$, which like a miracle turns out to be quasi-isomorphic to $\mathcal{C}_\varphi^\bullet(\Gamma, \mathcal{D}(V))$ in the cyclotomic theory (cf. [12, Proposition 5.3.14, p. 103]). Moreover, Herr established that taking an appropriate dual of the Herr complex $\mathcal{C}_\varphi^\bullet(\Gamma, \mathcal{D}(V^*(1)))$ results - up to shifting by 2 - another complex quasi-isomorphic to $\mathcal{C}_\psi^\bullet(\Gamma, \mathcal{D}(V))$ giving rise to Tate's local duality.

Recently there has been quite some activity to develop a theory of Lubin-Tate (φ, Γ)-modules [4, 5, 3, 6, 31, 16], where the ground field now is a finite extension L of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_L$ and prime element π_L. Fixing a Lubin-Tate formal group $\mathcal{G}$ associated to π_L, we obtain the Galois extension L_∞ by adjoining the π_L^n-division points of $\mathcal{G}$ to L with L-analytic Galois group $\Gamma_L = G(L_\infty/L)$; we also set $H_L := G_{L_\infty}$. Following Fontaine's original ideas Kisin-Ren established an equivalence of categories [20, Theorem (1.6), p. 446]), which for the convenience of the reader we recall in section 3 adding some details concerning topologies etc. based on the very detailed account [30, Theorem 3.3.10, p. 134]). In particular, to any finitely generated $\mathcal{O}_L$-module V with linear and continuous action by G_K, where K is any finite extension of L, we may attach an étale (φ, Γ)-module $\mathcal{D}_{K|L}(V)$. In this context the Iwasawa

cohomology again has been successfully described in terms of $\mathcal{D}_{K|L}(V)$ in [31]. The purpose of this article is to add an explicit description of the Galois cohomology groups in terms of a φ- and ψ-Herr complex, respectively. To this end let V any such $\mathcal{O}_L$-representation of G_K as above. Then there is a complex of the corresponding (φ, Γ)-module, of which the cohomology is exactly the continuous group cohomology of G_K with coefficients in V as follows. By $C_{\mathrm{cts}}^{\bullet}(G, A)$ we denote the continuous cochain complex of a profinite group G with values in the abelian group A. Furthermore, for any (φ, Γ)-module M we introduce a generalized Herr complex $\mathcal{C}_{\varphi_{K|L}}^{\bullet}(\Gamma_K, M)$ as the total complex of the double complex

$$C_{\mathrm{cts}}^{\bullet}(\Gamma_K, M) \xrightarrow{C_{\mathrm{cts}}^{\bullet}(\Gamma_K, \varphi_M) - \mathrm{id}} C_{\mathrm{cts}}^{\bullet}(\Gamma_K, M)$$

and we denote by $\mathcal{H}_{\varphi_{K|L}}^{*}(\Gamma_K, M)$ its cohomology. If $\mathcal{C}^{\bullet}$ is a bounded below complex of abelian groups (or of R-modules for a suitable ring R), then we denote by $\mathbf{R}\Gamma(\mathcal{C}^{\bullet})$ the same complex viewed as object in the derived category $\mathbf{D}^{\mathrm{b}/+}(\mathbf{Ab})$ (respectively in $\mathbf{D}^{\mathrm{b}/+}(R\text{-}\mathbf{Mod})$).

Theorem A (cf. Theorem 5.1.11). *Let $V \in \mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg})}(G_K)$ and set $M = \mathcal{D}_{K|L}(V)$. Then there are isomorphisms*

$$H_{\mathrm{cts}}^{*}(G_K, V) \xrightarrow{\cong} \mathcal{H}_{\varphi_{K|L}}^{*}(\Gamma_K, M),$$

$$H_{\mathrm{cts}}^{*}(H_K, V) \xrightarrow{\cong} \mathcal{H}_{\varphi_{K|L}}^{*}(M).$$

These isomorphisms are functorial in V and compatible with restriction and corestriction. They stem from isomorphisms in $\mathbf{D}^{+}(\mathcal{O}_L\text{-}\mathbf{Mod})$

$$\mathbf{R}\Gamma(C_{\mathrm{cts}}^{\bullet}(G_K, V)) \xrightarrow{\cong} \mathbf{R}\Gamma(\mathcal{C}_{\varphi_{K|L}}^{\bullet}(\Gamma_K, M)),$$

$$\mathbf{R}\Gamma(C_{\mathrm{cts}}^{\bullet}(H_K, V)) \xrightarrow{\cong} \mathbf{R}\Gamma(\mathcal{C}_{\varphi_{K|L}}^{\bullet}(M)).$$

In our proof, we follow closely the approach of Scholl in [34, Theorem 2.2.1, p. 702–705]. Due to the lack of Hochschild-Serre spectral sequences for general continuous cohomology (see [40] for a discussion of this issue), the main technical difficulty consists of using Mittag-Leffler type arguments to reduce to cases in which the coefficients become discrete (and hence admit such a spectral sequence). We would like to mention that in the course of writing up our results we learned that independently Aribam and Kwatra have achieved a (partial) result of this kind, too, concerning torsion coefficients (cf. [1, Theorem 3.16, p. 10–11]).

The situation for a generalized ψ-Herr complex is more difficult. First of all there is no reason why one should obtain a quasi-isomorphic complex upon replacing φ by ψ. One reason is, that the integral operator ψ considered in [31] is no longer a left inverse to φ if $L \neq \mathbb{Q}_p$. Furthermore, the structure of the kernel $M^{\psi=0}$ of the ψ-operator of an étale -module M is difficult to analyse (but see [32, 5] for some aspects in this regard). Therefore, we decided to follow the path of dualizing as this was already successful in [31] for the purpose of Iwasawa cohomology: Since φ and ψ are related to each other under Pontrjagin duality (cf. [31, Remark 5.6, p. 27]), it seems to be the correct way, to dualize the complex of φ. One attempt would have been to imitate the methods of Herr (cf. [19, Lemme 5.6, p. 333]) to establish a quasi isomorphism between the complexes of (φ, Γ)-modules related to φ and ψ using Tate duality. This approach requires to show that all the differentials of the φ-Herr complex have closed

image, which implies that they are strict which then implies that the cohomology groups of the dualized complex coincide with the dual of the cohomology groups of the complex we started with. In his original work, Herr checked that the differentials have closed image for each differential separately (cf. [19, p. 334]). Unfortunately, in the general case we have to deal with direct products of Herr's differentials and modules and it is no longer clear, that the differentials have closed image.

Instead we imitate results of Nekovář (cf. [26, Sections (8.2) and (8.3), p. 157–160]) to replace the complex $C_{\text{cts}}^\bullet(H_K, A)$ with a complex $C_{\text{cts}}^\bullet(G_K, F_{\Gamma_K}(A))$ of $\Lambda_K = \mathcal{O}_L[\![\Gamma_K]\!]$-modules, where $A = V^\vee$ is the dual of some G_K-representation. Here "replace" means, that the two complexes are quasi isomorphic (cf. Proposition 5.2.21). This then has the advantage that we can apply the Matlis dual $\overline{D_K} = \text{Hom}_{\Lambda_K}(-, \Lambda_K^\vee)$ to this complex. Nekovář proved that this dualized complex is quasi isomorphic to a complex computing the Iwasawa cohomology (cf. Lemma 5.2.44). We then finally check, that the complex related to ψ is quasi isomorphic to this dualized complex. Using again a result of Nekovář , we then get the following statement.

Theorem B (cf. Theorem 5.2.53)**.** *Let $T \in \mathbf{Rep}_{\mathcal{O}_L}^{(\text{fg})}(G_K)$ and let $K \subseteq K' \subseteq K_\infty$ an intermediate field, finite over K, such that $\Gamma_{K'} := \text{Gal}(K_\infty | K')$ is isomorphic to some $\mathbb{Z}_p^r$. Then we have an isomorphism in $\mathbf{D}^+(\mathcal{O}_L\text{-}\mathbf{Mod})$*

$$\mathbf{R}\Gamma(\mathcal{C}_\psi^\bullet(\mathcal{D}_{K|L}(T(\tau^{-1})) \overset{\mathbf{L}}{\otimes}_{\Lambda_{K'}} \mathcal{O}_L \cong \mathbf{R}\Gamma_{\text{cts}}^\bullet(G_{K'}, T).$$

The left hand side of the isomorphism in the theorem should be considered as generalised ψ-Herr complex. For instance, by choosing a Koszul type complex associated with topological generators of $\Gamma_{K'}$ one obtains a quite explicit complex, which specialises to the ψ-Herr complex in the cyclotomic situation. Moreover, this result is crucial for descent calculations, see e.g. [32] in the context of pairings and regulator maps.

2. Preliminaries

By $\mathbb{N}$ we denote the natural numbers starting with 1 and we let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For a homomorphism $f \colon A \to B$ we denote by $\ker(f)$ its kernel, by $\text{im}(f)$ its image and by $\text{coker}(f)$ its cokernel.

2.1. **On Continuous Group Cohomology.** Continuous group cohomology has been introduced by Tate. For the convenience of the reader we recall the basic notions, but refer the reader to [39], [28, §2.7] and [23, §2.1] for further details.

For topological spaces X, Y we endow the set of continuous maps $\text{Map}_{\text{cts}}(X, Y)$ always with the compact open topology (cf. [8, Definition 1, Chapter X §3.4, p. 301]). Note, that in this topology $\text{Map}_{\text{cts}}(X, Y)$ is a Hausdorff space if Y is (cf. [8, Remarks (1), Chapter X §3.4, p. 301–302]). For $K \subseteq X$ compact and $U \subseteq Y$ open denote by $\text{M}(K, U)$ the set of all $f \in \text{Map}_{\text{cts}}(X, Y)$ with $f(K) \subseteq U$.

Remark 2.1.1. *We recall that for a profinite group G and an abelian topological group A on which G acts continuously we have the canonical acyclic complex*

$$0 \longrightarrow A \longrightarrow \text{Map}_{\text{cts}}(G, A) \longrightarrow \text{Map}_{\text{cts}}(G^2, A) \longrightarrow \text{Map}_{\text{cts}}(G^3, A) \longrightarrow \cdots$$

Let for $n \in \mathbb{N}_0$

$$X_{\text{cts}}^n(G, A) := \text{Map}_{\text{cts}}(G^{n+1}, A)$$

and $\partial_{\text{cts}}^n \colon X_{\text{cts}}^{n-1} \to X_{\text{cts}}^n$ be the differential , which is given by

$$\partial_{\text{cts}}^n(x)(\sigma_0, \ldots, \sigma_n) = \sum_{i=0}^n (-1)^i x(\sigma_0, \ldots, \hat{\sigma}_i, \ldots, \sigma_n),$$

where $" \hat{\ } "$ means that the corresponding element is omitted. Furthermore, we denote by $X_{\text{cts}}^\bullet(G, A)$ the corresponding complex, i.e.

$$X_{\text{cts}}^\bullet(G, A) = \cdots \xrightarrow{\partial_{\text{cts}}^{n-1}} X_{\text{cts}}^{n-1}(G, A) \xrightarrow{\partial_{\text{cts}}^n} X_{\text{cts}}^n(G, A) \xrightarrow{\partial_{\text{cts}}^{n+1}} \cdots.$$

As usual, we then set

$$C_{\text{cts}}^n(G, A) := X_{\text{cts}}^n(G, A)^G.$$

One checks that ∂_{cts}^n restricts to a homomorphism $C_{\text{cts}}^{n-1}(G, A) \to C_{\text{cts}}^n(G, A)$. We then let $C_{\text{cts}}^\bullet(G, A)$ be the complex

$$C_{\text{cts}}^\bullet(G, A) = \cdots \xrightarrow{\partial_{\text{cts}}^{n-1}} C_{\text{cts}}^{n-1}(G, A) \xrightarrow{\partial_{\text{cts}}^n} C_{\text{cts}}^n(G, A) \xrightarrow{\partial_{\text{cts}}^{n+1}} \cdots.$$

This complex is called the **continuous standard resolution** of G with coefficients in A. We denote its n-th cohomology group by $H_{\text{cts}}^n(G, A)$ and call it the n-th **continuous cohomology group** of G with coefficients in A.

Lemma 2.1.2. *Let G be a profinite group and let*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

be an exact sequence of topological G-modules such that the topology of A is induced by that of B and that $B \to C$ has a continuous set theoretical section $s \colon C \to B$. Then for all $n > 0$ the diagrams

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Map}_{\text{cts}}(G^{n-1}, A) & \longrightarrow & \operatorname{Map}_{\text{cts}}(G^{n-1}, B) & \longrightarrow & \operatorname{Map}_{\text{cts}}(G^{n-1}, C) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Map}_{\text{cts}}(G^n, A) & \longrightarrow & \operatorname{Map}_{\text{cts}}(G^n, B) & \longrightarrow & \operatorname{Map}_{\text{cts}}(G^n, C) & \longrightarrow & 0
\end{array}
$$

and

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Map}_{\text{cts}}(G^n, A)^G & \longrightarrow & \operatorname{Map}_{\text{cts}}(G^n, B)^G & \longrightarrow & \operatorname{Map}_{\text{cts}}(G^n, C)^G & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Map}_{\text{cts}}(G^{n+1}, A)^G & \longrightarrow & \operatorname{Map}_{\text{cts}}(G^{n+1}, B)^G & \longrightarrow & \operatorname{Map}_{\text{cts}}(G^{n+1}, C)^G & \longrightarrow & 0
\end{array}
$$

are commutative with exact rows and the latter diagram induces a long exact sequence of continuous cohomology

$$0 \longrightarrow A^G \longrightarrow B^G \longrightarrow C^G \longrightarrow H_{\text{cts}}^1(G, A) \longrightarrow \cdots$$

$$\cdots \longrightarrow H_{\text{cts}}^n(G, A) \longrightarrow H_{\text{cts}}^n(G, B) \longrightarrow H_{\text{cts}}^n(G, C) \longrightarrow H_{\text{cts}}^{n+1}(G, A) \longrightarrow \cdots$$

Furthermore, the topology of $\operatorname{Map}_{\text{cts}}(G^n, A)$ is induced by the topology of $\operatorname{Map}_{\text{cts}}(G^n, B)$ and the section $s \colon C \to B$ induces a continuous, set theoretical section $s_* \colon \operatorname{Map}_{\text{cts}}(G^n, C) \to \operatorname{Map}_{\text{cts}}(G^n, B)$.

2.2. **Monoid Cohomology.** As described in the introduction, the aim of Section 5 is to compute Galois cohomology using the theory of Lubin-Tate (φ, Γ)-modules. For this, we also compute the cohomology of complexes like $A \xrightarrow{f-1} A$, where A is a topological abelian group and f is a continuous endomorphism of A.

This can be embedded in the theory of monoid cohomology, which then allows us, in the case of discrete coefficients, to write this cohomological functor as derived functor. We then combine this with a usual group action, which commutes with the endomorphism and obtain spectral sequences on cohomology.

Let A be a topological abelian group and $f \in \text{End}(A)$ continuous. Then

$$(1) \qquad \qquad \cdot : \mathbb{N}_0 \times A \longrightarrow A, \ (n, a) \longmapsto f^n(a),$$

defines a continuous $\mathbb{N}_0$-action on A. As in the group case the following holds:

Let M be a topological monoid and A be a discrete abelian group with a continuous action of M. Then we have

$$(2) \qquad \qquad A^M \cong \text{Hom}_{\mathbb{Z}[M]}(\mathbb{Z}, A),$$

as $\mathbb{Z}[M]$-modules, where $\mathbb{Z}$ is considered as trivial $\mathbb{Z}[M]$-module.

We are mostly interested in the case of a discrete G-module A, where G is a profinite group, together with an $\mathbb{N}_0$-action (which then automatically is continuous since both, $\mathbb{N}_0$ and A are discrete), which comes from a G-homomorphism of A. To shorten notation, we make the following definitions.

Definition 2.2.1. Let G be a profinite group and M a topological monoid.

By $\mathfrak{Dis}_M$ we denote the category whose objects are discrete abelian groups with a continuous action of M and whose morphisms are the continuous group homomorphisms which respect the operation of M.

Similarly we denote by $\mathfrak{Dis}_G$ the category whose objects are discrete abelian groups with a continuous action of G and whose morphisms are the continuous group homomorphisms which respect the operation of G.

And finally we denote by $\mathfrak{Dis}_{G,M}$ the category whose objects are discrete abelian groups, together with commuting continuous actions of G and M and whose morphisms are the continuous group homomorphisms which respect the operations from G and M.

The corresponding categories, whose objects are abstract abelian groups, are denoted by $\mathfrak{Abs}_M$, $\mathfrak{Abs}_G$ and $\mathfrak{Abs}_{G,M}$.

Furthermore, by $\mathfrak{Top}_G$ we denote the category of topological abelian Hausdorff groups with a continuous action from G. The morphisms of this category are the continuous group homomorphisms which respect the action from G.

Analogously we denote by $\mathfrak{Top}_{G,M}$ the category of topological abelian Hausdorff groups with continuous actions from both, G and M, such that these actions commute. The morphisms of this category are the continuous group homomorphisms which respect the actions from G and M.

Remark 2.2.2. *Let G be a profinite group and M a topological monoid. Then the categories $\mathfrak{Dis}_{G,M}$ and $\mathfrak{Dis}_{G \times M}$ coincide, where $G \times M$ is considered as a topological monoid.*

Our aim now is to see that the category $\mathfrak{Dis}_{G,M}$ has enough injective objects. For this, we follow the idea of [29, (2.6.5) Lemma, Chapter I §6, p. 131] and outline some details.

Remark 2.2.3. *Let G be a group and M a monoid. Then the category $\mathfrak{Abs}_{G,M}$ coincides with the category of $\mathbb{Z}[G][M]$-modules. In particular, the category $\mathfrak{Abs}_{G,M}$ has enough injectives.*

Let G be a profinite group and M a discrete monoid. The usual arguments show that the category $\mathfrak{Dis}_{G,M}$ has enough injective objects (cf. [23, Prop. 2.2.12] for details).

Lemma 2.2.4. *Let G be a profinite group and M a discrete monoid. Then the functor*

$$(-)^{G,M} \colon \mathfrak{Dis}_{G,M} \to \mathbf{Ab}$$

is left exact and additive ($\mathbf{Ab}$ denotes the category of abelian groups).

Proof. Since $\mathfrak{Dis}_{G,M}$ and $\mathfrak{Dis}_{G \times M}$ coincide (cf. Remark 2.2.2) we can view the functor $(-)^{G,M}$ as $(-)^{G \times M}$. Then (2) says

$$(-)^{G \times M} = \operatorname{Hom}_{\mathbb{Z}[G \times M]}(\mathbb{Z}, -)$$

which immediately gives the claim, since $\operatorname{Hom}(\mathbb{Z}, -)$ is left exact and additive. $\square$

By the above the right derivations for $(-)^{G,M}$, where G is a profinite group and M a discrete monoid, exist (cf. [38, Tag 0156, Lemma 10.3.2 (2)]). This then leads us to the following definition.

Definition 2.2.5. Let G be a profinite group and M a discrete monoid. Then $H^n(G, M; -) \coloneqq R^n(-)^{G,M}$ denotes the n-th right derived functor of $(-)^{G,M}$ and is called the **n-th cohomology group**.

Proposition 2.2.6. *Let G be a profinite group, $N \lhd G$ a closed, normal subgroup and M a discrete monoid. Then for every $A \in \mathfrak{Dis}_{G,M}$ there are two cohomological spectral sequences converging to $H^n(G, M; A)$:*

$$H^a(G/N, H^b(N, M; A)) \Longrightarrow H^{a+b}(G, M; A)$$
$$H^a(G/N, M; H^b(N, A)) \Longrightarrow H^{a+b}(G, M; A).$$

Proof. Recall that the categories $\mathfrak{Dis}_{G,M}$, $\mathfrak{Dis}_{G/N,M}$ and $\mathfrak{Dis}_{G/N}$ have enough injectives. The functors $(-)^{N,M} \colon \mathfrak{Dis}_{G,M} \to \mathfrak{Dis}_{G/N}$ respectively $(-)^N \colon \mathfrak{Dis}_{G,M} \to \mathfrak{Dis}_{G/N,M}$ send injectives to injectives as is straightforward to check, see [23, Lem. 2.2.16]. Furthermore, since the actions of G and M on objects of $\mathfrak{Dis}_{G,M}$ commute, the compositions

$$\mathfrak{Dis}_{G,M} \xrightarrow{\ (-)^{N,M}\ } \mathfrak{Dis}_{G/N} \xrightarrow{\ (-)^{G/N}\ } \mathbf{Ab}$$

and

$$\mathfrak{Dis}_{G,M} \xrightarrow{\ (-)^N\ } \mathfrak{Dis}_{G/N,M} \xrightarrow{\ (-)^{G/N,M}\ } \mathbf{Ab}$$

both coincide with $(-)^{G,M}$. This then leads to the claimed Grothendieck spectral sequences. $\square$

As we now have accomplished the abstract theory for our goals, we want to discuss how to compute these cohomology groups when the monoid action arises from an endomorphism. First of all, we want to compare $\mathbb{N}_0$-actions with $\mathbb{Z}[X]$-modules.

Remark 2.2.7. *The category $\mathfrak{Abs}_{\mathbb{N}_0}$ coincides with the category of $\mathbb{Z}[X]$-modules.*

In the following, we will switch between these two concepts without further mentioning it.

Remark 2.2.8. *Let G be a profinite group, $A \in \mathfrak{Dis}_{G,\mathbb{N}_0}$. For every $n \in \mathbb{N}_0$ we can define an $\mathbb{N}_0$-action on $C^n_{\mathrm{cts}}(G, A)$ by operating on the coefficients:*

$$(X \cdot f)(\sigma) := X \cdot (f(\sigma)).$$

Remark 2.2.9. *Let $A^{\bullet,\bullet}$ be a (commutative) double complex of abelian groups. We write $\mathrm{Tot}(A^{\bullet,\bullet})$ for its total complex, by which we mean the complex with objects*

$$\mathrm{Tot}^n(A^{\bullet,\bullet}) := \bigoplus_{i+j=n} A^{i,j}$$

and differentials

$$\mathrm{d}^n_{\mathrm{Tot}(A^{\bullet,\bullet})} := \bigoplus_{i+j=n} \mathrm{d}^{i,j}_{\mathrm{hor}} \circ \mathrm{pr}_{i-1,j} \oplus (-1)^i \mathrm{d}^{i,j}_{\mathrm{vert}} \circ \mathrm{pr}_{i,j-1}.$$

If $f^{\bullet,\bullet} \colon A^{\bullet,\bullet} \to B^{\bullet,\bullet}$ is a morphism of (commutative) double complexes, then

$$
\begin{aligned}
\mathrm{Tot}^n(f^{\bullet,\bullet}) \colon \mathrm{Tot}^n(A^{\bullet,\bullet}) &\longrightarrow \mathrm{Tot}^n(B^{\bullet,\bullet}) \\
(a_{ij})_{i+j=n} &\longmapsto (f_{ij}(a_{ij}))_{i+j=n}
\end{aligned}
$$

defines a morphism of the corresponding total complexes.
If $X^{\bullet}$ and $Y^{\bullet}$ are complexes of abelian groups and $g^{\bullet} \colon X^{\bullet} \to Y^{\bullet}$ is a morphism of complexes, then it also is a double complex concentrated in degrees 0 and 1 and we again write $\mathrm{Tot}(g^{\bullet} \colon X^{\bullet} \to Y^{\bullet})$ for its total complex.

Remark 2.2.10. *Let G be a profinite group and $A \in \mathfrak{Dis}_G$. As in [29, p. 12–13] we omit the subscript "cts" for the notations introduced in* Remark 2.1.1, *i.e., we write*

$$X^n(G, A) := \mathrm{Map}_{\mathrm{cts}}(G^{n+1}, A),$$

∂^n for the differential $X^{n-1}(G, A) \to X^n(G, A)$ and

$$C^n(G, A) := X^n(G, A)^G.$$

Definition 2.2.11. *Let G be a profinite group and $A \in \mathfrak{Dis}_{G,\mathbb{N}_0}$. Then define*

$$
\begin{aligned}
\mathcal{C}^{\bullet}_X(G, A) &:= \mathrm{Tot}(C^{\bullet}(G, A) \xrightarrow{\ X-1\ } C^{\bullet}(G, A)), \\
\mathcal{H}^*_X(G, A) &:= H^*(\mathcal{C}^{\bullet}_X(G, A)).
\end{aligned}
$$

If the $\mathbb{N}_0$-action on A comes from an endomorphism $f \in \mathrm{End}_G(A)$ (cf. (1)), then we also write

$$
\begin{aligned}
\mathcal{C}^{\bullet}_f(G, A) &:= \mathrm{Tot}(C^{\bullet}(G, A) \xrightarrow{\ \mathcal{C}^{\bullet}(G,f)-\mathrm{id}\ } C^{\bullet}(G, A)), \\
\mathcal{H}^*_f(G, A) &:= H^*(\mathcal{C}^{\bullet}_f(G, A)).
\end{aligned}
$$

If $A \in \mathfrak{Abs}_{\mathbb{N}_0}$ then we also write $\mathcal{H}^*_X(A)$ for the cohomology of the complex $A \xrightarrow{X-1} A$ concentrated in the degrees 0 and 1.

The aim now is to see that the cohomology of the complex $\mathcal{C}^{\bullet}_X(G, A)$ coincides with the right derived functors of $(-)^{G,\mathbb{N}_0}$. Before proving this, we want to make a smaller step and explain first how to compute the right derived functors of $(-)^{\mathbb{N}_0}$ and that these coincide with the cohomology of the complex $A \xrightarrow{X-1} A$ concentrated in degrees 0 and 1.

Proposition 2.2.12. *Let $A \in \mathfrak{Abs}_{\mathbb{N}_0}$. Then we have*

$$H^0(\mathbb{N}_0; A) = A^{\mathbb{N}_0},$$
$$H^1(\mathbb{N}_0; A) = A_{\mathbb{N}_0},$$
$$H^i(\mathbb{N}_0; A) = 0 \text{ for all } i \in \mathbb{Z} \setminus \{0, 1\}.$$

In particular, the right derived functors of $(-)^{\mathbb{N}_0}$ coincide with the cohomology of the complex $A \xrightarrow{X-1} A$ concentrated in degrees 0 and 1. Using the notation from above, this means that for all $i \in \mathbb{Z}$ there are natural isomorphisms

$$H^i(\mathbb{N}_0; A) = \mathcal{H}^i_X(A).$$

Proof. Follows immediately from the projective resolution of $\mathbb{Z}$ by:

$$0 \longrightarrow \mathbb{Z}[X] \longrightarrow \mathbb{Z}[X] \longrightarrow \mathbb{Z} \longrightarrow 0$$
$$P(X) \longmapsto (X-1)P(X)$$
$$P(X) \longmapsto P(1).$$

$\square$

Proposition 2.2.13. *Let G be a profinite group and $A \in \mathfrak{Dis}_{G,\mathbb{N}_0}$. Then the double complex*

$$K^{\bullet,\bullet} : C^\bullet(G, A) \xrightarrow{X-1} C^\bullet(G, A)$$

gives rise to two spectral sequences converging to the cohomology $\mathcal{H}^*_X(G, A)$:

$$\mathcal{H}^a_X(H^b(G, A)) \Longrightarrow \mathcal{H}^{a+b}_X(G, A)$$
$$H^a(G, \mathcal{H}^b_X(A)) \Longrightarrow \mathcal{H}^{a+b}_X(G, A).$$

Proof. Since for every $n \in \mathbb{Z}$ the double complex $K^{\bullet,\bullet}$ has at most two nonzero entries $K^{p,q}$ with $p + q = n$, this is shown in [38, Tag 012X, Lemma 12.22.6]. $\square$

Lemma 2.2.14. *Let G be a profinite group and $f \colon A \to B$ be a morphism in $\mathfrak{Dis}_{G,\mathbb{N}_0}$. Then the diagram*

$$
\begin{array}{ccc}
\mathcal{C}^n_X(G, A) & \xrightarrow{C^n_X(G,f)} & C^n_X(G, B) \\
\downarrow{\partial_A} & & \downarrow{\partial_B} \\
\mathcal{C}^{n+1}_X(G, A) & \xrightarrow{C^{n+1}_X(G,f)} & C^{n+1}_X(G, B)
\end{array}
$$

is commutative for all $n \in \mathbb{N}_0$.

Proof. Left to the reader. $\square$

Lemma 2.2.15. *Let G be a profinite group and*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

be an exact sequence in $\mathfrak{Dis}_{G,\mathbb{N}_0}$. Then, the sequence

$$0 \longrightarrow \mathcal{C}^\bullet_X(G, A) \xrightarrow{\mathcal{C}^\bullet_X(G,\alpha)} \mathcal{C}^\bullet_X(G, B) \xrightarrow{\mathcal{C}^\bullet_X(G,\beta)} \mathcal{C}^\bullet_X(G, C) \longrightarrow 0$$

is exact.

Proof. Since A, B and C are discrete groups, we deduce from Corollary 2.1.2 that for all $n \in \mathbb{N}_0$ the sequence

$$0 \longrightarrow \mathcal{C}^n(G, A) \xrightarrow{\mathcal{C}^n(G,\alpha)} \mathcal{C}^n(G, B) \xrightarrow{\mathcal{C}^n(G,\beta)} \mathcal{C}^n(G, C) \longrightarrow 0$$

is exact. But since $\mathcal{C}_X^n(G, Z) = \mathcal{C}^n(G, Z) \oplus \mathcal{C}^{n-1}(G, Z)$ (where $\mathcal{C}^{-1}(G, Z) = 0$) and $\mathcal{C}_X^n(G, \eta) = \mathcal{C}^n(G, \eta) \oplus \mathcal{C}^{n+1}(G, \eta)$ for all $Z \in \mathfrak{Dis}_{G, \mathbb{N}_0}$ and any morphism η in $\mathfrak{Dis}_{G, \mathbb{N}_0}$, we immediately deduce that the sequence

$$0 \longrightarrow \mathcal{C}_X^n(G, A) \xrightarrow{\mathcal{C}_X^n(G,\alpha)} \mathcal{C}_X^n(G, B) \xrightarrow{\mathcal{C}_X^n(G,\beta)} \mathcal{C}_X^n(G, C) \longrightarrow 0$$

is also exact. $\square$

Lemma 2.2.16. *Let G be a profinite group. The functors $(\mathcal{H}_X^n(G, -))_n$ then form a cohomological δ-functor, i.e. if*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

is an exact sequence in $\mathfrak{Dis}_{G, \mathbb{N}_0}$ then, for every $n \in \mathbb{N}_0$, there is a group homomorphism

$$\delta^n \colon \mathcal{H}_X^n(G, C) \longrightarrow \mathcal{H}_X^{n+1}(G, A)$$

such that the sequence

$$\cdots \longrightarrow \mathcal{H}_X^n(G, B) \longrightarrow \mathcal{H}_X^n(G, C) \xrightarrow{\delta^n} \mathcal{H}_X^{n+1}(G, A) \longrightarrow \mathcal{H}_X^{n+1}(G, B) \longrightarrow \cdots$$

is exact.

Proof. The proof is the standard application for the snake lemma (cf. for example at [29, (1.3.2) Theorem, Chapter I §3, p. 27]). $\square$

Lemma 2.2.17. *Let G be a group. Then there holds*

$$\mathbb{Z}[G][X] \cong \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[X].$$

Lemma 2.2.18. *Let G be an abelian profinite group. Then, for every $n \in \mathbb{N}$ the functor $\mathcal{H}_X^n(G, -)$ is effaceable, i.e. for every $A \in \mathfrak{Dis}_{G, \mathbb{N}_0}$ there exists a $B \in \mathfrak{Dis}_{G, \mathbb{N}_0}$ and a monomorphism $u \colon A \to B$ in $\mathfrak{Dis}_{G, \mathbb{N}_0}$ such that $\mathcal{H}_X^n(G, u) = 0$.*

Proof. Left to the reader, or [23, Lem. 2.2.31]. $\square$

Corollary 2.2.19. *Let G be a profinite group. Then the family of functors $(\mathcal{H}_X^n(-))_n$ from $\mathfrak{Dis}_{G, \mathbb{N}_0}$ to $\mathbf{Ab}$ forms a universal delta functor.*

Theorem 2.2.20. *Let G be a profinite group. Then we have*

$$\mathcal{H}_X^n(G, A) = H^n(G, \mathbb{N}_0; A)$$

for all $n \in \mathbb{N}_0$ and $A \in \mathfrak{Dis}_{G, \mathbb{N}_0}$.

Proof. Since $(H^n(G, \mathbb{N}_0; -))_n$ are the right derived functors of $(-)^{G, \mathbb{N}_0}$ this is a universal delta functor and since $(\mathcal{H}_X^n(G, -))_n$ is also an universal delta functor (cf. Corollary 2.2.19), it remains to check that they coincide in degree 0. For this, let $A \in \mathfrak{Dis}_{G, \mathbb{N}_0}$. We have

$$H^0(G, \mathbb{N}_0; A) = A^{G, \mathbb{N}_0}$$

and

$$\mathcal{H}_X^0(G, A) = H^0(\mathcal{C}_X^\bullet(G, A))$$
$$= \ker(A \xrightarrow{d^0} \mathcal{C}^1(G, A)) \cap \ker(A \xrightarrow{X-1} A)$$
$$= A^G \cap A^{X=1}$$

Since, by definition, $A^{X=1} = A^{\mathbb{N}_0}$ it follows immediately that $A^{G,\mathbb{N}_0} = A^G \cap A^{\mathbb{N}_0}$. $\qquad\square$

Next we want to reformulate Proposition 2.2.6 with the above theorem, just to avoid confusions for latter applications.

Proposition 2.2.21. *Let G be a profinite group, $N \lhd G$ a closed, normal subgroup and $A \in$ $\mathfrak{Dis}_{G,\mathbb{N}_0}$. Then there are two cohomological spectral sequences converging to $\mathcal{H}_X^n(G, -)$:*

$$H^a(G/N, \mathcal{H}_X^b(N, A)) \Longrightarrow \mathcal{H}_X^{a+b}(G, M; A)$$
$$\mathcal{H}_X^a(G/N, H^b(N, A)) \Longrightarrow \mathcal{H}_X^{a+b}(G, M; A).$$

Proof. This is Proposition 2.2.6 using $H^n(G, \mathbb{N}_0; -) = \mathcal{H}_X^n(G, -)$ from Theorem 2.2.20. $\quad\square$

As for the standard continuous cohomology (cf. [29, (2.7.2) Lemma, Chapter II §7, p. 137]), we will also need a long exact sequence for $\mathcal{H}_X^*(G, -)$ in a slightly different setting as in Lemma 2.2.16.

Proposition 2.2.22. *Let G be a profinite group and let*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

be a short exact sequence in $\mathfrak{Top}_{G,\mathbb{N}_0}$ such that the topology of A is induced by that of B and such that β has a continuous, set theoretical section. Then there are continuous homomorphisms

$$\delta^n \colon \mathcal{H}_X^n(G, C) \longrightarrow \mathcal{H}_X^{n+1}(G, A)$$

such that the sequence

$$\cdots \longrightarrow \mathcal{H}_X^n(G, B) \longrightarrow \mathcal{H}_X^n(G, C) \xrightarrow{\delta^n} \mathcal{H}_X^{n+1}(G, A) \longrightarrow \mathcal{H}_X^{n+1}(G, B) \longrightarrow \cdots$$

is exact.

