On the Existence of certain Vector Valued Siegel Modular Forms

Thomas Wieber

June 1, 2014

Contents

Nomenclature

$\lceil x \rceil$ ceiling function $\min \{n \in \mathbb{Z} \mid n \geq x\}$, page 53

$\lfloor x \rfloor$ floor function $\max \{m \in \mathbb{Z} \mid m \leq x\}$, page 53

$[f]_{\mathcal{O}_{U,p}}$ germ of the holomorphic function f in $\mathcal{O}_{U,p}$, page 22

$\bigwedge^k (V)$ vector space of alternating tensors on V, page 10

$V^{\otimes n}$ k times tensor product of the C-vector space V, page 10

$v_1 \otimes_K \cdots \otimes_K v_k$ tensor product of the vectors $v_1, \ldots, v_k$, page 9

A^{-t} the transposed inverse matrix of A, i.e. $A^{-t} = (A^t)^{-1} = (A^{-1})^t$, page 12

A^t the transposed matrix of A, page 12

$\mathbb{A}_\alpha$ α-th affine space, page 21

$\mathrm{Adj}(A)$ the adjugate matrix of A, page 12

$\mathrm{Bihol}\,(U)$ the group of biholomorphic functions on U, page 15

$\mathbb{C}_d[X^1, \ldots, X^n]$ vector space of polynomials homogeneous of degree d, page 8

$\operatorname{codim} Y$ codimension of an analytic variety Y in a complex manifold M, i.e. $\operatorname{codim} Y = \dim M - \dim Y$, page 43

$\mathbb{E}$ the unit disc $B_1(0)$ in $\mathbb{C}$, page 44

$\mathbb{E}^n$ unit polycylinder or polydisc in $\mathbb{C}^n$, page 46

E_p fibre over p, page 30

${}_G\backslash^S$ orbit space, page 11

G_x stabiliser subgroup, page 11

$\mathcal{H}$ left half plane, i.e. $\{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$, page 48

$\mathcal{M}(M) \otimes_{\mathcal{O}} \Gamma(T^*M^{\otimes q})$ vector space of meromorphic tensors such that their associated holomorphic tensors belong to $\Gamma(T^*M^{\otimes q})$, page 38

$\mathcal{M}(M)$ algebra of meromorphic functions on a manifold M, page 37

$\mathcal{O}_U$ sheaf of holomorphic functions, page 22

$\mathcal{O}_{U,p}$ stalk of holomorphic functions at the point p, page 22

$(\Omega^\bullet)^{\otimes k}(M, D)$ vector space of generalised logarithmic tensors, page 54

$\Omega^q(M)$ vector space of q-forms, page 34

$\mathcal{O}(M, N)$ the set of holomorphic functions between the manifolds M and N, page 20

$\mathrm{ord}\,(f, Y, p)$ order of the singularity of f along Y in p, page 49

$\mathcal{O}(U)$ the algebra of complex valued holomorphic functions, page 15

$\mathcal{O}(U, \mathbb{C}^m)$ the vector space of holomorphic functions, page 15

$\mathcal{O}(U, Y)$ restriction of $\mathcal{O}(U, \mathbb{C}^m)$, page 15

$P(f)$ pole locus of f, page 37

$\mathbb{P}^n\mathbb{C}$ n-dimensional projective space, page 21

p_n^k k-th n-dimensional standard element, page 46

$p_{n,Q}^k$ k-th n-dimensional Q-standard element, page 46

Y_{reg} regular locus of an analytic subvariety Y, page 42

Y_{sing} singular locus of an analytic subvariety Y, page 42

$\operatorname{supp} D$ support of the divisor D, page 43

A^{-t} the transposed inverse matrix of A, i.e. $A^{-t} = (A^t)^{-1} = (A^{-1})^t$, page 12

A^t the transposed matrix of A, page 12

T_pM tangent space of M at p, page 23

$\mathfrak{w}^j$ j-th coordinate for a second chart on $\mathbb{P}^n\mathbb{C}$, page 37

$Z^*(f)$ zero locus of f without zero, page 7

$Z(f)$ zero locus of f, page 7

$Z((f_i)_{i\in I})$ common zero locus of $(f_i)_{i\in I}$, page 7

$\mathfrak{z}$ point in $\mathbb{P}^n\mathbb{C}$, without loss of generality in $\mathbb{A}_0$, page 22

$\mathfrak{z}^i$ i-th coordinate of $\mathfrak{z}$, page 22

1 Introduction

This is an excerpt of my unsubmitted diploma thesis. The broad presentation of differential geometry could and hopefully should be useful for students.

June 2014, Heidelberg

2 Algebraic preliminaries

2.1 Polynomials

[Zero locus of a function]We denote by $Z(f)$ the **zero locus** of $f : X \to \mathbb{C}^n$. For a family of functions $(f_i)_{i\in I}$ $Z((f_i)_{i\in I})$ denotes the common zero locus of them, i.e. $\bigcap_{i\in I} Z(f_i)$. If X is a vector space, then we are sometimes interested in $Z^*(f) := Z(f)\setminus\{0\}$.

