1 Sheaves of smooth functions

As we make extensive usage of connected subsets of the manifold X during this subsection we may assume without loss of generality that X is connected.

Definition 1 (Sheaf of smooth functions $\mathcal{C}_U^\infty$)
The **sheaf of smooth functions** $\mathcal{C}_U^\infty$ of an open subset U of X is the collection of all smooth functions mapping a domain contained in U to the real numbers, i.e.
$\mathcal{C}_U^\infty = \bigcup_{\mathcal{D}\subset U} C^\infty(\mathcal{D})$

For the sake of simplicity I sometimes write $f \in \mathcal{C}_U^\infty$, if $f \in C^\infty(\mathcal{D})$ with $\mathcal{D} \overset{\text{open}}{\subset} U$ was 100 percent correct.

Definition 2 (Smooth germ)
Given a point p in $V \overset{\text{open}}{\subset} U$ and a smooth function $f : V \to \mathbb{R}$ in $\mathcal{C}_U^\infty$ we define f's **germ** in p as $[f] = \left\{ g \in \mathcal{C}_U^\infty \quad (g \in C^\infty(\mathcal{D}), \mathcal{D} \overset{\text{open}}{\subset} U) : \exists\, W : p \in W \overset{\text{open}}{\subset} (V \cap \mathcal{D} \subset) U \;\&\; f|_W = g|_W \right\}$.
Two functions out of one germ are also sometimes referred to as being equivalent in signs $f \sim g$. The set of these equivalence classes is the **stalk of $\mathcal{C}_U^\infty$at p** denoted by $\mathcal{C}_{U,p}^\infty$.

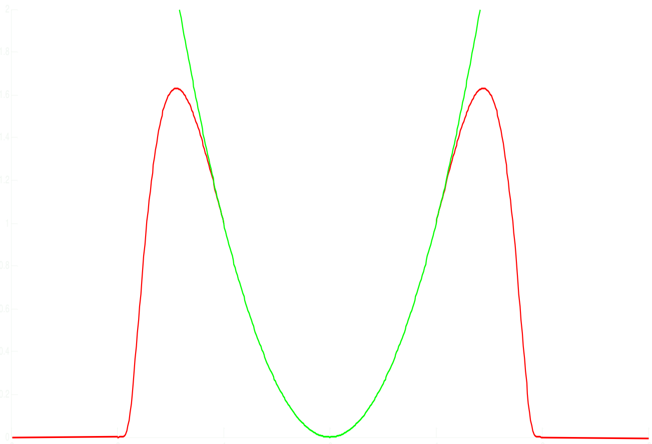

Figure 1: example for definition 2 : the parabola and an equivalent function

Lemma 3
$\mathcal{C}_{U,p}^\infty$ possesses an $\mathbb{R}$-algebra structure with addition $[f] + [g] := [f + g]$ and multiplication $[f] \cdot [g] := [fg]$.

Proof

These binary operations are well defined, as for $f_1|_{W_f} = f_2|_{W_f}$ and $g_1|_{W_g} = g_2|_{W_g}$ we conclude $f_1|_W = f_2|_W$ and $g_1|_W = g_2|_W$ with $p \in W = W_f \cap W_g$. Therefore $(f_1 + g_1)|_W = (f_2 + g_2)|_W$ and $(f_1 g_1)|_W = (f_2 g_2)|_W$. So all the algebra axioms for $\mathcal{C}^\infty_{U,p}$can be deduced from the algebras $C^\infty(W)$. ■

Corollary 4

The above lemma induces an algebra homomorphism

$$\begin{array}{rccc} \pi : & C^\infty(V) & \longrightarrow & \mathcal{C}^\infty_U \\ & f & \longmapsto & [f] \end{array}$$

for $V \subset U$. We can actually prove that π is surjective. □

Remark 5

Furthermore we can evaluate $[f]$ at p, as f coincides with every other representative g on an open neighbourhood of p and hence especially on p. □