Universität Heidelberg
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Diophantine Problems and p-adic Period mappings (after Lawrence and Venkatesh)

Oberseminar Arithmetische Geometrie im Sommersemester 2020

1. Introduction

In this seminar, we will study the two main results of the paper [LV18] by Brian Lawrence and Akshay Venkatesh.

Firstly, Lawrence and Venkatesh give a new proof of Falting's Theorem (Mordell's conjecture):

Theorem 1 (Faltings)**.** *Let K be a number field and let Y/K be a smooth projective curve of genus ≥ 2. Then the set $Y(K)$ of K-valued points is finite.*

The new proof utilizes the set-up of Faltings' original proof, but makes no usage of abelian varieties. Instead, Lawrence and Venkatesh use p-adic Hodge theory to construct a p-adic period map, which encodes the variation of p-adic Galois representations in a family of algebraic varieties. One important aspect of their argument then is the interplay between the p-adic period map and the complex period map: a crucial statement about the former can be verified on the latter, which the authors do by explicit topological computations of monodromy on the relevant Riemann surfaces. In [LV18, §1.5], the authors compare their proof to Faltings' and assess that it is not simpler or less difficult than the latter. The real gain of their method is its applicability to higher-dimensional varieties. This brings us to the second main result of their paper.

Lawrence and Venkatesh apply the same methods in a higher-dimensional situation to show that the set of hypersurfaces in a projective space, with good reduction away from a fixed set of primes, is contained in a proper Zariski-closed subset of the moduli space of all hypersurfaces. More precisely, they obtain:

Theorem 2 (Lawrence-Venkatesh)**.** *There exists a positive integer n_0 and a function $d_0(n)$ with the following property: If $n \geq n_0$ and $d \geq d_0(n)$ and if $X \to Y$ denotes the universal family of hypersurfaces in $\mathbb{P}^n$ of degree d, then $Y(\mathbb{Z}[S^{-1}])$ is not Zariski dense in Y for any finite set of primes S.*

Since the bigger part of our seminar will be devoted to understanding the proof of Faltings' Theorem in [LV18], we give an outline in the next section.

2. Outline of the arguments in [LV18]

The following is a rough sketch of the strategy of the proof, with some oversimplifications. See [LV18, §1.2] for a more detailed outline of the proof, with precise statements. Our presentation also follows the overview in [Gör]. Let K be a number field and Y/K a smooth projective curve of genus ≥ 2. Suppose that we have a smooth projective morphism $f\colon X \to Y$. Given a point $y \in Y(K)$, we denote by X_y the fiber of f over y. For every "good" prime number p, we have a Galois representation ρ_y of the absolute Galois group G_K on the étale cohomology $H^*_{\text{et}}(X_y \times_K \overline{K}, \mathbb{Q}_p)$. We write ρ_y^{ss} for the semisimplification of the G_K-representation ρ_y. Faltings' finiteness result on Galois representations (see [LV18, §2.3] for a suitable version of the statement) basically says that, as y varies through $Y(K)$, there occur only finitely many isomorphism classes ρ_y^{ss}. Thus, Faltings' Theorem is obtained by showing that both of the following statements hold *for a suitable choice of f and a suitable choice of a place v of K above p*:

(i) The representation ρ_y is semisimple for all but finitely many $y \in Y(K)$;

(ii) The map

$$\begin{aligned} Y(K) &\longrightarrow (\text{representations of } G_{K_v})/\cong \\ y &\longmapsto \rho_y|_{G_{K_v}} \end{aligned}$$

has finite fibers.

Instead of using the theory of heights of abelian varieties to prove the above (as Faltings did), Lawrence and Venkatesh proceed as follows. We focus on sketching their proof of (ii), see [LV18, §1.4] for a discussion of (i). To simplify notation, let us write $\rho_{y,v}$ for the restriction of ρ_y to G_{K_v}. By p-adic Hodge theory, we have a fully faithful embedding

$$(\text{crystalline representations of } G_{K_v}) \longrightarrow (\text{filtered } \phi\text{-modules over } K_v).$$