Proof. Algebraically this is exactly the same proof as Lemma 2.2.16. It then remains to check, that the occurring homomorphisms are continuous which is only for the δ^n a real question. But this can be answered using a topological version of the snake lemma, like [33, Proposition 4, p. 133]. $\qquad\square$

2.3. **Some Homological Algebra.** In this subsection we want to collect and prove some facts we will need later on.

Definition 2.3.1. Let $C^\bullet$ be a complex of abelian groups and $n \in \mathbb{Z}$. Then we denote by $C^\bullet[n]$ the shift of this complex by n. This means, that for all $i \in \mathbb{Z}$ we have $C^i[n] = C^{i+n}$.

Lemma 2.3.2. *Let $Y^\bullet$ and $Z^\bullet$ be complexes of abelian groups and let $g^\bullet \colon Y^\bullet \to Z^\bullet$ be a morphism of complexes, such that every g^i is surjective. Then there is a canonical, surjective homomorphism*

$$\ker(\mathrm{d}_Y^i) \cap \ker g^i \twoheadrightarrow H^i(\mathrm{Tot}(g^\bullet \colon Y^\bullet \to Z^\bullet)).$$

In particular, if all the g^i are bijective, we have

$$H^i(\mathrm{Tot}(g^\bullet \colon Y^\bullet \to Z^\bullet)) = 0.$$

Proof. Easy to check, see [23, Lem. 2.3.2] $\qquad\square$

Lemma 2.3.3. *Let*

$$0 \longrightarrow X^\bullet \xrightarrow{\ f^\bullet\ } Y^\bullet \xrightarrow{\ g^\bullet\ } Z^\bullet \longrightarrow 0$$

be a short exact sequence of complexes of abelian groups. Then the sequence

$$0 \longrightarrow X^\bullet \longrightarrow \mathrm{Tot}(Y^\bullet \to Z^\bullet) \longrightarrow \mathrm{Tot}(Y^\bullet/f^\bullet(X^\bullet) \to Z^\bullet) \longrightarrow 0$$

is also an exact sequence of complexes and for the cohomology we have

$$H^i(X^\bullet) \cong H^i(\mathrm{Tot}(g^\bullet \colon Y^\bullet \to Z^\bullet)).$$

Proof. Standard, see [23, Lem. 2.3.3]. $\qquad\square$

Corollary 2.3.4. *Let G be a profinite group, $A, B \in \mathfrak{Dis}_G$ and f a continuous endomorphism of B which respects the action of G such that the sequence*

$$0 \longrightarrow A \longrightarrow B \xrightarrow{\ f-1\ } B \longrightarrow 0$$

is exact. Then we have

$$H^i(G, A) = \mathcal{H}_f^i(G, B)$$

for all $i \geq 0$.

Proof. This is just the above Lemma 2.3.3 with Corollary 2.1.2 and the notation from Definition 2.2.11. $\qquad\square$

Corollary 2.3.5. *Let G be a profinite group and let*

$$0 \longrightarrow A \xrightarrow{\ \alpha\ } B \xrightarrow{\ \beta\ } C \longrightarrow 0$$

be an exact sequence in $\mathfrak{Top}_G$, such that the topology of A is induced by that of B and such that β has a continuous, set theoretical section. Then the exact sequence of complexes

$$0 \longrightarrow C_{\mathrm{cts}}^\bullet(G, A) \xrightarrow{\ C_{\mathrm{cts}}^\bullet(G,\alpha)\ } C_{\mathrm{cts}}^\bullet(G, B) \xrightarrow{\ C_{\mathrm{cts}}^\bullet(G,\beta)\ } C_{\mathrm{cts}}^\bullet(G, C) \longrightarrow 0$$

(cf. Corollary 2.1.2) induces

$$A^G = H_{\mathrm{cts}}^0(G, A) \cong H^0(\mathrm{Tot}(C_{\mathrm{cts}}^\bullet(G, \beta) \colon C_{\mathrm{cts}}^\bullet(G, B) \to C_{\mathrm{cts}}^\bullet(G, C)))$$

and

$$C^G \to H_{\mathrm{cts}}^1(G, A) \cong H^1(\mathrm{Tot}(C_{\mathrm{cts}}^\bullet(G, \beta) \colon C_{\mathrm{cts}}^\bullet(G, B) \to C_{\mathrm{cts}}^\bullet(G, C))).$$

Proof. This is an immediate consequence of the combination of the above Lemma 2.3.3 with Corollary 2.1.2. $\qquad\square$

Now let's turn to some facts about projective limits.

Remark 2.3.6. *Note that $C_{\mathrm{cts}}^\bullet(G, -)$ commutes with projective limits, since the functors* $\mathrm{Map}_{\mathrm{cts}}(G^n, -)$ *and* $(-)^G$ *commute with projective limits, i.e. if* $A = \varprojlim_n A_n$, *then*

$$C_{\mathrm{cts}}^\bullet(G, A) \cong \varprojlim_n C_{\mathrm{cts}}^\bullet(G, A_n).$$

Lemma 2.3.7. *Let G be a profinite group, $A \in \mathfrak{Top}_G$ and let $(A_n)_n$ be an inverse system in* $\mathfrak{Top}_G$ *such that $A = \varprojlim_n A_n$ in $\mathfrak{Top}_G$. Let furthermore $f \in \mathrm{End}_{\mathrm{cts},G}(A)$, such that $f = \varprojlim f_n$ with $f_n \in \mathrm{End}_{\mathrm{cts},G}(A_n)$. Then it holds*

$$\mathcal{C}_f^\bullet(G, A) \cong \varprojlim_n \mathcal{C}_f^\bullet(G, A_n).$$

Proof. First we want to note, that for groups $X = \varprojlim X_n$ and $Y = \varprojlim Y_n$ always holds $X \times Y = \varprojlim_n (X_n \times Y_n)$.

This means that the objects of the two complexes $\mathcal{C}_f^\bullet(G, A)$ and $\varprojlim \mathcal{C}_f^\bullet(G, A_n)$ coincide, so it remains to check that the differentials do as well. If we denote the i-th object of $C_{\mathrm{cts}}^\bullet(G, A)$ by C^i and the differential by d^i then it suffices to check that the following cube is commutative

This is a direct consequence from the assumption $f = \varprojlim f_n$ and that $C_{\mathrm{cts}}^\bullet(G, -)$ commutes with inverse limits. $\qquad \square$

Lemma 2.3.8. *Let G be a profinite group and $(A_n)_n$ be an inverse system in $\mathfrak{Top}_G$ such that the inverse system of complexes $(C_{\mathrm{cts}}^\bullet(G, A_n))_n$ has surjective transition maps and let $A := \varprojlim_n A_n$. If $f \in \mathrm{End}_{\mathrm{cts},G}(A)$ then also the system $(\mathcal{C}_f^\bullet(G, A_n))_n$ has surjective transition maps.*

Proof. By assumption, for every $k \in \mathbb{N}_0$, the transition map $C_{\mathrm{cts}}^k(G, A_n) \to C_{\mathrm{cts}}^k(G, A_{n-1})$ is surjective. But then also the transition map

$$C_{\mathrm{cts}}^k(G, A_n) \oplus C_{\mathrm{cts}}^{k-1}(G, A_n) \longrightarrow C_{\mathrm{cts}}^k(G, A_{n-1}) \oplus C_{\mathrm{cts}}^{k-1}(G, A_{n-1})$$
$$\| \qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$$
$$\mathcal{C}_f^k(G, A_n) \qquad\qquad\qquad\qquad\qquad \mathcal{C}_f^k(G, A_{n-1})$$

is surjective, since it's the direct sum of two surjective maps. $\qquad \square$

Definition 2.3.9. An inverse system (of abelian groups) $(X_n)_{n \in \mathbb{N}}$ is called **Mittag-Leffler** (**ML**) if for any $n \in \mathbb{N}$, there is an $m \geq n$ such that the image of the transition maps $X_k \to X_n$ coincide for all $k \geq m$ (cf. [29, p. 138]). An inverse system (of abelian groups) $(X_n)_{n \in \mathbb{N}}$ is called **Mittag-Leffler zero** (**ML-zero**) if for any $n \in \mathbb{N}$ there is an $m \geq n$ such that the transition map $X_k \to X_n$ is zero for all $k \geq m$ (cf. [29, p. 139]). A morphism $(X_n)_n \to (Y_n)_n$ of inverse

systems is called **Mittag-Leffler isomorphism** (**ML-isomorphism**) if the corresponding systems of kernels and cokernels are ML-zero.

By $\varprojlim^r$ we denote the r-th right derived functor of $\varprojlim$.

Proposition 2.3.10. *Let (X_n) and (Y_n) be inverse systems of abelian groups.*

1. *If (X_n) has surjective transition maps, then it is ML.*
2. *If (X_n) is ML then $\varprojlim_n^r X_n = 0$ for all $r \geq 0$.*
3. *If $f_n \colon X_n \to Y_n$ is a ML-isomorphism then for all $i \geq 0$ the homomorphism*

$$\varprojlim_n^i f_n \colon \varprojlim_n^i X_n \longrightarrow \varprojlim_n^i Y_n$$

   is an isomorphism.

Proof.

1. Let $\alpha_{nm} \colon X_m \to X_n$ denote the transition map for $m \geq n$. Then it is $\operatorname{im}(\alpha_{nm}) = X_n$ for all $m \geq n$, i.e. the system X_n is ML.
2. [29, Chapter II §7, (2.7.4) Proposition, p. 140]
3. [38, Tag 0918, Lem. 15.79.2.]

$\square$

Proposition 2.3.11. *Let $(X_n^\bullet)$ and $(Y_n^\bullet)$ be inverse systems of complexes of abelian groups such that the transition maps $X_{n+1}^i \to X_n^i$ and $Y_{n+1}^i \to Y_n^i$ are surjective for all $i \in \mathbb{Z}$ and $n \geq 0$.*

1. *For all $i \in \mathbb{Z}$ we get a short exact sequence*

$$0 \longrightarrow \varprojlim_n^1 H^{i-1}(X_n^\bullet) \longrightarrow H^i(\varprojlim_n X_n^\bullet) \longrightarrow \varprojlim_n H^i(X_n^\bullet) \longrightarrow 0.$$

2. *Let $(f_n^\bullet) \colon (X_n^\bullet) \to (Y_n^\bullet)$ be a morphism of inverse systems of complexes. If the induced map on cohomology $H^i(f_n^\bullet) \colon H^i(X_n^\bullet) \to H^i(Y_n^\bullet)$ is a ML-isomorphism for all $i \in \mathbb{Z}$, then $\varprojlim_n (f_n^\bullet) \colon \varprojlim_n X_n^\bullet \to \varprojlim_n Y_n^\bullet$ is a quasi isomorphism.*

Proof.

1. [24, Chapter 3, Proposition 1, p. 531; Corollary 1.1, p. 535–536]
2. From the first part of the proposition we obtain for every $i \in \mathbb{Z}$ a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \varprojlim_n^1 H^{i-1}(X_n^\bullet) & \longrightarrow & H^i(\varprojlim_n X_n^\bullet) & \longrightarrow & \varprojlim_n H^i(X_n^\bullet) & \longrightarrow & 0 \\
& & \downarrow{\varprojlim_n^1 H^{i-1}(f_n^\bullet)} & & \downarrow{H^i(\varprojlim_n f_n^\bullet)} & & \downarrow{\varprojlim_n H^i(f_n^\bullet)} & & \\
0 & \longrightarrow & \varprojlim_n^1 H^{i-1}(Y_n^\bullet) & \longrightarrow & H^i(\varprojlim_n Y_n^\bullet) & \longrightarrow & \varprojlim_n H^i(Y_n^\bullet) & \longrightarrow & 0.
\end{array}
$$

   The assumption that $H^i(f_n^\bullet)$ is a ML-isomorphism for all $i \in \mathbb{Z}$ then says that the left and the right horizontal maps in the above diagram are isomorphisms (cf. Proposition 2.3.10). The 5-Lemma then implies that also $H^i(\varprojlim_n f_n^\bullet)$ is an isomorphism for all $i \in \mathbb{Z}$, i.e. $\varprojlim_n f_n^\bullet$ is a quasi isomorphism.

$\square$

Remark 2.3.12. *Since isomorphisms of inverse systems are always ML-isomorphisms, the above Proposition also states, that if $(f_n^\bullet) \colon (X_n^\bullet) \to (Y_n^\bullet)$ is a quasi isomorphism of inverse*

systems of complexes, for which the transition maps $X^i_{n+1} \to X^i_n$ and $Y^i_{n+1} \to Y^i_n$ are surjective for all $i \in \mathbb{Z}$ and $n \geq 0$, then also $\varprojlim_n (f^\bullet_n) \colon \varprojlim_n X^\bullet_n \to \varprojlim_n Y^\bullet_n$ is a quasi isomorphism.

Remark 2.3.13. In the above Proposition 2.3.11 and Remark 2.3.12 one cannot easily drop the assumption that the transition maps are surjective. There exist examples (see [23, Rem. 2.3.13.]) of two inverse systems of complexes which are quasi isomorphic, but their projective limits are not. Hence in the proof of [34, Theorem 2.2.1, p. 702–705] right before [34, Proposition 2.2.7, p. 703–705], an explanation is missing why it really is enough to prove this proposition.

3. Lubin-Tate (φ, Γ)-modules

The goal in this section is to state in the Lubin-Tate case the equivalence of categories from [20, Thm. 1.6], which follows closely the original result [14, 3.4.3] for (ϕ, Γ)-modules in the cyclotomic case, in the style and with similar notation as in [10] or [15, Theorem 4.22, p. 82]. Namely, if $K|L|\mathbb{Q}_p$ are finite extensions, we want to describe an equivalence of categories between the category of continuous $\mathcal{O}_L$-representations of the absolute Galois group G_K and a yet to be defined category of étale (φ_L, Γ_K)-modules. While there is only a sketch of proof in [20, Thm. 1.6], for the case $K = L$ a very detailed proof can be found in the book [30]. In his thesis [23] the first named author checked and comments how to adjust Schneider's proof in the general case. One more useful source will be [35].

3.1. **Preparations and Notations.** Let p be a prime number and let $\overline{\mathbb{Q}_p}$ be a fixed algebraic closure of the p-adic numbers $\mathbb{Q}_p$ and let as usual $\mathbb{Z}_p$ be the integral p-adic numbers. Each finite extension of $\mathbb{Q}_p$ is considered to be a subfield of $\overline{\mathbb{Q}_p}$. Let $\mathbb{C}_p$ be the completion of $\overline{\mathbb{Q}_p}$ with respect to the valuation v_p with $v_p(p) = 1$ and let $\mathcal{O}_{\mathbb{C}_p}$ be the ring of integers of $\mathbb{C}_p$.

Let furthermore $L|\mathbb{Q}_p$ be a finite extension, d_L its degree over $\mathbb{Q}_p$, $\mathcal{O}_L$ the ring of integers, $\pi_L \in \mathcal{O}_L$ a prime element, k_L the residue class field, $q_L = p^r$ its cardinality, L_0 the maximal unramified extension of $\mathbb{Q}_p$ in L with ring of integers $\mathcal{O}_{L_0}$.

Let furthermore $K|L$ be a finite extension, d_K its degree over $\mathbb{Q}_p$, $\mathcal{O}_K$ its ring of integers, $\pi_K \in \mathcal{O}_K$ a prime element, k_K the residue class field, q_K its cardinality, K_0 the maximal unramified extension of $\mathbb{Q}_p$ in K with ring of integers $\mathcal{O}_{K_0}$.

We will denote the absolute Galois groups of $\mathbb{Q}_p$, L and K by $G_{\mathbb{Q}_p}$, G_L and G_K, respectively. For any k_L-algebra B we will denote by $W(B)_L$ the ring of **ramified Witt vectors** with values in B. (cf. [30, Section 1.1, p. 6–21]). If B happens to be perfect, these are standard Witt vectors tensored with $\mathcal{O}_L$ (cf. [30, Proposition 1.1.26, p. 23–24]).

A **perfectoid** field $\mathcal{K} \subseteq \mathbb{C}_p$ is a complete field, such that its value group $|\mathcal{K}^\times|$ is dense in $\mathbb{R}^\times_+$ and which satisfies $(\mathcal{O}_\mathcal{K}/p\mathcal{O}_\mathcal{K})^p = \mathcal{O}_\mathcal{K}/p\mathcal{O}_\mathcal{K}$ (cf. [30, p. 42]).

Let $\mathcal{K}$ be a perfectoid field. The **tilt** $\mathcal{K}^\flat$ of $\mathcal{K}$ is the fraction field of the ring

$$\mathcal{O}_{\mathcal{K}^\flat} \coloneqq \varprojlim_{x \mapsto x^{q_L}} \mathcal{O}_\mathcal{K}/\varpi\mathcal{O}_\mathcal{K},$$

where ϖ is an element in $\mathcal{O}_\mathcal{K}$ such that $|\varpi| \geq |\pi_L|$. In fact, this definition is independent from the choice of the element ϖ (cf. [30, Lemma 1.4.5, p. 43–44]). The field $\mathcal{K}^\flat$ is perfect and complete and has characteristic p (cf. [30, Proposition 1.4.7, p. 45]). Moreover, the field $\mathbb{C}_p^\flat$ is algebraically closed (cf. [30, Proposition 1.4.10, p. 46–47]). The theory of perfectoid fields was originally established by Peter Scholze (cf. [35]) but Schneider's book covers all of the theory we do need here.

Let from now on, as in [30, Definition 1.3.2, p. 29], $\phi \in R[\![X_1, \ldots X_n]\!]$ be a fixed Frobenius power series associated to π_L, i.e. we have

$$\phi(X) \equiv \pi_L X \bmod \deg 2$$
$$\phi(X) = X^{q_L} \bmod \pi_L \mathcal{O}_L[\![X]\!].$$

Let furthermore $\mathcal{G}_\phi \in \mathcal{O}_L[\![X, Y]\!]$ be the Lubin-Tate formal group which belongs to ϕ (cf. [30, Proposition 1.3.4, p. 31]). For $a \in \mathcal{O}_L$ denote by $[a]_\phi \in \mathcal{O}_L[\![X]\!]$ the corresponding endomorphism of $\mathcal{G}_\phi$ (cf. [30, Proposition 1.3.6, p. 32]). Note that we then have $[a]_\phi(X) \equiv aX \bmod \deg 2$ and $[\pi_L]_\phi = \phi$ (loc. cit.). We then set $\mathfrak{M} := \{x \in \overline{\mathbb{Q}_p} \mid |x| < 1\}$ and obtain that the operation

$$\mathcal{O}_L \times \mathfrak{M} \longrightarrow \mathfrak{M}$$
$$(a, x) \longmapsto [a]_\phi(z)$$

makes $\mathfrak{M}$ into an $\mathcal{O}_L$-module (cf. [30, p. 33]). Then, for every $a \in \mathcal{O}_L$, we can view $[a]_\phi$ as endomorphism of $\mathfrak{M}$ and therefore are able to define

$$\mathcal{G}_{\phi,n} := \ker([\pi_L^n]_\phi \colon \mathfrak{M} \to \mathfrak{M}) = \{x \in \mathfrak{M} \mid [\pi_L^n]_\phi(x) = 0\}.$$

Note that $(\mathcal{G}_{\phi,n})_n$ is via $[\pi_L]_\phi$ an inverse system and we let

$$\mathcal{T}\mathcal{G}_\phi := \varprojlim_n \mathcal{G}_{\phi,n}$$

be the projective limit of this system. (cf. [30, p. 50]). $\mathcal{T}\mathcal{G}_\phi$ is also called the **Tate module** of the group $\mathcal{G}_\phi$. From [30, Proposition 1.3.10, p. 34] we can deduce that $\mathcal{T}\mathcal{G}_\phi$ is a free $\mathcal{O}_L$-module of rank one.

Following [30, (1.3.9), p. 33] we let $L_n = L(\mathcal{G}_\phi[\pi_L^n])$ and $L_\infty = \cup_n L_n$. Denote as there the Galois group $\mathrm{Gal}(L_\infty|L)$ by Γ_L, set $\Gamma_{L_n|L} = \mathrm{Gal}(L_n|L)$ and $H_L = \mathrm{Gal}(\overline{\mathbb{Q}_p}|L_\infty)$. Define furthermore $K_n := K(\mathcal{G}_\phi[\pi_L^n]) = KL_n$ and $K_\infty := \cup_n K_n = KL_\infty$ as well as $\Gamma_K = \mathrm{Gal}(K_\infty|K)$ and $H_K = \mathrm{Gal}(\overline{\mathbb{Q}_p}|K_\infty)$. These definitions can be summarized in the following diagram:
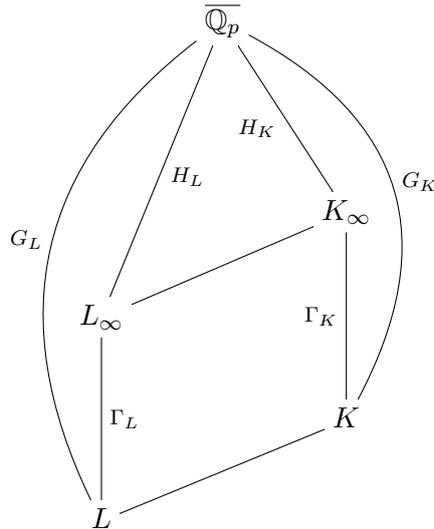
Remark 3.1.1. *The group Γ_L is isomorphic to $\mathcal{O}_L^\times$ via the Lubin-Tate character χ_{LT}. Furthermore, Γ_L acts continuously on $\mathfrak{TG}_\phi$ via χ_L, i.e. for all $\gamma \in \Gamma_L$ and $t \in \mathfrak{TG}_\phi$ we have*

$$\gamma \cdot t = \chi_L(\gamma) \cdot t = [\chi_L(\gamma)]_\phi(t).$$

Proof. For the first assertion see [30, (1.3.12),p. 36], the second follows immediately from [30, (1.3.11),p. 34–35] and is also stated at [30, (1.4.17),p. 51]. □

Remark 3.1.2. *One can view Γ_K as an open subgroup of Γ_L. If, in addition, $K|L$ is unramified, then we have $\Gamma_K \cong \Gamma_L$.*

3.2. **The coefficient ring.** We first want to recall the definition of the coefficient ring used in [30] and then deduce the coefficient ring in the general case.
First we recall the ring

$$\mathscr{A}_L := \varprojlim_n \mathcal{O}_L/\pi_L^n \mathcal{O}_L((X)),$$

from [30, p. 75]. This ring will be prototypical for our coefficients once we bring the variable X to life. $\mathscr{A}_L$ carries an action from Γ_L by

$$\begin{aligned} \Gamma_L \times \mathscr{A}_L &\longrightarrow \mathscr{A}_L \\ (\gamma, f) &\longmapsto f([\chi_L(\gamma)]_\phi(X)). \end{aligned}$$

and possesses an injective $\mathcal{O}_L$-algebra endomorphism

$$\begin{aligned} \varphi_L \colon \mathscr{A}_L &\longrightarrow \mathscr{A}_L \\ f &\longmapsto f([\pi_L]_\phi(X)) \end{aligned}$$

(cf. [30, p. 78]). At [30, p. 79] Schneider defines a **weak topology** on $\mathscr{A}_L$, for which the $\mathcal{O}_L[\![X]\!]$-submodules

$$U_m := X^m \mathcal{O}_L[\![X]\!] + \pi_L^m \mathscr{A}_L$$

form a fundamental system of open neighbourhoods of $0 \in \mathscr{A}_L$. As $\varphi_L(\mathscr{A}_L)$-module $\mathscr{A}_L$ is free with basis $1, X, \ldots, X^{q_L-1}$ ([30, Proposition 1.7.3, p. 78]), with respect to the weak topology $\mathscr{A}_L$ is a complete Hausdorff topological $\mathcal{O}_L$-algebra ([30, Lemma 1.7.6, p. 79–80]) and both the endomorphism φ_L and the Γ_L-action are continuous for the weak topology ([30, Proposition 1.7.8, p. 80–82]).

Following Colmez [11, §9.2] one can find an element $\omega \in \mathcal{O}_{\mathbb{C}_p^\flat}$, such that $X \mapsto \omega$ defines an inclusion $k_L((X)) \hookrightarrow \mathbb{C}_p^\flat$ (respecting important properties). As in [30, p. 50] we denote the image of this inclusion by $\mathbf{E}_L$ and we want to recall from loc. cit. that $\mathbf{E}_L$ is a complete nonarchimedean discretely valued field, with uniformizer ω and residue class field k_L. Let in addition $\mathbf{E}_L^+$ denote the ring of integers inside $\mathbf{E}_L$. Furthermore, $\mathbf{E}_L$ carries a continuous operation by Γ_L, for which we have $\gamma \cdot \omega = [\chi_L(\gamma)]_\phi(\omega) \bmod \pi_L$ (cf. [30, Lemma 1.4.15, p. 51]). By raising elements to its q_L-th power, it is clear that $\mathbf{E}_L$ also carries a Frobenius homomorphism, which is continuous and the reduction modulo p of φ_L. Let furthermore $\mathbf{E}_L^{\mathrm{sep}}$ denote the separable closure of $\mathbf{E}_L$ inside $\mathbb{C}_p^\flat$ and let $\mathbf{E}_L^{\mathrm{sep},+}$ denote the integral closure of $\mathbf{E}_L^+$ inside $\mathbf{E}_L^{\mathrm{sep}}$. The Galois group $\mathrm{Gal}(\mathbf{E}_L^{\mathrm{sep}}|\mathbf{E}_L)$ is isomorphic to H_L by [30, Section 1.6, p. 68–75] and [30, Theorem 1.6.7, p. 73–74]. Then Schneider lifts ω to $W(\mathbf{E}_L)_L \subseteq W(\mathcal{O}_{\mathbb{C}_p^\flat})_L$ and calls this lift ω_ϕ (cf. [30, Section 2.1, p. 84–98; in particular p. 93]). Here one cannot just take the

Teichmüller lift, because one wants that the lift fulfills the following relations

$$\mathrm{Fr}(\omega_\phi) = [\pi_L]_\phi(\omega_\phi)$$

$$\gamma \cdot \omega_\phi = [\chi_L(\gamma)]_\phi(\omega_\phi)$$

for all $\gamma \in \Gamma_L$ and where Fr is the Frobenius on $W(\mathbb{C}_p^\flat)_L$ (cf. [30, Lemma 2.1.13, p. 92–93] for the Frobenius and [30, Lemma 2.1.15, p. 95] for the Γ_L-action). Similar to the construction of $\mathbf{E}_L$, sending X to ω_ϕ then defines an inclusion $\mathscr{A}_L \hookrightarrow W(\mathbf{E}_L)_L$ (cf. [30, p. 94]). In Particular, it gives us a commutative square (loc. cit.)

$$
\begin{array}{ccc}
\mathscr{A}_L & \xrightarrow{\ X \mapsto \omega_\phi\ } & W(\mathbf{E}_L)_L \\
\downarrow & & \downarrow \\
k_L((X)) & \xrightarrow{\ X \mapsto \omega\ } & \mathbf{E}_L.
\end{array}
$$

Let $\mathbf{A}_L$ denote the image of the inclusion $\mathscr{A}_L \hookrightarrow W(\mathbf{E}_L)_L$. In addition, define

$$\mathbf{A}_L^+ := \mathcal{O}_L[\![\omega_\phi]\!] = \mathbf{A}_L \cap W(\mathbf{E}_L^+)_L.$$

$\mathbf{A}_L$ is endowed with the weak topology, i.e., induced by that from $W(\mathbb{C}_p^\flat)_L$ and the isomorphism $\mathscr{A}_L \cong \mathbf{A}_L$ is topological for the weak topologies on both sides (cf. [30, Proposition 2.1.16, p. 95–96]). Furthermore, this topological isomorphism respects the Γ_L-actions on both sides, where $\mathbf{A}_L$-carries a Γ_L-action induced from the G_L-action of $W(\mathbb{C}_p^\flat)_L$ (cf. [30, p. 94]) and what is φ_L on $\mathscr{A}_L$ is the Frobenius on $\mathbf{A}_L$, which again is induced from the Frobenius on $W(\mathbb{C}_p^\flat)_L$ (cf. [30, Proposition 2.1.16, p. 95–96]). We therefore denote the Frobenius on $\mathbf{A}_L$ also by φ_L. An immediate consequence then is, that the Γ_L-action and φ_L are continuous on $\mathbf{A}_L$.

This is the coefficient ring for Schneider's (φ_L, Γ_L)-modules (cf. [30, Definition 2.2.6, p. 100–101]) but since we want to establish (φ, Γ)-modules over a finite extension $K|L$ as it was done in the classical way (cf. [15, Definition 4.21, p. 81]) for finite extensions of $\mathbb{Q}_p$, we transfer this construction to our situation. Let for this $\mathbf{A}_L^{\mathrm{nr}} \subseteq W(\mathbf{E}_L^{\mathrm{sep}})_L$ be the maximal unramified extension of $\mathbf{A}_L$ inside $W(\mathbf{E}_L^{\mathrm{sep}})_L$. In particular [30, Lemma 3.1.3, p. 112–113] says that for every finite, separable extension $F|\mathbf{E}_L$ inside $\mathbf{E}_L^{\mathrm{sep}}$, there exists a unique ring $\mathbf{A}_L(F) \subseteq W(\mathbf{E}_L^{\mathrm{sep}})_L$ containing $\mathbf{A}_L$ such that $\mathbf{A}_L^{\mathrm{nr}}$ is the colimit of the family $\mathbf{A}_L(F)$. The ring $\mathbf{A}$ is defined as the closure of $\mathbf{A}_L^{\mathrm{nr}}$ inside $W(\mathbf{E}_L^{\mathrm{sep}})$ with respect to the π_L-adic topology and one has (cf. [30, p. 113 and Remark 3.14, p. 114])

$$\mathbf{A} \cong \varprojlim_n \mathbf{A}_L^{\mathrm{nr}}/\pi_L^n \mathbf{A}_L^{\mathrm{nr}}.$$

$\mathbf{A}_L^{\mathrm{nr}}$ and $\mathbf{A}$ have an action from G_L, the Frobenius on $W(\mathbf{E}_L^{\mathrm{sep}})$ preserves both rings, they are discrete valuation rings with prime element π_L, where $\mathbf{A}$ is even complete and their residue class field is $\mathbf{E}_L^{\mathrm{sep}}$ (cf. [30, p. 113–114]). In fact, the G_L-action on both $\mathbf{A}_L^{\mathrm{nr}}$ and $\mathbf{A}$ is continuous for the weak topologies, since the G_L action on $W(\mathbb{C}_p^\flat)_L$ is continuous for the weak topology (cf. [30, Lemma 1.4.13, p. 48–49] and [30, Lemma 1.5.3, p. 65–66]) and both, the weak topology and the G_L action on $\mathbf{A}_L^{\mathrm{nr}}$ respectively $\mathbf{A}$, are induced form $W(\mathbb{C}_p^\flat)_L$. Furthermore we have the relation (cf. [30, Lemma 3.1.6, p. 115–116])

$$(\mathbf{A})^{H_L} = \mathbf{A}_L.$$

This leads to the definition

$$\mathbf{A}_{K|L} := (\mathbf{A})^{H_K}.$$

In addition, define

$$\mathbf{A}^+ := \mathbf{A} \cap W(\mathbf{E}_L^{\mathrm{sep},+})_L$$
$$\mathbf{A}_L^{\mathrm{nr},+} := \mathbf{A}_L^{\mathrm{nr}} \cap W(\mathbf{E}_L^{\mathrm{sep},+})_L$$
$$\mathbf{A}_{K|L}^+ := \mathbf{A}_{K|L} \cap W(\mathbf{E}_L^{\mathrm{sep},+})_L.$$

Then, since by definition it is $\mathbf{A}_L \subseteq \mathbf{A}_{K|L} \subseteq W(\mathbf{E}_L^{\mathrm{sep}})_L$, the ring $\mathbf{A}_{K|L}$ is a complete nonarchimedean discrete valuation ring with prime element π_L and the restriction of the Frobenius from $W(\mathbf{E}_L^{\mathrm{sep}})_L$ gives a ring endomorphism $\varphi_{K|L}$ of $\mathbf{A}_{K|L}$, which then also commutes with φ_L (cf. [30, Lemma 3.1.3, p. 112–113]). Furthermore, since $\mathbf{A}$ carries an action from G_L and therefore also one from G_K, the ring $\mathbf{A}_{K|L}$ carries an action from Γ_K. Next, we want to define a weak topology on $\mathbf{A}_{K|L}$, deduce some properties and see that $\varphi_{K|L}$ and the action from Γ_K are continuous for this topology.

Definition 3.2.1. The **weak topology** on any of the rings $\mathbf{A}$, $\mathbf{A}_L^{\mathrm{nr}}$, $\mathbf{A}_{K|L}$ and $\mathbf{A}_L$ is defined as the induced topology of the weak topology of $W(\mathbb{C}_p^\flat)_L$ (for the latter see [30, p. 64–65]).

Remark 3.2.2. *The weak topology on $W(\mathbb{C}_p^\flat)_L$ is complete and Hausdorff (cf. [30, Lemma 1.5.5, p. 67–68]) and $W(\mathbb{C}_p^\flat)_L$ is a topological ring with respect to its weak topology (cf. [30, Lemma 1.5.4, p. 66–67]). Therefore, the induced topology on any of the rings $\mathbf{A}$, $\mathbf{A}_L^{\mathrm{nr}}$, $\mathbf{A}_{K|L}$ and $\mathbf{A}_L$ is Hausdorff and these rings are topological rings.*

The question now is, wether $\varphi_{K|L}$ and the action from Γ_K are continuous for the weak topology on $\mathbf{A}_{K|L}$. For this, we want to recall a well-known fact.

Lemma 3.2.3. *Let X and Y be topological spaces, $f \colon X \to Y$ be a continuous map and let $Z \subseteq Y$ be a subspace with $\mathrm{im}(f) \subseteq Z$. Then $f \colon X \to Z$ is continuous.*

Proposition 3.2.4. *The from $W(\mathbf{E}_L^{\mathrm{sep}})_L$ induced Γ_K-action and the induced Frobenius $\varphi_{K|L}$ on $\mathbf{A}_{K|L}$ are continuous.*

Proof. This now is an immediate consequence of Lemma 3.2.3 and the fact, that G_L acts continuously on $W(\mathbf{E}_L^{\mathrm{sep}})_L$ (cf. [30, Lemma 1.5.3, p. 65–66]) as well as that Fr is continuous on $W(\mathbf{E}_L^{\mathrm{sep}})$ with respect to the weak topology:
Since the maps

$$G_L \times \mathbf{A}_{K|L} \longrightarrow G_L \times W(\mathbf{E}_L^{\mathrm{sep}})_L \longrightarrow W(\mathbf{E}_L^{\mathrm{sep}})_L$$

and

$$\mathbf{A}_{K|L} \hookrightarrow W(\mathbf{E}_L^{\mathrm{sep}})_L \xrightarrow{\ \mathrm{Fr}\ } W(\mathbf{E}_L^{\mathrm{sep}})_L$$

are continuous as composite maps of continuous maps and their image is inside $\mathbf{A}_{K|L}$ (for the latter see [30, Lemma 3.1.3, p. 112–113]) the claim follows. $\square$

We want to end this subsection by fixing some notation, defining weak topologies on modules over any of the above rings and calculating the residue class field of $\mathbf{A}_{K|L}$. We denote by $\mathbf{B}_L$, $\mathbf{B}$, $\mathbf{B}_{K|L}$ and $\mathbf{B}_L^{\mathrm{nr}}$ the quotient fields of $\mathbf{A}_L$, $\mathbf{A}$, $\mathbf{A}_{K|L}$ and $\mathbf{A}_L^{\mathrm{nr}}$, respectively. Furthermore, set $\mathbf{E}_{K|L} := (\mathbf{E}_L^{\mathrm{sep}})^{H_K}$ and let $\mathbf{E}_{K|L}^+$ denote the integral closure of $\mathbf{E}_L^+$ inside $\mathbf{E}_L$. In Lemma

3.2.11 we will see that $\mathbf{E}_{K|L}$ is the residue class field of $\mathbf{A}_{K|L}$. Beforehand, we define weak topologies for modules.

