Topological Realisations of Absolute Galois Groups

Oberseminar Arithmetische Geometrie im Sommersemester 2019

The goal of this term's Oberseminar is to discuss the same-named article of Kucharczyk and Scholze, cf. [KS16]. As the title says we will construct topological spaces whose fundamental group coincides with the absolute Galois group of an extension of $\mathbb{Q}$. In particular, we will prove the following theorem.

Theorem 1.
Let F be an extension of $\mathbb{Q}$ containing all roots of unity. Then there exists a compact Hausdorff space X_F whose étale fundamental fundamental group agrees with the absolute Galois group of F.

In order to do this, we will recall some topological constructions (Talks 2 and 3), especially the theory of étale fundamental groups which we will compare to the classical ones (Talk 2). As a first step, we will prove an analogous theorem in the world of schemes (Talk 4). After we proved Theorem 1 (Talk 5), we will study what extra information the topological space X_F carries.

The first interesting object, we will study, is the classical fundamental group of X_F which we will denote with $\pi_1(X_F)$ for this exposé. We then will prove the following theorem (Talk 6).

Theorem 2.
Let F be an abelian extension of $\mathbb{Q}$ containing all roots of unity. Then X_F is path-connected, the map $\pi_1(X_F) \to \mathrm{Gal}(\overline{F}|F)$ is injective and has dense image. It also is continuous although $\pi_1(X_F)$ does not carry the subspace topology of $\mathrm{Gal}(\overline{F}|F)$.
Moreover, $\pi_1(X_F)$ can be written as an inverse limit of discrete infinite groups.

Next, we will head towards cohomology groups and prove the following theorem (Talk 7).

Theorem 3.
Let $i \geq 0$ and $n \geq 1$. Then there is a natural isomorphism

$$H^i(X_F, \mathbb{Z}/n\mathbb{Z}) \cong H^i(\mathrm{Gal}(\overline{F}|F), \mathbb{Z}/n\mathbb{Z}),$$

where $H^i(X_F, \mathbb{Z}/n\mathbb{Z})$ denotes the sheaf cohomology with coefficients in the constant sheaf $\mathbb{Z}/n\mathbb{Z}$. Moreover, $H^i(X_F, \mathbb{Z})$ is torsion free and there is a canonical isomorphism

$$H^i(X_F, \mathbb{Z})/(n) \cong H^i(\mathrm{Gal}(\overline{F}|F), \mathbb{Z}/n\mathbb{Z}).$$

Last we will discuss if the general case can be handled by some descent technique. Although this is not automatic there are some structures on X_F we did not use so far. The seminar's end will then be proving the following statements (Talk 8).

Theorem 4.
Let l be a prime and F be a perfect field of characteristic different from l such that $\mathrm{Gal}(\overline{F}|F)$ is pro-l and let $n \leq \infty$ be maximal such that $\mu_{l^n} \subseteq F$. Then exists a compact Hausdorf space $Y_{l^n,F}$ with an action of $U_{l^n} = 1 + l^n\mathbb{Z}_{(l)}$ such that its étale fundamental group is isomorphic to $\mathrm{Gal}(\overline{F}|F(\zeta_{l^\infty}))$ and for every $m \geq 1$ there is a natural isomorphism

$$H^i(Y_{l^n,F}, \mathbb{Z}/l^m\mathbb{Z}) \cong H^i(\mathrm{Gal}(\overline{F}|F(\zeta_{l^\infty})), \mathbb{Z}/l^m\mathbb{Z}).$$

1. Time and Place

We meet on *Thursdays, 11ct* in *Seminarraum 4, INF 205.* The first meeting will be on April 18. Please contact me if you would like to give a talk.

2. Contact

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3. Talks

The original article [KS16] is well divided, so the talks are just the sections of the article and therefore are same-named. If enough people are interested, we can split the larger talks.

Talk 1: **Introduction and Classical fundamental groups.** - 1 Session
Give a short introduction to the topic ([KS16, Chapter 1, p. 2–7]) , then recall some basic topology and discuss the examples in [KS16, Section 2.1, p. 7–12]. The focus should be on this last part.

Talk 2: **Étale fundamental groups of topological spaces and comparison to the classical ones.** - 2 Sessions
Follow [KS16, Section 2.2,p. 12–21] to define the étale fundamental group of topological spaces ([KS16, Definition 2.21, p. 20]). Then compare it to the classical fundamental group of topological groups, discuss some of the examples ([KS16, Sections 2.3, p. 21–26]) and the result for étale fundamental groups of schemes ([KS16, Lemma 2.32, p. 26]).

Talk 3: **Topological invariants of Pontryagin duals.** - 1 Session
Recall some basic facts of Pontryagin duals and compute the étale fundamental group of a

Pontryagin dual for torsion-free discrete abelian groups. Then head towards group algebras over $\mathbb{C}$ and compute their étale fundamental groups ([KS16, Chapter 3, p. 26–32]).

Talk 4: **Galois groups as étale fundamental groups of $\mathbb{C}$-schemes.** - 1 Session
Define rational Witt vectors and prove Theorem 1 for schemes ([KS16, Section 4.1,p. 32–34]). Then, prove some statements we will use in the coming talks ([KS16, Section 4.2 and 4.3, p. 34–39]).

Talk 5: **Galois groups as étale fundamental groups of topological spaces.** - 1 Session
Prove Theorem 1 ([KS16, Section 5.1, p. 40–41]) and discuss the relation between the topological space constructed for this proof and the scheme constructed in Talk 4 ([KS16, Section 5.2, p. 41–44]).

Talk 6: **Classical fundamental groups inside Galois groups.** - 2 Sessions
Prove the results ([KS16, Section 6.1 and 6.2, p. 44–52]) which we will need for the proof of Theorem 2 ([KS16, Section 6.3, p. 52–56]).

Talk 7: **Cohomology.** - 1 Session
Introduce the Cartan-Leray spectral sequence ([KS16, Section 7.1, p. 56–58]) and use it to prove Theorem 3 ([KS16, Section 7.2, p. 58–62]).

Talk 8: **The cyclotomic character.** - 2 Sessions
Give the variant of the preceding constructions, which also works if we do not have all roots of unity ([KS16, Section 8.1, p. 62–65]). Then, introduce the three actions on cohomology ([KS16, Section 8.2, p. 65–67]), compare them to each other and prove Theorem 4 on the way ([KS16, Sections 8.3 - 8.5, p. 67–73]).

4. References

[KS16] Robert Kucharczyk and Peter Scholze. *Topological Realisations of Absolute Galois Groups*. http://www.math.uni-bonn.de/people/scholze/GaloisTop.pdf, 2016.