2.1.0. $Z^*(f)$ really necessary ?

As we need a Corollary of Hilbert's Nullstellensatz we begin with

[Hilbert's Nullstellensatz] Let $P_1, \ldots, P_k \in \mathbb{C}[X^1, \ldots, X^n]$. If $\mathfrak{a} = \mathfrak{a}(Z(P_1, \ldots, P_k))$ is the ideal of polynomials vanishing on $Z(P_1, \ldots, P_k)$, i.e. $\mathfrak{a} = \left\{P \in \mathbb{C}[X^1, \ldots, X^n] : P(z) = 0\ \forall z \in Z(P_1, \ldots, P_k)\right\}$, then $\mathfrak{a} = rad(P_1, \ldots, P_k) := \left\{P \in \mathbb{C}[X^1, \ldots, X^n] : \exists\, m > 0 : P^m \in (P_1, \ldots, P_k)\right\}$. For a proof please have a look in [Lan02, Theorem 1.5., p.380].

If the zero locus of a polynomial Q is a subset of the zero locus of another polynomial P, then Q divides a power of P. We just have to observe that P lies in $\mathfrak{a}(Z(Q))$ and hence there exists an element A of $\mathbb{C}[X^1, \ldots, X^n]$ such that $P^m = AQ$.

In a unique factorization domain(UFD) we denote by $(f_i)_{i\in I_\mathcal{P}}$ the collection of representatives of equivalence classes of prime elements.

An isomorphism between UFDs $\phi : R \to S$ maps primes onto primes and leaves prime factorizations invariant, i.e. $\phi(\epsilon_R \cdot \prod_{i\in I_\mathcal{P}} f_i^{\nu_i}) = \epsilon_S \cdot \prod_{i\in I_\mathcal{P}} g_i^{\nu_i}$.

2.1.0. prove this.

[Square-free element]In a unique factorization domain an element x with a factorization $x = \epsilon \cdot \prod_{i\in I_\mathcal{P}} f_i^{\nu_i}$ is **square-free** if $\nu_i \le 1\ \forall\ i \in I_\mathcal{P}$.

If the zero locus of a square-free polynomial Q is a subset of the zero locus of another polynomial P, then Q divides P. We deduce from section 2.1 on this page that

$P^m = AQ$. Expressing P and Q by $\epsilon_P \cdot \prod_{i\in I_\mathcal{P}} f_i^{\nu_i(P)}$ and $\epsilon_Q \cdot \prod_{i\in I_\mathcal{P}} f_i^{\nu_i(Q)}$, respectively, leads to $m \cdot \nu_i(P) \geq \nu_i(Q)\ \forall\ i \in I_\mathcal{P}$. Hence $\nu_i(Q) = 1$ implies $\nu_i(P) \geq 1 = \nu_i(Q)$.

[Homogeneous function] Let C be a complex cone, i.e. for all t in $\mathbb{C}^*$ $z \in C$ implies $tz \in C$. We call a function $f : C \to \mathbb{C}$ **homogeneous of degree** d if it holds $f(tz) = t^d f(z)$ for every z in $\mathbb{C}^n$ and t in $\mathbb{C}^*$.

[Homogeneous polynomials] The set of **polynomials homogeneous of degree** d in n variables is denoted by $\mathbb{C}_d[X^1,\dots,X^n]$. As 0 lies in each of these $\mathbb{C}_d[X^1,\dots,X^n]$s they become vector spaces collapsing for negative ds.

2.1.0. Add reference.

For positive d $\mathbb{C}_d[X^1,\dots,X^n]$ is a vector space of dimension $\frac{(d+n-1)!}{d!(n-1)!}$.

If the product of two non-zero polynomials P and Q is homogeneous, then so are P and Q. Let $n = \deg(P)$ and $m = \deg(Q)$, then it holds $t^{m+n}P(z)Q(z) = P(tz)Q(tz)$ for any $z \in \mathbb{C}^n$ and $t \in \mathbb{C}$. Expressing $P(tz)$ and $Q(tz)$ as a power series in t with coefficients equaling the homogenous parts of P and Q, respectively, gives $t^{m+n}P(z)Q(z) = a_\alpha t^\alpha b_\beta t^\beta = \sum_{k=0}^{n+m} t^k \sum_{l=0}^k a_l b_{k-l}$. Comparing the coefficients leads to $\sum_{l=0}^k a_l b_{k-l} = 0$ for $k < m+n$ especially $a_0 = 0$ or $b_0 = 0$. We prove this lemma algorithmically and denote by A_i the index for which we have already calculated after the i-th step that $a_j = 0\ \forall j \leq A_i < n$. $\sum_{l=0}^k a_l b_{k-l} = \sum_{l=0}^{A_i} a_l b_{k-l} + a_{A_i} b_{k-B_i} + \sum_{l=k-B_i}^k a_l b_{k-l} = a_{A_i} b_{k-B_i}$. Hence either A_i or B_i can be increased.