Lemma 3.2.5. *Let $R \in \{\mathbf{A}, \mathbf{A}_L^{\mathrm{nr}}, \mathbf{A}_{K|L}, \mathbf{A}_L\}$ and M be a finitely generated R-module. If $k, l \in \mathbb{N}$ such that $R^k \twoheadrightarrow M$ and $R^l \twoheadrightarrow M$ are surjective homomorphisms, then the induced quotient topologies on M coincide (where R^k and R^l carry the product topology of the weak topology on R).*

Proof. This is [21, Lemma 3.2.2 (i), p. 100–102]. There, in fact, is no proof for $\mathbf{A}_{K|L}$, but in his proof, the author only uses that the coefficient ring is a topological ring with respect to the weak topology, what we stated in the above Remark 3.2.2. $\square$

Definition 3.2.6. Let $R \in \{\mathbf{A}, \mathbf{A}_L^{\mathrm{nr}}, \mathbf{A}_{K|L}, \mathbf{A}_L\}$ and M be a finitely generated R-module. The **weak topology** on M is defined as the quotient topology for any surjective homomorphism $R^k \twoheadrightarrow M$, where R^k carries the product topology of the weak topology on R.

Lemma 3.2.7. *Let $R \in \{\mathbf{A}, \mathbf{A}_L^{\mathrm{nr}}, \mathbf{A}_{K|L}, \mathbf{A}_L\}$ and M be a finitely generated R-module. Then M with its weak topology is a topological R-module and if $M = M_1 \oplus M_2$, then the weak topology on M coincides with the direct product of the weak topologies on the M_1 and M_2. Furthermore, if N is another finitely generated R-module and $f \colon M \to N$ is an R-module homomorphism, then f is continuous with respect to the weak topologies on both M and N.*

Proof. This is [21, Lemma 3.2.2 (ii)-(iv), p. 100–102]. Again, there is no proof for $\mathbf{A}_{K|L}$, but the property used is that of a discrete valuation ring, which $\mathbf{A}_{K|L}$ also fulfills. $\square$

Proposition 3.2.8 (Relative Ax-Sen-Tate)**.** *Let $\mathcal{K}$ be a nonarchimedean valued field of characteristic 0, $\overline{\mathcal{K}}$ an algebraic closure with completion $\mathbb{C}$ and $\mathcal{L}|\mathcal{K}$ a Galois extension within $\overline{\mathcal{K}}$ with completion $\widehat{\mathcal{L}}$. Let furthermore $H \le \mathrm{Gal}(\mathcal{L}|\mathcal{K})$ be a closed subgroup. Then it holds*

$$(\widehat{\mathcal{L}})^H = (\mathcal{L}^H)^{\wedge}.$$

Proof. This is an immediate consequence of the usual Ax-Sen-Tate theorem (cf. [15, Proposition 3.8, p. 43–44]): Since $\mathcal{L}|\mathcal{K}$ is algebraic, $\overline{\mathcal{K}}$ is also an algebraic closure for $\mathcal{L}$ and then we deduce (loc. cit.)

$$\mathbb{C}^{G_{\mathcal{L}}} = \widehat{\mathcal{L}}.$$

Infinite Galois theory then says that we have $H = \mathrm{Gal}(\mathcal{L}|\mathcal{L}^H) \cong G_{\mathcal{L}^H}/G_{\mathcal{L}}$. Together with Ax-Sen-Tate we then deduce

$$(\mathcal{L}^H)^{\wedge} = \mathbb{C}^{G_{\mathcal{L}^H}} = (\mathbb{C}^{G_{\mathcal{L}}})^H = (\widehat{\mathcal{L}})^H.$$

$\square$

For our purposes the following integral version of the above Relative Ax-Sen-Tate Theorem will be the interesting one.

Corollary 3.2.9. *Let $\mathcal{K}$ be a nonarchimedean valued field of characteristic 0, $\overline{\mathcal{K}}$ an algebraic closure with completion $\mathbb{C}$ and $\mathcal{L}|\mathcal{K}$ a Galois extension within $\overline{\mathcal{K}}$ with completion $\widehat{\mathcal{L}}$. Denote by $\mathcal{O}_?$ the ring of integers of any of the above fields ?. Let furthermore $H \le \mathrm{Gal}(\mathcal{L}|\mathcal{K})$ be a closed subgroup. Then it holds*

$$(\mathcal{O}_{\widehat{\mathcal{L}}})^H = ((\mathcal{O}_{\mathcal{L}})^H)^{\wedge}.$$

Proof. For an element $x \in \mathcal{C}$ we have

$$x \in (\mathcal{O}_{\widehat{\mathcal{L}}})^H \Leftrightarrow x \in (\widehat{\mathcal{L}})^H \text{ with } |x| \le 1 \overset{3.2.8}{\Leftrightarrow} x \in (\mathcal{L}^H)^\wedge \text{ with } |x| \le 1 \Leftrightarrow x \in ((\mathcal{O}_{\mathcal{L}})^H)^\wedge,$$

where the last equivalence comes from the fact that the integers of the completion are the completion of the integers. $\qquad\square$

Lemma 3.2.10. *It holds* $(\mathbf{A}_L^{\mathrm{nr}})^{H_K} = \mathbf{A}_{K|L}$.

Proof. This is a direct consequence of the above Corollary 3.2.9. This namely says that

$$\mathbf{A}_{K|L} = (\mathbf{A})^{H_K} = ((\mathbf{A}_L^{\mathrm{nr}})^{H_K})^\wedge.$$

But since $(\mathbf{A}_L^{\mathrm{nr}})^{H_K}|\mathbf{A}_L$ is finite and $\mathbf{A}_L$ is complete, $(\mathbf{A}_L^{\mathrm{nr}})^{H_K}$ itself is complete, i.e. it is

$$(\mathbf{A}_L^{\mathrm{nr}})^{H_K} = ((\mathbf{A}_L^{\mathrm{nr}})^{H_K})^\wedge = \mathbf{A}_{K|L}.$$

$\qquad\square$

Lemma 3.2.11. $\mathbf{E}_{K|L}$ *is the residue class field of* $\mathbf{A}_{K|L}$.

Proof. We have an exact sequence

$$0 \longrightarrow \mathbf{A}_L^{\mathrm{nr}} \overset{\cdot \pi_L}{\longrightarrow} \mathbf{A}_L^{\mathrm{nr}} \longrightarrow \mathbf{A}_L^{\mathrm{nr}}/\pi_L \mathbf{A}_L^{\mathrm{nr}} \longrightarrow 0.$$

By taking H_K-invariants and using $(\mathbf{A}_L^{\mathrm{nr}})^{H_K} = \mathbf{A}_{K|L}$ from Lemma 3.2.10 we obtain the exact sequence

$$0 \longrightarrow \mathbf{A}_{K|L} \overset{\cdot \pi_L}{\longrightarrow} \mathbf{A}_{K|L} \longrightarrow (\mathbf{E}_L^{\mathrm{sep}})^{H_K} \longrightarrow H^1(H_K, \mathbf{A}_L^{\mathrm{nr}}).$$

Since $\mathbf{B}_L^{\mathrm{nr}}|\mathbf{B}_L$ is unramified, and therefore also tamely ramified, we get from [29, (6.1.10) Theorem, p. 342–342] that $\mathbf{A}_L^{\mathrm{nr}}$ is a cohomologically trivial H_L-module. Therefore the right term in the latter sequence is equal to zero and we get the exact sequence

$$0 \longrightarrow \mathbf{A}_{K|L} \overset{\cdot \pi_L}{\longrightarrow} \mathbf{A}_{K|L} \longrightarrow \mathbf{E}_{K|L} \longrightarrow 0$$

which ends the proof. $\qquad\square$

3.2.1. *Concrete description of Weak Topologies.* As the title says, the goal of this chapter is to give a concrete description of both, the ring $\mathbf{A}_{K|L}$ and its weak topology. We will start with the topology and first we want the recall the description of the weak topology of $\mathbf{A}_L$ and recall that a similar description holds true on $W(\mathbb{C}_p^\flat)_L$

Remark 3.2.12. [30, Proposition 2.1.16 (i), p. 95–96] *says that the weak topology* $\mathbf{A}_L$ *has an analogous description as the description above. Concretely, a fundamental system of open neighbourhoods of* 0 *for the weak topology on* $\mathbf{A}_L$ *is given by*

$$\omega_\phi^m \mathbf{A}_L^+ + \pi_L^m \mathbf{A}_L, \ m \ge 1.$$

Remark 3.2.13. *A fundamental system of open neighbourhoods of* 0 *for the weak topology on* $W(\mathbb{C}_p^\flat)_L$ *is given by the* $W(\mathcal{O}_{\mathbb{C}_p^\flat})_L$*-submodules*

$$\omega_\phi^m W(\mathcal{O}_{\mathbb{C}_p^\flat})_L + \pi_L^m W(\mathbb{C}_p^\flat)_L, \ m \ge 1.$$

Proof. Because of $|\Phi_0(\omega_\phi)|_\flat = |\omega|_\flat = |\pi_L|^{q_L/q_L - 1} < 1$ (cf. [30, Lemma 2.1.13 (i), p. 92–93] for the first equality and [30, Lemma 1.4.14, p. 50] for the second) this is exactly [30, Remark 2.1.5 (ii), p. 86–87]. $\qquad\square$

In the following Proposition we will show, that the above description of the weak topology on $\mathbf{A}_L$ extends to unramified, integral extensions. Its proof is a generalization of [30, Proposition 2.1.16 (i), p. 95–96].

Proposition 3.2.14. *Let $B|\mathbf{B}_L$ be an unramified extension, $A \subseteq B$ the integral closure of $\mathbf{A}_L$ in B and set $A^+ := A \cap W(\mathbf{E}_L^{\mathrm{sep},+})_L$.*
Then the family

$$\omega_\phi^m A^+ + \pi_L^m A, \ m \geq 1$$

of A^+-submodules of A forms a fundamental system of open neighbourhoods of 0 for the weak topology on A.

Proof. Since we have $\omega_\phi^m A^+ \subseteq \omega_\phi^m W(\mathcal{O}_{\mathbb{C}_p^\flat})_L$ and $\pi_L^m A \subseteq \pi_L^m W(\mathbb{C}_p^\flat)_L$ for all $m \geq 1$, we also get

$$\omega_\phi^m A^+ + \pi_L^m A \subseteq (\omega_\phi^m W(\mathcal{O}_{\mathbb{C}_p^\flat})_L + \pi_L^m W(\mathbb{C}_p^\flat)_L) \cap A$$

for all $m \geq 1$, i.e. the topology on A generated by the family $(\omega_\phi^m A^+ + \pi_L^m A)_m$ is finer then the topology induced from $W(\mathbb{C}_p^\flat)_L$.
To see that it is also coarser, let $E|\mathbf{E}_L$ be the residue class field of A and E^+ be the integral closure of $\mathbf{E}_L^+$ in E and consider the following families of $W(\mathcal{O}_{\mathbb{C}_p^\flat})_L$-submodules of $W(\mathbb{C}_p^\flat)_L$:

$$V_{n,m} := \left\{ (b_0, b_1, \dots) \in W(\mathcal{O}_{\mathbb{C}_p^\flat})_L \mid b_0, \dots, b_{m-1} \in \omega^n \mathcal{O}_{\mathbb{C}_p^\flat} \right\},$$
$$U_{n,m} := \left\{ (b_0, b_1, \dots) \in W(\mathbb{C}_p^\flat)_L \mid b_0, \dots, b_{m-1} \in \omega^n \mathcal{O}_{\mathbb{C}_p^\flat} \right\}.$$

These are introduced in [30, Section 1.5, p. 64–68] to define the weak topology on $W(\mathbb{C}_p^\flat)_L$. In particular, the $U_{n,m}$ give a fundamental system of open neighbourhoods of 0 in $W(\mathbb{C}_p^\flat)_L$ (loc. cit.) and the $V_{n,m}$ give one of $W(\mathcal{O}_{\mathbb{C}_p^\flat})_L$. Since ω_ϕ is topologically nilpotent (cf. [30, Lemma 2.1.6, p. 87]) we can find for any $k \in \mathbb{N}$ an element $n \in \mathbb{N}$ such that $\omega_\phi^n \in V_{k,m}$. But since $\Phi_0(\omega_\phi) = \omega$, i.e. $\omega_\phi = (\omega, \dots)$, the condition $\omega_\phi^n \in V_{k,m}$ implies $n \geq k$. Therefore we can find an increasing sequence of natural numbers $m \leq l_1 < \cdots < l_m$ such that

$$\omega_\phi^{q_L^{l_i}} \in V_{q_L^{l_{i-1}+1}, m} \quad \text{for all } 2 \leq i \leq m.$$

Since A^+ only contains positive powers of ω_ϕ, this then implies, that for all $2 \leq i \leq m$ we have

$$\omega_\phi^{q_L^{l_i}} A^+ \subseteq V_{q_L^{l_{i-1}+1}, m}.$$

We will now show that

$$U_{q_L^{l_m}, m} \cap A \subseteq \omega_\phi^m A^+ + \pi_L^m A.$$

For this let $f_m \in U_{q_L^{l_m}, m} \cap A$. We then have

$$\Phi_0(f_m) \in \omega^{q_L^{l_m}} \mathcal{O}_{\mathbb{C}_p^\flat} \cap E = \omega^{q_L^{l_m}} E^+.$$

Since by [30, Lemma 3.1.3 (b), p. 112–113] the diagram

commutes, we can find $g_m \in \omega_\phi^{q_L^{lm}} A^+$ and $f_{m-1} \in A$ such that

$$f_m = g_m + \pi_L f_{m-1}.$$

Recall $\omega_\phi^{q_L^{lm}} A^+ \subseteq V_{q_L^{l_{m-1}+1},m}$ from above and obtain

$$\pi_L f_{m-1} = f_m - g_m \in (U_{q_L^{lm},m} + V_{q_L^{l_{m-1}+1},m}) \cap A = U_{q_L^{l_{m-1}+1},m} \cap A.$$

Then [30, Proposition 1.1.18 (i), p. 16–17] says that, if $f_{m-1} = (b_0, b_1, \dots)$ for some $b_j \in \mathbb{C}_p^\flat$ then we have $\pi_L f_{m-1} = (0, b_0^{q_L}, b_1^{q_L}, \dots)$. This then immediately implies $f_{m-1} \in U_{q_L^{l_{m-1}},m} \cap A$. This means that we can do a decreasing induction for $m \geq i \geq 1$ and find for every such i elements $g_i \in \omega_\phi^{q_L^{l_i}} A^+$ and $f_{i-1} \in A$ such that

$$f_i = g_i + \pi_L f_{i-1}.$$

Putting all this together, we get

$$f_m = \sum_{i=1}^m \pi_L^{m-i} g_m + \pi_L^m f_0.$$

In particular we have

$$\sum_{i=1}^m \pi_L^{m-i} g_m \in \omega_\phi^{q_L^{l_1}} A^+ \subseteq \omega_\phi^m A^+.$$

Therefore we have $f_m \in \omega_\phi^m A^+ + \pi_L^m A$ which was exactly the statement we wanted to see to end the proof. $\square$

Corollary 3.2.15. *A fundamental system of open neighbourhoods of* 0 *for the weak topology on* $\mathbf{A}_{K|L}$ *(resp.* $\mathbf{A}_L^{\mathrm{nr}}$*) is given by the* $\mathbf{A}_{K|L}^+$*- (resp.* $\mathbf{A}_L^{\mathrm{nr},+}$*-) submodules*

$$\omega_\phi^m \mathbf{A}_{K|L}^+ + \pi_L^m \mathbf{A}_{K|L}, \, m \geq 1, \text{ respectively}$$
$$\omega_\phi^m \mathbf{A}_L^{\mathrm{nr},+} + \pi_L^m \mathbf{A}_L^{\mathrm{nr}}, \, m \geq 1.$$

Proof. This is an application of Proposition 3.2.14. $\square$

Proposition 3.2.16. *The weak topology on* $\mathbf{A}_{K|L}$ *coincides with the weak topology of* $\mathbf{A}_{K|L}$ *considered as* $\mathbf{A}_L$*-module.*

Proof. If $(u_i)_i$ is an $\mathbf{A}_L$-basis of $\mathbf{A}_{K|L}$, then $(\omega_\phi^k u_i)_i$ is so for all $k \geq 0$. Therefore $\mathbf{A}_{K|L}$ has an $\mathbf{A}_L$-basis consisting of elements of $\mathbf{A}_{K|L}^+$. The claim then follows from the above Corollary 3.2.15 together with Corollary 3.2.12. $\square$

Proposition 3.2.17. *The canonical inclusion* $\mathbf{A}_{K|L} \hookrightarrow \mathbf{A}$ *is a topological embedding. Furthermore, for every* $n \in \mathbb{N}$ *the induced inclusion* $\mathbf{A}_{K|L}/\pi_L^n \mathbf{A}_{K|L} \hookrightarrow \mathbf{A}/\pi_L^n \mathbf{A}$ *is a topological embedding as well.*

Proof. Because of

$$\mathbf{A}_{K|L} \cap \mathbf{A} = \mathbf{A}_{K|L} \cap \mathbf{A} \cap W(\mathcal{O}_{\mathbb{C}_p^\flat})_L = \mathbf{A}_{K|L} \cap W(\mathcal{O}_{\mathbb{C}_p^\flat})_L$$

the first part of the assertion follows from the definition of the weak topology. The second then follows from the commutative diagram

$$
\begin{array}{ccc}
\mathbf{A}_{K|L} & \hookrightarrow & \mathbf{A} \\
\downarrow & & \downarrow \\
\mathbf{A}_{K|L}/\pi_L^n \mathbf{A}_{K|L} & \hookrightarrow & \mathbf{A}/\pi_L^n \mathbf{A}.
\end{array}
$$

$\square$

Proposition 3.2.18. *The weak topology on* $\mathbf{A}$ *coincides with the topology of the projective limit* $\varprojlim_n \mathbf{A}_L^{\mathrm{nr}}/\pi_L^n \mathbf{A}_L^{\mathrm{nr}}$ *where each factor carries the quotient topology of the weak topology on* $\mathbf{A}_L^{\mathrm{nr}}$. *Moreover, a fundamental system of open neighbourhoods of* 0 *for the weak topology on* $\mathbf{A}$ *is given by the sets*

$$
\omega_\phi^m \mathbf{A}_L^{\mathrm{nr},+} + \pi_L^m \mathbf{A}, \ m \geq 1.
$$

Note that, by definition, $\mathbf{A}^+ = \mathbf{A}_L^{\mathrm{nr},+}$.

Proof. For this proof, we will refer to the latter topology of the Proposition's formulation as the *projective limit topology.*

As in the above Proposition 3.2.17 the inclusion $\mathbf{A}_L^{\mathrm{nr}} \hookrightarrow \mathbf{A}$ clearly is a topological embedding and since the diagram

$$
\begin{array}{ccc}
\mathbf{A}_L^{\mathrm{nr}} & \hookrightarrow & \mathbf{A} \\
\downarrow & & \downarrow \\
\mathbf{A}_L^{\mathrm{nr}}/\pi_L^n \mathbf{A}_L^{\mathrm{nr}} & = & \mathbf{A}/\pi_L^n \mathbf{A}.
\end{array}
$$

for every $n \in \mathbb{N}$ is commutative, the quotient topology on $\mathbf{A}_L^{\mathrm{nr}}/\pi_L^n \mathbf{A}_L^{\mathrm{nr}}$ with respect to the weak topology on $\mathbf{A}_L^{\mathrm{nr}}$ coincides with its quotient topology with respect to the weak topology on $\mathbf{A}$. Therefore the canonical projections

$$
\mathbf{A} = \varprojlim_n \mathbf{A}_L^{\mathrm{nr}}/\pi_L^n \mathbf{A}_L^{\mathrm{nr}} \longrightarrow \mathbf{A}_L^{\mathrm{nr}}/\pi_L^n \mathbf{A}_L^{\mathrm{nr}}
$$

are continuous for the weak topology on $\mathbf{A}$. This means that the weak topology of $\mathbf{A}$ is finer than its projective limit topology.

From Proposition 3.2.14 we deduce that a fundamental system of open neighbourhoods of 0 for the quotient topology of the weak topology on $\mathbf{A}_L^{\mathrm{nr}}/\pi_L^n \mathbf{A}_L^{\mathrm{nr}}$ is given by the sets

$$
\omega_\phi^m \mathbf{A}_L^{\mathrm{nr},+} + \pi_L^n \mathbf{A}_L^{\mathrm{nr}}, \ m \geq 1.
$$

Then the sets

$$
\omega_\phi^m \mathbf{A}_L^{\mathrm{nr},+} + \pi_L^n \mathbf{A}, \ m,n \geq 1
$$

form a fundamental system of open neighbourhoods of 0 for the projective limit topology on $\mathbf{A}$. But clearly the sets with $m = n$ define the same topology. Since the weak topology is defined by the sets

$$
\left(\omega_\phi^m W(\mathcal{O}_{\mathbb{C}_p^\flat})_L + \pi_L^m W(\mathbb{C}_p^\flat)_L \right) \cap \mathbf{A}, \ m \geq 1
$$

(cf. Remark 3.2.13) and we clearly have

$$
\omega_\phi^m \mathbf{A}_L^{\mathrm{nr},+} + \pi_L^m \mathbf{A} \subseteq \left(\omega_\phi^m W(\mathcal{O}_{\mathbb{C}_p^\flat})_L + \pi_L^m W(\mathbb{C}_p^\flat)_L \right) \cap \mathbf{A}
$$

for all $m \geq 1$, the projective limit topology is finer than the weak topology. $\square$

Lemma 3.2.19. *Let k be a finite field and $E|k((X))$ be a finite, separable extension. Then there exists a finite extension $\kappa|k$ and $Y \in E$ such that $E \cong \kappa((Y))$.*

Proof. This is [22, Lemma 1.38, p. 20]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 3.2.20. *Let $k'|k$ be an extension of finite fields and $k'((Y))|k((X))$ be a finite, separable extension. Then the Y-adic and the X-adic topologies on $k'((Y))$ coincide.*
In particular, there exists an $l \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ it holds

$$X^{ln}k'[\![Y]\!] \subseteq Y^{ln}k'[\![Y]\!] \subseteq X^n k'[\![Y]\!].$$

Proof. Since $k[\![X]\!]$ is a discrete valuation ring with respect to its X-adic topology and $k'[\![Y]\!]$ is so as well with respect to its Y-adic topology, we deduce from usual ramification theory, that there exists an $l \in \mathbb{N}$ such that

$$Y^l k'[\![Y]\!] = X k'[\![Y]\!].$$

Since $Y k'[\![Y]\!]$ is the maximal ideal of $k'[\![Y]\!]$ it clearly is $X k'[\![Y]\!] \subseteq Y k'[\![Y]\!]$ and therefore we get for all $n \in \mathbb{N}$

$$X^{ln}k'[\![Y]\!] \subseteq Y^{ln}k'[\![Y]\!] \subseteq X^n k'[\![Y]\!].$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 3.2.21. *Let $E|\mathbf{E}_L$ be a finite and separable extension. Then the subspace topology on E induced from the topology of $\mathbb{C}_p^\flat$ coincides with the extension from the ω-adic topology on $\mathbf{E}_L$. Note that the latter topology is the ω-adic topology on E, due to the above* Lemma 3.2.20.
In particular, the integral closure E^+ of $\mathbf{E}_L^+$ inside E consists of exactly those elements of E whose absolute value in $\mathbb{C}_p^\flat$ is less or equal to 1.

Proof. We denote the absolute value induced from $\mathbb{C}_p^\flat$ by $|\cdot|_\flat$ as in [30, Lemma 1.4.6, p. 44–45] and we use the identifications $E \cong \kappa((Y))$ as well as $\mathbf{E}_L \cong k_L((X))$ (cf. Lemma 3.2.19), where $\kappa|k_L$ is a finite extension.
The maximal unramified intermediate field of $\kappa((Y))|k_L((X))$ is $\kappa((X))$ and therefore it exists an $l \in \mathbb{N}$ and $g_i \in \kappa[\![X]\!]$ for $0 \le i < l$ with $X \mid g_i$ and $X^2 \nmid g_0$ such that (cf. [37, Chapter I, §6, Proposition 17, p. 19])

$$\sum_{i=0}^{l-1} g_i Y^i + Y^l = 0.$$

Since $|X|_\flat < 1$ and $|x|_\flat = 1$ for $x \in \kappa$ (in particular, every nonzero element coming from a finite field has absolute value 1 in $\mathcal{O}_{\mathbb{C}_p^\flat}$ with respect to $|\cdot|_\flat$) we have $|g_i|_\flat \le 1$ for all $0 \le i < l$ and we can deduce

$$|Y^l|_\flat \le \max_{0 \le i < l} |g_i|_\flat |Y^i|_\flat \le \max_{0 \le i < l} |Y^i|_\flat$$

and therefore $|Y|_\flat \le 1$. Furthermore, since we have $Y^l \kappa[\![Y]\!] = X \kappa[\![Y]\!]$ we can find a $g \in \kappa[\![Y]\!]$ such that $Y^l = Xg$ and since $|Y|_\flat \le 1$ we then deduce $|g|_\flat \le 1$ and

$$|Y^l|_\flat = |X|_\flat |g|_\flat \le |X|_\flat < 1.$$

But this then immediately implies

$$|Y|_\flat < 1.$$

Since $X \mid g_0$ and $X^2 \nmid g_0$ it is $|g_0|_\flat = |X|_\flat$ and because of $X \mid g_i$ for all $0 \le i < l$ we also have

$$|g_0|_\flat \ge |g_i|_\flat \quad \text{for all} \ \ 0 < i < l.$$

Since $|Y|_\flat < 1$ we deduce from the above

$$|g_0|_\flat > |g_i|_\flat |Y^i|_\flat \quad \text{for all } \ 0 < i < l$$

and therefore

$$|Y^l|_\flat = |g_0|_\flat = |X|_\flat$$

because $|\cdot|_\flat$ is a nonarchimedean absolute value.

Denote by $|\cdot|$ the extension of the absolute value of $\mathbf{E}_L$ (which corresponds with the ω-adic topology) to E. Then we deduce from [37, Chapter 2, §2, Corollary 4, p. 29] that

$$|Y^l| = |\text{Nor}(Y)|_\flat,$$

where Nor denotes the norm of the extension $\kappa((Y))|\kappa((X))$. From the polynomial we started with we then can deduce $\text{Nor}(Y) = g_0$ and therefore

$$|Y^l| = |g_0|_\flat = |X|_\flat.$$

This means that $|\cdot|$ and $|\cdot|_\flat$ coincide on E.

From the identification above we deduce $E^+ = \kappa[\![Y]\!]$. But since $|Y|_\flat < 1$, these are exactly the elements of E whose absolute value is less or equal to 1. $\qquad\square$

Corollary 3.2.22. $\mathbf{E}_{K|L}$ *is, with respect to the topology induced from $\mathbb{C}_p^\flat$, a complete, nonarchimedean discretely valued field of characteristic p with residue class field k_K and ring of integers $\mathbf{E}_{K|L}^+$.*

Lemma 3.2.23. *Let X be a topological space and $(Y_n)_n$ a family of subsets of X with $Y_n \subseteq Y_{n+1}$. Set $Y := \varinjlim_n Y_n = \bigcup_n Y_n$. Then, the subset topology on Y coincides with the final topology of the inductive limit with respect to the subset topologies on the Y_n.*

Proof. First we show that the canonical injections $f_n \colon Y_n \hookrightarrow Y$ are continuous for the subset topology on Y. This then implies that the subspace topology on Y is coarser than the projective limit topology since the latter is the finest topology such that all injections f_n are continuous (cf. [7, Chapter I, §2.4, Proposition 6, p. 32]).

Let $U \subseteq Y$ be open and $V \subseteq X$ open such that $U = V \cap Y$. Then it is

$$f_n^{-1}(U) = U \cap Y_n = V \cap V \cap Y_n = V \cap Y_n,$$

i.e. $f_n^{-1}(U) \subseteq Y_n$ is open.

It is left to show, that the subspace topology is finer then the direct limit topology. For this, let $U \subseteq Y$ be open with respect to the direct limit topology, i.e. it is $U = \bigcup_n f_n^{-1}(U)$, where for every $n \in \mathbb{N}$ it exists an open $V_n \subseteq X$ such that $f^{-1}(U) = V_n \cap Y_n$. We set $V := \bigcup V_n$ and claim $U = V \cap Y$. To see this, let $u \in U$. Then it exists $n \in \mathbb{N}$ such that $u \in V_n \cap Y_n$ and in particular $u \in V$. Conversely let $u \in V \cap Y$. Then, by definition, there exist $n_1, n_2 \in \mathbb{N}$ such that $u \in V_{n_1}$ and $u \in Y_{n_2}$. For $n := \max\{n_1, n_2\}$ we then deduce $u \in V_n \cap Y_n$ and therefore $u \in U$. $\qquad\square$

Proposition 3.2.24. *The integral closure $\mathbf{E}_L^{\text{sep},+}$ of $\mathbf{E}_L^+$ inside $\mathbf{E}_L^{\text{sep}}$ consists of exactly those elements with absolute value $|\cdot|_\flat$ less or equal to 1.*

Furthermore, the topology on $\mathbf{E}_L^{\text{sep}}$ induced from $\mathbb{C}_p^\flat$ coincides with the final topology with respect to the colimit

$$\mathbf{E}_L^{\text{sep}} = \bigcup_{\substack{E|\mathbf{E}_L \\ \text{fin, sep}}} E$$

where each E carries the topology induced from $\mathbb{C}_p^\flat$.

In particular, the $\mathbf{E}_L^{\mathrm{sep},+}$-submodules

$$\omega^n \mathbf{E}_L^{\mathrm{sep},+}$$

form a fundamental system of open neighbourhoods of 0 for this topology on $\mathbf{E}_L^{\mathrm{sep}}$.

Proof. This now is an immediate consequence of Lemma 3.2.21 and Lemma 3.2.23. $\qquad\square$

3.2.2. *Structure of Coefficient Rings (unramified case).* For this subsection, let $K|L$ be an unramified extension. Then this a Galois extension and its Galois group is isomorphic to the Galois group of the respective residue class fields. It therefore is cyclic and generated by the lift of the q_L-Frobenius $x \mapsto x^{q_L}$. We will denote this lift by $\sigma_{K|L}$ and call it Frobenius on K. Recall also from Remark 3.1.2 that the groups Γ_L and Γ_K are isomorphic and for every $n \in \mathbb{N}$ the groups $\Gamma_{L_n|L}$ and $\Gamma_{K_n|K}$ are isomorphic as well.

Remark 3.2.25. *We have $(H_L : H_K) = [K : L]$.*

Proof. Since $\Gamma_L \cong \Gamma_K$ (cf. Remark 3.1.2) we have $(H_L : H_K) = [K_\infty : L_\infty] = [K : L]$. $\qquad\square$

Lemma 3.2.26. *We have $k_K \mathbf{E}_L = \mathbf{E}_{K|L}$.*

Proof. Since k_K is fixed by H_K it clearly is $k_K \mathbf{E}_L \subseteq \mathbf{E}_{K|L}$. Since $K|L$ is unramified we have $[K : L] = [k_K : k_L]$ and therefore

$$[k_K \mathbf{E}_L : \mathbf{E}_L] = [k_K : k_K \cap \mathbf{E}_L] = [k_K : k_L] = [K : L] = (H_L : H_K) = [\mathbf{E}_{K|L} : \mathbf{E}_L].$$

$\square$

Lemma 3.2.27. *We have $\mathbf{A}_{K|L} = \mathcal{O}_K \otimes_{\mathcal{O}_L} \mathbf{A}_L$ and $\mathbf{B}_{K|L} = K\mathbf{B}_L$.*

Proof. Since $K|L$ is unramified $\mathcal{O}_K \otimes_{\mathcal{O}_L} \mathbf{A}_L$ is unramified over $\mathbf{A}_L$ and since K is fixed by H_K we deduce $\mathcal{O}_K \otimes_{\mathcal{O}_L} \mathbf{A}_L \subseteq \mathbf{A}_{K|L}$. Since both are free $\mathbf{A}_L$-modules of rank $[K : L] = (H_L : H_K)$ they coincide.

The statement for the fields of fractions then follows immediately. $\qquad\square$

In order to understand how the operations of Γ_K and the Frobenius look on $\mathbf{A}_{K|L}$ respectively $\mathbf{B}_{K|L}$ it now suffices to understand the corresponding operations on $\mathcal{O}_K$ respectively K. Note, that since $K|L$ is unramified, we clearly have $W(k_K)_L = \mathcal{O}_K$.

Lemma 3.2.28. *Let Fr denote the (restriction of the) q_L-Frobenius on k_K. Then the automorphism $\sigma_{K|L}$ on $\mathcal{O}_K$ coincides with the restriction of $W(\mathrm{Fr})_L$.*

Proof. Due to the functoriality of the Witt construction, $W(\mathrm{Fr})_L$ is an automorphism on $\mathcal{O}_K$ which fixes $\mathcal{O}_L$, it induces also an automorphism on K which fixes L and it's reduction modulo π_L is Fr. The first observation says, that the restriction of $W(\mathrm{Fr})_L$ is an element of $\mathrm{Gal}(K|L)$ and since $\mathrm{Gal}(K|L)$ and $\mathrm{Gal}(k_K|k_L)$ are isomorphic via $\sigma \mapsto \sigma \bmod \pi_L$, the second observation says that the restriction of $W(\mathrm{Fr})_L$ is a lift of Fr. Since this lift is unique we get the desired equality $W(\mathrm{Fr})_L = \sigma_{K|L}$ on K respectively $\mathcal{O}_K$. $\qquad\square$

Before we give explicit descriptions of the operations on $\mathbf{A}_{K|L}$ we want to fix some notation.