Furthermore, the representation $\rho_{y,v}$ is crystalline and is mapped to $(H_{\text{dR}}(X_y/K_v), \text{Fil}^\bullet, \varphi)$ under the above functor, where $\text{Fil}^\bullet$ is the Hodge filtration inside the de Rham cohomology $H_{\text{dR}}(X_y/K_v)$ and the semilinear automorphism φ is the Frobenius coming from crystalline cohomology via a suitable comparison theorem between crystalline and de Rham cohomology. Hence it suffices to show that

$$\begin{aligned} Y(K_v) &\longrightarrow (\text{filtered } \phi\text{-modules over } K_v)/\cong \\ y &\longmapsto (H_{\text{dR}}(X_y/K_v), \text{Fil}^\bullet, \varphi) \end{aligned} \tag{$*$}$$

has finite fibers. At this point it is useful to interpret the de Rham cohomology group $H_{\text{dR}}(X_y/K_v)$ as the fiber of the *relative de Rham cohomology vector bundle* over y. This bundle comes equipped with a connection - *the Gauss-Manin connection* - which allows us to identify nearby fibers in a canonical way. Fixing $y_0 \in Y(K_v)$, the Gauss-Manin connections gives us identifications $H_{\text{dR}}(X_y/K_v) \cong H_{\text{dR}}(X_{y_0}/K_v)$ for all $y \in \Omega$, where $\Omega \subseteq Y(K_v)$ is a small p-adic disk around y_0. These identifications respect the Frobenius morphisms, but not necessarily the Hodge filtrations $\text{Fil}^\bullet$. The variation of the filtrations is described by the p-adic period map

$$\begin{aligned} \text{Period}_v\colon \Omega &\longrightarrow \mathcal{F}(K_v) \\ y &\longmapsto (\text{Fil}^\bullet H_{\text{dR}}(X_y/K_v) \text{ transported to } H_{\text{dR}}(X_{y_0}/K_v)) \end{aligned}$$

where $\mathcal{F}$ is a suitable flag variety parametrising certain chains of subspaces in $H_{\text{dR}}(X_{y_0}/K_v)$ $=: H$. The period map is a K_v-analytic map. Two points y, y' have the same image under the map $(*)$ precisely when the triples $(H, \text{Period}_v(y), \varphi)$ and $(H, \text{Period}_v(y'), \varphi)$ are isomorphic. This is the case if and only if there exists an element in the centralizer

$Z(\varphi)$ of φ transforming the filtration $\mathrm{Period}_v(y')$ into $\mathrm{Period}_v(y)$, i.e. when $\mathrm{Period}_v(y') \in Z(\varphi) \cdot \mathrm{Period}_v(y)$. One observes thus that the fiber of $(*)$ over $(H, \mathrm{Period}_v(y), \varphi)$ is contained in

$$\mathrm{Period}_v^{-1}(\mathrm{Period}_v(\Omega) \cap Z(\varphi) \cdot \mathrm{Period}_v(y)).$$

Assuming that we have shown that the $Z(\varphi)$-orbit is a proper subvariety and that the image of Period_v is Zariski dense, then their intersection is contained in the zero set of a non-zero K_v-analytic function. Hence our fiber is contained in the zero set of a non-zero analytic function (obtained by pullback via Period_v) on the disk Ω, which as such is finite.

It remains to construct a specific family $f: X \to Y$ for which $Z(\varphi)$ is small and the image of Period_v is large. Such a family is constructed in [LV18, §7] and is given the name *the Kodaira-Parshin family*. See [LV18, §1.3] for some details on controlling the centralizer. To check Zariski density of the image of the p-adic period map, the crucial point is the passage to the complex period map, which is possible because the Gauss-Manin connection is defined over K. But over the complex numbers, Zariski density can be verified by studying the monodromy action of $\pi_1(Y)$ on the $\mathbb{C}$-points of a flag variety and by other topological methods.

3. Time and Place

The seminar takes place Thursdays at 11:00 o'clock in Seminarraum 4, INF 205. The first talk will be on the 23rd of April.

4. Contact

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Please contact me if you are interested in giving a talk. Since the subject matter combines methods of number theory, algebraic geometry, complex geometry, and differential geometry/topology, we hope for a broad range of participants.

5. Talks

We note that [LV18, §1.6] gives a very neat summary of the structure of the paper.

Talk 1. Introduction
Give an overview of the seminar following the introduction in [LV18].

Talk 2. De Rham cohomology over $\mathbb{C}$ and the Gauss-Manin connection
Expand on [LV18, §3.2] by recalling de Rham cohomology, its comparison with singular cohomology and the monodromy representation. Introduce the Gauss-Manin connection and discuss how it allows us to compare the de Rham cohomology of nearby fibers; [Lit] might be one useful reference for this. Introduce the complex period map and prove [LV18, Lemma 3.1]. Strive to supplement the phenomena you discuss with some examples (e.g. on elliptic curves).

Talk 3. Algebraic de Rham cohomology and the p-adic period map
Recall de Rham cohomology for a general variety over an arbitrary field of characteristic zero, as in e.g. [Ked]. Discuss the Gauss-Manin connection in this setting ([LV18, §3.1 and §3.3]). For a thorough discussion of the diagram (3.9) in [LV18], state the necessary results on crystalline cohomology that Lawrence and Venkatesh refer to. Finally, prove [LV18, Lemma 3.2 and Lemma 3.3].