2.1.0. smoothen proof

The factorization of a homogeneous polynomial Q consists of homogeneous polynomials.

2.2 Tensor products

[Lan02, Chapter XVI]

[Tensor product of modules] The **tensor product** $M_1 \otimes_R \cdots \otimes_R M_k$ of the modules $M_1,\dots,M_k$ over a commutative ring R is the module uniquely determined (up to isomorphisms) by the *universal property*, i.e. there is a multilinear map $\mathrm{ten} : M_1 \times \cdots \times M_k \longrightarrow M_1 \otimes_R \cdots \otimes_R M_k$ and for each *multilinear* map $f : M_1 \times \cdots \times M_k \longrightarrow N$ exists exactly one *linear* map $f_* : M_1 \otimes_R \cdots \otimes_R M_k \longrightarrow N$ satisfying $f = f_* \circ \mathrm{ten}$.

$$
\begin{tikzcd}
 & M_1 \otimes_R \cdots \otimes_R M_k \arrow[d, dashed, "\exists!\ f_*"] \\
M_1 \times \cdots \times M_k \arrow[ur, "\mathrm{ten}"] \arrow[r, "f"'] & N
\end{tikzcd}
$$

1. We denote by $v_1 \otimes_R \cdots \otimes_R v_k$ the image of $(v_1, \ldots, v_k) \in M_1 \times \cdots \times M_k$ under the map ten.

2. When it is clear which ring is used $v_1 \otimes_R \cdots \otimes_R v_k$ and even $M_1 \otimes_R \cdots \otimes_R M_k$ can be abbreviated by $v_1 \otimes \cdots \otimes v_k$ and $M_1 \otimes \cdots \otimes M_k$, respectively.

In order to present the first example of a tensor product we fix some further notation.

[Dual space and basis]Let V be a finite dimensional vector space over field K of characteristic 0. We denote by V^* its **dual space** $\mathrm{Hom}_K(V, K)$. The **dual basis** $\left\{e^{1*}, \ldots, e^{n*}\right\}$ is the basis of V^* specified with respect to a basis $\{e_1, \ldots, e_n\}$ of V by $e^{i*}(e_j) = \delta^i_j \ \forall i, j$.

[Tensor product of vector spaces] If R equals $\mathbb{C}$ (or any other field of characteristic 0) then $\mathbb{C}$-modules are $\mathbb{C}$-vector spaces and the tensor product $V_1 \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} V_k$ coincides with the multilinear maps from $V_1^* \times \cdots \times V_k^*$ to $\mathbb{C}$, i.e. $\mathrm{Mult}\left(V_1^* \times \cdots \times V_k^*, \mathbb{C}\right)$. And ten maps $(v_1, \ldots, v_k)$ to

$$
\begin{array}{rccc}
v_1 \otimes_K \cdots \otimes_K v_k : & V_1^* \times \cdots \times V_k^* & \longrightarrow & K \\
 & (\phi_1, \ldots, \phi_k) & \longmapsto & \prod \phi_i(v_i).
\end{array}
$$

We deduce from the universal property :

1. There is a natural isomorphism between $(M_1 \otimes_R \cdots \otimes_R M_l) \otimes_R (M_{l+1} \otimes_R \cdots \otimes_R M_k)$ and $M_1 \otimes_R \cdots \otimes_R M_k$.

2. The modules $M \otimes_R N$ and $N \otimes_R M$ are naturally isomorphic.

3. A tuple of linear maps $\Psi_i : M_i \to N_i$ induces
$\Psi = \Psi_1 \otimes \ldots \otimes \Psi_k \ : (M_1 \otimes_R \cdots \otimes_R M_k) \to (N_1 \otimes_R \cdots \otimes_R N_k)$.

From now on we concentrate on vector spaces over the complex numbers $\mathbb{C}$.

Given the tensor product $V_1 \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} V_k$ then the vector spaces' basis $\left(e^j_{i_j}\right)_{1 \le i_j \le \dim V_j}$ induce a basis $\left(e^1_{i_1} \otimes \cdots \otimes e^k_{i_k}\right)_{1 \le i_j \le \dim V_j}$ of $V_1 \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} V_k$.

We shorten $V_1 \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} V_k$ to $V^{\otimes n}$ when V equals each V_i. In this case we also shorten the basis elements $e^1_{j_1} \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} e^n_{j_n}$ to $e_{j_1,\dots,j_n}$.

[Group of permutations $\mathfrak{S}_n$] The group of permutations on $\{1,\dots,n\}$ is denoted by $\mathfrak{S}_n$.

[Alternating tensor] We call a tensor T **alternating** if it holds for all permutations σ in $\mathfrak{S}_n$ and all vectors in V $T(v_1,\dots,v_n) = \operatorname{sgn}(\sigma)\, T(v_{\sigma(1)},\dots,v_{\sigma(n)}) \equiv \operatorname{sgn}(\sigma)\, T^\sigma(v_1,\dots,v_n)$.