Definition 3.2.29. Let ϑ be an $\mathcal{O}_L$-linear endomorphism of $\mathcal{O}_K$ and $f \in \mathbf{A}_{K|L}$ we denote by f^ϑ the element, on which ϑ is applied to the coefficients of f, i.e. if $f(\omega_\phi) = \sum a_i \omega_\phi^i$ then

$$f^\vartheta(\omega_\phi) = \sum_{i \in \mathbb{N}_0} \vartheta(a_i) \omega_\phi^i.$$

Proposition 3.2.30. *Let $f = f(\omega_\phi) = \sum a_i \omega_\phi^i \in \mathbf{A}_{K|L}$ and $\gamma \in \Gamma_K$. We then have*

$$\gamma \cdot f = \sum_{i \in \mathbb{Z}} a_i [\chi_{\mathrm{LT}}(\gamma)]_\phi(\omega_\phi^i).$$

For the Frobenius $\varphi_{K|L}$ we have

$$\varphi_{K|L}(f) = \sum_{i \in \mathbb{Z}} \sigma_{K|L}(a_i) [\pi_L]_\phi(\omega_\phi^i).$$

Together with the above Definition 3.2.29, *we then have the description*

$$\varphi_{K|L}(f(\omega_\phi)) = f^{\sigma_{K|L}}(\varphi_{K|L}(\omega_\phi)).$$

Proof. This is an immediate consequence of Remark 3.1.2, Lemma 3.2.27 and Lemma 3.2.28. $\square$

3.2.3. *Structure of Coefficient Rings (general case).*

Proposition 3.2.31. *Let $B|\mathbf{B}_L$ be a finite, unramified extension and $A \subseteq B$ be the integral closure of $\mathbf{A}_L$. Then there exists a finite, unramified extension $E|L$ and an element $\nu_\phi \in W(\mathbf{E}_L^{\mathrm{sep}})_L$ with $\nu_\phi^j = \omega_\phi$ for some $j > 0$ such that*

$$A \cong \varprojlim_n \mathcal{O}_E / \pi_L^n \mathcal{O}_E((\nu_\phi)).$$

Proof. Let κ be the residue class field of A and recall that the residue class field of $\mathbf{A}_L$ is $\mathbf{E}_L = k_L((\omega))$. Since $B|\mathbf{B}_L$ is unramified, we then have

$$[B : \mathbf{B}_L] = [\kappa : k_L((\omega))].$$

Since $\kappa|k_L((\omega))$ is finite and separable ($B|\mathbf{B}_L$ is unramified), we deduce from Lemma 3.2.19 that $\kappa \cong k((\nu))$ for some finite extension $k|k_L$ and $\nu \in \mathbf{E}_L^{\mathrm{sep}}$ with $\nu^j = \omega$ for some $j > 0$. But then there exists a unique finite and unramified extension $E|L$ with $k_E = k$. In particular, we have $j[E : L] = [B : \mathbf{B}_L]$. Furthermore, since $\mathbf{B}_L$ is a complete discrete valuation field, and $B|\mathbf{B}_L$ is a finite extension, the henselian lemma in the sense of [27, II §4, (4.6) Henselsches Lemma, p. 135–136] holds true and therefore we can find a $\nu_\phi \in B \subseteq W(\mathbf{E}_L^{\mathrm{sep}})_L$ which is a root of the polynomial $X^j - \omega_\phi$ and for which we have

$$\nu_\phi \bmod \pi_L = \nu.$$

Since $X^j - \omega_\phi$ is irreducible over $\mathbf{B}_L$ and $E|L$ is unramified, we deduce

$$\begin{aligned}
[E\mathbf{B}_L(\nu_\phi) : \mathbf{B}_L] &= [E\mathbf{B}_L(\nu_\phi) : \mathbf{B}_L(\nu_\phi)] \cdot [\mathbf{B}_L(\nu_\phi) : \mathbf{B}_L] \\
&= [E : E \cap \mathbf{B}_L(\nu_\phi)] \cdot j \\
&= [E : L] \cdot j
\end{aligned}$$

and therefore

$$B = E\mathbf{B}_L(\nu_\phi).$$

In particular, we have

$$B = \left\{ \sum_{i \in \mathbb{Z}} a_i \nu_\phi^i \,\middle|\, a_i \in E, \ \lim_{i \to -\infty} a_i = 0 \text{ and it exists } n \in \mathbb{N} \right.$$

$$\left. \text{such that } \pi_L^n a_i \in \mathcal{O}_E \text{ for all } i \in \mathbb{Z} \right\}$$

Then A consists of those elements of B with π_L-adic absolute value ≤ 1. Since this absolute value is nonarchimedean, these are exactly those elements $\sum_{i \in \mathbb{Z}} a_i \nu_\phi^i \in B$ with $a_i \in \mathcal{O}_E$ for all $i \in \mathbb{Z}$, i.e. $A \cong \varprojlim_n \mathcal{O}_E / \pi_L^n \mathcal{O}_E((\nu_\phi))$. $\qquad\square$

3.3. $(\varphi_{K|L}, \Gamma_K)$-**modules and Galois representations.** If not otherwise stated, all continuity statements refer to the corresponding weak topology.

Definition 3.3.1. Let M be an $\mathbf{A}_{K|L}$-module. We regard M as a left-$\mathbf{A}_{K|L}$-module and $\mathbf{A}_{K|L}$ itself as a right-$\mathbf{A}_{K|L}$-module via $\varphi_{K|L}$. For the tensor product in this situation we write $\mathbf{A}_{K|L}\ _{\varphi_{K|L}}\otimes_{\mathbf{A}_{K|L}} M$, which is per definition an abelian group, but since $\mathbf{A}_{K|L}$ is also a left- $\mathbf{A}_{K|L}$-module (with the standard multiplication) this tensor product is also a (left)-$\mathbf{A}_{K|L}$-module.

Definition 3.3.2. Let M be a finitely generated $\mathbf{A}_{K|L}$-module equipped with a $\varphi_{K|L}$-linear endomorphism φ_M. Then φ_M^{lin} denotes the homomorphism

$$\varphi_M^{\mathrm{lin}}\colon \mathbf{A}_{K|L}\ _{\varphi_{K|L}}\otimes_{\mathbf{A}_{K|L}} M \longrightarrow M$$
$$f \otimes m \longmapsto f\varphi_M(m).$$

Definition 3.3.3. A finitely generated $\mathbf{A}_{K|L}$-module M is called $(\varphi_{K|L}, \Gamma_K)$-**module** if it is equipped with a $\varphi_{K|L}$-linear endomorphism φ_M and a continuous, semilinear action from Γ_K, which commutes with the endomorphism φ_M. A $(\varphi_{K|L}, \Gamma_K)$-module is called **étale** if the homomorphism φ_M^{lin} is bijective.
A morphism of $(\varphi_{K|L}, \Gamma_K)$-modules $f\colon M \to N$ is an $\mathbf{A}_{K|L}$-module homomorphism, which respects the actions from Γ_K and the endomorphisms φ_M and φ_N. We will denote the category of étale $(\varphi_{K|L}, \Gamma_K)$-modules by $\mathbf{Mod}_{\varphi,\Gamma}^{\mathrm{\acute{e}t}}(\mathbf{A}_{K|L})$.

Theorem 3.3.4. *The exact tensor categories* $\mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg})}(G_K)$ *and* $\mathbf{Mod}_{\varphi,\Gamma}^{\mathrm{\acute{e}t}}(\mathbf{A}_{K|L})$ *are equivalent to each other. The equivalence is given by the quasi inverse functors*

$$\mathcal{D}_{K|L}\colon \mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg})}(G_K) \longrightarrow \mathbf{Mod}_{\varphi,\Gamma}^{\mathrm{\acute{e}t}}(\mathbf{A}_{K|L})$$
$$V \longmapsto (\mathbf{A} \otimes_{\mathcal{O}_L} V)^{H_K}$$

and

$$\mathcal{V}_{K|L}\colon \mathbf{Mod}_{\varphi,\Gamma}^{\mathrm{\acute{e}t}}(\mathbf{A}_{K|L}) \longrightarrow \mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg})}(G_K)$$
$$M \longmapsto \left(\mathbf{A} \otimes_{\mathbf{A}_{K|L}} M\right)^{\mathrm{Fr}\otimes\varphi_M=1}.$$

4. Iwasawa Cohomology

In this section we want to list the results from [31, Section 5, p. 23–31] which we will need later on. Note that [31, Remark 5.1, p. 23] was also proven here (cf. Lemma 5.1.1).

Definition 4.0.1. Let M be a topological $\mathcal{O}_L$-module. The **Pontrjagin dual** of M is defined as

$$M^\vee := \mathrm{Hom}_{\mathcal{O}_L}^{\mathrm{cts}}(M, L/\mathcal{O}_L) \cong \mathrm{Hom}_{\mathcal{O}_K}^{\mathrm{cts}}(M, K/\mathcal{O}_K).$$

It is always equipped with the compact-open topology.

Proposition 4.0.2 (Pontrjagin duality)**.** *The functor $-^\vee$ defines an involuntary contravariant autoequivalence of the category of (Hausdorff) locally compact linear-topological $\mathcal{O}_L$-modules. In particular, for such a module M there is a canonical isomorphism*

$$M \xrightarrow{\cong} (M^\vee)^\vee.$$

Proof. This is explained in [31, Proposition 5.4, p. 25–26]. □

Remark 4.0.3. *Let $M_0 \xrightarrow{\alpha} M \xrightarrow{\beta} M_1$ be a sequence of locally compact linear-topological $\mathcal{O}_K$-modules such that $\mathrm{im}(\alpha) = \ker(\beta)$ and β is topologically strict with closed image. Then the dual sequence*

$$M_1^\vee \xrightarrow{\beta^\vee} M^\vee \xrightarrow{\alpha^\vee} M_0^\vee$$

is exact as well.

Proof. The proof is similar to the one of [31, Remark 5.5, p. 27]. □

Remark 4.0.4. *Let $V \in \mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg})}(G_K)$ of finite length and $n \geq 1$ such that $\pi_L^n V = 0$. Then there is a natural isomorphism of topological groups:*

$$\mathcal{D}_{K|L}(V)^\vee \cong \mathcal{D}_{K|L}(V^\vee(\chi_{\mathrm{LT}})).$$

This isomorphism identifies $\psi_{\mathcal{D}_{K|L}(V^\vee(\chi_{\mathrm{LT}}))}$ with $\varphi_{\mathcal{D}_{K|L}(V)}^\vee$.

Proof. This is [31, Remark 5.6, p. 27] □

Proposition 4.0.5 (Local Tate duality)**.** *Let $V \in \mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg})}(G_K)$, $n \geq 1$ such that $\pi_L^n V = 0$ and E a finite extension of K. Then the cup product and the local invariant map induce perfect pairings of finite $\mathcal{O}_L$-modules*

$$H^i(G_E, V) \times H^{2-i}(G_E, \mathrm{Hom}_{\mathbb{Z}_p}(V, \mathbb{Q}_p/\mathbb{Z}_p(1))) \to H^2(G_E, \mathbb{Q}_p/\mathbb{Z}_p(1)) = \mathbb{Q}_p/\mathbb{Z}_p$$

and

$$H^i(G_E, V) \times H^{2-i}(G_E, \mathrm{Hom}_{\mathcal{O}_L}(V, L/\mathcal{O}_L(1))) \to H^2(G_E, L/\mathcal{O}_L(1)) = L/\mathcal{O}_L.$$

There $-(1)$ denotes the twist by the cyclotomic character.
This means that there are conical isomorphisms

$$H^i(G_E, V) \cong H^{2-i}(K, V^\vee(1))^\vee.$$

Proof. This is [31, Proposition 5.7, p. 27–28], where [36, Theorem 2, p. 91–92] is applied. □

Definition 4.0.6. *Let $V \in \mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg})}(G_K)$.* The **generalized Iwasawa cohomology of** V is defined by

$$H_{\mathrm{Iw}}^i(K_\infty|K, V) \coloneqq \varprojlim_{K \subseteq E \subseteq K_\infty} H^i(G_E, V).$$

We always consider these modules as Γ_K-modules.

Remark 4.0.7. *Let $E|K$ be a finite extension contained in K_∞. Then there is an isomorphism of $\mathcal{O}_L$-modules:*

$$\varprojlim_{E \subseteq E' \subseteq K_\infty} H^i(G_{E'}, V) \cong H_{\mathrm{Iw}}^i(K_\infty|K, V).$$

Proof. The claim follows immediately from the fact, that the set $\{E'|E \text{ finite} \mid E' \subseteq K_\infty\}$ is cofinal in the set $\{E'|K \text{ finite} \mid E' \subseteq K_\infty\}$. □

Lemma 4.0.8. *Let $V \in \mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg})}(G_K)$. Then we have*

$$H_{\mathrm{Iw}}^i(K_\infty|K, V) \cong H^i(G_K, \mathcal{O}_L[\![\Gamma_K]\!] \otimes_{\mathcal{O}_L} V).$$

Proof. The proof is similar to the one of [31, Lemma 5.8, p. 28–29].   □

Lemma 4.0.9. $V \mapsto H_{\mathrm{Iw}}(K_\infty|K, V)$ *defines a δ-functor on $\mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg})}(G_K)$.*

Proof. Replace Γ_L by Γ_K in the proof of [31, Lemma 5.9, p. 29].   □

Remark 4.0.10. *Let $V, V_0 \in \mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg})}(G_K)$ such that V_0 is $\mathcal{O}_L$-free and G_K acts through its factor Γ_K on V_0. Then there is a natural isomorphism*

$$H_{\mathrm{Iw}}^i(K_\infty|K, V \otimes_{\mathcal{O}_L} V_0) \cong H_{\mathrm{Iw}}^i(K_\infty|K, V) \otimes_{\mathcal{O}_L} V_0.$$

Remark 4.0.11. *Let $V \in \mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg})}(G_K)$ be of finite length. Then there is an isomorphism*

$$H_{\mathrm{Iw}}^i(K_\infty|K, V) \cong H^i(H_K, V^\vee(1))^\vee.$$

Note that $H_K = G_{K_\infty}$.

Proof. From Proposition 4.0.5 we deduce

$$H^i(G_{K_n}, V) \cong H^{2-i}(G_{K_n}, V^\vee(1))^\vee$$

for every $n \in \mathbb{N}$. Taking projective limits gives us

$$\begin{aligned}
H_{\mathrm{Iw}}^i(K_\infty|K, V) &= \varprojlim H^i(G_{K_n}, V) \\
&= \varprojlim H^{2-i}(G_{K_n}, V^\vee(1))^\vee \\
&= \varprojlim \mathrm{Hom}_{\mathcal{O}_L}^{\mathrm{cts}}(H^{2-i}(G_{K_n}, V^\vee(1)), L/\mathcal{O}_L) \\
&= \mathrm{Hom}_{\mathcal{O}_L}^{\mathrm{cts}}(\varinjlim H^{2-i}(G_{K_n}, V^\vee(1)), L/\mathcal{O}_L) \\
&= \mathrm{Hom}_{\mathcal{O}_L}^{\mathrm{cts}}(H^{2-i}(\varprojlim G_{K_n}, V^\vee(1)), L/\mathcal{O}_L) \\
&= \mathrm{Hom}_{\mathcal{O}_L}^{\mathrm{cts}}(H^{2-i}(H_K, V^\vee(1)), L/\mathcal{O}_L) \\
&= H^{2-i}(H_K, V^\vee(1))^\vee.
\end{aligned}$$

□

Lemma 4.0.12.
  1. $H_{\mathrm{Iw}}^i(K_\infty|K, V) = 0$ *for $i \neq 1, 2$.*
  2. $H_{\mathrm{Iw}}^2(K_\infty|K, V)$ *is finitely generated as $\mathcal{O}_L$-module.*
  3. $H_{\mathrm{Iw}}^1(K_\infty|K, V)$ *is finitely generated as $\mathcal{O}_L[\![\Gamma_K]\!]$-module.*

Proof. The proof is similar to the one of [31, Lemma 5.12, p. 29–30].   □

Theorem 4.0.13 ([31, Theorem 5.13]). *Let V be in $\mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg})}(G_K)$, $\tau = \chi_{\mathrm{cyc}}\chi_{\mathrm{LT}}^{-1}$ and $\psi = \psi_{\mathcal{D}_{K|L}(V(\tau^{-1}))}$. Then we have an exact sequence*

$$0 \longrightarrow H_{\mathrm{Iw}}^1(K_\infty|K, V) \longrightarrow \mathcal{D}_{K|L}(V(\tau^{-1})) \xrightarrow{\psi-1} \mathcal{D}_{K|L}(V(\tau^{-1})) \longrightarrow H_{\mathrm{Iw}}^2(K_\infty|K, V) \longrightarrow 0,$$

which is functorial in V. Furthermore, each occurring map is continuous and $\mathcal{O}_L[\![\Gamma_K]\!]$-equivariant.

Remark 4.0.14. *A version in the derived category is shown in Proposition 5.2.51 and, unfortunately, is not a direct consequence of this theorem, of course.*

5. GALOIS COHOMOLOGY IN TERMS OF LUBIN-TATE (φ, Γ)-MODULES

We keep the notation from Section 3. Recall from Theorem 3.2 (resp. from [30, p. 113-114]) that $\mathbf{E}_L^{\mathrm{sep}}$ is the residue class field of $\mathbf{A}$ and $\mathbf{E}_L^{\mathrm{sep},+}$ is the residue class field of $\mathbf{A}^+$.

5.1. **Description with φ.** The goal of this subsection is to compute Galois cohomology from the generalized φ-Herr complex, which is related to $\varphi_{K|L}$ and Γ_K.

Lemma 5.1.1.

1. *The following sequences are exact:*

$$0 \longrightarrow \mathcal{O}_L \longrightarrow \mathbf{A} \xrightarrow{\mathrm{Fr}-\mathrm{id}} \mathbf{A} \longrightarrow 0.$$

$$0 \longrightarrow \mathcal{O}_L \longrightarrow \mathbf{A}^+ \xrightarrow{\mathrm{Fr}-\mathrm{id}} \mathbf{A}^+ \longrightarrow 0.$$

2. *Let $E \mid L$ be a finite extension. For every $n \in \mathbb{N}$ the maps*

$$\varphi_{E|L} - \mathrm{id} \colon \omega_\phi^n \mathbf{E}_E^+ \longrightarrow \omega_\phi^n \mathbf{E}_E^+,$$

$$\mathrm{Fr} - \mathrm{id} \colon \omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+} \longrightarrow \omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+}$$

   are isomorphisms.
3. *For every $n \in \mathbb{N}$ the map*

$$\mathrm{Fr} - \mathrm{id} \colon \omega_\phi^n \mathbf{A}^+ \longrightarrow \omega_\phi^n \mathbf{A}^+$$

   is an isomorphism.

Proof.

1. We start with the sequence

$$0 \longrightarrow k_L \longrightarrow \mathbf{E}_L^{\mathrm{sep}} \xrightarrow{x \mapsto x^{q_L}-x} \mathbf{E}_L^{\mathrm{sep}} \longrightarrow 0,$$

   and claim that it is exact. Recall that $\mathrm{Fr}(x) \equiv x^{q_L} \bmod \pi_L$ holds for all $x \in \mathbf{A}$ by definition. The inclusion $\mathcal{O}_L \hookrightarrow \mathbf{A}$ induces the inclusion $k_L \hookrightarrow \mathbf{E}_L^{\mathrm{sep}}$ and we have

$$\ker(\mathrm{Fr} - \mathrm{id}) = \{x \in \mathbf{E}_L^{\mathrm{sep}} \mid x^{q_L} - x\} = k_L.$$

   It remains to check that $\mathrm{Fr} - \mathrm{id}$ is surjective on $\mathbf{E}_L^{\mathrm{sep}}$. But since the polynomial $X^{q_L} - X - \alpha$ is separable for every $\alpha \in \mathbf{E}_L^{\mathrm{sep}}$ and $\mathbf{E}_L^{\mathrm{sep}}$ is separably closed by definition this follows immediately.
   Now suppose that the sequence

$$0 \longrightarrow \mathcal{O}_L/\pi_L^n\mathcal{O}_L \longrightarrow \mathbf{A}/\pi_L^n\mathbf{A} \xrightarrow{\varphi_L-\mathrm{id}} \mathbf{A}/\pi_L^n\mathbf{A} \longrightarrow 0$$

   is exact for $n \geq 1$ and consider the following commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_L/\pi_L^n\mathcal{O}_L & \longrightarrow & \mathbf{A}/\pi_L^n\mathbf{A} & \xrightarrow{\varphi_L-\mathrm{id}} & \mathbf{A}/\pi_L^n\mathbf{A} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_L/\pi_L^{n+1}\mathcal{O}_L & \longrightarrow & \mathbf{A}/\pi_L^{n+1}\mathbf{A} & \xrightarrow{\varphi_L-\mathrm{id}} & \mathbf{A}/\pi_L^{n+1}\mathbf{A} & \longrightarrow & 0.
\end{array}
$$

Our aim is to show that the second sequence is exact. The kernel of the homomorphism $\mathcal{O}_L \hookrightarrow \mathbf{A} \twoheadrightarrow \mathbf{A}/\pi_L^{n+1}\mathbf{A}$ is $\pi_L^{n+1}\mathcal{O}_L$, i.e. we have exactness at the first position. Since we have $\varphi_L(x) = x$ for all $x \in \mathcal{O}_L$, we also have $\mathcal{O}_L/\pi_L^{n+1}\mathcal{O}_L \subseteq \ker(\mathrm{Fr} - \mathrm{id})$. So let $x \in \mathbf{A}$ such that $\mathrm{Fr}(x) - x \equiv 0 \bmod \pi_L^{n+1}\mathbf{A}$. Then we also have $\mathrm{Fr}(x) - x \equiv 0 \bmod \pi_L^n\mathbf{A}$ and because the first sequence is exact, we obtain a $y \in \mathcal{O}_L$ such that $y \equiv x \bmod \pi_L^n\mathbf{A}$. Then there is an $\alpha \in \mathbf{A}$ such that $x - y = \pi_L^n\alpha$, especially we have $x - y \equiv \pi_L^n\alpha \bmod \pi_L^{n+1}\mathbf{A}$. Since $\mathrm{Fr}(X) \equiv X^{q_L} \bmod \pi_L$ we get $\mathrm{Fr}(\alpha) \equiv \alpha^{q_L} \bmod \pi_L\mathbf{A}$ and therefore $\mathrm{Fr}(\pi_L^n\alpha) \equiv \pi_L^n\alpha^{q_L} \bmod \pi_L^{n+1}\mathbf{A}$ since Fr is $\mathcal{O}_L$-linear. Then we also get

$$0 \equiv (\mathrm{Fr} - \mathrm{id})(x - y) \equiv (\mathrm{Fr} - \mathrm{id})(\pi_L^n\alpha) \equiv \pi_L^n(\alpha^{q_L} - \alpha) \bmod \pi_L^{n+1}\mathbf{A}.$$

Since $\mathbf{A}$ is a domain, this then implies $\alpha^{q_L} \equiv \alpha \bmod \pi_L\mathbf{A}$. Since the sequence in question is exact for $n = 1$ by the start of the proof, we then get a $z \in \mathcal{O}_L$ such that $z \equiv \alpha \bmod \pi_L\mathbf{A}$, i.e. it exists $\beta \in \mathbf{A}$ such that $\alpha = z + \pi_L\beta$. We then get

$$x \equiv y + \pi_L^n\alpha \equiv y + \pi_L^n(z + \pi_L\beta) \equiv y + \pi_L^n z \bmod \pi_L^{n+1}\mathbf{A},$$

i.e. $\ker(\mathrm{Fr} - \mathrm{id}) \subseteq \mathcal{O}_L/\pi_L^{n+1}\mathcal{O}_L$.

It remains to check that $\mathrm{Fr} - \mathrm{id}$ is surjective on $\mathbf{A}/\pi_L^{n+1}\mathbf{A}$. So let $x \in \mathbf{A}$. Because $\mathrm{Fr} - \mathrm{id}$ is surjective on $\mathbf{A}/\pi_L^n\mathbf{A}$ we get a $y \in \mathbf{A}$ such that $\varphi_L(y) - y \equiv x \bmod \pi_L^n\mathbf{A}$. As before there is now an $\alpha \in \mathbf{A}$ such that $\varphi_L(y) - y \equiv x + \pi_L^n\alpha \bmod \pi_L^{n+1}\mathbf{A}$. Again, since the sequence for $n = 1$ is exact we can find $z \in \mathbf{A}$ such that $\varphi_L(z) - z \equiv \alpha \bmod \pi_L\mathbf{A}$ and therefore we can find $\beta \in \mathbf{A}$ such that $\varphi_L(z) - z + \pi_L\beta = \alpha$. We then get

$$\mathrm{Fr}(y - \pi_L^n z) - (y - \pi_L^n z) = \mathrm{Fr}(y) - y - \pi_L^n(\mathrm{Fr}(z) - z)$$
$$\equiv x + \pi_L^n\alpha - \pi_L^n\alpha + \pi_L^{n+1}\beta \equiv x \bmod \pi_L^{n+1}\mathbf{A},$$

i.e. $y - \pi_L^n z$ is $\bmod \pi_L^{n+1}\mathbf{A}$ a preimage of x under $\varphi_L - \mathrm{id}$.

Since the transition maps $\mathcal{O}_L/\pi_L^{n+1}\mathcal{O}_L \to \mathcal{O}_L/\pi_L^n\mathcal{O}_L$ are surjective, the inverse system $(\mathcal{O}_L/\pi_L^n\mathcal{O}_L)_n$ is a Mittag-Leffler System and therefore we have $\varprojlim^1 \mathcal{O}_L/\pi_L^n\mathcal{O}_L = 0$ (cf. Remark 2.3.9). By taking the inverse limit of the sequence

$$0 \longrightarrow \mathcal{O}_L/\pi_L^n\mathcal{O}_L \longrightarrow \mathbf{A}/\pi_L^n\mathbf{A} \xrightarrow{\mathrm{Fr}-\mathrm{id}} \mathbf{A}/\pi_L^n\mathbf{A} \longrightarrow 0$$

we then get the exact sequence

$$0 \longrightarrow \mathcal{O}_L \longrightarrow \mathbf{A} \xrightarrow{\mathrm{Fr}-\mathrm{id}} \mathbf{A} \longrightarrow 0.$$

The proof of the exactness of the second sequence is similar to the prove above. Just replace $\mathbf{E}_L^{\mathrm{sep}}$ by $\mathbf{E}_L^{\mathrm{sep},+}$ which is the separable closure of $\mathbf{E}_L^+$ in $\mathbf{E}_L^{\mathrm{sep}}$.

2. As before we have $\mathrm{Fr}(x) = x^{q_L}$ for all $x \in \mathbf{E}_L^{\mathrm{sep}}$. Especially this equation holds for elements in $\mathbf{E}_L^{\mathrm{sep},+}$ and $\mathbf{E}_E^+$. The injectivity of the above maps then is easy to see: Let $0 \neq x \in \omega_\phi^n\mathbf{E}_L^{\mathrm{sep},+}$. So, in particular we have $\deg_{\omega_\phi}(x) \geq n > 0$ and therefore also $\deg_{\omega_\phi}(\mathrm{Fr}(x)) > \deg_{\omega_\phi}(x)$, i.e. $\mathrm{Fr}(x) - x \neq 0$ and so $\mathrm{Fr} - \mathrm{id}$ is injective on $\omega_\phi^n\mathbf{E}^+$. Because of $\mathbf{E}_E^+ \subseteq \mathbf{E}_L^{\mathrm{sep}}$ the homomorphism $\varphi_{E|L} - \mathrm{id}$ is also injective on $\omega_\phi^n\mathbf{E}_E^+$.

For the surjectivity let α be an element of $\omega_\phi^n\mathbf{E}_L^{\mathrm{sep},+}$ or of $\omega_\phi^n\mathbf{E}_E^+$. Then the series

$(\mathrm{Fr}(\alpha)^i)_i$ converges to zero and therefore

$$\beta := \sum_{i=0}^{\infty} -\mathrm{Fr}(\alpha)^i$$

is also an element of $\omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+}$ or of $\omega_\phi^n \mathbf{E}_E^+$ and clearly is a preimage of α under $\mathrm{Fr} - \mathrm{id}$.

3. Let $n, l \in \mathbb{N}$ be fixed and note that there is a canonical identification

$$(\omega_\phi^n \mathbf{A}^+)/(\pi_L^l \omega_\phi^n \mathbf{A}^+) \cong \omega_\phi^n(\mathbf{A}^+/\pi_L^l \mathbf{A}^+)$$

since ω_ϕ^n is not a zero divisor in both $\mathbf{A}^+$ and $\mathbf{A}^+/\pi_L^l \mathbf{A}^+$. Now assume that

$$\mathrm{Fr} - \mathrm{id} \colon \omega_\phi^n(\mathbf{A}^+/\pi_L^k \mathbf{A}^+) \longrightarrow \omega_\phi^n(\mathbf{A}^+/\pi_L^k \mathbf{A}^+)$$

for all natural numbers $k \leq l$ is an isomorphism. Note that we just proved this for $l = 1$. Consider the commutative diagram:

$$
\begin{array}{ccc}
\mathrm{Fr} - \mathrm{id} \colon \omega_\phi^n(\mathbf{A}^+/\pi_L^l \mathbf{A}^+) & \longrightarrow & \omega_\phi^n(\mathbf{A}^+/\pi_L^l \mathbf{A}^+) \\
\uparrow & & \uparrow \\
\mathrm{Fr} - \mathrm{id} \colon \omega_\phi^n(\mathbf{A}^+/\pi_L^{l+1} \mathbf{A}^+) & \longrightarrow & \omega_\phi^n(\mathbf{A}^+/\pi_L^{l+1} \mathbf{A}^+)
\end{array}
$$

Our aim is to show, that the latter horizontal homomorphism is also an isomorphism. Let $x \in \mathbf{A}^+$ such that $\omega_\phi^n x \not\equiv 0 \bmod \pi_L^{l+1} \mathbf{A}^+$. The degree n-term (with respect to ω_ϕ) of $\mathrm{Fr}(\omega_\phi^n x) - \omega_\phi^n x$ is $\omega_\phi^n(\pi_L - 1)x$ and therefore it is unequal to zero modulo π_L^{l+1}. To see this, we assume $\omega_\phi^n(\pi_L - 1)x \equiv 0 \bmod \pi_L^{n+1}$ and let j be the smallest integer such that $2^j \geq n + 1$ and multiply this congruence with $(1 + \pi_L)(1 + \pi_L^2) \cdots (1 + \pi_L^{2^{j-1}})$. Then we get

$$0 \equiv \omega_\phi^n(\pi_L^j - 1)x \equiv -\omega_\phi^n x \bmod \pi_L^{n+1} \mathbf{A}^+$$

what we excluded, i.e. it has to be $\omega_\phi^n(\pi_L - 1)x \not\equiv 0 \bmod \pi_L^{n+1} \mathbf{A}^+$ and therefore $\mathrm{Fr} - \mathrm{id}$ is injective on $\omega_\phi^n(\mathbf{A}^+/\pi_L^{l+1} \mathbf{A}^+)$.

Let $x \in \omega_\phi^n \mathbf{A}^+$. Then there exists $y \in \omega_\phi^n \mathbf{A}^+$ such that $\varphi_L(y) - y \equiv x \bmod \pi_L^l \mathbf{A}^+$ (because we assumed the surjectivity for all values $\leq l$), i.e. there exists $\alpha \in \omega_\phi^n \mathbf{A}^+$ such that $\mathrm{Fr}(y) - y = x + \pi_L^l \alpha$. Then again there exists $\beta \in \omega_\phi^n \mathbf{A}^+$ such that $\mathrm{Fr}(\beta) - \beta \equiv \alpha \bmod \pi_L$, i.e. there exists some $\eta \in \omega_\phi^n \mathbf{A}^+$ such that $\mathrm{Fr}(\beta) - \beta = \alpha + \pi_L \eta$. We then get

$$
\begin{aligned}
(\mathrm{Fr} - \mathrm{id})(y - \pi_L^l \beta) &= (\mathrm{Fr} - \mathrm{id})(y) - \pi_L^l(\mathrm{Fr} - \mathrm{id})(\beta) \\
&= x + \pi_L^l \alpha - \pi_L^l(\alpha + \pi_L \eta) \equiv x \bmod \pi_L^{l+1} \mathbf{A}^+,
\end{aligned}
$$

i.e. the map $\mathrm{Fr} - \mathrm{id}$ is surjective on $\omega_\phi^n(\mathbf{A}^+/\pi_L^{l+1} \mathbf{A}^+)$. Since these maps are all isomorphisms, passing to the projective limit gives that the map $\mathrm{Fr} - \mathrm{id}$ is an isomorphism on $\omega_\phi^n \mathbf{A}^+$ □

Corollary 5.1.2. *For every $n \in \mathbb{N}$ the following sequence is exact:*

$$0 \longrightarrow \mathcal{O}_L \longrightarrow \mathbf{A}/\omega_\phi^n \mathbf{A}^+ \xrightarrow{\mathrm{Fr} - \mathrm{id}} \mathbf{A}/\omega_\phi^n \mathbf{A}^+ \longrightarrow 0.$$

Proof. In Lemma 5.1.1 we showed that

$$0 \longrightarrow \mathcal{O}_L \longrightarrow \mathbf{A} \xrightarrow{\mathrm{Fr}-\mathrm{id}} \mathbf{A} \longrightarrow 0.$$

is an exact sequence and that

$$\mathrm{Fr}-\mathrm{id}\colon \omega_\phi^n \mathbf{A}^+ \longrightarrow \omega_\phi^n \mathbf{A}^+$$

is an isomorphism for every $n \in \mathbb{N}$. Since every element of the image of $\mathcal{O}_L \hookrightarrow \mathbf{A}$ has degree 0 (with respect to ω_ϕ) the homomorphism $\mathcal{O}_L \to \mathbf{A}/\omega_\phi^n \mathbf{A}^+$ is still injective. Since Fr fixes $\mathcal{O}_L$ it is clear that we have $\mathcal{O}_L \subseteq \ker(\mathrm{Fr}-\mathrm{id})$. For the other inclusion let $x \in \ker(\mathrm{Fr}-\mathrm{id})$. Then there exists an $\alpha \in \mathbf{A}$ such that $\alpha \bmod \omega_\phi^n \mathbf{A}^+ = x$ and $\mathrm{Fr}(\alpha) - \alpha \in \omega_\phi^n \mathbf{A}^+$. But since $\mathrm{Fr}-\mathrm{id}$ is an isomorphism on $\omega_\phi^n \mathbf{A}^+$ there exists also a $\beta \in \omega_\phi^n \mathbf{A}^+ \subseteq \mathbf{A}$ such that $\mathrm{Fr}(\beta) - \beta = \mathrm{Fr}(\alpha) - \alpha$. Because of the exactness of

$$0 \longrightarrow \mathcal{O}_L \longrightarrow \mathbf{A} \xrightarrow{\mathrm{Fr}-\mathrm{id}} \mathbf{A} \longrightarrow 0$$

it then exists $\eta \in \mathcal{O}_L$ such that $\eta = \alpha - \beta$. This implies $\eta \equiv \alpha \bmod \omega_\phi^n \mathbf{A}^+$, i.e. $\eta = x$ which means $\ker(\mathrm{Fr}-\mathrm{id}) \subseteq \mathcal{O}_L$. This proves the exactness in the middle. For the surjectivity of $\mathrm{Fr}-\mathrm{id}$ recall that $\mathbf{A} \twoheadrightarrow \mathbf{A}/\omega_\phi^n \mathbf{A}^+$ and $\mathrm{Fr}-\mathrm{id}\colon \mathbf{A} \to \mathbf{A}$ are surjective and consider the commutative diagram

$$
\begin{array}{ccc}
\mathbf{A} & \xrightarrow{\ \mathrm{Fr}-\mathrm{id}\ } & \mathbf{A} \\
\downarrow & & \downarrow \\
\mathbf{A}/\omega_\phi^n \mathbf{A}^+ & \xrightarrow[\mathrm{Fr}-\mathrm{id}]{} & \mathbf{A}/\omega_\phi^n \mathbf{A}^+.
\end{array}
$$

This implies that the homomorphism $\mathrm{Fr}-\mathrm{id}\colon \mathbf{A}/\omega_\phi^n \mathbf{A}^+ \to \mathbf{A}/\omega_\phi^n \mathbf{A}^+$ is also surjective. $\qquad\square$

Lemma 5.1.3. *Let $A|\mathbf{A}_L$ be a finite, unramified extension. Then, for every $m \in \mathbb{N}$, the canonical projection $A/\pi_L^{m+1}A \to A/\pi_L^m A$ has a continuous, set theoretical section with respect to the weak topology on A.*

Proof. From Proposition 3.2.31 we deduce that

$$A \cong \varprojlim_n \mathcal{O}_E/\pi_L^n \mathcal{O}_E((X))$$

for some finite, unramified extension $E|L$. Therefore we have

$$A/\pi_L^m A = \mathcal{O}_E/\pi_L^m \mathcal{O}_E((X))$$

for every $m \in \mathbb{N}$. Therefore it is enough to give a continuous set theoretical section of the canonical projection $\mathcal{O}_E/\pi_L^{m+1}\mathcal{O}_E((X)) \to \mathcal{O}_E/\pi_L^m \mathcal{O}_E((X))$ with respect to the X-adic topology. Since the $\mathcal{O}_E/\pi_L^m \mathcal{O}_E$ are finite discrete, there exists for every $m \in \mathbb{N}$ a continuous map

$$\iota_m\colon \mathcal{O}_E/\pi_L^m \mathcal{O}_E \longrightarrow \mathcal{O}_E/\pi_L^{m+1}\mathcal{O}_E$$

which is a set theoretical section of the canonical projection. We then define a map

$$\alpha_m\colon \mathcal{O}_E/\pi_L^m \mathcal{O}_E((X)) \longrightarrow \mathcal{O}_E/\pi_L^{m+1}\mathcal{O}_E((X)),$$

$$\sum_{i>>-\infty} \lambda_i X^i \longmapsto \sum_{i>>-\infty} \iota_m(\lambda_i) X^i.$$

This then clearly is a set theoretical section of the canonical projection. We have to check continuity.

