A crash introduction to Gromov hyperbolic spaces

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Geodesic metric spaces

Definition

A metric space (X, d) is **proper** if for every $r > 0$ the ball $\overline{B(x, r)}$ is compact. It is **geodesic** if every two points of X are joined by a geodesic.

- $\mathbb{R}^n$ with the Euclidean distance d_{Eucl}.
- the infinite tree T with its length distance (every edge has length 1);

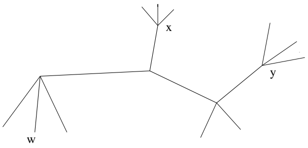


Figure: The infinite tree T

Geodesic metric spaces: the hyperbolic space $\mathbb{H}^n$

Disk model $\mathbb{D}^n$

$\mathbb{D}^n := \{x \in \mathbb{R}^n \mid |x| < 1\}$ with the Riemannian metric induced by $g_x := \frac{4}{(1-||x||^2)^2} g_{Eucl}$

Upper half plane $\mathbb{H}^n$

$\mathbb{H}^n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$ with the Riemannian metric induced by $g_x := \frac{1}{x_n} g_{Eucl}$

- $\forall x, y \in \mathbb{D}^n$ there exists a unique geodesic $\overline{xy}$
- every geodesic segment can be extended indefinitely

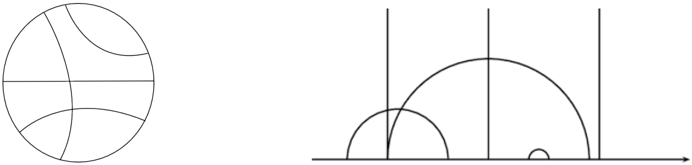


Figure: radii/arcs $\perp \partial\mathbb{D}^2$ and half-circles/lines $\perp \partial\mathbb{H}^2$

Geodesic metric spaces: the hyperbolic space $\mathbb{H}^n$

The **boundary of** $\mathbb{H}^n$ is defined as the space:
$\partial\mathbb{H}^n = \{$ geodesic rays $c : [0, \infty) \to \mathbb{H}^n\}/\sim$

$$c \sim c' \text{ if and only if } d(c(t), c'(t)) < M$$

It can be topologized so that $\partial\mathbb{H}^n = S^{n-1}$ and $\overline{\mathbb{H}^n} = \mathbb{H}^n \cup \partial\mathbb{H}^n$ is compact.

Two geodesics can be: incident (1 common point in $\mathbb{H}^n$, asymptotic (1 common point on $\partial\mathbb{H}^n$), or ultraparallel (no common points).

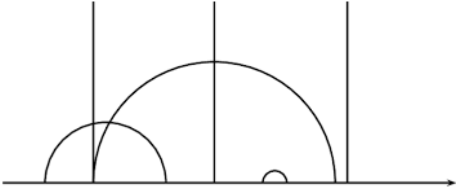


Figure: $\partial\mathbb{H}^2$ =x -axis $\cup\{\infty\}$

Motivation: Negative curvature without formulas

Riemannian manifold $(M^n, g) \longrightarrow$ intrinsic notion of curvature defined by g

- (flat geometry) $\mathbb{R}^2$ has constant sectional curvature $= 0$
- (spherical geometry) $\mathbb{S}^2$ has constant sectional curvature $= 1$
- (hyperbolic geometry) $\mathbb{H}^2$ has constant sectional curvature $= -1$: geodesics diverge exponentially fast; geodesic triangles are thin...

This talk:

- negative curvature for metric spaces using only geometric notions
- analogies between trees and $\mathbb{H}^n \longrightarrow$ Gromov hyperbolic metric spaces

No Riemannian geometry, no tensors, no painful formulas like this:

$$K = -\frac{1}{E}\left(\frac{\partial}{\partial u}\Gamma_{12}^2 - \frac{\partial}{\partial v}\Gamma_{11}^2 + \Gamma_{12}^1\Gamma_{11}^2 - \Gamma_{11}^1\Gamma_{12}^2 + \Gamma_{12}^2\Gamma_{12}^2 - \Gamma_{11}^2\Gamma_{22}^2\right)$$

(Gaussian curvature $n = 2$)

Geodesic metric spaces

Let (X, d_X) and (Y, d_Y) be proper geodesic metric spaces. A map $f : X \to Y$ is

- an **isometry** when $d_Y(f(x), f(y)) = d_X(x, y)$ for all $x, y \in X$;
- a (λ, ϵ)**quasi-isometric embedding** when $d_Y(f(x), f(y)) \sim d_X(x, y)$:

$$\frac{1}{\lambda} d_X(x, y) - \epsilon \leq d_Y(f(x), f(y)) \leq \lambda d_X(x, y) + \epsilon \text{ for all } x, y \in X \ ;$$

- a (λ, ϵ) **quasi-isometry** when f is a (λ, ϵ)-q.isometric embedding and $\exists C \geq 0$ such that every point of Y lies in a C-neighborhood of Imf.