[The vector space of alternating tensors $\bigwedge^n(V)$] The set of alternating tensors $\bigwedge^n(V)$ is a vector space giving rise to a vector space epimorphism

$$\begin{array}{rccl} alt: & V^{\otimes n} & \twoheadrightarrow & \bigwedge^n(V) \\ & T & \longmapsto & \frac{1}{|\mathfrak{S}_n|}\sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma)\, T^\sigma \end{array}$$

that equals the identity map on $\bigwedge^n(V)$.

$\bigwedge^k(V)$ has got the basis $(alt\,(e_{j_1,\dots,j_k}))_{1\le j_1<j_2<\cdots<j_{k-1}<j_k\le n}$ and consequently dimension $\binom{n}{k}$.

We fix further notations. $alt\,(v_1 \otimes \ldots \otimes v_k)$ is shortened to $v_1 \wedge \cdots \wedge v_k$. If $j_1 < j_2 < \cdots < j_{k-1} < j_k$ then $alt\,(e_{j_1} \otimes \ldots \otimes e_{j_k})$ is abbreviated by $e_{\{j_1,\dots,j_k\}} \equiv e_J$.

[Exterior algebra] We denote by $\mathrm{Alt}^\bullet(V)$ the direct sum $\bigoplus_{k=0}^\infty \bigwedge^k(V)$ which is finite due to item 3, i.e. $\bigoplus_{k=0}^n \bigwedge^k(V)$.

A linear map $\Psi : V \to W$ induces a map $\bigwedge^k \Psi : \bigwedge^k(V) \to \bigwedge^k(W)$ satisfying $\left(\bigwedge^k \Psi\right)(v^I e^V_I) = v^I \det(\Psi^J_I) e^W_J$.

2.2.0. finish this section.

2.3 Group actions

[Group action] A **group action** of a group G on a set S is a group homomorphism $\rho : G \to (Aut(S), \circ)$ of G into the group of automorphisms, i.e. bijective self-maps.

We normally denote $\rho(g)(x)$ by gx or $g(x)$.
The natural questions arising from definition 2.3 are which elements of G leave an element x of S unchanged and where is x mapped by all different g in G ?

[Stabiliser subgroup G_x] For a given point $x \in S$ and a group G acting on S the group $G_x := \{g \in G : g(x) = x\}$ is called **stabiliser subgroup** G_x of x.

[Orbit space ${}_G\backslash^S$] The **orbit** Gx of x under G is an equivalence class in S of the form $\{y \in S : \exists\, g \in G : y = gx\} = \bigcup_{g \in G} \{gx\}$. The **orbit space** ${}_G\backslash^S$ is the collection of all orbits.

We characterize group actions by their stabiliser subgroups, similarly as [BBI01, p.83].

[Free group action] If each stabiliser subgroup G_x only consists of the identity element in G then the group action is **free**.

If the group is acting on a set with an additional structure, then we focus on the automorphisms of this structure. We do this twice, once in section 3.1 with topological spaces and now with vector spaces.

absichtlich drin gelassen

2.4 Group representations

[Group representation] A **representation** of a group G on a vector space V is a group homomorphism $\rho : G \to GL(V)$. If G is equipped with an additional structure then sharper definitions are possible, e.g. $\mathrm{GL}(n, \mathbb{C})$ is a linear algebraic group.

[Polynomial map] A map $\varphi : \mathrm{GL}(n, \mathbb{C}) \to \mathbb{C}$ is **polynomial** if there exists a polynomial $P \in \mathbb{C}[X^1, \ldots, X^{n^2}]$ such that it holds $\varphi(A) = P(a_1^1, \ldots, a_n^n)$ for all $A = (a_j^i)$ in $\mathrm{GL}(n, \mathbb{C})$.

[Vector valued polynomial map] A map φ from $\mathrm{GL}(n, \mathbb{C})$ to a finite dimensional vector space V is called **polynomial** if for a given basis (e_i) the associated coordinate functions φ^i are polynomial. This definition is independent of the chosen basis because coordinate functions belonging to different basis transform linearly by the corresponding basis change matrix. According to [Spr77, definition 1.4.8, p.7] we pick as vector space the set of endomorphisms $W = \mathrm{End}\,(V)$ and define

[Rational representation of $\mathrm{GL}(n, \mathbb{C})$] A representation of $\mathrm{GL}(n, \mathbb{C})$ on a vector space V is **rational**, if there is a natural number k for which $\det A^k \cdot \rho(A)$ is polynomial.

[Weight of a rational group representation] Due to the fact that $\mathbb{C}\left[X^1, \ldots, X^{n^2}\right]$ is a UFD there is a minimal *integer* for which $\det A^k \cdot \rho(A)$ is still polynomial. It is referred to as the **weight** of ρ.

[Reduced group representation] A rational representation ρ is called **reduced** if it has got zero weight.

1. Building blocks for a lot of representations are the standard $\rho_e(A) = A$, contragradient $\rho_c(A) = A^{-t}$ and the determinant representation $\det(A)$.