So let $f \in \mathcal{O}_E/\pi_L^{m+1}\mathcal{O}_E((X))$ and $n \in \mathbb{N}_0$. If then $\alpha_m^{-1}(f + X^n\mathcal{O}_E/\pi_L^{m+1}\mathcal{O}_E[\![X]\!])$ is empty, there is nothing to prove. So assume there is $g \in \alpha_m^{-1}(f + X^n\mathcal{O}_E/\pi_L^{m+1}\mathcal{O}_E[\![X]\!])$ and let $h \in X^n\mathcal{O}_E/\pi_L^m\mathcal{O}_E[\![X]\!]$. Then g and $g+h$ coincide in degrees $< n$ and therefore, by definition, also $\alpha_m(g)$ and $\alpha_m(g+h)$ coincide in degrees $< n$, i.e.

$$\alpha_m(g+h) \in \alpha_m(g) + X^n\mathcal{O}_E/\pi_L^{m+1}\mathcal{O}_E[\![X]\!] = f + X^n\mathcal{O}_E/\pi_L^{m+1}\mathcal{O}_E[\![X]\!]$$

since $\alpha_m(g) \in f + X^n\mathcal{O}_E/\pi_L^{m+1}\mathcal{O}_E[\![X]\!]$. It then follows

$$g + X^n\mathcal{O}_E/\pi_L^m\mathcal{O}_E[\![X]\!] \subseteq \alpha_m^{-1}(f + X^n\mathcal{O}_E/\pi_L^{m+1}\mathcal{O}_E[\![X]\!])$$

and therefore that α_m is continuous. $\qquad\square$

Corollary 5.1.4. *For every $m \in \mathbb{N}$ the canonical projection $\mathbf{A}/\pi_L^{m+1}\mathbf{A} \to \mathbf{A}/\pi_L^m\mathbf{A}$ has a continuous, set theoretical section.*

Proof. Since $\mathbf{A}$ is the π_L-adic completion of $\mathbf{A}_L^{\mathrm{nr}}$ it is

$$\mathbf{A}/\pi_L^m\mathbf{A} = \mathbf{A}_L^{\mathrm{nr}}/\pi_L^m\mathbf{A}_L^{\mathrm{nr}}$$

for every $m \in \mathbb{N}$. Since colimits are exact it is

$$\mathbf{A}_L^{\mathrm{nr}}/\pi_L^m\mathbf{A}_L^{\mathrm{nr}} = \bigcup_{A|\mathbf{A}_L \text{ fin, nr}} A/\pi_L^m A$$

for every $m \in \mathbb{N}$ and since we have for every $A|\mathbf{A}_L$ finite and unramified and every $m \in \mathbb{N}$ a continuous, set theoretical section of the canonical projection $A/\pi_L^{m+1}A \to A/\pi_L^m A$ (cf. Lemma 5.1.3) this induces for every $m \in \mathbb{N}$ a set theoretical section of the canonical projection $\mathbf{A}_L^{\mathrm{nr}}/\pi_L^{m+1}\mathbf{A}_L^{\mathrm{nr}} \to \mathbf{A}_L^{\mathrm{nr}}/\pi_L^m\mathbf{A}_L^{\mathrm{nr}}$, which then is continuous, since $\mathbf{A}_L^{\mathrm{nr}}$ carries the topology of the colimit and then so does $\mathbf{A}_L^{\mathrm{nr}}/\pi_L^m\mathbf{A}_L^{\mathrm{nr}}$ for every $m \in \mathbb{N}$. $\qquad\square$

Lemma 5.1.5. *Let $V \in \mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg})}(G_K)$, set $M := \mathcal{D}_{K|L}(V)$ and $V_m := V/\pi_L^m V$ as well as $M_m := M/\pi_L^m M$ for $m \in \mathbb{N}$. Then the transition maps of the inverse systems $(V_m)_m$, $(M_m)_m$ and $(\mathbf{A} \otimes_{\mathcal{O}_L} V_m)_m$ are surjective and they have a continuous, set theoretical section. In particular, the short sequences*

$$0 \longrightarrow \mathbf{A} \otimes_{\mathcal{O}_L} V_m \xrightarrow{\mathrm{id}\otimes\cdot\pi_L} \mathbf{A} \otimes_{\mathcal{O}_L} V_{m+1} \longrightarrow \mathbf{A} \otimes_{\mathcal{O}_L} V_1 \longrightarrow 0,$$
$$0 \longrightarrow M_m \xrightarrow{\cdot\pi_L} M_{m+1} \longrightarrow M_1 \longrightarrow 0$$

are exact and have continuous, set theoretical sections.

Proof. Since $\mathcal{D}_{K|L}$ is exact as an equivalence of categories (cf. Theorem 3.3.4) and the tensor product is right exact, it is immediately clear that the transition maps of the systems $(M_m)_m$ and $(\mathbf{A} \otimes_{\mathcal{O}_L} V_m)_m$ are surjective since the transition maps of $(V_m)_m$ are.

Since the V_m are finite and discrete one can define a set theoretical section of the canonical projection $V_{m+1} \to V_m$ by choosing a preimage for every element in V_m. Since M_m is a finitely generated $\mathbf{A}_{K|L}$-module, there are for every $m \in \mathbb{N}$ isomorphisms of topological $\mathbf{A}_{K|L}$-modules

$$M_m \cong \bigoplus_{i=1}^{n^{(m)}} \mathbf{A}_{K|L}/\pi_L^{n_i^{(m)}}\mathbf{A}_{K|L}$$

such that $n_i^{(m)} \leq n_{i+1}^{(m)}$ and the canonical projection $M_{m+1} \to M_m$ maps the i-th component of $\oplus_{i=1}^{n^{(m+1)}} \mathbf{A}_{K|L}/\pi_L^{n_i^{(m+1)}} \mathbf{A}_{K|L}$ to the i-th component of $\oplus_{i=1}^{n^{(m)}} \mathbf{A}_{K|L}/\pi_L^{n_i^{(m)}} \mathbf{A}_{K|L}$ for $i \geq n^{(m)}$ and is zero on the i-th component with $i > n^{(m)}$. With Lemma 5.1.3 we then obtain a continuous, set theoretical section for every component, which then also gives a continuous set theoretical section for $M_{m+1} \to M_m$.

As topological $\mathcal{O}_L$-module we have

$$\mathbf{A} \otimes_{\mathcal{O}_L} V_m \cong \bigoplus_{i=0}^{k^{(m)}} \mathbf{A}/\pi_L^{k_i^{(m)}} \mathbf{A}$$

and therefore we see that there exists a continuous, set theoretical section of the canonical projection $\mathbf{A} \otimes_{\mathcal{O}_L} V_{m+1} \to \mathbf{A} \otimes_{\mathcal{O}_L} V_m$ as above using Corollary 5.1.4 instead of Lemma 5.1.3. The statement on the short exact sequences then follows immediately. $\square$

Lemma 5.1.6. *Let $E|L$ be a finite extension and $H_E = \mathrm{Gal}(\overline{\mathbb{Q}_p}|E_\infty)$ as usual. Then the operation of H_E on $\mathbf{E}_L^{\mathrm{sep}}$ is continuous with respect to the discrete topology on $\mathbf{E}_L^{\mathrm{sep}}$.*

Proof. Let $x \in \mathbf{E}_L^{\mathrm{sep}}$. Then there exists a finite extension $\mathbb{F}|\mathbf{E}_E$ such that $x \in \mathbb{F}$. Then x is fixed by $U := \mathrm{Gal}(\mathbf{E}_L^{\mathrm{sep}}|\mathbb{F})$ which is an open subgroup of H_E. If then $\tau \in H$ and $y \in \mathbf{E}_L^{\mathrm{sep}}$ are such that $\tau(y) = x$, then $U\tau \times \{y\}$ is an open neighbourhood of $\{\tau\} \times \{y\}$ in $H_E \times \mathbf{E}_L^{\mathrm{sep}}$ with $\sigma(\tau(y)) = x$ for all $\sigma \in U$. $\square$

Lemma 5.1.7. *Let V be a finite dimensional k_L-representation of G_K. Then there exists a finite Galois extension $E|K$ such that H_E acts trivially on V.*

Proof. Since the action of G_K on V is continuous, the homomorphism $G_K \to \mathrm{Aut}_{k_L}(V)$ is continuous and since V is a finite dimensional k_L-vector space, it is finite and so $\mathrm{Aut}_{k_L}(V)$ carries the discrete topology, i.e. the kernel of the upper homomorphism is open, which means that there exists a finite Galois extension $E|K$ such that G_E acts trivially on V. With G_E also H_E acts trivially on V. $\square$

Lemma 5.1.8. *Let V be a finite dimensional k_L-representation of G_K and $E|K$ a finite Galois extension, such that H_E acts trivially on V and set $\Delta := \mathrm{Gal}(E_\infty|K_\infty)$. Then Δ acts on the short exact sequence*

$$0 \to \omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V \to \mathbf{E}_E \otimes_{k_L} V \to \mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V \to 0$$

and it holds

1. $H^j(\Delta, \mathbf{E}_E \otimes_{k_L} V) = 0$ *for all $j > 0$.*
2. *There exists $r \geq 0$ such that $\omega_\phi^r H^j(\Delta, \omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V) = 0$ for all $j > 0$ and $n \in \mathbb{Z}$.*

Proof. The proof is literally the same as the one of [34, Lemma 2.2.10, p.20] $\square$

Lemma 5.1.9. *Let V be a finite dimensional k_L-representation of G_K and $E|K$ a finite Galois extension, such that H_E acts trivially on V and set $\Delta := \mathrm{Gal}(E_\infty|K_\infty)$. Then we have*

1. $(\mathbf{E}_L^{\mathrm{sep}} \otimes_{k_L} V)^{H_K} \cong (\mathbf{E}_E \otimes_{k_L} V)^\Delta$.
2. $(\omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+} \otimes_{k_L} V)^{H_K} \cong (\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V)^\Delta$ *for all $n \geq 0$.*

Proof. In both cases the proof is the same. So let X be $\mathbf{E}_L^{\mathrm{sep}}$ or $\omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+}$ for some $n \geq 0$. Note that $H_K/H_E \cong \Delta$. We then get

$$(X \otimes_{k_L} V)^{H_K} = ((X \otimes_{k_L} V)^{H_E})^{H_K/H_E} = (X^{H_E} \otimes_{k_L} V)^\Delta,$$

where the last equation is true, since H_E acts trivial on V. $\qquad\square$

Before stating a corollary, we should introduce some notation. Since all projective systems which appear here are indexed by the natural numbers, we will make the following definitions only for projective systems indexed by natural numbers.

Proposition 5.1.10. *Let V be a finite dimensional k_L-representation of G_K and $E|K$ a finite Galois extension, such that H_E acts trivially on V and set $\Delta := \mathrm{Gal}(E_\infty|K_\infty)$. Let in addition $M = \mathcal{D}_{K|L}(V)$ and*

$$M_n := M/\left(\omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+} \otimes_{k_L} V\right)^{H_K}.$$

Then we have

1. *The inverse systems $(H^j(\Delta, \omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V))_n$ and $(H^j(\Delta, \mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V))_n$ are ML-zero for all $j > 0$.*
2. *The map of inverse systems $(M_n)_n \to (H^0(\Delta, \mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V))_n$ is an ML-isomorphism.*

Proof.

1. Since V is a finite dimensional k_L-vector space, it's flat and therefore the homomorphism $\omega_\phi^{n+1} \mathbf{E}_E^+ \otimes_{k_L} V \subseteq \omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V$ is injective and induces a homomorphism

$$H^j(\Delta, \omega_\phi^{n+1} \mathbf{E}_E^+ \otimes_{k_L} V) \to H^j(\Delta, \omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V).$$

The image of this last homomorphism is a subset of $\omega_\phi H^j(\Delta, \omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V)$, i.e. the maps $H^j(\Delta, \omega_\phi^k \mathbf{E}_E^+ \otimes_{k_L} V) \to H^j(\Delta, \omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V)$ are zero for $k \geq n + r$ (cf. Lemma 5.1.8, 2.), i.e. the inverse system $(H^j(\Delta, \omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V))_n$ is ML-zero for $j > 0$.

Since every class in $\mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+$ has a unique representative of highest degree $\leq n-1$ in ω_ϕ the homomorphism $\mathbf{E}_E \to \mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+$ has a set theoretical splitting (by sending a class to this representative). This map is continuous, since the preimage of a subset of $\mathbf{E}_E$ in $\mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+$ is equal to the image under the canonical projection, which is open by definition. Since V is flat, the sequence

$$0 \to \omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V \to \mathbf{E}_E \otimes_{k_L} V \to \mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V \to 0$$

is exact and we can deduce a long exact cohomology sequence (cf. [28, (2.3.2) Lemma, p.106]) and since $H^j(\Delta, \mathbf{E}_E \otimes_{k_L} V) = 0$ for $j > 0$ (cf. Lemma 5.1.8, 1.), the homomorphism

$$H^j(\Delta, \mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V) \to H^{j+1}(\Delta, \omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V)$$

is an isomorphism for all $j > 0$ and the diagram

$$
\begin{array}{ccc}
H^j(\Delta, \mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V) & \longrightarrow & H^{j+1}(\Delta, \omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V) \\
\uparrow & & \uparrow \\
H^j(\Delta, \mathbf{E}_E/\omega_\phi^{n+1} \mathbf{E}_E^+ \otimes_{k_L} V) & \longrightarrow & H^{j+1}(\Delta, \omega_\phi^{n+1} \mathbf{E}_E^+ \otimes_{k_L} V)
\end{array}
$$

commutes. This means that the transition map

$$H^j(\Delta, \mathbf{E}_E/\omega_\phi^k \mathbf{E}_E^+ \otimes_{k_L} V) \to H^j(\Delta, \mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V)$$

is zero for $k \geq n + r$ and therefore the inverse system $(H^j(\Delta, \mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V))_n$ is ML-zero.

2. As seen before, for every $n \geq 0$ we have an exact sequence

$$0 \to \omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V \to \mathbf{E}_E \otimes_{k_L} V \to \mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V \to 0.$$

Taking Δ-invariants then gives an exact sequence

$$0 \longrightarrow (\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V)^\Delta \longrightarrow (\mathbf{E}_E \otimes_{k_L} V)^\Delta \longrightarrow \cdots$$

$$\cdots \longrightarrow (\mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V)^\Delta \longrightarrow H^1(\Delta, \omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V) \longrightarrow 0,$$

where the last term is zero because $H^j(\Delta, \mathbf{E}_E \otimes_{k_L} V) = 0$ for $j > 0$ (cf. Lemma 5.1.8, 1.). With Lemma 5.1.9 this sequences becomes

$$0 \longrightarrow (\omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+} \otimes_{k_L} V)^{H_K} \longrightarrow (\mathbf{E}_L^{\mathrm{sep}} \otimes_{k_L} V)^{H_K} \longrightarrow \cdots$$

$$\cdots \longrightarrow (\mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V)^\Delta \longrightarrow H^1(\Delta, \omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V) \longrightarrow 0$$

and then gives the following short exact sequence

$$0 \longrightarrow (\mathbf{E}_L^{\mathrm{sep}} \otimes_{k_L} V)^{H_K}/(\omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+} \otimes_{k_L} V)^{H_K} \longrightarrow \cdots$$

$$\cdots \longrightarrow (\mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V)^\Delta \longrightarrow H^1(\Delta, \omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V) \to 0.$$

In particular, $H^1(\Delta, \omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V)$ is the cokernel of the homomorphism $M_n \to (\mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V)^\Delta$. According to the first part of the proof the inverse system $(H^1(\Delta, \omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V))_n$ is ML-zero, and since the kernel of $M_n \to (\mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes V)^\Delta$ is zero it is also ML-zero, which then ends the proof.

$\square$

Theorem 5.1.11. *Let $V \in \mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg})}(G_K)$ and set $M = \mathcal{D}_{K|L}(V)$. Then there are isomorphisms*

$$H^*_{\mathrm{cts}}(G_K, V) \xrightarrow{\cong} \mathcal{H}^*_{\varphi_{K|L}}(\Gamma_K, M),$$

$$H^*_{\mathrm{cts}}(H_K, V) \xrightarrow{\cong} \mathcal{H}^*_{\varphi_{K|L}}(M).$$

These isomorphisms are functorial in V and compatible with restriction and corestriction. On the level of complexes these isomorphisms are induced from quasi-isomorphisms

$$C^\bullet_{\mathrm{cts}}(G_K, V) \longrightarrow \varprojlim_{n,m} \mathcal{C}^\bullet_{\mathrm{Fr}}(G_K, (\mathbf{A}/\omega_\phi^n \mathbf{A}^+) \otimes_{\mathcal{O}_L} V/\pi_L^m V) \longleftarrow \mathcal{C}^\bullet_{\varphi_{K|L}}(\Gamma_K, M),$$

$$C^\bullet_{\mathrm{cts}}(H_K, V) \longrightarrow \varprojlim_{n,m} \mathcal{C}^\bullet_{\mathrm{Fr}}(H_K, (\mathbf{A}/\omega_\phi^n \mathbf{A}^+) \otimes_{\mathcal{O}_L} V/\pi_L^m V) \longleftarrow \mathcal{C}^\bullet_{\varphi_{K|L}}(M).$$

If A denotes a cofinitely generated $\mathcal{O}_L$-module with continuous G_K-action, we obtain similar quasi-isomorphisms

$$C^\bullet_{\mathrm{cts}}(G_K, A) \longrightarrow \varinjlim_m \varprojlim_n \mathcal{C}^\bullet_{\mathrm{Fr}}(G_K, (\mathbf{A}/\omega_\phi^n \mathbf{A}^+) \otimes_{\mathcal{O}_L} A_m) \longleftarrow \mathcal{C}^\bullet_{\varphi_{K|L}}(\Gamma_K, M)$$

where $A_m := A[\pi_L^m]$ denotes the kernel of multiplication by π_L^m and $M = \varinjlim_m \mathcal{D}_{K|L}(A_m)$. An analogous statement for H_K is stated in (4), but compare also with Proposition 5.2.24 below.

Proof. In this proof, we follow closely the proof of [34, Theorem 2.2.1, p.702-706].

Step 1: Explaining the strategy.

First, for $m \in \mathbb{N}$ set $V_m := V/\pi_L^m V$ and $M_m := M/\pi_L^m M$. Since $\mathcal{D}_{K|L}$ is an equivalence of categories (cf. Theorem 3.3.4) it is exact and therefore we have $M_m = \mathcal{D}_{K|L}(V_m)$. The open subgroups

$$M_m \cap \left(\omega_\phi^n \mathbf{A}^+ \otimes_{\mathcal{O}_L} V_m \right) = \left(\omega_\phi^n \mathbf{A}^+ \otimes_{\mathcal{O}_L} V_m \right)^{H_K}$$

form a basis of neighbourhoods of 0 in M_m. These subgroups are clearly stable under the operation of Γ_K and since $\varphi_{K|L}$ commutes with the operation of G_K on $\left(\omega_\phi^n \mathbf{A}^+ \otimes_{\mathcal{O}_L} V_m \right)$ these subgroups are also stable under $\varphi_{K|L}$. We then set

$$M_{m,n} := M_m / \left(\omega_\phi^n \mathbf{A}^+ \otimes_{\mathcal{O}_L} V_m \right)^{H_K}.$$

Since $\left(\omega_\phi^n \mathbf{A}^+ \otimes_{\mathcal{O}_L} V_m \right)^{H_K}$ is an open subgroup, this is a discrete Γ_K-module and we have topological isomorphisms

$$M_m \cong \varprojlim_n M_{m,n}$$
$$M \cong \varprojlim_m M_m.$$

In Corollary 5.1.2 we proved that the sequence

$$0 \longrightarrow \mathcal{O}_L \longrightarrow \mathbf{A}/\omega_\phi^n \mathbf{A}^+ \xrightarrow{\mathrm{Fr}-\mathrm{id}} \mathbf{A}/\omega_\phi^n \mathbf{A}^+ \longrightarrow 0$$

is exact and since $\mathbf{A}/\omega_\phi^n \mathbf{A}^+$ is a free $\mathcal{O}_L$-module, it is flat and therefore the sequence

$$0 \longrightarrow V_m \longrightarrow \mathbf{A}/\omega_\phi^n \mathbf{A}^+ \otimes_{\mathcal{O}_L} V_m \xrightarrow{\mathrm{Fr}-\mathrm{id}} \mathbf{A}/\omega_\phi^n \mathbf{A}^+ \otimes_{\mathcal{O}_L} V_m \longrightarrow 0$$

is also exact. Then Lemma 2.3.3 says that for every $m, n \geq 1$ we have a quasi isomorphism

$$C_{\mathrm{cts}}^\bullet(G_K, V_m) \longrightarrow \mathcal{C}_{\mathrm{Fr}}^\bullet(G_K, (\mathbf{A}/\omega_\phi^n \mathbf{A}^+) \otimes_{\mathcal{O}_L} V_m).$$

The inverse systems $(V_m)_m$ and $((\mathbf{A}/\omega_\phi^n \mathbf{A}^+) \otimes_{\mathcal{O}_L} V_m)_{n,m}$ have surjective transition maps. From Corollary 2.1.2 we then can deduce that also the inverse systems of complexes $(C_{\mathrm{cts}}^\bullet(G_K, V_m))_m$ and $C_{\mathrm{cts}}^\bullet(G_K, ((\mathbf{A}/\omega_\phi^n \mathbf{A}^+) \otimes_{\mathcal{O}_L} V_m))_{n,m}$ have surjective transition maps and Lemma 2.3.8 then says that the system $\mathcal{C}_{\mathrm{Fr}}^\bullet(G_K, ((\mathbf{A}/\omega_\phi^n \mathbf{A}^+) \otimes_{\mathcal{O}_L} V_m))_{n,m}$ has surjective transition maps as well. From the quasi isomorphism $C_{\mathrm{cts}}^\bullet(G_K, V_m) \to \mathcal{C}_{\mathrm{Fr}}^\bullet(G_K, (\mathbf{A}/\omega_\phi^n \mathbf{A}^+ \otimes V_m))$ we then can deduce with Proposition 2.3.11 that the cohomologies of the complexes $\varprojlim_{n,m} \mathcal{C}_{\mathrm{Fr}}^\bullet(G_K, (\mathbf{A}/\omega_\phi^n \mathbf{A}^+ \otimes_{\mathcal{O}_L} V_m))$ and $\varprojlim_m C_{\mathrm{cts}}^\bullet(G_K, V_m)$ coincide. Since $\varprojlim_m C_{\mathrm{cts}}^\bullet(G_K, V_m) \cong C_{\mathrm{cts}}^\bullet(G_K, V)$, the cohomology of $\varprojlim_m C_{\mathrm{cts}}^\bullet(G_K, V_m)$ is $H_{\mathrm{cts}}^*(G_K, V)$, which then is also computed by $\varprojlim_{n,m} \mathcal{C}_{\mathrm{Fr}}^\bullet(G_K, (\mathbf{A}/\omega_\phi^n \mathbf{A}^+ \otimes_{\mathcal{O}_L} V_m))$.

On the other hand, since the canonical inclusion $\iota \colon M_{m,n} \hookrightarrow (\mathbf{A}/\omega_\phi^n \mathbf{A}^+) \otimes_{\mathcal{O}_L} V_m$ commutes with $\varphi_{K|L}$ and since together with the canonical projection $\mathrm{pr} \colon G_K \twoheadrightarrow \Gamma_K$ it holds

$$\iota(\mathrm{pr}(\sigma)x) = \sigma\iota(x)$$

for all $\sigma \in G_K$ and $x \in M_{m,n}$ and since the operations of $\varphi_{K|L}$ and G_K respectively Γ_K commute wo get an induced morphism of complexes

$$\alpha_{m,n} \colon \mathcal{C}_{\varphi_{K|L}}^\bullet(\Gamma_K, M_{m,n}) \to \mathcal{C}_{\mathrm{Fr}}^\bullet(G_K, (\mathbf{A}/\omega_\phi^n \mathbf{A}^+) \otimes_{\mathcal{O}_L} V_m)$$

(cf. [29, I §5, p45], the additional properties concerning $\varphi_{K|L}$ we noted above, ensure that we get the morphism of the above total complex with respect to $\varphi_{K|L}$ on the left hand side and Fr on the right hand side).

We now want to see that $\varprojlim_{n,m} \alpha_{m,n}$ is a quasi isomorphism. Because of $\varprojlim_{n,m} \mathcal{C}^{\bullet}_{\varphi_{K|L}}(\Gamma_K, M_{m,n}) = \mathcal{C}^{\bullet}_{\varphi_{K|L}}(\Gamma_K, M)$ (cf. Lemma 2.3.7), this then says that the cohomology of $\mathcal{C}^{\bullet}_{\varphi_{K|L}}(\Gamma_K, M)$ and $\varprojlim_{n,m} \mathcal{C}^{\bullet}_{\mathrm{Fr}}(G_K, (\mathbf{A}/\omega_\phi^n \mathbf{A}^+ \otimes_{\mathcal{O}_L} V_m))$ coincide. But then the cohomologies of $\mathcal{C}^{\bullet}_{\varphi_{K|L}}(\Gamma_K, M)$ and $C^{\bullet}_{\mathrm{cts}}(G_K, V)$ coincide, what is exactly what we want to prove.

To see that $\varprojlim_{n,m} \alpha_{m,n}$ is a quasi isomorphism, it is enough to see, that $\varprojlim_n \alpha_{m,n}$ is a quasi isomorphism for every $m \geq 1$. Because if this is shown, one knows that the inverse systems of complexes $(\mathcal{C}^{\bullet}_{\varphi_{K|L}}(\Gamma_K, M_m))_m$ and $(\mathcal{C}^{\bullet}_{\varphi_{K|L}}(G_K, \mathbf{A} \otimes_{\mathcal{O}_L} V_m))_m$ are quasi isomorphic. Since the transition maps $M_{m+1} \to M_m$ as well as $\mathbf{A} \otimes_{\mathcal{O}_L} V_{m+1} \to \mathbf{A} \otimes_{\mathcal{O}_L} V_m$ are surjective and have a continuous section (cf. Lemma 5.1.5), one can see as before, using Corollary 2.1.2 and Lemma 2.3.8, that the inverse systems of complexes $(\mathcal{C}^{\bullet}_{\varphi_{K|L}}(\Gamma_K, M_m))_m$ and $(\mathcal{C}^{\bullet}_{\varphi_{K|L}}(G_K, \mathbf{A} \otimes_{\mathcal{O}_L} V_m))_m$ have surjective transition maps. As before with Proposition 2.3.11 respectively Remark 2.3.12 one then sees that $\varprojlim_m \mathcal{C}^{\bullet}_{\varphi_{K|L}}(\Gamma_K, M_m)$ and $\varprojlim_m \mathcal{C}^{\bullet}_{\varphi_{K|L}}(G_K, \mathbf{A} \otimes_{\mathcal{O}_L} V_m)$ are quasi isomorphic. So, what is still to show, is that $\varprojlim_n \alpha_{m,n}$ is a quasi isomorphism for every $m \geq 1$. This will be the rest of the proof.

Step 2: Reduction to the case $m = 1$.
Since for every $m \geq 1$ the sequence

$$0 \longrightarrow V_m \longrightarrow V_{m+1} \longrightarrow V_1 \longrightarrow 0.$$

is exact and $\mathcal{D}_{K|L}$ is an exact functor (since it is an equivalence), this implies that for every $m \geq 1$ there is a short exact sequence

$$0 \longrightarrow M_m \longrightarrow M_{m+1} \longrightarrow M_1 \longrightarrow 0.$$

By the definition of the topology on the M_m it is clear, that the topology of M_m is induced from that of M_{m+1} and from Lemma 5.1.5 we deduce that it has a continuous set theoretical section. Therefore Proposition 2.2.22 says that we get a long exact sequence of cohomology. Now assume the result is shown for $m = 1$. Then $\mathcal{H}^*_{\varphi_{K|L}}(\Gamma_K, M) \to H^*_{\mathrm{cts}}(G_K, V)$ is an isomorphism for every V with $\pi_L V = 0$. Induction on m and the 5-lemma applied to the following diagram which arises from the long exact cohomology sequences (where we write $\Gamma = \Gamma_K$ and $G = G_K$ and $\varphi = \varphi_{K|L}$)

$$\begin{array}{ccccccccc}
\mathcal{H}^{l-1}_\varphi(\Gamma, M_1) & \xrightarrow{\delta} & \mathcal{H}^l_\varphi(\Gamma, M_m) & \longrightarrow & \mathcal{H}^l_\varphi(\Gamma, M_{m+1}) & \longrightarrow & \mathcal{H}^l_\varphi(\Gamma, M_1) & \xrightarrow{\delta} & \mathcal{H}^{l+1}_\varphi(\Gamma, M_m) \\
\cong \downarrow & & \cong \downarrow & & \downarrow & & \cong \downarrow & & \cong \downarrow \\
H^{l-1}_{\mathrm{cts}}(G, V_1) & \xrightarrow{\delta} & H^l_{\mathrm{cts}}(G, V_m) & \longrightarrow & H^l_{\mathrm{cts}}(G, V_{m+1}) & \longrightarrow & H^l_{\mathrm{cts}}(G, V_1) & \xrightarrow{\delta} & H^{l+1}_{\mathrm{cts}}(G, V_m)
\end{array}$$

then implies the result for all $m \geq 1$.

Step 3: Splitting $\alpha_{1,n}$ up.
For the rest of the proof we may assume $\pi_L V = 0$ and therefore also $\pi_L M = 0$, but we will still write $M_{1,n}$ to avoid confusion. Note that this implies

$$\mathbf{A} \otimes_{\mathcal{O}_L} V \cong \mathbf{E}^{\mathrm{sep}}_L \otimes_{k_L} V,$$

$$\omega_\phi^n \mathbf{A}^+ \otimes_{\mathcal{O}_L} V \cong \omega_\phi^n \mathbf{E}^{\mathrm{sep},+}_L \otimes_{k_L} V$$

as well as the correspondingly isomorphism with respect to the fixed modules of H_K.

Now fix a finite Galois extension $E|K$ such that H_E acts trivially on V (cf. Lemma 5.1.7). Then, the canonical inclusion

$$M_{1,n} = \frac{(\mathbf{E}_L^{\mathrm{sep}} \otimes_{k_L} V)^{H_K}}{(\omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+} \otimes_{k_L} V)^{H_K}} \hookrightarrow \frac{(\mathbf{E}_L^{\mathrm{sep}} \otimes_{k_L} V)^{H_E}}{(\omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+} \otimes_{k_L} V)^{H_E}} = \mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V$$

induces together with the canonical projection $\mathrm{Gal}(E_\infty|K) \twoheadrightarrow \Gamma_K$, as in step 1 for $\alpha_{m,n}$, for all $n \in \mathbb{N}$ a morphism of complexes

$$\beta_n \colon \mathcal{C}_{\varphi_{K|L}}^\bullet(\Gamma_K, M_{1,n}) \longrightarrow \mathcal{C}_{\mathrm{Fr}}^\bullet(\mathrm{Gal}(E_\infty|K), \mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V).$$

Simultaneously, the canonical inclusion $\mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V \hookrightarrow \mathbf{E}_L^{\mathrm{sep}}/\omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+} \otimes_{k_L} V$ together with the canonical projection $G_K \twoheadrightarrow \mathrm{Gal}(E_\infty|K)$ induces for all $n \in \mathbb{N}$ a morphism of complexes

$$\gamma_n \colon \mathcal{C}_{\mathrm{Fr}}^\bullet(\mathrm{Gal}(E_\infty|K), \mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V) \longrightarrow \mathcal{C}_{\mathrm{Fr}}^\bullet(G_K, \mathbf{E}_L^{\mathrm{sep}}/\omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+} \otimes_{k_L} V).$$

Since both diagrams

$$\begin{array}{ccc}
M_{1,n} \hookrightarrow \mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{kL} V & \qquad & \Gamma_K \longleftarrow \mathrm{Gal}(E_\infty|K) \\
\searrow \qquad \downarrow & & \nwarrow \qquad \uparrow \\
\mathbf{E}_L^{\mathrm{sep}}/\omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+} \otimes_{k_L} V, & & G_K
\end{array}$$

are commutative, where all the arrows in the left diagram are canonical inclusions and the ones in the right diagram are canonical projections, it is immediately clear that also the diagram

$$\begin{array}{ccc}
\mathcal{C}_{\varphi_{K|L}}^\bullet(\Gamma_K, M_{1,n}) & \xrightarrow{\ \beta_n\ } & \mathcal{C}_{\mathrm{Fr}}^\bullet(\mathrm{Gal}(E_\infty|K), \mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V) \\
& \searrow_{\alpha_{1,n}} & \downarrow_{\gamma_n} \\
& & \mathcal{C}_{\mathrm{Fr}}^\bullet(G_K, \mathbf{E}_L^{\mathrm{sep}}/\omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+} \otimes_{k_L} V)
\end{array}$$

commutes. So, to prove that $\varprojlim_n \alpha_{1,n}$ is a quasi-isomorphism it is enough to prove that $\varprojlim_n \beta_n$ and $\varprojlim_n \gamma_n$ are quasi-isomorphisms. In addition, we will also show that γ_n is a quasi-isomorphism for every $n \geq 1$.