Quasi-isometries: Examples

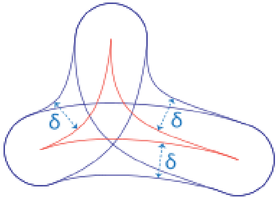
- Every finite diameter space X is quasi-isometric to a point.
- All finite degree trees are quasi-isometric to each other.
- $\mathbb{R}^2$ is not quasi-isometric to $\mathbb{R}$.
- $\mathbb{R}^2$ is not quasi-isometric to $\mathbb{H}^2$

A **quasigeodesic** is a quasi-isometric embedding $\gamma : I \to X$ where $I \subset \mathbb{R}$.

- Every geodesic in X is as quasi-geodesic
- The curve $t \mapsto (t, \log(1+t))$ is a quasi-geodesic in $(\mathbb{R}^2, d_E)$ but not a geodesic.

Gromov hyperbolicity I : the Rips condition

Let $\delta > 0$. A geodesic triangle is called δ-**slim** if each of its sides is contained in the δ-neighborhood of the union of the other two sides.



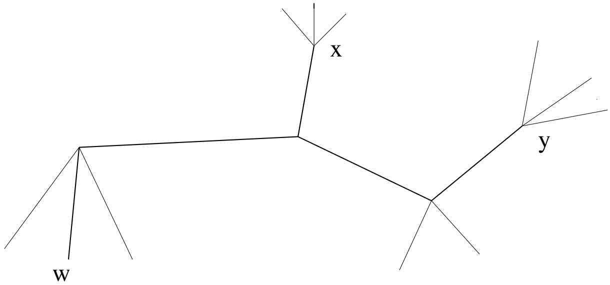
Definition (Criterion I)

A space (X, d) is **Gromov-hyperbolic** (or δ-hyperbolic) if there exists a $\delta \geq 0$ such that every geodesic triangle is δ-slim.

Gromov hyperbolicity I: Examples

Gromov hyperbolic spaces are "thickened" versions of trees:

- $\mathbb{H}^n$ (any complete Riemannian manifold with sectional curvature $\leq \epsilon < 0$).
- Trees are 0-hyperbolic: geodesic triangles are tripods



- $\mathbb{R}^2$ is **not** Gromov-hyperbolic

Gromov hyperbolicity II: Gromov product

Definition (Gromov Product)

The **Gromov product** of $x, y \in X$ with respect to $w \in X$ is defined as

$$(x, y)_w = \frac{1}{2}(d(x, w) + d(y, w) - d(x, y))$$

- it measures the "defect" of triangle inequality: $d(x, y) \leq d(x, w) + d(y, w)$
- $(x, y)_w$ measures how long the geodesics $\overline{wx}$ and $\overline{wy}$ travel together before diverging

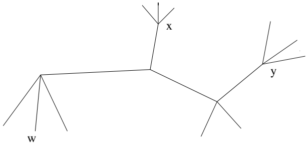


Figure: In a tree $(x, y)_w$ is the distance from w to $\overline{xy}$

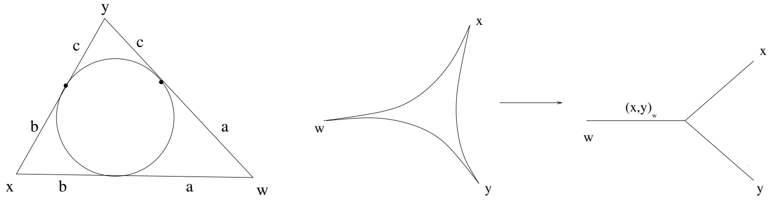
Gromov hyperbolicity II: Geometric interpretation

- $(x,y)_w$ measures how long $\overline{wx}$ and $\overline{wy}$ travel together before diverging

There exists unique $a, b, c \geq 0$ such that:

$$\begin{aligned} d(x,w) &= a+b \\ d(x,y) &= b+c \\ d(y,w) &= a+c \end{aligned}$$

The solutions are $a = (x,y)_w$; $b = (y,w)_x$ and $c = (x,w)_y$.