2. The representation defined by

$$\begin{array}{rccc} \rho_e \otimes \rho_e : & \mathrm{GL}(n, \mathbb{C}) & \longrightarrow & \mathrm{Aut}\,(\mathrm{M}(n, \mathbb{C})) \\ & A & \longmapsto & \left\{X \mapsto AXA^t\right\} \end{array}$$

   is reduced.

3. The representation defined by

$$\begin{array}{rccc} \rho_c \otimes \rho_c : & \mathrm{GL}(n, \mathbb{C}) & \longrightarrow & \mathrm{Aut}\,(\mathrm{M}(n, \mathbb{C}))) \\ & A & \longmapsto & \left\{X \mapsto A^{-t}XA^{-1}\right\} \end{array}$$

   is rational of weight 2. Indeed Cramer's rule yields that A^{-1} is $\frac{1}{\det(A)}\,\mathrm{Adj}(A)$ where Adj denotes the polynomial map sending a matrix to its adjugate. Therefore ρ has got a weight of at most 2 . But observing $\det(c \cdot I_n)\rho(c \cdot I_n) = \frac{1}{c} \cdot I_{M(n,\mathbb{C})}$ shows that the weight has to be strictly greater than 1.

4. Both representations given above can be restricted to the vector space of symmetric

matrices $(\mathbb{C}^n)^{\odot 2}$, i.e.

$$\begin{array}{rccl} \rho_e \odot \rho_e : & \mathrm{GL}(n,\mathbb{C}) & \longrightarrow & \mathrm{Aut}\left((\mathbb{C}^n)^{\odot 2}\right) \\ & A & \longmapsto & \left\{X \mapsto AXA^t\right\} \end{array}$$

and $\rho_c \odot \rho_c$ analogously.

3 Analytic preliminaries

[Proper map]A map $p : X \to Y$ between two locally compact Hausdorff spaces is **proper**, if the preimage of a compact subset in Y is compact in X.

[Complex differentiable function]Let U be an open subset of $\mathbb{C}^n$. A function $f : U \to \mathbb{C}^m$ is **complex differentiable**, if it is Fréchet differentiable, i.e there is an associated function $Df : U \to L(\mathbb{C}^n, \mathbb{C}^m)$ satisfying

$$f(z) = f(z_0) + Df\,(z_0)\,(z - z_0) + o(\|z - z_0\|)$$

in every $z_0 \in U$, where $o(\|z - z_0\|)$ is the Landau Small O symbol.

In honour of Oka Kiyoshi we denote the algebra of complex differentiable functions by $\mathcal{O}(U, \mathbb{C}^m)$. Complex differentiable functions are usually called **holomorphic**.

Given a subset Y of $\mathbb{C}^m$ we define $\mathcal{O}(U, Y)$ to be $\{f \in \mathcal{O}(U, \mathbb{C}^m) : f(U) \subset Y\}$ and proceed similarly with $\mathbb{C}^m = \mathbb{C}$, i.e. $\mathcal{O}(U) := \mathcal{O}(U, \mathbb{C})$.
The partial derivative $D_j f(z_0)$ and the transformation matrix of $Df(z_0)$ are denoted by $\frac{\partial f}{\partial z^j}(z_0)$ and $\mathrm{Jac}(f, z_0) = \left(\frac{\partial f^i}{\partial z^j}(z_0)\right)_{1 \le i \le m,\ 1 \le j \le n}$ respectively.

[Biholomorphic function]We call a bijective holomorphic function $f : U \to V$ with holomorphic inverse $f^{-1} : V \to U$ **biholomorphic**. If U and V coincide then the biholomorphic functions form the group $\mathrm{Bihol}\,(U) \subsetneq \mathcal{O}(U, U)$.

We shall show in section 3.9 on page 40 that it is redundant to claim separately that f^{-1} is holomorphic, because it follows from the two other properties.

3.1 Continuous group actions

[Topological group] A **topological group** is a group with a topology on the ensemble of its elements such that the functions $(g,h) \mapsto g \cdot h$ and $g \mapsto g^{-1}$ are continuous.

It is worthwhile to mention that we can equip every group with the discrete topology making it a topological group.

[Continuous group action] The action of a topological group G on a topological space X is **continuous**, if

$$\begin{array}{rccc} \rho : & G \times X & \longrightarrow & X \\ & (g,x) & \longmapsto & gx \end{array}$$

is continuous with respect to the product topology.

Consider a group G with discrete topology and a topological space X. Then any homomorphism from G into the group of X's homeomorphisms is a continuous group action. It follows immediately that $\{g\} \times g^{-1}(\Omega)$ is open in the product topology for any open subset Ω of X.

[Topology on the orbit space ${}_G\backslash^X$] Suppose G acts continuously on a topological space X. Then the projection

$$\begin{array}{rccc} \pi_G : & X & \twoheadrightarrow & {}_G\backslash^X \\ & x & \longmapsto & Gx \end{array}$$

3.1.0. really useful ?

induces the quotient topology on ${}_G\backslash^X$, i.e. U is open in ${}_G\backslash^X$ iff $\pi_G^{-1}(U)$ is open in X. A useful fact is that any $A \subset {}_G\backslash^X$ equals $\bigcup_{x \in \pi_G^{-1}(A)} \{Gx\}$.