Step 4: $\varprojlim_n \gamma_n$ is a quasi-isomorphism.

Due to Lemma 2.2.21 there is an E_2-spectral sequence converging to the cohomology of the source of γ_n

$$H^a(\mathrm{Gal}(E_\infty|K), \mathcal{H}_{\mathrm{Fr}}^b(\mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V)) \Longrightarrow$$

$$\mathcal{H}_{\mathrm{Fr}}^{a+b}(\mathrm{Gal}(E_\infty|K), \mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V)$$

as well as en E_2-spectral sequence converging to the target of γ_n

$$H^a(\mathrm{Gal}(E_\infty|K), \mathcal{H}_{\mathrm{Fr}}^b(H_E, \mathbf{E}_L^{\mathrm{sep}}/\omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+} \otimes_{k_L} V)) \Longrightarrow$$

$$\mathcal{H}_{\mathrm{Fr}}^{a+b}(G_K, \mathbf{E}_L^{\mathrm{sep}}/\omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+} \otimes_{k_L} V).$$

The canonical inclusion $\mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V \hookrightarrow \mathbf{E}_L^{\mathrm{sep}}/\omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+} \otimes_{k_L} V$ together with the trivial map $H_E \to 1$ then induces a homomorphism on the above E_2-pages. Together with the from γ_n induced map on cohomology this then gives a morphism of spectral sequences. So, to show that γ_n induces an isomorphism on cohomology it is enough to show that the induced homomorphism on the above E_2 pages is an isomorphism. And for this it is enough, that the homomorphism between the coefficients $\mathcal{H}_{\mathrm{Fr}}^b(\mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V)$ and $\mathcal{H}_{\mathrm{Fr}}^b(H_E, \mathbf{E}_L^{\mathrm{sep}}/\omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+} \otimes_{k_L} V)$ is an isomorphism. Since H_E acts trivially on V it is

$$\mathcal{H}_{\mathrm{Fr}}^b(\mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V) = \mathcal{H}_{\mathrm{Fr}}^b(\mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+) \otimes_{k_L} V$$
$$\mathcal{H}_{\mathrm{Fr}}^b(H_E, \mathbf{E}_L^{\mathrm{sep}}/\omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+} \otimes_{k_L} V) = \mathcal{H}_{\mathrm{Fr}}^b(H_E, \mathbf{E}_L^{\mathrm{sep}}/\omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+}) \otimes_{k_L} V$$

by [26, (3.4.4) Proposition, p. 66–67]. Therefore it is enough to show that there is an isomorphism between $\mathcal{H}_{\mathrm{Fr}}^b(\mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+)$ and $\mathcal{H}_{\mathrm{Fr}}^b(H_E, \mathbf{E}_L^{\mathrm{sep}}/\omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+})$. To see this, consider the commutative square

$$
\begin{array}{ccc}
\mathcal{H}_{\mathrm{Fr}}^b(\mathbf{E}_E) & \longrightarrow & \mathcal{H}_{\mathrm{Fr}}^b(\mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+) \\
\downarrow & & \downarrow \\
\mathcal{H}_{\mathrm{Fr}}^b(H_E, \mathbf{E}_L^{\mathrm{sep}}) & \longrightarrow & \mathcal{H}_{\mathrm{Fr}}^b(H_E, \mathbf{E}_L^{\mathrm{sep}}/\omega_\phi^n \mathbf{E}_L^{\mathrm{sep},+}),
\end{array}
$$

where $\mathbf{E}_L^{\mathrm{sep}}$ is regarded as discrete H_E-module (cf. Lemma 5.1.6) and where the horizontal maps are induced from the respective canonical projections and the vertical maps from the respective canonical inclusions.

First we want to see, that the upper horizontal map is an isomorphism. $\mathcal{H}_{\mathrm{Fr}}^b(\mathbf{E}_E)$ is computed by $\mathbf{E}_E \xrightarrow{\varphi_L - \mathrm{id}} \mathbf{E}_E$ and $\mathcal{H}_{\mathrm{Fr}}^b(\mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+)$ by the corresponding complex and the square

$$
\begin{array}{ccc}
\mathbf{E}_E & \xrightarrow{\mathrm{Fr}-\mathrm{id}} & \mathbf{E}_E \\
\downarrow & & \downarrow \\
\mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ & \xrightarrow{\mathrm{Fr}-\mathrm{id}} & \mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+
\end{array}
$$

is commutative. Denote the kernel and image of the upper horizontal map by κ_1 and im_1 and the ones of the lower vertical map by κ_2 and im_2 respectively. By Lemma 5.1.1 the map $\omega_\phi^n \mathbf{E}_E^+ \xrightarrow{\mathrm{Fr}-\mathrm{id}} \omega_\phi^n \mathbf{E}_E^+$ is an isomorphism, especially is $\omega_\phi^n \mathbf{E}_E^+ \subseteq \mathrm{im}_1$ and so we see immediately $\mathrm{im}_2 \subseteq \mathrm{im}_1 /\omega_\phi^n \mathbf{E}_E^+$. For the other inclusion let $\overline{x} \in \mathrm{im}_1 /\omega_\phi^n \mathbf{E}_E^+$ and $x \in \mathbf{E}_E$ a preimage under the canonical projection. Because of $\omega_\phi^n \mathbf{E}_E^+ \subseteq \mathrm{im}_1$ we deduce $x \in \mathrm{im}_1$. If $y \in \mathbf{E}_E$ is a preimage of x under $\mathrm{Fr} - \mathrm{id}$, then because of the commutativity of the latter diagram we get $(\mathrm{Fr} - \mathrm{id})(\overline{y}) = \overline{x}$, i.e. $\overline{x} \in \mathrm{im}_2$. Therefore $\mathcal{H}_{\mathrm{Fr}}^1(\mathbf{E}_E)$ and $\mathcal{H}_{\mathrm{Fr}}^1(\mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+)$ coincide.

For the term in degree zero let $x \in \kappa_1$ such that $x \in \omega_\phi^n \mathbf{E}_E^+$. Since $\mathrm{Fr} - \mathrm{id}$ is an isomorphism on $\omega_\phi^n \mathbf{E}_E^+$ and $(\mathrm{Fr} - \mathrm{id})(x) = 0$, x itself is zero, i.e. the canonical homomorphism $\kappa_1 \to \kappa_2$ is injective. Let now $\eta \in \kappa_2$ and $y' \in \mathbf{E}_E$ be a preimage under the canonical projection. By commutativity it is $\overline{(\mathrm{Fr} - \mathrm{id})(y')} = 0$ and therefore $(\mathrm{Fr} - \mathrm{id})(y') \in \omega_\phi^n \mathbf{E}_E^+$. Again since $\mathrm{Fr} - \mathrm{id}$ is an isomorphism on $\omega_\phi^n \mathbf{E}_E^+$ we find an element $y'' \in \omega_\phi^n \mathbf{E}_E^+$ with $(\mathrm{Fr} - \mathrm{id})(y') = (\mathrm{Fr} - \mathrm{id})(y'')$. Set $y := y' - y''$. Then $\overline{y} = \overline{y'} - \overline{y''} = \overline{y'} = \eta$ and $(\mathrm{Fr} - \mathrm{id})(y) = 0$, i.e. $\kappa_1 \to \kappa_2$ is also surjective

and therefore an isomorphism. Since every other cohomology group is zero, we conclude that

$$\mathcal{H}^b_{\mathrm{Fr}}(\mathbf{E}_E) \cong \mathcal{H}^b_{\mathrm{Fr}}(\mathbf{E}_E/\omega^n_\phi \mathbf{E}^+_E)$$

for all $b \geq 0$.

For the lower horizontal map in the upper square, recall that Lemma 5.1.1 also says that $\mathrm{Fr}-\mathrm{id}$ is on $\omega^n_\phi \mathbf{E}^{\mathrm{sep},+}_L$ an isomorphism. Therefore one sees with a similar argument as above that the canonical projection $\mathbf{E}^{\mathrm{sep}}_L \twoheadrightarrow \mathbf{E}^{\mathrm{sep}}_L/\omega^n_\phi \mathbf{E}^{\mathrm{sep},+}_L$ induces an isomorphism between the cohomology groups $\mathcal{H}^{b'}_{\mathrm{Fr}}(\mathbf{E}^{\mathrm{sep}}_L)$ and $\mathcal{H}^{b'}_{\mathrm{Fr}}(\mathbf{E}^{\mathrm{sep}}_L/\omega^n_\phi \mathbf{E}^{\mathrm{sep},+}_L)$ for all $b' \geq 0$. Lemma 2.2.13 states that there are two E_2-spectral sequences converging to $\mathcal{H}^*_{\mathrm{Fr}}(H_E, \mathbf{E}^{\mathrm{sep}}_L)$ respectively $\mathcal{H}^*_{\mathrm{Fr}}(H_E, \mathbf{E}^{\mathrm{sep}}_L/\omega^n_\phi \mathbf{E}^{\mathrm{sep},+}_L)$ (recall from the beginning of Step 4 that $\mathbf{E}^{\mathrm{sep}}_L$ is considered as discrete H_E-module):

$$H^{a'}(H_E, \mathcal{H}^{b'}_{\mathrm{Fr}}(\mathbf{E}^{\mathrm{sep}}_L)) \Rightarrow \mathcal{H}^{a'+b'}_{\mathrm{Fr}}(H_E, \mathbf{E}^{\mathrm{sep}}_L)$$

$$H^{a'}(H_E, \mathcal{H}^{b'}_{\mathrm{Fr}}(\mathbf{E}^{\mathrm{sep}}_L/\omega^n_\phi \mathbf{E}^{\mathrm{sep},+}_L)) \Rightarrow \mathcal{H}^{a'+b'}_{\mathrm{Fr}}(H_E, \mathbf{E}^{\mathrm{sep}}_L/\omega^n_\phi \mathbf{E}^{\mathrm{sep},+}_L).$$

We conclude as before: The canonical projection $\mathbf{E}^{\mathrm{sep}}_L \twoheadrightarrow \mathbf{E}^{\mathrm{sep}}_L/\omega^n_\phi \mathbf{E}^{\mathrm{sep},+}_L$ induces a morphism of spectral sequences and since the induced homomorphism is an isomorphism on the E_2-pages, we obtain an isomorphism between the limit terms $\mathcal{H}^b_{\mathrm{Fr}}(H_E, \mathbf{E}^{\mathrm{sep}}_L)$ and $\mathcal{H}^b_{\mathrm{Fr}}(H_E, \mathbf{E}^{\mathrm{sep}}_L/\omega^n_\phi \mathbf{E}^{\mathrm{sep},+}_L)$ for all $b \geq 0$.

To see that the left vertical arrow in the first square is an isomorphism we consider the E_2-spectral sequence (cf. Lemma 2.2.13)

$$\mathcal{H}^{a'}_{\mathrm{Fr}}(H^{b'}(H_E, \mathbf{E}^{\mathrm{sep}}_L)) \Rightarrow \mathcal{H}^{a'+b'}_{\mathrm{Fr}}(H_E, \mathbf{E}^{\mathrm{sep}}_L).$$

Since $\mathbf{E}^{\mathrm{sep}}_L$ is a separabel closure of $\mathbf{E}_E$ with Galois group isomorphic to H_E it is $H^{b'}(H_E, \mathbf{E}^{\mathrm{sep}}_L) = 0$ for all $b' > 0$. Then [29, Chapter II §1, (2.1.4) Proposition, p.100] says that we have an isomorphism $\mathcal{H}^b_{\mathrm{Fr}}(\mathbf{E}_E) \cong \mathcal{H}^b_{\mathrm{Fr}}(H_E, \mathbf{E}^{\mathrm{sep}}_L)$ for all $b \geq 0$ (here we identified $H^0(H_E, \mathbf{E}^{\mathrm{sep}}_L) = (\mathbf{E}^{\mathrm{sep}}_L)^{H_E} = \mathbf{E}_E$), which is induced from the canonical inclusion, i.e. the left vertical arrow in the first square also is an isomorphism. Then also the right vertical arrow is an isomorphism (since all other arrows are isomorphisms) and so is the map on E_2-terms from which we started. Hence γ_n is a quasi-isomorphism for all n.

To see that $\varprojlim_n \gamma_n$ is an isomorphism, it remains to check that the transition maps are surjective (cf. Proposition 2.3.11 respectively Remark 2.3.12). Since the transition maps

$$\mathbf{E}_E/\omega^{n+1}_\phi \mathbf{E}^+_E \otimes_{k_L} V \longrightarrow \mathbf{E}_E/\omega^n_\phi \mathbf{E}^+_E \otimes_{k_L} V,$$

$$\mathbf{E}^{\mathrm{sep}}_L/\omega^{n+1}_\phi \mathbf{E}^{\mathrm{sep},+}_L \otimes_{k_L} V \longrightarrow \mathbf{E}^{\mathrm{sep}}_L/\omega^n_\phi \mathbf{E}^{\mathrm{sep},+}_L \otimes_{k_L} V$$

are surjective and the groups carry the discrete topology, Corollary 2.1.2 says that also the transition maps

$$C^\bullet_{\mathrm{cts}}(\mathrm{Gal}(E_\infty|K), \mathbf{E}_E/\omega^{n+1}_\phi \mathbf{E}^+_E \otimes_{k_L} V) \longrightarrow C^\bullet_{\mathrm{cts}}(\mathrm{Gal}(E_\infty|K), \mathbf{E}_E/\omega^n_\phi \mathbf{E}^+_E \otimes_{k_L} V),$$

$$C^\bullet_{\mathrm{cts}}(G_K, \mathbf{E}^{\mathrm{sep}}_L/\omega^{n+1}_\phi \mathbf{E}^{\mathrm{sep},+}_L \otimes_{k_L} V) \longrightarrow C^\bullet_{\mathrm{cts}}(G_K, \mathbf{E}^{\mathrm{sep}}_L/\omega^n_\phi \mathbf{E}^{\mathrm{sep},+}_L \otimes_{k_L} V)$$

are surjective. But then Lemma 2.3.8 says that the transition maps

$$\mathcal{C}^\bullet_{\mathrm{Fr}}(\mathrm{Gal}(E_\infty|K), \mathbf{E}_E/\omega^{n+1}_\phi \mathbf{E}^+_E \otimes_{k_L} V) \longrightarrow \mathcal{C}^\bullet_{\mathrm{Fr}}(\mathrm{Gal}(E_\infty|K), \mathbf{E}_E/\omega^n_\phi \mathbf{E}^+_E \otimes_{k_L} V),$$

$$\mathcal{C}^\bullet_{\mathrm{Fr}}(G_K, \mathbf{E}^{\mathrm{sep}}_L/\omega^{n+1}_\phi \mathbf{E}^{\mathrm{sep},+}_L \otimes_{k_L} V) \longrightarrow \mathcal{C}^\bullet_{\mathrm{Fr}}(G_K, \mathbf{E}^{\mathrm{sep}}_L/\omega^n_\phi \mathbf{E}^{\mathrm{sep},+}_L \otimes_{k_L} V)$$

are surjective, too. Then Proposition 2.3.11 respectively Remark 2.3.12 say that $\varprojlim_n \gamma_n$ is a quasi isomorphism.

Step 5: $\varprojlim_n \beta_n$ is a quasi-isomorphism.
Now let $\Delta := \mathrm{Gal}(E_\infty \mid K_\infty)$. Lemma 2.2.21 then says that there is an E_2-spectral sequence of inverse systems of abelian groups given by

$$\mathcal{H}^i_{\mathrm{Fr}}(\Gamma_K, H^j(\Delta, \mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V)) \Longrightarrow \mathcal{H}^{i+j}_{\mathrm{Fr}}(\mathrm{Gal}(E_\infty|K), \mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V).$$

We will write $_n\mathscr{E}_2^{ij}$ for second page of this E_2-spectral sequence, $_n\mathscr{E}^k$ for its limit term and $\mathscr{E}_2^{ij} = \varprojlim_n \, _n\mathscr{E}_2^{ij}$ as wells as $\mathscr{E}^k = \varprojlim_n \, _n\mathscr{E}^k$. Proposition 5.1.10 says that the system $(H^j(\Delta, \mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V))_n$ is ML-zero for $j > 0$, i.e. for every $n \in \mathbb{N}$ there is an $m(n) \in \mathbb{N}$ such that the transition map

$$H^j(\Delta, \mathbf{E}_E/\omega_\phi^{m(n)} \mathbf{E}_E^+ \otimes_{k_L} V) \longrightarrow H^j(\Delta, \mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V)$$

is the zero map. For fixed $n \in \mathbb{N}$ and $m(n) \in \mathbb{N}$ as above, we then obtain that the transition map

$$C^i_{\mathrm{cts}}(\Gamma_K, H^j(\Delta, \mathbf{E}_E/\omega_\phi^{m(n)} \mathbf{E}_E^+ \otimes_{k_L} V)) \longrightarrow C^i_{\mathrm{cts}}(\Gamma_K, H^j(\Delta, \mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V))$$

is also zero for all $i \geq 0$ and $j > 0$. Then clearly the transition map

$$\mathcal{C}^i_{\mathrm{Fr}}(\Gamma_K, H^j(\Delta, \mathbf{E}_E/\omega_\phi^{m(n)} \mathbf{E}_E^+ \otimes_{k_L} V)) \longrightarrow \mathcal{C}^i_{\mathrm{Fr}}(\Gamma_K, H^j(\Delta, \mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V))$$

is zero for all $i \geq 0$ and $j > 0$, too. And so is the induced map on cohomology, i.e the inverse systems $(_n\mathscr{E}_2^{ij})_n$ are ML-zero for all $i \geq 0$ and $j > 0$. But then the edge homomorphism $\mathscr{E}_2^{i0} \to \mathscr{E}^i$ is an isomorphism, since $\mathscr{E}_2^{ij} = 0$ for all $i \geq 0$ and $j > 0$ (cf. [29, Chapter II, §1, (2.1.4) Corollary, p.100]). Recall that this edge homomorphism is induced from both, the canonical projection $\mathrm{Gal}(E_\infty|K) \twoheadrightarrow \Gamma_K$ and the canonical inclusion $(\mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V)^\Delta \hookrightarrow \mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V$.
Proposition 5.1.10 says that $(\eta_n)_n \colon (M_{1,n})_n \to (H^0(\Delta, \mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V))_n$ is an ML-isomorphism. Therefore the inverse systems $(\ker(\eta_n))_n$ and $(\mathrm{coker}(\eta_n))_n$ are ML-zero. As above, we then deduce that also the systems $(\mathcal{C}^i_{\varphi_{K|L}}(\Gamma_K, \ker(\eta_n)))_n$ and $(\mathcal{C}^i_{\mathrm{Fr}}(\Gamma_K, \mathrm{coker}(\eta_n)))_n$ are ML-zero for all $i \in \mathbb{N}_0$. Since $H^0(\Delta, \mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V)$ and $M_{1,n}$ carry the discrete topology for all $n \in \mathbb{N}$, we deduce from Lemma 2.2.15, which says that $\mathcal{C}^i_{\mathrm{Fr}}(\Gamma_K, -)$ is for discrete modules an exact functor, the exact sequence

$$0 \longrightarrow \mathcal{C}^i_{\varphi_{K|L}}(\Gamma_K, \ker(\eta_n)) \longrightarrow \mathcal{C}^i_{\varphi_{K|L}}(\Gamma_K, M_{1,n}) \xrightarrow{\mathcal{C}^i_{\mathrm{Fr}}(\Gamma_K, \eta_n)} \cdots$$

$$\cdots \longrightarrow \mathcal{C}^i_{\mathrm{Fr}}(\Gamma_K, H^0(\Delta, \mathbf{E}_E/\omega_\phi^n \mathbf{E}_E^+ \otimes_{k_L} V)) \to \mathcal{C}^i_{\mathrm{Fr}}(\Gamma_K, \mathrm{coker}(\eta_n)) \longrightarrow 0.$$

Taking inverse limits then gives us an isomorphism of complexes

$$\mathfrak{C}^\bullet_{\varphi_{K|L}}(\Gamma_K, M_1) \cong \mathcal{C}^\bullet_{\mathrm{Fr}}(\Gamma_K, H^0(\Delta, \mathbf{E}_E \otimes_{k_L} V)),$$

which, by construction, is induced from the canonical inclusion $M_1 \hookrightarrow \mathbf{E}_E \otimes_{k_L} V$ and which then prolongs to an isomorphism of its respective cohomology groups, i.e. for all $i \in \mathbb{N}_0$ we get

$$\mathcal{H}^i_{\varphi_{K|L}}(\Gamma_K, M_1) \cong \mathcal{H}^i_{\mathrm{Fr}}(\Gamma_K, H^0(\Delta, \mathbf{E}_E \otimes_{k_L} V)).$$

Together with the observation from above, that the edge homomorphism $\mathscr{E}_2^{i0} \to \mathscr{E}^i$ is an isomorphism for all $i \in \mathbb{N}_0$ we deduce for all $i \in \mathbb{N}_0$ the isomorphism

$$\mathcal{H}_{\varphi_{K|L}}^i(\Gamma_K, M_1) \cong \mathcal{H}_{\mathrm{Fr}}^i(\mathrm{Gal}(E_\infty|K), \mathbf{E}_E \otimes_{k_L} V),$$

which by construction is $\varprojlim_n \beta_n$.

$\square$

5.2. **Description with ψ.** In this subsection we want to give a description of the Galois cohomology groups of a representation using a ψ-operator.

Definition 5.2.1. Let A be an $\mathcal{O}_L$-module. We say that A is **cofinitely generated** if its Pontrjagin dual $A^\vee = \mathrm{Hom}_{\mathcal{O}_L}^{\mathrm{cts}}(A, L/\mathcal{O}_L)$ is finitely generated.

Remark 5.2.2.

1. *Since finitely generated $\mathcal{O}_L$-modules together with their natural topology are compact, cofinitely generated $\mathcal{O}_L$-modules are discrete, which means that $\mathrm{Hom}_{\mathcal{O}_L}^{\mathrm{cts}}(-, L/\mathcal{O}_L) = \mathrm{Hom}_{\mathcal{O}_L}(-, L/\mathcal{O}_L)$ for both, finitely and cofinitely generated $\mathcal{O}_L$-modules.*

2. *For $n \in \mathbb{N}$ we have an isomorphism*

$$\mathcal{O}_L/\pi_L^n \mathcal{O}_L \xrightarrow{\hspace{2cm}} (\mathcal{O}_L/\pi_L^n \mathcal{O}_L)^\vee$$

$$x \bmod \pi_L^n \mathcal{O}_L \longmapsto [1 \bmod \pi_L^n \mathcal{O}_L \mapsto \pi_L^{-n} x \bmod \mathcal{O}_L]$$

which then also implies a non-canonical isomorphism $T \cong T^\vee$ for a finitely generated torsion $\mathcal{O}_L$-module, since $(-)^\vee$ is compatible with finite direct sums. These isomorphisms are clearly topological, since all these objects carry the discrete topology.

3. *Due to Pontrjagin duality (cf. Proposition 4.0.2) a cofinitely generated $\mathcal{O}_L$-module is always the Pontrjagin dual of a finitely generated $\mathcal{O}_L$-module.*

4. *If $T \in \mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg})}(G_K)$ is torsion, then $T^\vee$ also is a finitely generated torsion $\mathcal{O}_L$-module with a continuous action from G_K.*

Definition 5.2.3. Let A be a cofinitely generated $\mathcal{O}_L$-module and $n \in \mathbb{N}$. We denote by A_n the kernel of the multiplication $m_{\pi_L^n}$ with π_L^n on A, i.e.

$$A_n = \ker(m_{\pi_L^n} \colon A \to A).$$

Proposition 5.2.4. *Let A be a cofinitely generated $\mathcal{O}_L$-module. Then we have $A = \varinjlim_n A_n$. In particular, if A is torsion, say with $\pi_L^m A = 0$ for some $m \in \mathbb{N}$, then we have $A = A_m$.*

Proof. Let T be a finitely generated $\mathcal{O}_L$-module such that $A = \mathrm{Hom}_{\mathcal{O}_L}^{\mathrm{cts}}(T, L/\mathcal{O}_L)$, let $e_1, \ldots, e_m$ be a set of generators of T and let $f \in A$. Then for every $i \in \{1, \ldots, m\}$ there exists an $n_i \in \mathbb{N}$ such that $\pi_L^{n_i} f(e_i) = 0$. Set $n := \max_i n_i$. Then it is $\pi_L^n f(\alpha) = 0$ for every $\alpha \in A$, i.e. $f \in A_n$. In particular, if there exists $m \in \mathbb{N}$ such that $\pi_L^m g = 0$ for every $g \in A$, then the above shows $A = A_m$. $\square$

Lemma 5.2.5. *Let $T \in \mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg})}(G_K)$ such that $\pi_L^m T = 0$. Then H_K acts continuously on $\mathbf{A} \otimes_{\mathcal{O}_L} T$ equipped with the discrete topology.*

Proof. Recall from page 17 that

$$\mathbf{A} \cong \varprojlim_n \mathbf{A}_L^{\mathrm{nr}}/\pi_L^n \mathbf{A}_L^{\mathrm{nr}}$$

and that H_L is the Galois group of $\mathbf{A}_L^{\mathrm{nr}}|\mathbf{A}_L$. The latter means, that H_L acts continuously on $\mathbf{A}_L^{\mathrm{nr}}$ with respect to the discrete topology because if $x \in \mathbf{A}_L^{\mathrm{nr}}$, then $\mathbf{B}_L(x)|\mathbf{B}_L$ is a finite extension and therefore it exists an open subgroup $U \leq H_L$ which fixes x. But then (U, x) is an open subset of the preimage of x under the operation

$$H_L \times \mathbf{B}_L \to \mathbf{B}_L.$$

Then H_L also clearly acts continuously on $\mathbf{A}_L^{\mathrm{nr}}/\pi_L^n \mathbf{A}_L^{\mathrm{nr}}$ for all $n \in \mathbb{N}$ equipped with the discrete topology. Since H_K is an open subgroup of H_L it then also acts continuously on $\mathbf{A}_L^{\mathrm{nr}}/\pi_L^n \mathbf{A}_L^{\mathrm{nr}}$ for all $n \in \mathbb{N}$ equipped with the discrete topology. Because of $\pi_L^n T = 0$ we have $T = T \otimes_{\mathcal{O}_L} \mathcal{O}_L/\pi_L^n \mathcal{O}_L$ and therefore

$$\mathbf{A} \otimes_{\mathcal{O}_L} T = \mathbf{A} \otimes_{\mathcal{O}_L} \mathcal{O}_L/\pi_L^n \mathcal{O}_L \otimes_{\mathcal{O}_L} T = \mathbf{A}/\pi_L^n \mathbf{A} \otimes_{\mathcal{O}_L} T = \mathbf{A}_L^{\mathrm{nr}}/\pi_L^n \mathbf{A}_L^{\mathrm{nr}} \otimes_{\mathcal{O}_L} T.$$

Since H_K acts continuously on both T and $\mathbf{A}_L^{\mathrm{nr}}/\pi_L^n \mathbf{A}_L^{\mathrm{nr}}$ with respect to the discrete topology it does so on the tensor product equipped with the linear topological structure, which then again is discrete. $\qquad\square$

Lemma 5.2.6. *Let* $T \in \mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg})}(G_K)$ *such that* $\pi_L^m T = 0$. *Then we have* $H_{\mathrm{cts}}^i(H_K, \mathbf{A} \otimes_{\mathcal{O}_L} T) = 0$ *for all* $i > 0$.

Proof. This is [31, Lemma 5.2, p. 23–24], since it is even $H_{\mathrm{cts}}^i(U, \mathbf{E}_L^{\mathrm{sep}}) = 0$ for all $i > 0$ and open subgroups $U \leq H_L$. $\qquad\square$

Corollary 5.2.7. *Let* A *be a cofinitely generated* $\mathcal{O}_L$-*module with a continuous action from* G_K. *Then* H_K *acts continuously on* $\mathbf{A} \otimes_{\mathcal{O}_L} A$ *equipped with the discrete topology and we have* $H_{\mathrm{cts}}^i(H_K, \mathbf{A} \otimes_{\mathcal{O}_L} A) = 0$ *for all* $i > 0$.

Proof. If A is torsion, then Remark 5.2.2 says that this is just Lemma 5.2.5 and Lemma 5.2.6. If A is general, then with Proposition 5.2.4 we can write $A = \varinjlim_n A_n$, where the A_n are torsion $\mathcal{O}_L$-modules. Since tensor products commute with colimits we have

$$\varinjlim_n \mathbf{A} \otimes_{\mathcal{O}_L} A_n \cong \mathbf{A} \otimes_{\mathcal{O}_L} A$$

algebraically. But the direct limit topology of $\varinjlim_n \mathbf{A} \otimes_{\mathcal{O}_L} A_n$ again is discrete and so the above isomorphism is also topological. Then, $\mathbf{A} \otimes_{\mathcal{O}_L} A$ is a discrete H_L-module and therefore we deduce from [29, (1.5.1) Proposition, p. 45–46]

$$H^i(H_K, \mathbf{A} \otimes_{\mathcal{O}_L} A) = \varinjlim_n H^i(H_K, \mathbf{A} \otimes_{\mathcal{O}_L} A_n)$$

for all $i \geq 0$. Since $H^i(H_K, \mathbf{A} \otimes_{\mathcal{O}_L} A_n) = 0$ for all $i > 0$ and $n \in \mathbb{N}$ we also have $H^i(H_K, \mathbf{A} \otimes_{\mathcal{O}_L} A) = 0$ for all $i > 0$. $\qquad\square$

Lemma 5.2.8. *Let* A *be a cofinitely generated* $\mathcal{O}_L$-*module. Then the sequence*

$$0 \longrightarrow A \longrightarrow \mathbf{A} \otimes_{\mathcal{O}_L} A \xrightarrow{\mathrm{Fr}\otimes\mathrm{id}-\mathrm{id}} \mathbf{A} \otimes_{\mathcal{O}_L} A \longrightarrow 0.$$

is exact and has a continuous set theoretical splitting, where all terms are equipped with the discrete topology.

Proof. Since $\mathbf{A}$ is a flat $\mathcal{O}_L$-module the first assertion comes from Lemma 5.1.1, the second is obvious since all terms carry the discrete topology. $\qquad\square$

Proposition 5.2.9. *Let A be a cofinitely generated $\mathcal{O}_L$-module with a continuous action from G_K. Then the exact sequence*

$$0 \longrightarrow A \longrightarrow \mathbf{A} \otimes_{\mathcal{O}_L} A \xrightarrow{\mathrm{Fr}\otimes\mathrm{id}-\mathrm{id}} \mathbf{A} \otimes_{\mathcal{O}_L} A \longrightarrow 0.$$

and the canonical homomorphism

$$(\mathbf{A} \otimes_{\mathcal{O}_L} A)^{H_K} \hookrightarrow C^\bullet_{\mathrm{cts}}(H_K, \mathbf{A} \otimes_{\mathcal{O}_L} A)$$

induce quasi isomorphisms

$$(4) \qquad C^\bullet_{\mathrm{cts}}(H_K, A) \xrightarrow{\simeq} \mathcal{C}^\bullet_{\mathrm{Fr}}(H_K, \mathbf{A} \otimes_{\mathcal{O}_L} A) \xleftarrow{\simeq} \mathcal{C}^\bullet_{\varphi_{K|L}}(\mathcal{D}_{K|L}(A)).$$

Proof. Since Fr commutes with the action from H_K, the exact sequence

$$0 \longrightarrow A \longrightarrow \mathbf{A} \otimes_{\mathcal{O}_L} A \xrightarrow{\mathrm{Fr}\otimes\mathrm{id}-\mathrm{id}} \mathbf{A} \otimes_{\mathcal{O}_L} A \longrightarrow 0.$$

clearly is an exact sequence of (discrete) H_K-modules. Then Corollary 2.3.4 says that

$$H^i_{\mathrm{cts}}(H_K, A) \cong \mathcal{H}^i_{\mathrm{Fr}}(H_K, \mathbf{A} \otimes_{\mathcal{O}_L} A),$$

which is exactly the first quasi isomorphism. For the second quasi isomorphism it is with Proposition 2.2.13 enough to show

$$H^i_{\mathrm{cts}}(H_K, \mathbf{A} \otimes_{\mathcal{O}_L} A) = \begin{cases} \mathcal{D}_{K|L}(A) & , \text{ if } i = 0 \\ 0 & , \text{ else .} \end{cases}$$

But this is exactly the above Corollary 5.2.7. $\qquad\square$

Corollary 5.2.10. *Let A be a cofinitely generated $\mathcal{O}_L$-module with a continuous action from G_K. Then the following sequence is exact*

$$0 \longrightarrow H^0_{\mathrm{cts}}(H_K, A) \longrightarrow \mathcal{D}_{K|L}(A) \xrightarrow{\varphi_{K|L}-\mathrm{id}} \mathcal{D}_{K|L}(A) \longrightarrow H^1_{\mathrm{cts}}(H_K, A) \longrightarrow 0.$$

Proof. This is the long exact cohomology sequence of

$$0 \longrightarrow A \longrightarrow \mathbf{A} \otimes_{\mathcal{O}_L} A \xrightarrow{\mathrm{Fr}\otimes\mathrm{id}-\mathrm{id}} \mathbf{A} \otimes_{\mathcal{O}_L} A \longrightarrow 0.$$

combined with $H^1_{\mathrm{cts}}(H_K, \mathbf{A} \otimes_{\mathcal{O}_L} A) = 0$ from Corollary 5.2.7. $\qquad\square$

In the next step, we want to replace the above exact sequence with a sequence of $\Lambda_K = \mathcal{O}_L[\![\Gamma_K]\!]$-modules. An idea how to do this gives Nekovář in [26, (8.3.3) Corollary, p. 159] but unfortunately the modules we are working with are not ind-admissible, since $\mathbf{A}$ is no direct limit of finitely generated $\mathcal{O}_L[G_K]$-modules. As in the proof of Theorem 5.1.11 we use limits and colimits to reduce to the case of discrete coefficients.

We want to recall the notation from [26, (8.1.1), p. 148; (8.2.1), p. 157] and from the beginning of [26, (8.3) Infinite extensions, p. 158–159].