Talk 4. Galois representations and Hodge structures
Carefully discuss and prove Faltings' finiteness result on Galois representations ([LV18, Lemma 2.3]), e.g. by following the references given in [LV18]. Then discuss [LV18, §3.5]. You may take the classification of crystalline representations by filtered ϕ-modules for granted. Carefully discuss and prove [LV18, Proposition 3.4], which is a preliminary form of the main result of the paper. This method of proof will be used again and again.

Talk 5. The S-unit equation
As a first application, [LV18, §4] gives an alternative proof of the finiteness of the set of solutions to the S-unit equation. Talk a bit about the history and applications of the S-unit equation. The proof in [LV18, §4] serves as a warm-up and introduction to the new proof of Faltings' theorem, which follows similar lines. All of the following refers to [LV18]. Explain the reduction of the theorem to Lemma 4.2. Then introduce the modified Legendre family. Carefully explain why we choose this family for the application of Proposition 3.4. Prove Lemma 4.3 and then (assuming Lemma 4.4) prove Lemma 4.2. If time permits, sketch the proof of Lemma 4.4.

Talk 6. Proof of Falting's theorem, assuming the ingredients
Cover the whole of [LV18, §5]. In particular, state Proposition 5.3 without proof and give an outlook on the key properties of the Kodaira-Parshin family (which will be introduced in a later talk). Explain how Theorem 5.4 (Falting's theorem) is obtained by applying Proposition 5.3 to the Kodaira-Parshin family.

Talks 7-10 elaborate on the ingredients used in Talk 6.

Talk 7. Rational points on the base of an abelian-by-finite family
The goal of this talk is to prove [LV18, Proposition 5.3]. This takes up the whole of [LV18, §6]. Prove as much as you can. One suggestion is to focus on the reduction to Lemma 6.1, for which you may then merely give a sketch of the main ideas.

Talk 8. The Kodaira-Parshin family
Cover the whole of [LV18, §7]. In particular, discuss the construction of the Kodaira-Parshin family and explain how properties (ii) and (iii) in the list after Proposition 5.3 follow from the construction.

Talk 9. The monodromy of Kodaira-Parshin families (Part 1)
This talk and the next are purely topological. The goal is to show the last missing ingredient in the proof of Faltings' theorem: the Kodaira-Parshin family satisfies property (i) in the list after Proposition 5.3, i.e. it has *full monodromy.* Cover the first half of [LV18, §8], i.e. §8.1-§8.4. Aside from 8.2.3, the main result should be Proposition 8.5. Recall background on mapping class groups and Dehn twists along the way.

Talk 10. The monodromy of Kodaira-Parshin families (Part 2)
Cover the second half of [LV18, §8], i.e. §8.5-§8.6. Prove as much as you can. Argue by picture if necessary.

At this point we have proven Faltings' theorem! The next talks turn to higher-dimensional cases.

Talk 11. The Ax-Schanuel theorem of Bakker and Tsimerman
The goal is to make preparations for Talk 12. Cover the whole of [LV18, §9]. In particular, deduce the transcendence property of period mappings (Corollary 9.2) from the theorem of Bakker and Tsimerman, and transfer it to a p-adic setting (§9.2).

Talk 12. Higher dimensions
Give a survey of [LV18, §10]. The main result should be Proposition 10.2, which in combination with Theorem 10.1 implies Theorem 2 (Lawrence-Venkatesh) in our introduction. Moreover, sketch the strategy of the proof of Theorem 10.1. Explain how the results of Talk 11 factor into the proof of Lemma 10.5.

References

[Gör] Ulrich Görtz. *Seminar on Diophantine problems and the p-adic period morphism.* Available at https://www.esaga.uni-due.de/ws1819/alggeo/.

[Ked] Kiran S. Kedlaya. *p-adic cohomology: from theory to practice.* AWS Lecture notes (revised version). Available at http://swc.math.arizona.edu/aws/2007/KedlayaNotes11Mar.pdf.

[Lit] Daniel Litt. *Variation of Hodge structures.* Notes for Number Theory Learning Seminar on Shimura Varieties. Available at http://virtualmath1.stanford.edu/~conrad/shimsem/2013Notes/Littvhs.pdf.

[LV18] Brian Lawrence, Akshay Venkatesh. *Diophantine problems and p-adic period mappings.* Preprint, 2018. Available at https://arxiv.org/abs/1807.02721v3.