[Totally discontinuously group action] A group G acts **totally discontinuously** on a locally compact Hausdorff space, if

- for any two compact subsets K_1 and K_2 the set $\{g \in G : g \circ K_1 \cap K_2 \neq \emptyset\}$ is finite
- and G is acting continuously.

?? on page ?? displays an example for a totally discontinuously group action.

Totally discontinuously group actions have pleasant properties as stated in the following lemma.

If G acts totally discontinuously on X then

1. for every p in X there exists a neighbourhood $\tilde{U}$ such that $\left\{g \in G : g \circ \tilde{U} \cap \tilde{U} \neq \emptyset\right\}$ equals G_p,
2. the orbit space ${}_{G}\backslash^{X}$ is Hausdorff,
3. it holds $G_q \subset G_p$ for all q in the aforementioned $\tilde{U} = \tilde{U}_p$,
4. G_p is finite for every p in X,
5. therefore it exists a neighbourhood U of p, such that G_p is acting on U and ${}_{G_p}\backslash^{U}$ is homeomorphic to an open neighbourhood of Gp in ${}_{G}\backslash^{X}$,
6. if the action of G is also free, then the induced projection $X \to {}_{G}\backslash^{X}$ is locally a homeomorphism.

i.&ii. The first two statements are proven in Proposition 1.7 and 1.8 of [Shi71][p.3] respectively.

iii. All elements g of G_q satisfy $g \circ \tilde{U} \cap \tilde{U} \supset \{q\} \neq \emptyset$ per definitionem.

iv. Setting $\{p\} = K_1 = K_2$ leads to the third statement.

v. We refine $\tilde{U}$ from 1) by setting $U := \bigcap_{g \in G_p} g \circ \tilde{U} \subset \tilde{U}$ and hence it still satisfies

(3.1) $G_p = \{g \in G : g \circ U \cap U \neq \emptyset\}$.

Even more U is G_p-invariant by construction implying that the action of G_p on U is well defined, leading to

$$\begin{array}{rccc} \iota : & {}_{G_p}\backslash^{U} & \twoheadrightarrow & \pi_G(U) \subset {}_{G}\backslash^{X} \\ & G_p x & \longmapsto & Gx. \end{array}$$

This map is bijective, because $Gx = Gy$ implies $x = h(y)$ for a certain $h \in G$ and therefore $h \circ U \cap U \neq \emptyset$. We deduce from eq. (3.1) that h lies in G_p and so $G_p x = G_p y$.
In order to prove that ι is open it suffices to show that $\pi_G^{-1}(\iota(\Omega))$ is open for an arbitrary open set $\Omega \subset {}_{G_p}\backslash^{U}$ with $V = \pi_{G_p}^{-1}(\Omega)$. Indeed this preimage equals $\pi_G^{-1}\left(\bigcup_{x \in V} Gx\right) = \bigcup_{x \in V} \pi_G^{-1}(Gx) = \bigcup_{x \in V} \bigcup_{h \in G} \{hx\}$. As G is acting continuously

this is the union of the open sets $h(V)$ for $h \in G$.
ι is continuous because $\pi_{G_p}^{-1}(\iota^{-1}(\Omega))$ is $\{x \in U : Gx \in \Omega\} = U \cap \pi_G^{-1}(\Omega)$.

vi. As G is acting freely ${}_{G_x}\backslash^{U} \xrightarrow{\cong} \pi_G(U)$ can be extended to $U \xrightarrow{\cong} {}_{\{id\}}\backslash^{U} = {}_{G_x}\backslash^{U} \xrightarrow{\cong} \pi_G(U)$.

3.2 Topological and complex manifolds

The aim of this subsection - an adaption of [Wie10, Section 2.1] - is to introduce the concept of a complex manifold.

[C^0-atlas] A C^0**-atlas** on a topological space X consists of an open cover $(U_i)_{i \in I}$ of X and a family (also over I) of homeomorphisms $\phi_i : U_i \to V_i \overset{\text{open}}{\subset} \mathbb{R}^{n_i}$. The maps are known as **chart**s or **coordinate function**s **of the specific atlas**.

[Topological manifold] A **topological manifold** M is a second countable Hausdorff space admitting a C^0-atlas on it.

[Transition functions] The change of coordinates as visualised in fig. 3.1 on the next page is described by the **transition function**

$$\begin{array}{rccc} \tau_{j \to i}: & \underbrace{\phi_j(U_i \cap U_j)}_{\subset \mathbb{R}^{n_j}} & \longrightarrow & \underbrace{\phi_i(U_i \cap U_j)}_{\subset \mathbb{R}^{n_i}} \\ & x & \longmapsto & \phi_i \circ \phi_j^{-1}(x). \end{array}$$

As the transition functions are homeomorphisms, [Bro12] implies, that their domain and image both lie in the same $\mathbb{R}^k$ if non-void. Hence if one chart maps in $\mathbb{C}^k$ any other chart does so too.