Gromov hyperbolicity II: the 4-point condition

Definition (Criterion II: the 4-point condition)

X is δ-hyperbolic if and only if for all $x, y, z, w \in X$ we have:

$$(x, y)_w \geq \min((y, z)_w, (x, z)_w) - \delta \ .$$

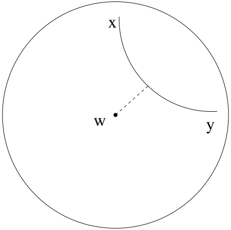
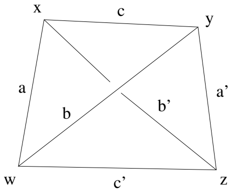


Figure: $(x, y)_w \sim d(w, [x, y])$ in $\mathbb{H}^2$

Gromov hyperbolicity II: the 4-point condition

Consider the tetrahedron $T(x, y, w, z)$:



- sum the lengths of the opposite sides $a + a'$, $b + b'$, $c + c'$;
- order the sums: $S \leq M \leq L$.

The 4-point condition is equivalent to the following condition on T:

$$L \leq M + 2\delta \ .$$

Geodesics and quasi-geodesics

Definition (Hausdorff distance)

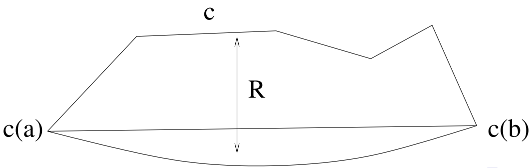
Let $A, B \subset X$. The **Hausdorff distance** between A, B is defined by

$$d_H(A,B) = \inf\{\epsilon \mid A \subseteq N_\epsilon(B) \text{ and } B \subseteq N_\epsilon(A)\}$$

Theorem (Stability of quasi-geodesics)

Let X be δ-hyperbolic. For every $\lambda \geq 1$ and $\epsilon \geq 0$ there exists $R_{\lambda,\epsilon}$ such that for every (λ,ϵ)-quasigeodesic $c : [a,b] \to X$ is R-close to every geodesic segment $[c(a), c(b)]$ we have:

$$d_H(\mathrm{Im}\, c, [c(a), c(b)]) \leq R_{\lambda,\epsilon} \ .$$



∂X Gromov boundary: asymptotic geodesic rays

Definition (Gromov-boundary)

We define the **Gromov boundary** of X as

$$\partial_{r,w}X := \{c : [0, \infty) \to X \text{ geodesic ray }, c(0) = w\}/\sim$$

where $c \sim c'$ if and only if $d_H(\mathrm{Im}\, c, \mathrm{Im}\, c') < \infty$.

A **basis of neighborhoods** for $p \in \partial_{r,w}X$ is given by

$$V(p, R) = \{q \in \partial X \mid \exists c \in [p], c' \in [q] \text{ with } \liminf_{s,t\to\infty}(c(s), c'(t))_w \geq R\} \; .$$

- (Visibility) For every $x \neq y \in \partial X$ there exists a bi-infinite geodesic connecting them

Properties of ∂X

A few facts (theorems):

- $\partial_{r,w}X$ does not depend on w;
- ∂X admits a metrizable topology and $\overline{X} := X \cup \partial X$ is compact.

Theorem

Let X, X' be hyperbolic spaces. If $f : X \to X'$ is a q.i.-embedding then the map

$$\begin{aligned} f_\partial : \partial X &\to \partial X' \\ [c] &\mapsto [f \circ c] \end{aligned}$$

is a topological embedding. If f is a quasi-isometry then f_∂ is a homeomorphism.

Gromov-boundaries: Hall of Fame

- The Gromov boundary of the tree T_n is a Cantor set.
- The Gromov boundary of $\mathbb{H}^n$ is $\partial \mathbb{H}^n = \mathbb{S}^{n-1}$.

Corollary

$\mathbb{H}^n$ and $\mathbb{H}^m$ are quasi-isometric if and only if $n = m$.

∂X Gromov boundary: converging sequences

Fix $w \in X$. Let $\{x_n\} \subseteq X$ be a sequence. We say $x_n \to \infty$ if

$$\liminf_{n,m \to \infty} (x_n, x_m)_w = +\infty.$$

(The definition actually does not depend on w.)

When X is Gromov-hyperbolic we define an equivalence relation $\sim$:

$$\{x_n\} \sim \{y_n\} \text{ if } \liminf_{i,j \to \infty} (x_i, y_j)_w = \infty \ .$$

The space $\partial_{s,w} X := \{$ sequences $x_n \to \infty\}/\sim$ is the **Gromov boundary** of X.

A **basis of neighborhoods** for $p \in \partial_{s,w} X$ is given by

$$U(p, R) = \{q \in \partial X \mid \exists (x_n) \in p, (y_n) \in q \text{ with } \liminf_{i,j \to \infty} (x_i, y_j)_w \geq R\}$$

Generalized Gromov product of $\overline{X}$

We can extend the Gromov product to $\overline{X} = X \cup \partial X$ as follows:

Definition (Gromov product on $\overline{X}$)

Let X be δ-hyperbolic space with basepoint $p \in X$. We extend the Gromov product to $\overline{X} = X \cup \partial X$ by:

$$(x, y)_p = \sup \liminf_{i,j\to\infty} (x_i, y_j)_p \ ,$$

where the supremum is taken over all sequences $X \ni x_i \to x$ and $X \ni y_j \to y$.

