Iwasawa theory of p-adic Lie extensions

Summary

One of the most challenging topics in modern number theory is the mysterious relation between *special values of L-functions* and *Galois cohomology*: they are the "shadows" in the two completely different worlds of complex and p-adic analysis of one and the same geometric object, viz the space of solutions for a given diophantine equation over the integral numbers, or more generally a motive M. The main idea of Iwasawa theory is to study manifestations of this principle such as the class number formula or the Birch and Swinnerton Dyer Conjecture simultaneously for whole p-adic families of such motives, which arise e.g. by considering towers of number fields or by (Hida) families of modular forms. The aim of this project is to supply further evidence for

I. the existence of p-adic L-functions and for main conjectures in (non-commutative) Iwasawa theory,

II. the (equivariant) ϵ-conjecture of Fukaya and Kato as well as

III. the 2-variable main conjecture of Hida families.

In particular, we hope to construct the first genuine "non-commutative" p-adic L-function as well as to find (non-commutative) examples fulfilling the expectation that the ϵ-constants, which are determined by the functional equations of the corresponding L-functions, build p-adic families themselves. In the third item a systematic study of Lie groups over pro-p-rings and Big Galois representations is planned with applications to the arithmetic of Hida families.

Zusammenfassung

Eines der herausfordernsten Themen der modernen Zahlentheorie ist der mysteriöse Zusammenhang zwischen *speziellen Werten von L-Funktionen* und *Galoiskohomologie*: Sie sind sozusagen die "Schatten" in den beiden völlig verschiedenen Welten der komplexen und p-adischen Analysis von ein und demselben geometrischen Objekt, nämlich dem Lösungsraum einer gegebenen diophantischen Gleichung über den ganzen Zahlen oder allgemeiner einem Motiv M. Die Hauptidee der Iwasawa-Theorie besteht darin, Manifestationen dieses Prinzips, wie etwa die Klassenzahlformel oder die Birch und Swinnerton Dyer Vermutung, gleichzeitig für ganze p-adische Familien solcher Motive zu untersuchen, die z.B. beim Betrachten von Zahlköpertürmen oder bei (Hida-)Familien von Modulformen auftreten. Das Ziel dieses Projektes besteht darin, weitere Evidenzen für

I. p-adische L-Funktionen und Hauptvermutungen in der (nicht-kommutativen) Iwasawa-Theorie,

II. die (äquivariante) ϵ-Vermutung von Fukaya und Kato sowie

III. die 2-Variablen Hauptvermutung von Hida-Familien

zu erbringen. Insbesondere hoffen wir, die erste echte "nicht-kommutative" p-adische L-Funktion sowie (nicht-kommutative) Beispiele für die Vermutung zu finden, dass die

durch die Funktionalgleichung der entsprechenden L-Funktionen bestimmten ϵ-Konstanten selbst auch in p-adische Familien variieren. In dem dritten Punkt ist eine systematische Untersuchung von Lie-Gruppen über pro-p-Ringen und von Großen Galois-Darstellungen mit Anwendungen auf die Arithmetik von Hida-Familien geplant.

State of the art (in January 2007)