This allows us to specify certain atlases.

[Holomorphic atlas] We call an C^0-atlas **holomorphic atlas**, if every transition function $\tau_{j \to i}$ is a holomorphic map between two open subsets of $\mathbb{C}^k$.

As $\tau_{j \to i}^{-1} = \tau_{i \to j}$ the above implies that the transition functions of holomorphic atlases are biholomorphic functions.

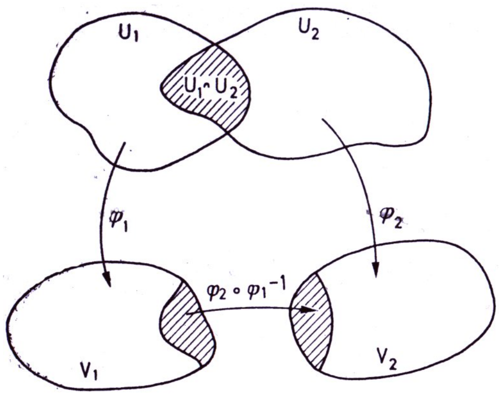

Figure 3.1: the charts and the transition function, picture retrieved from [For77]

[Equivalent atlases] Two holomorphic atlases $(U_i, \phi_i)_{i\in I}$ and $(\Omega_j, \psi_j)_{j\in J}$ are **equivalent** if their union is still a holomorphic atlas or equivalently every transition function

$$\begin{array}{rccc} \tau_{j\to i}: & \psi_j(U_i \cap \Omega_j) & \longrightarrow & \phi_i(U_i \cap \Omega_j) \\ & z & \longmapsto & \phi_i \circ \psi_j^{-1}(z) \end{array}$$

is holomorphic, cf. [GHL87, Def 1.7, p.5].

[Holomorphic structure] A **holomorphic structure** on a topological manifold M is an equivalence class of holomorphic atlases.

Every holomorphic atlas is contained in an equivalence class and induces a holomorphic structure in this way.

[Complex manifold] A **complex manifold** is a topological manifold with a holomorphic structure on it.

1. Any open subset U of $\mathbb{C}^n$ is a complex manifold because $id : U = M \longrightarrow U \subset \mathbb{C}^n$ is a homeomorphism. And as $\mathcal{A} = \{id_U\}$ only consists of this chart, the compatibility condition is trivially satisfied giving rise to a holomorphic structure.

2. Every open subset Ω of a complex manifold M is a complex manifold. We just have to restrict a holomorphic atlas of M $\mathcal{A} = (U_i, \phi_i)_{i\in I}$ to the family $\left(U_i \cap \Omega,\, \phi_i|_{U_i\cap\Omega}\right)_{i\in I}$ which is still satisfying the properties of a holomorphic atlas. Obviously any other

holomorphic atlas of $\mathcal{A}$'s equivalence class would have induced an equivalent holomorphic atlas on Ω.

3. We dedicate the whole section 3.3 to an important example the projective space which is extensively used in chapter 5.

[Charts of complex manifolds] A **chart of a complex manifold** is a map that is a chart in one of the holomorphic structure's atlases.

[Dimension of a manifold] Consider a point p on a topological or complex manifold M and two charts (U_1, ϕ_1) and (U_2, ϕ_2) with $p \in U_1 \cap U_2$. We shall use the notation $\mathbb{K}$ for $\mathbb{C}$ or $\mathbb{R}$ if it is clear from the context (topological vs. complex manifolfd) which one is meant. Then we conclude that ϕ_1 and ϕ_2 map into Euclidean spaces $\mathbb{K}^{n_i}$ of the same dimension because of fig. 3.1 on page 22. Therefore the map

$$\begin{array}{rccl} \dim_{\mathbb{K}} : & M & \longrightarrow & \mathbb{N} \\ & p & \longmapsto & n_i \text{ for } p \in U_i \end{array}$$

is well-defined. Furthermore it is constant on each U_i and hence locally constant. So we assign a distinct number to every connected component M_i, the **dimension of** M_i $\dim_{\mathbb{K}} M_i$. The **dimension of** M is the supremum over the dimensions of the connected components of M. We only observe manifolds of finite dimension. If all connected components have got the same dimension n we say M is of **pure dimension**. A (complex) n-dimensional manifold is denoted by $M = M^n$.

[Holomorphic functions between manifolds] A continuous function f between two topological manifolds M and N is called **holomorphic**, if for any charts (z, U) and (w, V) of M and N, respectively, satisfying $f(U) \subset V$ $w \circ f \circ z^{-1}$ lies in $\mathcal{O}(U, V)$. We denote the set of all these functions by $\mathcal{O}(M, N)$.

1. Here we used z and w as symbols for the charts of manifolds $M = M^n$ and $N = N^m$. We do this if we are only working with 2 or 3 different charts. Especially if we want to use the coordinates in $z(U)$ and $w(V)$ which we denote by $(z^1, \dots, z^n)$ and $(w^1, \dots, w^m)$ respectively.