- $(x, y)_p = \infty$ if and only if $x = y \in \partial X$
- $\forall x, y \in \overline{X} \quad \exists x_n \to x$ and $y_n \to y$ such that $(x, y)_p = \lim_{n\to\infty} (x_n, y_n)_p$
- $\forall x, y, z \in \overline{X} \quad (x, y)_p \geq \min\{(x, z)_p, (z, y)_p\} - \delta$

Generalized Gromov product on $\partial\mathbb{H}^n$

Fix a base point p and let $x, y \in \partial\mathbb{H}^n$. Then the generalized Gromov product

$$(x, y)_p = \log\csc(\theta/2) \ ,$$

where θ is the angle between the geodesic rays $\overline{px}$ and $\overline{py}$.

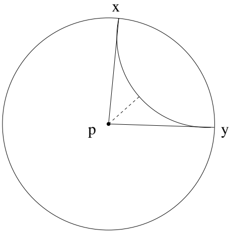


Figure: $(x, y)_p \sim d(p, [x, y])$

Visual metric on ∂X

Definition (Visual metric on ∂X)

Let X be a hyperbolic space with basepoint p. A metric d on ∂X is called a *visual metric* with parameter a if $\exists A, B > 0$ such that for all $x, y \in X$:

$$Aa^{-(x,y)_p} \leq d(x,y) \leq Ba^{-(x,y)_p} \ .$$

Theorem (Existence of visual metrics on ∂X)

Let X δ-hyperbolic. There exists ϵ_0 such that for every $0 \leq \epsilon < \epsilon_0$ then $d_{p,\epsilon}$ is a visual metric on ∂X:

$$K_\epsilon \rho_{p,\epsilon}(x,y) \leq d_{p,\epsilon}(x,y) \leq \rho_{p,\epsilon}(x,y) \ .$$

The topology induced on $(\partial X, d_{p,\epsilon})$ coincides with the one defined before.

Visual metric on ∂X: Construction

Fix $\epsilon > 0$. There is a natural "measure of separation" of points on ∂X:

$$\rho_{p,\epsilon}(x,y) = e^{-\epsilon (x,y)_p}$$

(Notice that $\rho_{p,\epsilon}(x,y) = 0$ if and only if $x = y$ and $\rho_{p,\epsilon}$ is symmetric).

Definition (Pseudo-metric associated to $\rho_{p,\epsilon}$)

The **pseudo-metric** associated to it is defined as:

$$d_{p,\epsilon}(x,y) := \inf \sum_{i=1}^{n} \rho_{p,\epsilon}(x_{i-1},x_i)$$

The infimum is over the chains $x = x_0, \ldots, x_n = y$ with no bound on n.
(When $\epsilon < \epsilon_0$ then $d_{p,\epsilon}$ is actually a metric: $d_{p,\epsilon}(x,y) > 0$ when $x \neq y$).

- $\partial \mathbb{H}^n$ is isometric to $\partial \mathbb{S}^{n-1}$.

Visual metric on ∂X

Theorem (Dependence on p, ϵ)

Let X be a δ-hyperbolic space.

1. *if $d_{p,\epsilon}$ and $d_{q,\epsilon}$ are visual metrics on ∂X then they are Lipschitz equivalent.*
2. *if $d_{p,\epsilon}$ and $d_{q,\epsilon'}$ are visual metrics on ∂X then they are Hölder equivalent:*

$$\exists C > 0 \quad \frac{[d_{q,\epsilon'}(x,y)]^{\alpha}}{C} \leq d_{p,\epsilon}(x,y) \leq C[d_{q,\epsilon'}(x,y)]^{\alpha} \quad \forall x, y \in \partial X$$

(with $\alpha = \log(\epsilon)/\log(\epsilon')$).

Corollary

Let $f : X \to Y$ be a quasi-isometry between δ-hyperbolic spaces. Then

$$f_\partial : (\partial X, d) \to (\partial Y, d)$$
$$[c] \mapsto [f \circ c]$$

is a Hölder homeomorphism.

Conclusion

General frame for the study of negative curvature spaces:

- Capture curvature without use of heavy differential geometry and formulas.
- Captures the analogies between the geometry of $\mathbb{H}^n$ and metric trees.
- Boundaries are nice.

- Other tensor-free approaches to negative curvature: geometry of $CAT(0)$ spaces .