Definition 5.2.11. Let G be a profinite group, $U \leq G$ an open subgroup and M a discrete $\mathcal{O}_L[U]$-module. We then define the induced module to be

$$\mathrm{Ind}^G_U(M) := \{f \colon G \to X \mid f(ug) = uf(g) \text{ for all } u \in U, g \in G\}.$$

$\mathrm{Ind}^G_U(M)$ carries a G-action by $(g \cdot f)(\sigma) := f(\sigma g)$. Furthermore, if M is a discrete $\mathcal{O}_L[G]$-module define

$$_U M := \mathrm{Hom}_{\mathcal{O}_L}(\mathcal{O}_L[G/U], M).$$

$_U M$ then again carries a G-action by $(\sigma \cdot (f))(x) := \sigma(f(\sigma^{-1}(x)))$. Let now $H \triangleleft G$ be a closed, normal subgroup and $\mathcal{U}(G; H)$ be the open subgroups of G containing H. Then, for $V, U \in \mathcal{U}(G; H)$ with $V \subseteq U$ the canonical map $G/V \twoheadrightarrow G/U$ induces $\mathcal{O}_L$-linear maps $_U M \hookrightarrow {}_V M$ under which the system $(_U M)_{U \in \mathcal{U}(G;H)}$ becomes a filtered directed system. We then set

$$F_{G/H}(M) := \varinjlim_{U \in \mathcal{U}(G;H)} {}_U M.$$

Similar as above, $F_{G/H}(M)$ then also carries an action from G. If $H = \{1\}$ we write $\mathcal{U}(G)$ instead of $\mathcal{U}(G; H)$ and $F_G(M)$ instead of $F_{G/\{1\}}(M)$. Furthermore, we set $\mathcal{U}_K := \mathcal{U}(G_K; H_K)$ and we write $F_{\Gamma_K}(M)$ instead of F_{G_K/H_K}. This can lead to an abuse of notation, but it will be clear from the context, which construction is chosen.

Remark 5.2.12. *For the above situation, Nekovář proves in [26, (8.1.3), p. 149] that*

$$\mathrm{Ind}_U^G(M) \longrightarrow {}_U M, \ f \longmapsto [gU \mapsto g(f(g^{-1}))]$$

is a G-equivariant isomorphism.

Remark 5.2.13. *In the above situation, if $f \in F_{G/H}(M)$ then it exists $U \in \mathcal{U}(G; H)$ such that $f \in {}_U M$. If then $V \in \mathcal{U}(G; H)$ with $V \subseteq U$ we also have $f \in {}_V M$ as well as*

$$f(gV) = f(gU)$$

for all $g \in G$.

Remark 5.2.14. *Let G be a group and $H \triangleleft G$ a normal subgroup such that G/H is abelian. Then every subgroup $U \leq G$ with $H \subseteq U$ is normal as well. In particular, if additionally G is profinite and H is closed, then the elements of $\mathcal{U}(G; H)$ are normal, open subgroups of G containing H. This is of great interest for us, since our application of this theory will be $G = G_K$ and $H = H_K$ and $G = \Gamma_K$ and $H = \{1\}$. In both cases, the factor G/H is Γ_K which is abelian.*

Proposition 5.2.15. *Let G be a profinite group, $H \triangleleft G$ a closed, normal subgroup, M a discrete $\mathcal{O}_L[G]$-module and let $U \in \mathcal{U}(G; H)$. Then the compact-open topology on $_U M$ is discrete and the G-action on $_U M$ is again continuous with respect to this topology.*
Furthermore, the transition maps $_V M \to {}_{V'} M$ for $V, V' \in \mathcal{U}(G; H)$ with $V' \subseteq V$ are injective, the direct limit topology on $F_{G/H}(M)$ is discrete and its G-action is continuous.

Proof. Since $U \leq G$ is an open subgroup, the set of cosets G/U is finite and therefore $\mathcal{O}_L[G/U]$ is a finitely generated free $\mathcal{O}_L$-module. So in particular, $\mathcal{O}_L[G/U]$ is compact. Then $_U M = \mathrm{Hom}_{\mathcal{O}_L}(\mathcal{O}_L[G/U], M)$ is discrete with respect to the compact open topology since M is discrete. To see that the action from G is continuous on $_U M$ it is enough to see that for every $f \in {}_U M$ there exists an open subset $V \subseteq G$ under which f is fixed. Note also that G acts by left multiplication on G/U. So, let $f \in {}_U M$ and let $g_1, \dots, g_n \in G$ be a set of representatives of the cosets of G/U. Since the action of G on M is continuous and M carries the discrete topology, there exist open subsets $V_1, \dots, V_n \subseteq G$ such that g_i is fixed by V_i for all $1 \leq i \leq n$. Then f is fixed by $V := \cap_i V_i$.

The statements on $F_{G/H}(M)$ follow immediately by taking the direct limit. So the statement on the transition maps is left. Let $V, V' \in \mathcal{U}(G; H)$ with $V' \subseteq V$. Then the canonical map $G/V' \to G/V$ is surjective. Then $\mathcal{O}_L[G/V'] \to \mathcal{O}_L[G/V]$ is a surjective $\mathcal{O}_L$-linear homomorphism and since $\mathrm{Hom}_{\mathcal{O}_L}(-, M)$ is left exact, the induced homomorphism $_{V'} M \to {}_V M$ is injective. $\square$

In the above situation, under the additional assumption that U is normal in G, Nekovář introduces in [26, (8.1.6.3) Conjugation, p. 151] an action from G/U on $_U M$ which will be important for us. We recall this action in the following Remark and we prove the statements.

Remark 5.2.16. *Let G be a profinite group, $U \triangleleft G$ be an open, normal subgroup and M a discrete $\mathcal{O}_L[G]$-module. For $g \in G$ and $f \in \mathrm{Ind}_U^G(M)$ we define $\widetilde{\mathrm{Ad}}(g)(f)$ to be*

$$(\widetilde{\mathrm{Ad}}(g)(f))(\sigma) := g(f(g^{-1}\sigma)).$$

This is an action from G on $\mathrm{Ind}_U^G(M)$ which is trivial on U, i.e. it induces an action from G/U on $\mathrm{Ind}_U^G(M)$ which we will denote also by $\widetilde{\mathrm{Ad}}$. Since both, $\mathrm{Ind}_U^G(M)$ and G/U carry the discrete topology, this action is continuous.
Furthermore, this action commutes with the standard action from G and under the isomorphism $\mathrm{Ind}_U^G(M) \cong {}_U M$ from Remark 5.2.12 it corresponds to the G/U-action

$$(\widetilde{\mathrm{Ad}}(gU)(f))(\sigma U) = f(\sigma g U)$$

on $_U M$. Then clearly the G-action on $_U M$ commutes with this action from G/U and the latter is again continuous.

Lemma 5.2.17. *Let G be a profinite group and $H \triangleleft G$ a closed, normal subgroup, such that G/H is abelian. Then $\widetilde{\mathrm{Ad}}$ induces a continuous action from G/H on $F_{G/H}(M)$.*
In particular, with this action $F_{G/H}(M)$ becomes an $\mathcal{O}_L[\![G/H]\!]$-module.

Proof. The action from G/H on $F_{G/H}(M)$ is given as follows: For $f \in F_{G/H}(M)$ and $U \in \mathcal{U}(G; H)$ such that $f \in {}_U M$ and $g \in G$ we have

$$\widetilde{\mathrm{Ad}}(gH)(f) = \widetilde{\mathrm{Ad}}(gU)(f).$$

This is well defined, since if $V \in \mathcal{U}(G; H)$ such that $V \subseteq U$ then $f \in {}_V M$ and for $\sigma \in G$ we have

$$\widetilde{\mathrm{Ad}}(gU)(f)(\sigma U) = f(\sigma g U) = f(\sigma g V) = \widetilde{\mathrm{Ad}}(gV)(f)(\sigma V).$$

The action is continuous since the above f is fixed under U/H, which is an open subgroup of G/H.
If f is as above, $x \in \mathcal{O}_L[\![G/H]\!]$ and $\mathrm{pr}_U \colon \mathcal{O}_L[\![G/H]\!] \to \mathcal{O}_L[G/U]$ denotes the canonical projection, then we have

$$\widetilde{\mathrm{Ad}}(x)(f) = \widetilde{\mathrm{Ad}}(\mathrm{pr}_U(x))(f).$$

This again is well defined and makes $F_{G/H}(M)$ into an $\mathcal{O}_L[\![G/H]\!]$-module. $\square$

Proposition 5.2.18. *Let G be a profinite group and $H \triangleleft G$ a closed, normal subgroup such that G/H is abelian. Then $F_{G/H}$ is an exact functor, viewed as functor from discrete $\mathcal{O}_L[G]$-modules to discrete $\mathcal{O}_L[\![G/H]\!][G]$-modules.*

Proof. The above Lemma 5.2.17 says that $F_{G/H}$ is a functor from discrete $\mathcal{O}_L[G]$-modules to discrete $\mathcal{O}_L[\![G/H]\!][G]$-modules. So it is left to check that it is exact. For fixed $U \in \mathcal{U}(G; H)$ the functor $M \mapsto {}_U M$ from discrete $\mathcal{O}_L[G]$-modules to discrete $\mathcal{O}_L[G/U][G]$-modules is exact since $\mathcal{O}_L[G/U]$ is a finitely generated, free $\mathcal{O}_L$-module. Since taking direct limits is exact as well, $F_{G/H}$ is exact. $\square$

Definition 5.2.19. If $\mathbf{C}$ is an abelian category, we denote by $\mathbf{D}(\mathbf{C})$ the corresponding derived category. As usual, we denote by $\mathbf{D}^+(\mathbf{C})$ the full subcategory whose objects are the complexes, which have no nonnegative entries and by $\mathbf{D}^b(\mathbf{C})$ the full subcategory whose objects are the

bounded below complexes.

If $\mathfrak{C}^\bullet$ is a complex in an abelian category $\mathbf{C}$, we denote as in [26] by $\mathbf{R}\Gamma(\mathfrak{C}^\bullet)$ the corresponding complex as an object in the derived category $\mathbf{R}(\mathbf{C})$.

In particular, if G is a profinite group and M is a topological G-module we set

$$\mathbf{R}\Gamma_{\mathrm{cts}}^\bullet(G, M) \coloneqq \mathbf{R}\Gamma(C_{\mathrm{cts}}^\bullet(G, M))$$

as an object in $\mathbf{R}(\mathbf{Ab})$.

Remark 5.2.20. *Let G be a profinite group, $H \triangleleft G$ a closed, normal subgroup, and M a discrete $\mathcal{O}_L[G]$-module. As in [26, (3.6.1.4), p. 72] we define an action from G on $C_{\mathrm{cts}}^\bullet(H, M)$ by*

$$\mathrm{Ad}(g)(c)(h_0, \ldots, h_n) \coloneqq g(c(g^{-1}h_0 g, \ldots, g^{-1}h_n g)),$$

where $c \in C_{\mathrm{cts}}^n(H, M)$. In loc. cit. Nekovář also proves that for $h \in H$ this action is homotopic to the identity and therefore induces an action from G/H on $\mathbf{R}\Gamma_{\mathrm{cts}}^\bullet(H, M)$ and $H^*(H, M)$ respectively.

Similarly, by

$$C_{\mathrm{cts}}^\bullet(G, F_{G/H}(M)) \xrightarrow{\widetilde{\mathrm{Ad}(g)}_*} C_{\mathrm{cts}}^\bullet(G, F_{G/H}(M)) \xrightarrow{\mathrm{Ad}(g)} C_{\mathrm{cts}}^\bullet(G, F_{G/H}(M))$$

we can define an action from G on $C_{\mathrm{cts}}^\bullet(G, F_{G/H}(M))$. Note that in this situation $\mathrm{Ad}(g) \colon C_{\mathrm{cts}}^\bullet(G, F_{G/H}(M)) \to C_{\mathrm{cts}}^\bullet(G, F_{G/H}(M))$ is homotopic to the identity and so the complex $\mathbf{R}\Gamma_{\mathrm{cts}}^\bullet(G, F_{G/H}(M))$ becomes a complex of $\mathcal{O}_L[\![G/H]\!]$-modules. See also Remark 5.2.23 below.

Proposition 5.2.21. *Let G be a profinite group, $H \triangleleft G$ a closed, normal subgroup and M a discrete $\mathcal{O}_L[G]$-module. Then there is a canonical morphism of complexes*

$$C_{\mathrm{cts}}^\bullet(G, F_{G/H}(M)) \to C_{\mathrm{cts}}^\bullet(H, M),$$

which is a quasi isomorphism. Moreover, for $g \in G$ the diagram

$$
\begin{array}{ccc}
C_{\mathrm{cts}}^\bullet(G, F_{G/H}(M)) & \longrightarrow & C_{\mathrm{cts}}^\bullet(H, M) \\
{\scriptstyle \widetilde{\mathrm{Ad}(g)}} \downarrow & & \\
C_{\mathrm{cts}}^\bullet(G, F_{G/H}(M)) & & \downarrow {\scriptstyle \mathrm{Ad}(g)} \\
{\scriptstyle \mathrm{Ad}(g)} \downarrow & & \\
C_{\mathrm{cts}}^\bullet(G, F_{G/H}(M)) & \longrightarrow & C_{\mathrm{cts}}^\bullet(H, M)
\end{array}
$$

is commutative. So in particular, the corresponding isomorphism $\mathbf{R}\Gamma_{\mathrm{cts}}^\bullet(G, F_{G/H}(M)) \to \mathbf{R}\Gamma_{\mathrm{cts}}^\bullet(H, M)$ in the derived category $\mathbf{D}^+(\mathcal{O}_L\text{-}\mathbf{Mod})$ is G/H-linear.

Proof. For the proof set $\mathcal{U} \coloneqq \mathcal{U}(G; H)$. [29, (1.5.1) Proposition, p. 45–46] says that we have

$$C_{\mathrm{cts}}^\bullet(G, F_{G/H}(M)) \cong C_{\mathrm{cts}}^\bullet(G, \varinjlim_{U \in \mathcal{U}} {}_U M) \cong \varinjlim_{U \in \mathcal{U}} C_{\mathrm{cts}}^\bullet(G, {}_U M).$$

With Remark 5.2.12 we then obtain

$$\varinjlim_{U \in \mathcal{U}} C_{\mathrm{cts}}^\bullet(G, {}_U M) \cong \varinjlim_{U \in \mathcal{U}} C_{\mathrm{cts}}^\bullet(G, \mathrm{Ind}_U^G(M)).$$

Shapiro's Lemma (cf. [29, (1.6.4) Proposition, p. 62–63]) and again [29, (1.5.1) Proposition, p. 45–46] then give us

$$\varinjlim_{U \in \mathcal{U}} C^\bullet_{\mathrm{cts}}(G, \mathrm{Ind}^G_U(M)) \simeq \varinjlim_{U \in \mathcal{U}} C^\bullet_{\mathrm{cts}}(U, M) \cong C^\bullet_{\mathrm{cts}}(\varprojlim_{U \in \mathcal{U}} U, M) = C^\bullet_{\mathrm{cts}}(H, M).$$

[26, (8.1.6.3), p. 151] says that for $U \in \mathcal{U}(G; H)$ and $g \in G$ the diagram

$$
\begin{array}{ccc}
C^\bullet_{\mathrm{cts}}(G, \mathrm{Ind}^G_U(M)) & \longrightarrow & C^\bullet_{\mathrm{cts}}(U, M) \\
\widetilde{\mathrm{Ad}(g)} \downarrow & & \\
C^\bullet_{\mathrm{cts}}(G, \mathrm{Ind}^G_U(M)) & & \Big\downarrow \mathrm{Ad}(g) \\
\mathrm{Ad}(g) \downarrow & & \\
C^\bullet_{\mathrm{cts}}(G, \mathrm{Ind}^G_U(M)) & \longrightarrow & C^\bullet_{\mathrm{cts}}(U, M)
\end{array}
$$

is commutative. Taking direct limits then proves the commutativity of the desired diagram. $\square$

Corollary 5.2.22. *Let $M \in \mathbf{Mod}^{\text{ét}}_{\varphi, \Gamma}(\mathbf{A}_{K|L})$ such that M is discrete as $\mathcal{O}_L[G]$-module. Then the above* Proposition 5.2.21 *together with* Proposition 2.2.13 *induces the Γ_K-linear isomorphism*

$$\mathbf{R}\Gamma(\mathcal{C}^\bullet_{\varphi_{K|L}}(\Gamma_K, F_{\Gamma_K}(M))) \xrightarrow{\;\cong\;} \mathbf{R}\Gamma(\mathcal{C}^\bullet_{\varphi_{K|L}}(M)).$$

Remark 5.2.23. *In the situation of* Proposition 5.2.21*, the morphism*

$$\mathrm{Ad}(g) \colon C^\bullet_{\mathrm{cts}}(G, F_{G/H}(M)) \longrightarrow C^\bullet_{\mathrm{cts}}(G, F_{G/H}(M))$$

for $g \in G$ is homotopic to the identity (cf. [26, (3.6.1.4), p. 72] respectively Remark 5.2.20*) and therefore the diagram*

$$
\begin{array}{ccc}
\mathbf{R}\Gamma^\bullet_{\mathrm{cts}}(G, F_{G/H}(M)) & \longrightarrow & \mathbf{R}\Gamma^\bullet_{\mathrm{cts}}(H, M) \\
\widetilde{\mathrm{Ad}(g)}_* \Big\downarrow & & \Big\downarrow \mathrm{Ad}(g) \\
\mathbf{R}\Gamma^\bullet_{\mathrm{cts}}(G, F_{G/H}(M)) & \longrightarrow & \mathbf{R}\Gamma_{\mathrm{cts}}(H, M)
\end{array}
$$

is commutative. The corresponding diagram for cohomology groups

$$
\begin{array}{ccc}
H^*_{\mathrm{cts}}(G, F_{G/H}(M)) & \longrightarrow & H^*_{\mathrm{cts}}(H, M) \\
\widetilde{\mathrm{Ad}(g)}_* \Big\downarrow & & \Big\downarrow \mathrm{Ad}(g) \\
H^*_{\mathrm{cts}}(G, F_{G/H}(M)) & \longrightarrow & H^*_{\mathrm{cts}}(H, M)
\end{array}
$$

then also is commutative. This then explains that the statement from [29, p. 65] coincides with the theory from Nekovář .

Proposition 5.2.24. *Let* $A = \varinjlim_m A_m$ *be a cofinitely generated* $\mathcal{O}_L$-*module, where* $A_m = \ker(\mu_{\pi_L^m})$ *as usual, with a continuous action from* G_K *and set*

$$\mathcal{A}_{mn} := (\mathbf{A} \otimes_{\mathcal{O}_L} A_m) / \left(\pi_L^n \mathbf{A}^+ \otimes_{\mathcal{O}_L} A_m \right)$$

$$M_{mn} := (\mathbf{A} \otimes_{\mathcal{O}_L} A_m)^{H_K} / \left(\pi_L^n \mathbf{A}^+ \otimes_{\mathcal{O}_L} A_m \right)^{H_K}.$$

Then the following diagram is commutative and each arrow in it is a quasi isomorphism. Moreover, the vertical arrows on the right hand side are homomorphisms of Λ_K-*modules.*

$$
\begin{array}{ccc}
C^\bullet_{\mathrm{cts}}(H_K, A) & \xleftarrow{\;\simeq\;} & C^\bullet_{\mathrm{cts}}(G_K, F_{\Gamma_K}(A)) \\
\Big\uparrow{\scriptstyle\simeq} & & \Big\uparrow{\scriptstyle\cong} \\
\varinjlim_{m\in\mathbb{N}} C^\bullet_{\mathrm{cts}}(H_K, A_m) & \xleftarrow{\;\simeq\;} & \varinjlim_{m\in\mathbb{N}} C^\bullet_{\mathrm{cts}}(G_K, F_{\Gamma_K}(A_m)) \\
\Big\uparrow{\scriptstyle\simeq} & & \Big\downarrow{\scriptstyle\simeq} \\
\varinjlim_{m\in\mathbb{N}}\varprojlim_{n\in\mathbb{N}} \mathcal{C}^\bullet_{\mathrm{Fr}}(H_K, \mathcal{A}_{mn}) & \xleftarrow{\;\simeq\;} & \varinjlim_{m\in\mathbb{N}}\varprojlim_{n\in\mathbb{N}} \mathcal{C}^\bullet_{\mathrm{Fr}}(G_K, F_{\Gamma_K}(\mathcal{A}_{mn})) \\
\Big\uparrow{\scriptstyle\simeq} & & \Big\uparrow{\scriptstyle\simeq} \\
\varinjlim_{m\in\mathbb{N}}\varprojlim_{n\in\mathbb{N}} \mathcal{C}^\bullet_{\varphi_{K|L}}(M_{mn}) & \xleftarrow{\;\simeq\;} & \varinjlim_{m\in\mathbb{N}}\varprojlim_{n\in\mathbb{N}} \mathcal{C}^\bullet_{\varphi_{K|L}}(\Gamma_K, F_{\Gamma_K}(M_{mn})) \\
\Big\uparrow{\scriptstyle\cong} & & \\
\varinjlim_{m\in\mathbb{N}} \mathcal{C}^\bullet_{\varphi_{K|L}}(\mathcal{D}_{K|L}(A_m)) & & \\
\Big\downarrow{\scriptstyle\simeq} & & \\
\mathcal{C}^\bullet_{\varphi_{K|L}}(\mathcal{D}_{K|L}(A)). & &
\end{array}
$$

In particular, the induced isomorphism $\mathbf{R}\Gamma(\mathcal{C}^\bullet_{\varphi_{K|L}}(\mathcal{D}_{K|L}(A))) \cong \mathbf{R}\Gamma^\bullet_{\mathrm{cts}}(G_K, F_{\Gamma_K}(A))$ *in* $\mathbf{D}^+(\mathcal{O}_L\text{-}\mathbf{Mod})$ *is* Λ_K-*linear, i.e. it is an isomorphism in* $\mathbf{D}^+(\Lambda_K\text{-}\mathbf{Mod})$.

Proof. We start with the left column and we consider the following diagram

$$C_{\mathrm{cts}}^{\bullet}(H_K, A)$$

$$(1) \Big\uparrow \simeq$$

$$\varinjlim_{m\in\mathbb{N}} C_{\mathrm{cts}}^{\bullet}(H_K, A_m)$$

$$(2) \Big\downarrow \simeq$$

$$\varinjlim_{m\in\mathbb{N}} \mathcal{C}_{\mathrm{Fr}}^{\bullet}(H_K, \mathbf{A} \otimes_{\mathcal{O}_L} A_m)$$

$$\xrightarrow[\cong]{\quad(5)\quad} \varinjlim_{m\in\mathbb{N}}\varprojlim_{n\in\mathbb{N}} \mathcal{C}_{\mathrm{Fr}}^{\bullet}(H_K, \mathcal{A}_{mn})$$

$$(3) \Big\updownarrow \simeq \qquad\qquad \simeq \Big\uparrow (7)$$

$$\varinjlim_{m\in\mathbb{N}}\varprojlim_{n\in\mathbb{N}} \mathcal{C}_{\varphi_{K|L}}^{\bullet}(M_{mn})$$

$$\xrightarrow[\cong]{\quad(6)\quad}$$

$$\varinjlim_{m\in\mathbb{N}} \mathcal{C}_{\varphi_{K|L}}^{\bullet}(\mathcal{D}_{K|L}(A_m))$$

$$(4) \Big\downarrow \simeq$$

$$\mathcal{C}_{\varphi_{K|L}}^{\bullet}(\mathcal{D}_{K|L}(A)).$$

That the morphisms (1) and (4) are quasi isomorphisms is well known (cf. eg. [29, (1.5.1) Proposition, p. 45–46]). (2) and (3) are quasi isomorphisms by Proposition 5.2.9. Proposition 2.3.7 says that (5) and (6) are isomorphisms of complexes. But then (7) is also a quasi isomorphism. So, all the morphisms in the left column of the original diagram are at least quasi isomorphisms. The horizontal morphisms are quasi isomorphisms by Proposition 5.2.21 and therefore the morphisms in the right column are also quasi isomorphisms. So it is left to check that the induced isomorphism $\mathbf{R}\Gamma(\mathcal{C}_{\varphi_{K|L}}^{\bullet}(\mathcal{D}_{K|L}(A))) \cong \mathbf{R}\Gamma_{\mathrm{cts}}^{\bullet}(G_K, F_{\Gamma_K}(A))$ is Λ_K-linear. But the morphisms

$$\varinjlim_{m\in\mathbb{N}} \mathcal{C}_{\varphi_{K|L}}^{\bullet}(\mathcal{D}_{K|L}(A_m)) \longrightarrow \mathcal{C}_{\varphi_{K|L}}^{\bullet}(\mathcal{D}_{K|L}(A))$$

and

$$\varinjlim_{m\in\mathbb{N}}\varprojlim_{n\in\mathbb{N}} \mathcal{C}_{\varphi_{K|L}}^{\bullet}(M_{mn}) \longrightarrow \varinjlim_{m\in\mathbb{N}} \mathcal{C}_{\varphi_{K|L}}^{\bullet}(\mathcal{D}_{K|L}(A_m))$$

are clearly Λ_K-linear and so are all the morphisms in the right column of the original diagram with respect to the Λ_K-action induced by $\widetilde{\mathrm{Ad}}$ (which is the correct action in the derived category according to Remark 5.2.23). Finally, the morphism

$$\varinjlim_{m\in\mathbb{N}}\varprojlim_{n\in\mathbb{N}} \mathbf{R}\Gamma(\mathcal{C}_{\varphi_{K|L}}^{\bullet}(\Gamma_K, F_{\Gamma_K}(M_{mn}))) \longrightarrow \varinjlim_{m\in\mathbb{N}}\varprojlim_{n\in\mathbb{N}} \mathbf{R}\Gamma(\mathcal{C}_{\varphi_{K|L}}^{\bullet}(M_{mn}))$$

is Λ_K-linear by Corollary 5.2.22. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

This description now has the advantage that the objects of the complexes are Λ_K-modules which allows us to apply the theory of Matlis duality. We give a brief overview of this theory.

Remark 5.2.25. *We have to consider different types of group actions on Λ_K. First, Γ_K acts by multiplication and G_K acts by multiplication through the natural projection* $\mathrm{pr}\colon G_K \twoheadrightarrow \Gamma_K$. *Sometimes we also have to consider Λ_K as Λ_K-module via the involution ι, i.e. Γ_K then acts by $\gamma \cdot x \coloneqq \gamma^{-1}x$. If this is the case, we write Λ_K^ι. Note that this does also affect the action from G_K, i.e. G_K acts on Λ_K^ι by $g \cdot x = \mathrm{pr}(g)^{-1}x$ and Γ_K acts by $\gamma \cdot x = \gamma^{-1}x$.*

Additionally, if M is a Λ_K-module, we denote by M^ι the Λ_K-module M where Γ_K acts via the involution ι, i.e. for all $\gamma \in \Gamma_K$ and $m \in M$ we have $\gamma \cdot m = \gamma^{-1}m$. If N is another Λ_K-module we clearly have

$$\mathrm{Hom}_{\Lambda_K}(M, N^\iota) = \mathrm{Hom}_{\Lambda_K}(M^\iota, N).$$

Definition 5.2.26. A Λ_K-module with a Λ_K-semilinear action of G_K is a Λ_K-module M with an action from G_K such that for all $\lambda \in \Lambda_K$, $m \in M$ and $g \in G_K$ we have

$$g(\lambda m) = g(\lambda)g(m) = \mathrm{pr}(g)\lambda g(m),$$

where $\mathrm{pr}\colon G_K \twoheadrightarrow \Gamma_K$ denotes the canonical projection (cf. Remark 5.2.25).

Remark 5.2.27. *For us it feels more natural to consider Λ_K-modules with a semilinear G_K-action instead of Λ_K-modules with a linear action from G_K, which are considered in [26]. The main reason for this is that if we consider modules with a linear action from G_K we would have to consider Λ_K with the trivial action from G_K. But this feels nonintuitive. In the text below we will always compare our results to the results of Nekovář in [26]. He considers Λ_K with the trivial action of G_K (cf. [26, (8.4.3.1) Lemma, p. 161–162]).*

Both concepts are linked in the following sense: If M is a Λ_K-module with a (linear or semilinear) action from G_K, then for $n \in \mathbb{Z}$ denote by $M<n>$ the Λ_K-module M with the G_K-action given by

$$g \cdot m = \mathrm{pr}(g)^n g(m),$$

with $g \in G_K$ and $m \in M$ and where $g(m)$ denotes the given action of G_K on M (cf. [26, (8.4.2), p. 161]). Then $M \mapsto M<1>$ induces a morphism from Λ_K-modules with a linear action from G_K to Λ_K-modules with a semilinear action from G_K. Its inverse clearly is $M \mapsto M<-1>$.

Remark 5.2.28. *Let M, N be Λ_K-modules with a Λ_K-semilinear action of G_K. Then $\mathrm{Hom}_{\Lambda_K}(M, N)$ also carries actions from both G_K and Γ_K (respectively Λ_K). The action from Γ_K is given by the multiplication of Λ_K on N (respectively M since the homomorphisms are Λ_K-linear). The action from G_K is given by*

$$(g \cdot f)(m) \coloneqq g_N(f(g_M^{-1}(m))),$$

for $f \in \mathrm{Hom}_{\Lambda_K}(M, N)$ and $m \in M$ and where g_M respectively g_N denote the actions from G_K on M and N.

Remark 5.2.29. *Let T be a topological $\mathcal{O}_L$-module with a continuous action from G_K and let M be a Λ_K-module with a Λ_K-semilinear action of G_K. Then Γ_K acts on $\mathrm{Hom}_{\mathcal{O}_L}(T, M)$ by multiplication on the coefficients and G_K as in the above Remark 5.2.28, i.e. by*

$$(g \cdot f)(t) \coloneqq g_M(f(g_T^{-1}(t))),$$

for $f \in \mathrm{Hom}_{\mathcal{O}_L}(T, M)$ and $m \in M$ and where g_T and g_M denote the actions from G_K on T and M respectively.

Lemma 5.2.30. *Let T be a topological $\mathcal{O}_L$-module with a continuous action from G_K and let M be a Λ_K-module with a Λ_K-semilinear action of G_K. Then the homomorphism of $\mathcal{O}_L$-modules*

$$\mathrm{Hom}_{\mathcal{O}_L}(T, M) \longrightarrow \mathrm{Hom}_{\Lambda_K}(T \otimes_{\mathcal{O}_L} \Lambda_K, M), \; f \longmapsto \beta_f := [t \otimes x \mapsto xf(t)]$$

is an isomorphism which respects the actions from Γ_K and G_K described in the above Remark 5.2.29 *for the left hand side and* Remark 5.2.25 *for the right hand side.*

Proof. The inverse homomorphism is given by

$$\mathrm{Hom}_{\Lambda_K}(T \otimes_{\mathcal{O}_L} \Lambda_K, M) \longrightarrow \mathrm{Hom}_{\mathcal{O}_L}(T, M), \; h \longmapsto [t \mapsto h(t \otimes 1)].$$

So it is left to check that the above homomorphisms respects the actions from Γ_K and G_K, which we leave to the reader. $\qquad\square$

Remark 5.2.31. *Let M be a Λ_K-module with a Λ_K-semilinear action of G_K. Then $M^\vee = \mathrm{Hom}_{\mathcal{O}_L}^{\mathrm{cts}}(M, L/\mathcal{O}_L)$ also carries actions from G_K and Γ_K. Both are given by*

$$(g \cdot f)(m) = f(g^{-1}(m)),$$

where $g \in G_K$ or in Γ_K, $f \in M^\vee$ and $m \in M$.
Note that Nekovář considers the Pontrjagin dual of M with the Γ_K-action without the involution, i.e. by $(\gamma \cdot f)(m) = f(\gamma(m))$ (cf. the proof respectively the result of [26, (8.4.3.1) Lemma, p. 161–162]). In our notation the Pontrjagin dual of Nekovář of M is $(M^\vee)^\iota = (M^\iota)^\vee$.

Remark 5.2.32. *Let M be a Λ_K-module with a Λ_K-semilinear action of G_K and $n \in \mathbb{Z}$. Then the identity of $M^\vee$ induces an isomorphism of Λ_K-modules with a Λ_K-semilinear action of G_K*

$$(M<n>)^\vee \cong M^\vee <n>.$$

Definition 5.2.33. Let M be a Λ_K-module. The **Matlis dual** of M is defined as

$$\overline{\mathrm{D}_K}(M) := \mathrm{Hom}_{\Lambda_K}(M, \Lambda_K^\vee).$$

This is a contravariant functor of Λ_K-modules and maps finitely generated Λ_K-modules to cofinitely generated and vice versa.
Λ_K acts on $\overline{\mathrm{D}_K}(M)$ by multiplication and if M has also a semilinear action from G_K, then G_K acts on $\overline{\mathrm{D}_K}(M)$ as described in the above Remark 5.2.28

Remark 5.2.34. *$\Lambda_K^\vee$ is an injective Λ_K-module. Moreover, it is an injective hull of the residue class field of Λ_K as Λ_K-module. Therefore $\overline{\mathrm{D}_K}$ is exact and for every finitely respectively cofinitely generated Λ_K-module the canonical homomorphism $M \to \overline{\mathrm{D}_K}(\overline{\mathrm{D}_K}(M))$ is an isomorphism.*

Proof. Since $\gamma \mapsto \gamma^{-1}$ defines an isomorphism of Λ_K-modules $\Lambda_K \to \Lambda_K^\iota$, the first statement is [26, (8.4.3.2) Corollary, p. 162]. For this, note that in [26, (8.4.3.1) Lemma, p. 161–162] Nekovář proves that $(\Lambda_K^\vee)^\iota = (\Lambda_K^\iota)^\vee$ and Nekovář's dualizing module coincide and with $(\Lambda_K^\iota)^\vee$ also $\Lambda_K^\vee$ is a dualizing module. The second statement is [9, Theorem 3.2.12, p. 105–107]. $\quad\square$

Remark 5.2.35. *As mentioned in* [26, (2.3.3, p. 41)] $L/\mathcal{O}_L$ *is an injective hull for* k_L. *Therefore we have a canonical isomorphism* $M \cong \operatorname{Hom}_{\mathcal{O}_L}(\operatorname{Hom}_{\mathcal{O}_L}(M, L/\mathcal{O}_L), L/\mathcal{O}_L)$ *for every finitely or cofinitely generated* $\mathcal{O}_L$-*module* M *and* $\operatorname{Hom}_{\mathcal{O}_L}(-, L/\mathcal{O}_L)$ *is an exact functor. As above, the proof for this is* [9, Theorem 3.2.12, p. 105–107].

We need some more notation from [26].

Remark 5.2.36. *Let* $T \in \mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg})}(G_K)$ *and* $U \in \mathcal{U}_K$. *Then we have two group actions on* $T \otimes_{\mathcal{O}_L} \mathcal{O}_L[G_K/U]$. *The first action, is the diagonal action from* G_K

$$g \cdot (a \otimes xU) = (ga) \otimes (gxU).$$

The second action is the following action from G_K/U:

$$\widetilde{\mathrm{Ad}}(gU)(a \otimes xU) \coloneqq a \otimes xg^{-1}U.$$

The homomorphism $\sum a_{xU} \otimes xU \mapsto \sum a_{xU} \delta_{xU}$ *where* δ_{xU} *is the Kronecker delta-function on* G_K/U *(i.e. it is 1 for* xU *and zero otherwise) defines an isomorphism between* $T \otimes_{\mathcal{O}_L} \mathcal{O}_L[G_K/U]$ *and* $_U T$ *(cf.* [26, (8.1.3), p. 149; (8.2.1) p. 157]*) under which the actions described above coincide with the corresponding actions on* $_U T$ *(cf.* [26, (8.1.6.3), p. 151]*).*

Definition 5.2.37. Let $T \in \mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg})}(G_K)$. We set

$$\mathcal{F}_{\Gamma_K}(T) \coloneqq \varprojlim_{U \in \mathcal{U}_K} T \otimes_{\mathcal{O}_L} \mathcal{O}_L[G_K/U]$$

together with the two actions from G_K and Γ_K described in the above Remark 5.2.36. With this, we define

$$\mathbf{R}\Gamma_{\mathrm{Iw}}^\bullet(K_\infty|K, T) \coloneqq \mathbf{R}\Gamma_{\mathrm{cts}}^\bullet(G_K, \mathcal{F}_{\Gamma_K}(T)).$$

Furthermore, by $\overset{\mathbf{L}}{\otimes}_R$ we denote the derived tensor product over the ring R.

