Filters, Nets and Tychnoff's Theorem

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Outline

Our problem

$$X = \prod_{i \in I} X_i$$

Our solution

- Nets
- Filters
- Equivalence of the theories derived from filters and nets
- Ultrafilters
- Subbases and the product topology
- Alexander subbase theorem
- Corollary Tychnoff's theorem

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Directed Sets and Nets

Definition (directed set)

A set A with a relation $\leq$ is directed, if

1. $\alpha \leq \alpha, \ \forall \alpha \in A$ (reflexive),
2. if $\alpha \leq \beta$ and $\beta \leq \gamma$, then $\alpha \leq \gamma$ (transitive),
3. $\forall \alpha, \beta \ \exists \gamma$, s.t. $\alpha \leq \gamma$ and $\beta \leq \gamma$.

Examples : $\mathbb{N}$ and $\mathbb{R}$.

Definition (net)

A net in X is a map x from a directed set A on another set X.
We write $x(\alpha) = x_\alpha$.

Example : if $A = \mathbb{N}$ and $X = \mathbb{R}$, then we get the sequences in $\mathbb{R}$.

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Filters

Definition (filter)

A filter in X $\mathcal{F}$ is a non-empty collection of subsets of X, s.t.

1. if $F \in \mathcal{F}$ and $F \subset G$, then $G \in \mathcal{F}$,
2. if $F_1, F_2 \in \mathcal{F}$, then $F_1 \cap F_2 \in \mathcal{F}$,
3. $\emptyset \notin \mathcal{F}$.

Example

- The collection of subsets of $\mathbb{R}$ containing 0,
- the collection of neighbourhoods of a point p in any space.

Equivalence of the Theories derived from Filters and Nets

Lemma

We can construct a filter from a net and vice versa.

Proof.

- For $(x_\alpha)_{\alpha \in A}$ define $F(\alpha) := \{x_\beta : \beta \geq \alpha\}$ and hence $\mathcal{F}\left((x_\alpha)_{\alpha \in A}\right)$ are the sets containing an $F(\alpha)$,
- For $\mathcal{F}$, just pick x_F in $F \in \mathcal{F}$.

□

Theorem

The theories derived from filters and nets are equivalent.

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Ultrafilters

Lemma (Existence of ultrafilters)

Every filter in X is contained in a maximal filter, called ultrafilter.

Proof.

Apply Zorn's Lemma on the collection of all filters X with the relation $\subset$. □

Definition (Subbase)

A collection of sets $(S_i)_{i\in I}$ is called a subbase, if any open set is the union of such sets $S_{i_1} \cap \cdots \cap S_{i_n}$.

Example : The collection of sets $(-\infty, b)$ and (a, ∞) with arbitrary a and b in $\mathbb{R}$.

Definition (product topology)

The product topology on $X = \prod_{i\in I} X_i$ has as subbase the collection $\{pr_i^{-1}(U) : U \text{ open in } X_i\}$.

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Subbases and the Product Topology

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Alexander Subbase Theorem and Tychonoff's Theorem

Theorem (Alexander subbase theorem)

A set X is compact, if for one subbase every cover of its elements has got a finite subcover.

Proof.

Only important point : we use ultrafilters! □

Theorem (Tychonoff's theorem, proven by E. Čech in 1937)

The product of compact sets over an arbitrary indexset is again compact in the product topology.

Applications

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- Banach Alaoglu uses that $\prod_{x \in X} \bar{B}_{\|x\|}(0)$ is compact,
- the p-adic integers are compact $\mathbb{Z}_p \subset \prod_k \mathbb{Z}/p^k\mathbb{Z}$.

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Further Ideas

Further ideas

- Tychnoff's Theorem is equivalent to Zorn' Lemma and the Axiom of Choice,
- we proved $X = \prod_{i \in I} X_i$ is compact, but $\bigcup_{i \in I} K_i$ isn't !

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