Attached to any motive M over a number field is the complex analytic L-function $L(M, s)$; e.g. for an elliptic curve E over $\mathbb{Q}$ its Hasse-Weil L-function, which encodes the number of solutions of the reduction of E at each prime p; other examples are the Riemann ζ-function or more generally Artin-L-functions. The Tamagawa Number Conjecture of Bloch and Kato, which generalizes the Birch and Swinnerton-Dyer Conjecture and the class number formula, describes the special values of $L(M, s)$ in terms of the Galois cohomology attached to the p-adic realisation of M. This conjecture has been generalized by Burns and Flach [18] to an equivariant version (ETNC), taking an additional action on M, e.g. by the Galois group of a finite extension of $\mathbb{Q}$, into account.
The aim of Iwasawa theory is to study this relationship over an infinite tower K_∞ of number fields K_n, where $K_\infty = \bigcup K_n$ is a Galois extension of $K = K_0$ with p-adic analytic Galois group G : Whenever the special values of the twists $M(\rho)$ of M by the Artin-characters ρ of G satisfy certain congruences they should give rise to an *analytic* p-adic L-function $\mathcal{L}^{an}_{M,K_\infty}$. On the other hand the Galois cohomology over K_∞, which now bears a module structure over the Iwasawa algebra $\Lambda(G) = \mathbb{Z}_p[\![G]\!]$ of G, gives rise to an *algebraic* p-adic L-function $\mathcal{L}^{alg}_{M,K_\infty}$ and the Iwasawa Main Conjecture (MC) claims that both p-adic L-functions are essentially the same. In contrast to finite extensions, conceptual progress relies on the fact that the simultaneous consideration of an infinite family of L-values (e.g. in terms of p-adic L-functions) contains much more information and makes the comparison with Galois-cohomology more rigid; this is also reflected in properties of the Iwasawa algebra, which - in contrast to group algebras of finite groups - is often a torsionfree regular ring, the module category of which admits nice structure theorems. This may explain why Iwasawa theory is the only known general method to prove exact formulae in number theory like those occurring in the Tamagawa Number Conjecture.
Abelian MCs, i.e. over (anti-)cyclotomic fields or over extensions with Galois group isomorphic to $\mathbb{Z}_p^d$, have been proved in various contexts by the work of many mathematicians (Mazur, Wiles, Rubin, Kato, Urban-Skinner, Bertolini-Darmon, Iovita-Pollack ...).

Although Iwasawa theory over arbitrary p-adic Lie groups was already begun in 1979 by Harris, it was not clear for a long time how the above philosophy could be made precise in this context. Only recently the theory was revived by Coates, Howson, Schneider, Sujatha (e.g. [22, 21]) and the applicant (e.g. [11, 8, 9]) undertaking an intensive study of the Euler characteristic of Selmer groups, the structure theory of $\Lambda(G)$-modules and the corresponding K-theory. On the basis of the applicant's habilitation thesis [12] a formalism was developed together with Coates, Fukaya, Kato, Sujatha [2] to also formulate a GL_2-main conjecture for elliptic curves without CM in the above spirit. In particular the algebraic p-adic L-function was constructed as an element of the first K-group $K_1(\Lambda(G)_S)$ of a certain localisation $\Lambda(G)_S$ of $\Lambda(G)$. However little is known about the analytic p-

adic L-function which also is supposed to be an element of $K_1(\Lambda(G)_S)$ and which can be interpreted as a function on the p-adic representations of G. Needless to say we expect similar MCs for p-adic Lie extensions different form GL_2.
In the false Tate curve case (G is isomorphic to the semi-direct product of $\mathbb{Z}_p$ with $\mathbb{Z}_p^*$) Kato [36] managed to determine $K_1(\Lambda(G))$ and $K_1(\Lambda(G)_S)$ in terms of the Iwasawa algebras of abelian groups isomorphic to $\mathbb{Z}_p$, and it turns out that the existence of the analytic p-adic L-function is equivalent to the validity of completely new types of congruences among certain p-adic L-functions from the cyclotomic theory. Some of these congruences were proved by Bouganis [15], see also the similar approach by Delbourgo and Ward [27]. Also in the situation of a *one dimensional* p-adic Lie group which is a product of a finite p-Heisenberg group with $\mathbb{Z}_p$ Kato (unpublished) again could calculate the corresponding K-groups and in fact he managed to verify the resulting congruences for the Tate-motive thereby showing the existence of the p-adic L-function and even the corresponding MC. See also the work of Ritter and Weiss [44] for another approach to congruences in this situation.
The main goal of the project I. is to establish the existence of a p-adic L-function for a non-commutative p-adic Lie group of *dimension at least two* following the above strategy verifying more and more congruences. But in order to verify the non-commutative MC one also needs to check that the Galois cohomology satisfies certain torsion properties and this will form another aspect of this project. Then, granted the existence of the p-adic L-function and this torsion property, the non-commutative MC usually follows from the validity of cyclotomic MCs for certain twists of the motive as shown by Burns, Kato, Ritter and Weiss.