Remark 5.2.38. *At* [26, p. 201] *Nekovář proves*

$$H_{\mathrm{Iw}}^*(K_\infty|K, T) \cong H^*(\mathbf{R}\Gamma_{\mathrm{Iw}}^\bullet(K_\infty|K, T)),$$

i.e. that the cohomology of the above complex coincides with the Iwasawa cohomology defined in Definition 4.0.6.

Remark 5.2.39. *Let* $T \in \mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg})}(G_K)$, *then we have an isomorphism of* Λ_K-*modules with a* Λ_K-*semilinear action of* G_K

$$\mathcal{F}_{\Gamma_K}(T) \cong T \otimes_{\mathcal{O}_L} \Lambda_K^\iota.$$

Proof. Since T is finitely generated and $\mathcal{O}_L$ is a discrete valuation ring, T is finitely presented. Therefore we have

$$\varprojlim_{U \in \mathcal{U}_K} T \otimes_{\mathcal{O}_L} \mathcal{O}_L[G_K/U] = T \otimes_{\mathcal{O}_L} \Lambda_K$$

as $\mathcal{O}_L$-modules. G_K acts on both sides diagonally and Γ_K acts on the left hand side via $\widetilde{\mathrm{Ad}}$ (which technically means via the involution) on the right hand term $\mathcal{O}_L[G_K/U]$. Since Γ_K acts on Λ_K^ι also via the involution, the claim follows. $\square$

Lemma 5.2.40. *We have an isomorphism of* Λ_K-*modules with a* Λ_K-*semilinear action of* G_K

$$(\Lambda_K^\iota)^\vee \cong F_{\Gamma_K}(L/\mathcal{O}_L) \left(= \varinjlim_{U \in \mathcal{U}_K} \operatorname{Hom}_{\mathcal{O}_L}(\mathcal{O}_L[G_K/U], L/\mathcal{O}_L) \right).$$

Proof. $\mathcal{O}_L[G_K/U]$ is compact for $U \in \mathcal{U}_K$, therefore $\mathrm{Hom}_{\mathcal{O}_L}(\mathcal{O}_L[G_K/U], L/\mathcal{O}_L)$ is discrete and so $\varinjlim_{U \in \mathcal{U}_K} \mathrm{Hom}_{\mathcal{O}_L}(\mathcal{O}_L[G_K/U], L/\mathcal{O}_L)$ is discrete too. This means that every map with source $\varinjlim_{U \in \mathcal{U}_K} \mathrm{Hom}_{\mathcal{O}_L}(\mathcal{O}_L[G_K/U], L/\mathcal{O}_L)$ into any topological space is continuous. We then compute (as $\mathcal{O}_L$-modules)

$$
\begin{aligned}
\mathrm{Hom}_{\mathcal{O}_L}^{\mathrm{cts}}(F_{\Gamma_K}(L/\mathcal{O}_L), L/\mathcal{O}_L) &= \mathrm{Hom}_{\mathcal{O}_L}^{\mathrm{cts}}(\varinjlim_{U \in \mathcal{U}_K} \mathrm{Hom}_{\mathcal{O}_L}(\mathcal{O}_L[G_K/U], L/\mathcal{O}_L), L/\mathcal{O}_L) \\
&= \mathrm{Hom}_{\mathcal{O}_L}(\varinjlim_{U \in \mathcal{U}_K} \mathrm{Hom}_{\mathcal{O}_L}(\mathcal{O}_L[G_K/U], L/\mathcal{O}_L), L/\mathcal{O}_L) \\
&\cong \varprojlim_{U \in \mathcal{U}_K} \mathrm{Hom}_{\mathcal{O}_L}(\mathrm{Hom}_{\mathcal{O}_L}(\mathcal{O}_L[G_K/U], L/\mathcal{O}_L), L/\mathcal{O}_L) \\
&\cong \varprojlim_{U \in \mathcal{U}_K} \mathcal{O}_L[G_K/U] \\
&= \Lambda_K.
\end{aligned}
$$

At the third equation, we used the identification

$$
\mathcal{O}_L[G_K/U] \cong \mathrm{Hom}_{\mathcal{O}_L}(\mathrm{Hom}_{\mathcal{O}_L}(\mathcal{O}_L[G_K/U], L/\mathcal{O}_L), L/\mathcal{O}_L)
$$

from Remark 5.2.35. Now we head towards the action from Γ_K. For $\gamma \in \Gamma_K$, $f \in \mathrm{Hom}_{\mathcal{O}_L}^{\mathrm{cts}}(F_{\Gamma_K}(L/\mathcal{O}_L), L/\mathcal{O}_L)$ and $h \in F_{\Gamma_K}(L/\mathcal{O}_L)$ we have

$$
(\gamma \cdot f)(h) = f(\gamma^{-1} \cdot h) = f(\widetilde{\mathrm{Ad}}(\gamma^{-1})h)
$$

for all $x \in F_{\Gamma_K}(L/\mathcal{O}_L)$. Going through the above isomorphisms shows that this results in an action from Γ_K on Λ_K via the involution, i.e. we have an isomorphism of Λ_K-modules

$$
\mathrm{Hom}_{\mathcal{O}_L}^{\mathrm{cts}}(F_{\Gamma_K}(L/\mathcal{O}_L), L/\mathcal{O}_L) \cong \Lambda_K^\iota.
$$

With the above notation, we have for $g \in G_K$

$$
(g \cdot f)(h) = f(g^{-1} \cdot h) = f(h \circ g),
$$

since G_K acts trivial on $L/\mathcal{O}_L$ by definition. Therefore the above isomorphism is also G_K-linear. $\qquad\square$

Remark 5.2.41. *The above result differs a bit from Nekovář's result in [26, (8.4.3.1) Lemma, p. 161–162] since Nekovář considers Λ_K-modules with a Λ_K-linear action from G_K and therefore he considers Λ_K with a trivial G_K action (cf. Remark 5.2.27). Furthermore, his Pontrjagin dual and ours for Λ_K-modules differ in the action of Γ_K by an involution (cf. Remark 5.2.31). For a better comparison, if we consider Λ_K with the trivial action from G_K the result of loc. cit in our notation is*

$$
(\Lambda_K^\vee)^\iota \cong F_{\Gamma_K}(L/\mathcal{O}_L) < 1 > .
$$

This is equivalent to

$$
(\Lambda_K^\vee)^\iota < -1 > \cong F_{\Gamma_K}(L/\mathcal{O}_L)
$$

and for the left hand side we obtain

$$
\begin{aligned}
(\Lambda_K^\vee)^\iota < -1 > &= (\Lambda_K^\iota)^\vee < -1 > \\
&= (\Lambda_K^\iota < -1 >)^\vee \\
&= ((\Lambda_K < 1 >)^\iota)^\vee.
\end{aligned}
$$

In the second line we used Remark 5.2.32. *But this means that Nekovář's result translate into ours since we considered* Λ_K *with the action from* G_K *given by the canonical projection* $\mathrm{pr}\colon G_K \twoheadrightarrow \Gamma_K$.

Lemma 5.2.42. *Let* $T \in \mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg})}(G_K)$. *Then we have an isomorphism of* Λ_K-*modules with a* Λ_K-*semilinear action of* G_K:

$$F_{\Gamma_K}(T^\vee) \cong \overline{\mathrm{D}_K}(\mathcal{F}_{\Gamma_K}(T)).$$

Proof. This proof follows similarly as in [26, (8.4.5.1) Lemma, p. 163], see [23, 5.2.42] for details.

$\square$

Remark 5.2.43. *Again, the above result differs slightly from the analogous result of Nekovář (cf.* [26, (8.4.5.1) Lemma, p. 163]*). This is a consequence of the difference pointed out in the above* Remark 5.2.41. *Translated to our notation, Nekovář's result from (loc. cit.) then is that there is an isomorphism of* Λ_K-*modules with a* Λ_K-*semilinear action of* G_K

$$F_{\Gamma_K}((T^\vee)^\iota) \cong \mathrm{Hom}_{\Lambda_K}(\mathcal{F}_{\Gamma_K}(T)^\iota, (\Lambda_K^\vee)^\iota).$$

Note that Nekovář's original result is formulated for Λ_K-*modules with a linear action from* G_K. *But as pointed out in* Remark 5.2.27 *both concepts are linked by the shifts* $< 1 >$ *and* $< -1 >$ *respectively. So to be precise, Nekovář's result is the above shifted by* $< -1 >$. *If we apply this shift, we would have to invert it below in order to compare Nekovář's result to our result. Since* Γ_K *acts trivially on* T *and therefore also on* $T^\vee$ *we have* $(T^\vee)^\iota = T^\vee$ *and we have a canonical isomorphism of* Λ_K-*modules with a* Λ_K-*semilinear action of* G_K

$$\mathrm{Hom}_{\Lambda_K}(\mathcal{F}_{\Gamma_K}(T)^\iota, (\Lambda_K^\vee)^\iota) = \mathrm{Hom}_{\Lambda_K}(\mathcal{F}_{\Gamma_K}(T), \Lambda_K^\vee) = \overline{\mathrm{D}_K}(\mathcal{F}_{\Gamma_K}(T)).$$

Combining the above identifications then gives us an isomorphism of Λ_K-*modules with a* Λ_K-*semilinear action of* G_K

$$(F_{\Gamma_K}(T^\vee)) \cong \overline{\mathrm{D}_K}(\mathcal{F}_{\Gamma_K}(T)),$$

which is exactly our result.

Lemma 5.2.44. *Let* $T \in \mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg})}(G_K)$. *We then have an isomorphism*

$$\mathbf{R}\Gamma_{\mathrm{Iw}}^\bullet(K_\infty|K, T) \cong \overline{\mathrm{D}_K}\left(\mathbf{R}\Gamma_{\mathrm{cts}}^\bullet(G_K, F_{\Gamma_K}(T^\vee)(1))\right)[-2].$$

For the cohomology groups we then have for all $i \geq 0$ *an isomorphism of* Λ_K-*modules*

$$\overline{\mathrm{D}_K}(H_{\mathrm{Iw}}^i(K_\infty|K, T)) \cong H_{\mathrm{cts}}^{2-i}(G_K, F_{\Gamma_K}(T^\vee)(1)) \cong H_{\mathrm{cts}}^{2-i}(H_K, T^\vee(1)).$$

Proof. This is [26, (8.11.2.2); (8.11.2.3), p. 201], but note that the shift of our complex is outside $\overline{\mathrm{D}_K}(-)$ and that we have $F_{\Gamma_K}(T^\vee) \cong \overline{\mathrm{D}_K}(\mathcal{F}_{\Gamma_K}(T))$ (cf. Lemma 5.2.42) since we have a slightly different convention for the involved action of Γ_K. In particular, this is Lemma 5.2.42 together with [26, (5.2.6) Lemma, p. 92]. The last isomorphism of the cohomology groups is Proposition 5.2.21.

$\square$

Proposition 5.2.45. *Let* $T \in \mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg})}(G_K)$. *Then the sequence*

$$0 \longrightarrow H_{\mathrm{Iw}}^1(K_\infty|K, T) \longrightarrow \overline{\mathrm{D}_K}(\mathcal{M}) \xrightarrow{\overline{\mathrm{D}_K}(\varphi_{K|L})-\mathrm{id}} \overline{\mathrm{D}_K}(\mathcal{M}) \longrightarrow H_{\mathrm{Iw}}^2(K_\infty|K, T) \longrightarrow 0$$

is exact, where $\mathcal{M} = \mathcal{D}_{K|L}(T^\vee(1))$.

Proof. With $A := T^\vee(1)$ we deduce from Proposition 5.2.10 and Proposition 5.2.21 that the sequence

$$0 \to H^0_{\mathrm{cts}}(G_K, F_{\Gamma_K}(A)) \to \mathcal{D}_{K|L}(A) \xrightarrow{\varphi_{K|L}-\mathrm{id}} \mathcal{D}_{K|L}(A) \to H^1_{\mathrm{cts}}(G_K, F_{\Gamma_K}(A)) \to 0$$

is exact and Proposition 5.2.24 says that it is a sequence of Λ_K-modules. Applying $\overline{\mathrm{D}_K}(-)$ then gives the exact sequence

$$0 \longrightarrow \overline{\mathrm{D}_K}(H^1_{\mathrm{cts}}(G_K, F_{\Gamma_K}(A))) \longrightarrow \overline{\mathrm{D}_K}(\mathcal{D}_{K|L}(A)) \xrightarrow{\overline{\mathrm{D}_K}(\varphi_{K|L})-\mathrm{id}} \cdots$$

$$\cdots \longrightarrow \overline{\mathrm{D}_K}(\mathcal{D}_{K|L}(A)) \longrightarrow \overline{\mathrm{D}_K}(H^0_{\mathrm{cts}}(G_K, F_{\Gamma_K}(A))) \longrightarrow 0$$

(cf. Remark 5.2.34). Lemma 5.2.44 translates this sequence into the desired one. $\qquad\square$

This sequence looks similar to the sequence

$$0 \to H^1_{\mathrm{Iw}}(K_\infty|K, T) \to \mathcal{D}_{K|L}(T(\tau^{-1})) \xrightarrow{\psi-\mathrm{id}} \mathcal{D}_{K|L}(T(\tau^{-1})) \to H^2_{\mathrm{Iw}}(K_\infty|K, T) \to 0$$

from Theorem 4.0.13 where $\tau^{-1} = \chi_{\mathrm{LT}}\chi_{\mathrm{cyc}}^{-1}$ and $T \in \mathbf{Rep}^{(\mathrm{fg})}_{\mathcal{O}_L}(G_K)$. In order to compare these sequences, we prove the following.

Lemma 5.2.46. *Let $n \in \mathbb{N}$. We have $\Omega^1_{\mathbf{A}_{K|L}}/\pi^n_L \Omega^1_{\mathbf{A}_{K|L}} = (\mathbf{A}_{K|L}/\pi^n_L \mathbf{A}_{K|L})^\vee$ and a Γ_K-linear inclusion*

$$\Omega^1_{\mathbf{A}_{K|L}}/\pi^n_L \Omega^1_{\mathbf{A}_{K|L}} \lhook\joinrel\longrightarrow \overline{\mathrm{D}_K}(\mathbf{A}_{K|L}/\pi^n_L \mathbf{A}_{K|L})$$

Proof. The isomorphism is a reformulation of an analogue of [31, Lemma 3.5, p. 11]. For the inclusion using the tensor-hom adjunction we obtain

$$\mathrm{Hom}_{\mathcal{O}_L}(\mathbf{A}_{K|L}/\pi^n_L \mathbf{A}_{K|L}, L/\mathcal{O}_L) \cong \mathrm{Hom}_{\mathcal{O}_L}(\mathbf{A}_{K|L}/\pi^n_L \mathbf{A}_{K|L} \otimes_{\Lambda_K} \Lambda_K, L/\mathcal{O}_L)$$
$$\cong \mathrm{Hom}_{\Lambda_K}(\mathbf{A}_{K|L}/\pi^n_L \mathbf{A}_{K|L}, \mathrm{Hom}_{\mathcal{O}_L}(\Lambda_K, L/\mathcal{O}_L)).$$

So we have to check that under this isomorphism $\mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}_L}(\mathbf{A}_{K|L}/\pi^n_L \mathbf{A}_{K|L}, L/\mathcal{O}_L)$ is sent to $\mathrm{Hom}_{\Lambda_K}(\mathbf{A}_{K|L}/\pi^n_L \mathbf{A}_{K|L}, (\Lambda_K)^\vee)$. For this, recall the above isomorphism precisely: Let $f \in \mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}_L}(\mathbf{A}_{K|L}/\pi^n_L \mathbf{A}_{K|L}, L/\mathcal{O}_L)$, then f is mapped to the element

$$[a \mapsto f_a := [\lambda \mapsto f(\lambda a)]]$$

in $\mathrm{Hom}_{\Lambda_K}(\mathbf{A}_{K|L}/\pi^n_L \mathbf{A}_{K|L}, \mathrm{Hom}_{\mathcal{O}_L}(\Lambda_K, L/\mathcal{O}_L))$. For $a \in \mathbf{A}_{K|L}/\pi^n_L \mathbf{A}_{K|L}$ the homomorphism f_a then is the composition

$$\Lambda_K \longrightarrow \mathbf{A}_{K|L}/\pi^n_L \mathbf{A}_{K|L} \xrightarrow{f} L/\mathcal{O}_L$$
$$\lambda \longmapsto \lambda a$$

of continuous maps, i.e. f_a is continuous too and we get the desired inclusion

$$\mathrm{Hom}^{\mathrm{cts}}_{\mathcal{O}_L}(\mathbf{A}_{K|L}/\pi^n_L \mathbf{A}_{K|L}, L/\mathcal{O}_L) \lhook\joinrel\longrightarrow \mathrm{Hom}_{\Lambda_K}(\mathbf{A}_{K|L}/\pi^n_L \mathbf{A}_{K|L}, (\Lambda_K)^\vee).$$

It is easy to check this inclusion is Γ_K-linear. $\qquad\square$

Definition 5.2.47. Let M be a topological $\mathbf{A}_{K|L}$-module with a continuous and semilinear action from Γ_K. We define

$$\mathrm{D}(M) := \mathrm{Hom}_{\mathbf{A}_{K|L}}(M, \Omega^1_{\mathbf{A}_{K|L}} \otimes_{\mathbf{A}_{K|L}} \mathbf{B}_{K|L}/\mathbf{A}_{K|L}).$$

And we define the Γ_K-action on $\mathrm{D}(M)$ to be

$$(\gamma \cdot f)(m) := \gamma(f(\gamma^{-1}(m))),$$

where Γ_K acts diagonal on the tensor product.

Remark 5.2.48. *Using the isomorphism* $\mathbf{A}_{K|L}(\chi_{\mathrm{LT}}) \to \Omega^1_{\mathscr{A}_{K|L}}$, $f \otimes t_0 \mapsto f g_{\mathrm{LT}} \mathrm{d}Z$ *we can identify* $\mathrm{D}(M)$, *for M as above, with*

$$\mathrm{Hom}_{\mathbf{A}_{K|L}}(M, \mathbf{B}_{K|L}/\mathbf{A}_{K|L}(\chi_{\mathrm{LT}})).$$

Lemma 5.2.49. *Let M be a discrete $\mathbf{A}_{K|L}$-module with a continuous and semilinear action from Γ_K such that $M = \varinjlim_m M_m$ where $M_m = \ker(\mu_{\pi_L^m})$. Then we have a Γ_K-linear inclusion*

$$\mathrm{D}(M) \lhook\joinrel\longrightarrow \overline{\mathrm{D}_K}(M).$$

Proof. For $m \in \mathbb{N}$ we obtain with the tensor-hom adjunction

$$\begin{aligned}
\overline{\mathrm{D}_K}(M_m) &\cong \mathrm{Hom}_{\Lambda_K}(M_m, (\Lambda_K)^\vee) \\
&\cong \mathrm{Hom}_{\Lambda_K}(M_m \otimes_{\mathbf{A}_{K|L}} \mathbf{A}_{K|L}/\pi_L^m \mathbf{A}_{K|L}, (\Lambda_K)^\vee) \\
&\cong \mathrm{Hom}_{\mathbf{A}_{K|L}}(M_m, \mathrm{Hom}_{\Lambda_K}(\mathbf{A}_{K|L}/\pi_L^m \mathbf{A}_{K|L}/, (\Lambda_K)^\vee)).
\end{aligned}$$

Lemma 5.2.46 then implies, that there is an inclusion

$$\mathrm{Hom}_{\mathbf{A}_{K|L}}(M_m, \Omega^1_{\mathbf{A}_{K|L}}/\pi_L^m \Omega^1_{\mathbf{A}_{K|L}}) \lhook\joinrel\longrightarrow \overline{\mathrm{D}_K}(M_m).$$

But since $\pi_L^m M_m = 0$ it is

$$\mathrm{Hom}_{\mathbf{A}_{K|L}}(M_m, \Omega^1_{\mathbf{A}_{K|L}}/\pi_L^m \Omega^1_{\mathbf{A}_{K|L}}) = \mathrm{Hom}_{\mathbf{A}_{K|L}}(M_m, \Omega^1_{\mathbf{A}_{K|L}} \otimes_{\mathbf{A}_{K|L}} \mathbf{B}_{K|L}/\mathbf{A}_{K|L}),$$

i.e. we have an inclusion $\mathrm{D}(M_m) \hookrightarrow \overline{\mathrm{D}_K}(M_m)$. Since $\mathrm{Hom}_R(-, X)$ commutes with limits for arbitrary rings R and R-modules X, we get the desired inclusion $\mathrm{D}(M) \hookrightarrow \overline{\mathrm{D}_K}(M)$ by applying limits. $\qquad\square$

Lemma 5.2.50. *Let A be a cofinitely generated $\mathcal{O}_L$-module with a continuous action from G_K. Then we have*

$$\mathrm{D}(\mathcal{D}_{K|L}(A)) \cong \mathcal{D}_{K|L}(A^\vee(\chi_{\mathrm{LT}})).$$

This isomorphism respects the action from Γ_K.

Proof. As usual we write $A = \varinjlim_m A_m$ with $A_m = \ker(\mu_{\pi_L^m})$. By an analogue of [31, Lemma 3.6, p. 11–12] we have an isomorphism

$$\mathrm{D}(\mathcal{D}_{K|L}(A_m)) \cong \mathcal{D}_{K|L}(A_m)^\vee,$$

which is Γ_K-linear by similar arguments as in (the proofs of) [31, Corollary 3.18, Proposition 3.19]. Remark 4.0.4 says that we have a Γ_K-linear isomorphism

$$\mathcal{D}_{K|L}(A_m)^\vee \cong \mathcal{D}_{K|L}((A_m)^\vee(\chi_{\mathrm{LT}})).$$

Combining these results gives us the Γ_K-linear isomorphism

$$\mathrm{D}(\mathcal{D}_{K|L}(A_m)) \cong \mathcal{D}_{K|L}((A_m)^\vee(\chi_{\mathrm{LT}})).$$

Applying limits now gives the desired result. □

Proposition 5.2.51. *Let $T \in \mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg})}(G_K)$ and set*

$$\mathcal{C}_\psi^\bullet(\mathcal{D}_{K|L}(T(\tau^{-1}))) := \mathcal{C}_{\mathrm{D}(\varphi)}^\bullet(\mathrm{D}(\mathcal{D}_{K|L}(T^\vee(1))))[-1].$$

Then the inclusion of complexes

$$\mathcal{C}_\psi^\bullet(\mathcal{D}_{K|L}(T(\tau^{-1}))) \lhook\joinrel\longrightarrow \mathcal{C}_{\overline{\mathrm{D}_K}(\varphi)}^\bullet(\overline{\mathrm{D}_K}(\mathcal{D}_{K|L}(T^\vee(1))))[-1]$$

$$\|$$

$$\overline{\mathrm{D}_K}(\mathcal{C}_\varphi^\bullet(\mathcal{D}_{K|L}(T^\vee(1))))[-2]$$

is a quasi isomorphism. So in particular we have an isomorphism in the derived category $\mathbf{D}^{\mathrm{b}}(\Lambda_K - \mathbf{Mod})$

$$\mathbf{R}\Gamma(\mathcal{C}_\psi^\bullet(\mathcal{D}_{K|L}(T(\tau^{-1})))) \cong \mathbf{R}\Gamma_{\mathrm{Iw}}^\bullet(K_\infty|K, T).$$

Proof. With $T^\vee(1) = T(-1)^\vee$, the above Lemma 5.2.49 and Lemma 5.2.50 imply

$$\mathcal{D}_{K|L}(T(\tau^{-1}) \xrightarrow{\cong} \mathrm{D}(\mathcal{D}_{K|L}(T^\vee(1)) \lhook\joinrel\longrightarrow \overline{\mathrm{D}_K}(\mathcal{D}_{K|L}(T^\vee(1))).$$

The cited lemmata also show that both homomorphisms are Γ_K-linear. Let $\mathcal{M} := \mathcal{D}_{K|L}(T^\vee(1))$ then Proposition 5.2.45 together with Theorem 4.0.13 implies the commutative diagram with exact rows and Λ_K-linear vertical homomorphisms

$$0 \to H_{\mathrm{Iw}}^1(K_\infty|K, T) \longrightarrow \overline{\mathrm{D}_K}(\mathcal{M}) \xrightarrow{\overline{\mathrm{D}_K}(\varphi)-\mathrm{id}} \overline{\mathrm{D}_K}(\mathcal{M}) \longrightarrow H_{\mathrm{Iw}}^2(K_\infty|K, T) \to 0$$

$$\| \qquad\qquad \uparrow \qquad\qquad \uparrow \qquad\qquad \|$$

$$0 \to H_{\mathrm{Iw}}^1(K_\infty|K, T) \to \mathcal{D}_{K|L}(T(\tau^{-1})) \xrightarrow{\psi-\mathrm{id}} \mathcal{D}_{K|L}(T(\tau^{-1})) \to H_{\mathrm{Iw}}^2(K_\infty|K, T) \to 0.$$

This gives the desired quasi isomorphism. The second statement then follows from Lemma 5.2.44 by using Proposition 5.2.24. □

Question 5.2.52. *It follows that, for A cofinitely generated over $\mathcal{O}_L$ and with continuous G_K-action, the complex*

$$0 \longrightarrow \overline{\mathrm{D}_K}(\mathcal{D}_{K|L}(A))/\mathrm{D}(\mathcal{D}_{K|L}(A)) \xrightarrow{\overline{\mathrm{D}_K}(\varphi)-\mathrm{id}} \overline{\mathrm{D}_K}(\mathcal{D}_{K|L}(A))/\mathrm{D}(\mathcal{D}_{K|L}(A)) \longrightarrow 0$$

is acyclic, in particular for $\mathrm{D}(\mathcal{D}_{K|L}(A)) = \mathbf{A}_{K|L}$. Can one show this directly, without going the intricate way using Matlis duality and the Nekovar's results? Moreover, is it realistically conceivable that even $\overline{\mathrm{D}_K}(\mathcal{D}_{K|L}(A)) = \mathrm{D}(\mathcal{D}_{K|L}(A))$ holds?

Theorem 5.2.53. *Let $T \in \mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg})}(G_K)$ and let $K \subseteq K' \subseteq K_\infty$ an intermediate field, finite over K, such that $\Gamma_{K'} := \mathrm{Gal}(K_\infty|K')$ is isomorphic to some $\mathbb{Z}_p^r$. Then we have an isomorphism in the derived category $\mathbf{D}^+(\mathcal{O}_L\text{-}\mathbf{Mod})$*

$$\mathbf{R}\Gamma_{\mathrm{Iw}}^\bullet(K_\infty|K, T) \overset{\mathbf{L}}{\otimes}_{\Lambda_{K'}} \mathcal{O}_L \cong \mathbf{R}\Gamma_{\mathrm{cts}}^\bullet(G_{K'}, T).$$

In particular, we have

$$\mathbf{R}\Gamma(\mathcal{C}_\psi^\bullet(\mathcal{D}_{K|L}(T(\tau^{-1})))) \overset{\mathbf{L}}{\otimes}_{\Lambda_{K'}} \mathcal{O}_L \cong \mathbf{R}\Gamma_{\mathrm{cts}}^\bullet(G_{K'}, T).$$

Proof. The first assertion is [26, (8.4.8.1) Proposition, p. 168]. Note that we have an isomorphism $\mathbf{R}\Gamma_{\mathrm{Iw}}^\bullet(K_\infty|K',T) \cong \mathbf{R}\Gamma_{\mathrm{Iw}}^\bullet(K_\infty|K,T)$ in $\mathbf{D}^+(\Lambda_{K'}\text{-}\mathbf{Mod})$ since the intermediate fields of $K_\infty|K'$ are cofinal in the intermediate fields of $K_\infty|K$. The second assertion then is an application of Proposition 5.2.51. □

Using [17, Prop. 1.6.5 (3)] we obtain the following variant.

Theorem 5.2.54. *Let* $T \in \mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg})}(G_K)$ *and let* $K \subseteq K' \subseteq K_\infty$ *any intermediate field, finite over* K. *Then we have an isomorphism in the derived category* $\mathbf{D}^+(\mathcal{O}_L\text{-}\mathbf{Mod})$

$$\mathbf{R}\Gamma_{\mathrm{Iw}}(K_\infty|K,T) \overset{\mathbf{L}}{\otimes}_{\Lambda_K} \mathcal{O}_L[\mathrm{Gal}(K'|K)] \cong \mathbf{R}\Gamma_{\mathrm{cts}}(G_{K'},T)$$

in particular

$$\mathbf{R}\Gamma(\mathcal{C}_\psi^\bullet(\mathcal{D}_{K|L}(T(\tau^{-1})))) \overset{\mathbf{L}}{\otimes}_{\Lambda_K} \mathcal{O}_L \cong \mathbf{R}\Gamma_{\mathrm{cts}}(G_K,T).$$

Proof. We have the following isomorphisms

$$\begin{aligned}
\mathbf{R}\Gamma_{\mathrm{cts}}(G_K, \mathcal{F}_{\Gamma_K}(T)) \overset{\mathbf{L}}{\otimes}_{\Lambda_K} \mathcal{O}_L[\mathrm{Gal}(K'|K)] &\cong \mathbf{R}\Gamma_{\mathrm{cts}}(G_K, \Lambda_K^\iota \otimes_{\mathcal{O}_L} T) \overset{\mathbf{L}}{\otimes}_{\Lambda_K} \mathcal{O}_L[\mathrm{Gal}(K'|K)] \\
&\cong \mathbf{R}\Gamma_{\mathrm{cts}}(G_K, \mathcal{O}_L[\mathrm{Gal}(K'|K)] \otimes_{\Lambda_K} (\Lambda_K^\iota \otimes_{\mathcal{O}_L} T)) \\
&\cong \mathbf{R}\Gamma_{\mathrm{cts}}(G_K, \mathcal{O}_L[\mathrm{Gal}(K'|K)]^\iota \otimes_{\mathcal{O}_L} T) \\
&\cong \mathbf{R}\Gamma_{\mathrm{cts}}(G_{K'}, T)
\end{aligned}$$

where the first isomorphisms comes from Remark 5.2.39, the second one from (loc. cit.), the third one is trivial while the last one is Shapiro's Lemma. □

Remark 5.2.55. *We want to give a more concrete statement of the above* Theorem 5.2.53. *So let as there* $T \in \mathbf{Rep}_{\mathcal{O}_L}^{(\mathrm{fg})}(G_K)$ *and* $K \subseteq K' \subseteq K_\infty$ *an intermediate field, finite over* K, *such that* $\Gamma_{K'} := \mathrm{Gal}(K_\infty|K')$ *is isomorphic to some* $\mathbb{Z}_p^r$. *Let furthermore* $\gamma_1, \ldots, \gamma_r$ *be a set of generators of* $\Gamma_{K'}$. *The* **Koszul-complex** $K_\bullet(\Lambda_{K'})$ *of* $\Lambda_{K'}$ *then is the complex*

$$0 \longrightarrow \textstyle\bigwedge^r \Lambda_{K'} \xrightarrow{\mathrm{d}_r} \bigwedge^{r-1} \Lambda_{K'} \xrightarrow{\mathrm{d}_{r-1}} \cdots \longrightarrow \Lambda_{K'} \xrightarrow{\mathrm{d}_1} \mathcal{O}_L \longrightarrow 0,$$

where $\bigwedge^i \Lambda_{K'}$ *denotes the* i-*th exterior algebra of* $\Lambda_{K'}$ *and*

$$d_i(x_1 \wedge \cdots \wedge x_i) = \sum_{j=1}^i (-1)^{j+1} \mathrm{pr}(x_j) x_1 \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge x_i.$$

Here $\widehat{(-)}$ *denotes that this entry is omitted and* pr *denotes the projection* $\Lambda_{K'} \twoheadrightarrow \Lambda_{K'}/(\gamma_1 - 1, \ldots, \gamma_r - 1) \cong \mathcal{O}_L$ *(cf.* [38, Section 15.28]*). Under the (uncanonical) isomorphism* $\Lambda_{K'} \to \mathcal{O}_L[\![X_1, \ldots, X_r]\!]$, $\gamma_i - 1 \mapsto X_i$ *the above projection becomes the projection to degree zero. Then by* [25, Theorem 16.5, p. 128–129] *the Koszul-complex* $K_\bullet(\Lambda_{K'})$ *of* $\Lambda_{K'}$ *is a free resolution of* $\mathcal{O}_L$ *and therefore (cf.* [38, Section 15.57, Definition 15.57.15]*)* $\mathbf{R}\Gamma(\mathcal{C}_\psi^\bullet(\mathcal{D}_{K|L}(T(\tau^{-1})))) \overset{\mathbf{L}}{\otimes}_{\Lambda_{K'}} \mathcal{O}_L$ *is represented by the complex*

$$(\mathcal{C}_\psi^\bullet(\mathcal{D}_{K|L}(T(\tau^{-1})))) \otimes_{\Lambda_{K'}} K_\bullet(\Lambda_{K'})$$

which then is isomorphic to the complex

$$\mathrm{Tot}\left(\mathcal{D}_{K|L}(T(\tau^{-1})) \otimes_{\Lambda_{K'}} K_\bullet(\Lambda_{K'}) \xrightarrow{(\psi-\mathrm{id})\otimes\mathrm{id}} \mathcal{D}_{K|L}(T(\tau^{-1})) \otimes_{\Lambda_{K'}} K_\bullet(\Lambda_{K'})\right) \cong$$

$$\mathrm{Tot}\left(K_\bullet(\mathcal{D}_{K|L}(T(\tau^{-1}))) \xrightarrow{K_\bullet(\psi)-\mathrm{id}} K_\bullet(\mathcal{D}_{K|L}(T(\tau^{-1})))\right).$$

Here $K_\bullet(\mathcal{D}_{K'|L}(T(\tau^{-1})))$ denotes the Koszul-complex of $\mathcal{D}_{K'|L}(T(\tau^{-1}))$ which is defined in an analogous way to the Koszul-complex of $\Lambda_{K'}$. This last complex then is the generalization of the ψ-Herr complex from the classical theory.

Using the self-duality of the Koszul-complex and an inspection of the complex in the last remark compared to the Pontrjagin dual of the (by local Tate-duality) corresponding φ-Herr-complex, one can indeed derive now that the differentials in the original φ-Herr-complex are strict with closed image (at least for finitely generated torsion coefficients). Indeed, its dual complex has the right cohomology groups, namely the duals of the cohomology groups of the original φ by the above results. It would be desirable to show these topological properties directly in order to get a genuine theory within the world of (φ, Γ)-modules.

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