Huber and Kings [35] generalized the ETNC to infinite Lie-extensions formulating a MC without a p-adic L-function. Perhaps the most general version of the ETNC is the ζ-(isomorphism)-conjecture of Fukaya and Kato [31]. They also explain that the existence of a p-adic L-function as in the MC of [2] would follow from the existence of this ζ-isomorphism together with their additional ϵ-(isomorhism)-conjecture, which we shall recall now.
The (equivariant) Local Tamagawa Number Conjecture (LTNC) expresses the belief that the (E)TNC should be compatible with the functional equation on the complex analytic side and with duality properties of the Galois cohomology on the p-adic side. Fukaya and Kato [31, 7] formulate an (equivariant) ϵ-conjecture at each prime which refines the LTNC: it states that the local epsilon factors as defined by Deligne together with the Bloch-Kato exponential map also behave well in p-adic families, i.e. they satisfy certain congruences. For $\ell \neq p$ this conjecture has been proved by Yasuda [47] while at p little is known in general.
In the cyclotomic theory, the interpolation of the Bloch-Kato exponential map was first described by Perrin-Riou [42], in fact it is a generalisation of Coleman's map [24, 25] in the spirit of Coates-Wiles [23]. Later other descriptions were given by Cherbonnier and Colmez [20]. Using the results on Perrin Riou's reciprocity law, Benois and Berger [17] proved recently the epsilon-conjecutre at p for crystalline representations. This in turn is used by Burns and Flach [19] to show the LTNC for Tate motives.

In project II. we intend to prove the ϵ-conjecture in cases of non-commutative p-adic Lie-extensions for the first time.
On the other hand Coleman maps often give an alternative approach to p-adic L-functions

in the presence of Euler systems. Though it is not at all clear whether a non-commutative Euler system exists we hope that generalisations of Coleman-maps shed new light both onto the espsilon conjecture and the existence of p-adic L-functions.
Fukaya [30, 29] extends Coleman's map to higher local fields using algebraic K-groups. Recently Zerbes generalized her work by combining it with techniques of [20]. In the context of Hida-families (compare III.) two-variable Coleman maps occur also in the work of Delbourgo, Ochiai [40, 41].

Following Greenberg, Mazur et al. classical Iwasawa theory should be considered as *cyclotomic* deformations of motives. Also Hida families of modular forms give rise to Galois deformations [33]. Both of these are examples of Big Galois representations in Nekovar's sense. In fact, from a technical point of view, the latter are the coefficients in the Galois cohomology for Iwasawa theory. In particular they are used in [31] for the formulation of their ζ- and ϵ-conjectures.
Thus having arithmetic applications in mind, it seems desirable to extend Lazard's classical results on p-adic Lie groups over $\mathbb{Z}_p$ and their cohomology to analytic groups over more general pro-p-rings $\mathcal{R}$ such as $\mathcal{R} = \mathbb{Z}_p[[X_1, \ldots, X_n]]$ and this is the starting point of project III. Some properties of analytic groups over such pro-p rings have been investigated already in [28]. Finally we mention that pro-p-subgroups of $SL_2(\mathcal{R})$ have been characterised by Pink [43].

Goals

Our research proposal has three aims:

I. Investigate the existence of p-adic L-functions and verify the non-commutative Main Conjecture (MC) in special cases.

II. Extend results concerning the ϵ-isomorphism and Coleman-maps.

III. Systematically study Lie groups over (big) pro-p-rings and their cohomology with applications towards the Iwasawa theory of Hida families.

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