2. The charts (ϕ_i, U_i) of a complex manifold $M = M^n$ are in $\mathcal{O}(U_i, \mathbb{C}^n)$ with U_i and $\mathbb{C}^n$ considered as complex manifolds. Indeed cocatenating ϕ with charts from U_i, i.e. $\phi_i|_{U_i \cap \Omega}$, and $\mathbb{C}^n$, i.e. id, as presented in item 3 gives $id \circ \phi_i \circ \phi_j^{-1} = \tau_{j \to i} \in \mathcal{O}(U_i, \mathbb{C}^n)$.

The construction undertaken in fig. 3.1 to item 3 can be adjusted to similar cases like smooth, k-times differentiable, etc. transition functions giving rise to smooth or C^k-manifolds.

3.3 The n-dimensional projective space

The topic of this subsection is the n-dimensional projective space, the manifold we are using in chapter 5 in order to prove the results for certain modular forms.

[The n-dimensional projective space $\mathbb{P}^n\mathbb{C}$] We define the **n-dimensional projective space** $\mathbb{P}^n\mathbb{C}$ to be the collection of lines in $\mathbb{C}^{n+1}$ through the origin. Each of these lines can be viewed as an equivalence class on $\mathbb{C}^{n+1}\backslash\{0\}$ for the relation $x \sim y \iff \exists\ \lambda \in \mathbb{C}^* : x = \lambda y$.

The n-dimensional projective space $\mathbb{P}^n\mathbb{C}$ is a complex manifold. Sticking with the notation of [GH78, Examples, 0 Foundational material, 2. Compl. Mfds, p.15] the map

$$\begin{array}{rccl} \pi : & \mathbb{C}^{n+1}\backslash\{0\} & \longrightarrow & \mathbb{P}^n\mathbb{C} = {}^{(\mathbb{C}^{n+1}\backslash\{0\})}/_{\sim} \\ & z = \left(z^0, \dots, z^n\right) & \longmapsto & [z] = \left[z^0, \dots, z^n\right] \end{array}$$

induces the quotient topology on $\mathbb{P}^n\mathbb{C}$. In [FG02, p. 209] it is shown that $\mathbb{P}^n\mathbb{C}$ is actually a Hausdorff space.
There are homeomorphisms between the α-th **affine space** $\mathbb{A}_\alpha = \{[z] : z^\alpha \neq 0\}$ and $\mathbb{C}^n$

$$\begin{array}{rccl} \phi_\alpha : & \mathbb{A}_\alpha & \longrightarrow & \mathbb{C}^n \\ & \left[z^0, \dots, z^n\right] & \longmapsto & \left(\frac{z^0}{z^\alpha}, \dots, \frac{z^{\alpha-1}}{z^\alpha}, \frac{z^{\alpha+1}}{z^\alpha}, \dots \frac{z^n}{z^\alpha}\right). \end{array}$$

These maps and open sets form the compatible holomorphic charts of $\mathbb{P}^n\mathbb{C}$ because

$$\begin{array}{rccl} \tau_{\alpha\beta} : & \mathbb{C}^n\backslash\left\{w^\beta = 0\right\} & \longrightarrow & \mathbb{C}^n\backslash\left\{w^\beta = 0\right\} \\ & (w^0, \dots, \widehat{w^\alpha}, \dots, w^n) & \longmapsto & \frac{1}{w^\beta}(w^0, \dots, \widehat{w^\beta}, \dots, w^{\alpha-1}, 1, w^{\alpha+1}, \dots, w^n) \end{array}$$

is holomorphic and $\pi^{-1}(\mathbb{A}_\alpha) = \left(\mathbb{C}^{n+1}\backslash\left(\mathbb{C}^{\alpha-1} \times \{0\} \times \mathbb{C}^{n-\alpha}\right)\right)$.

In computations and proofs we assume without loss of generality that the current chart is $(\mathbb{A}_0, \phi_0)$, so we introduce a special notation for this case : the coordinates of $[z^0, \ldots, z^n] =:$ $\mathfrak{z}$ are denoted by $\left(\frac{z^1}{z^0}, \ldots \frac{z^n}{z^0}\right) =: (\mathfrak{z}^1, \ldots, \mathfrak{z}^n)$.

It holds $Z^*(Q) = \pi^{-1}\pi(Z^*(Q))$ for every homogeneous polynomial in $n+1$ variables. The inclusion of the left hand side in the right hand side is obviously true for any map. As π is just the projection onto the equivalence classes $\pi^{-1}\pi(z)$ coincides with $\left\{w \in \mathbb{C}^{n+1} : \exists\, t \in \mathbb{C}^* : w = t \cdot z\right\}$. As Q is homogenous z in $Z^*(Q)$ implies $\pi^{-1}\pi(z) \subset Z^*(Q)$ and hence $\pi^{-1}\pi(Z^*(Q)) \subset Z^*(Q)$.

3.4 Stalks of holomorphic functions

The concept of sheaves, stalks and germs equips the tangent space with a chart independent vector space structure and is even easily extendable to varieties, as mentioned in [Ser65, page LG 3.8]. As we make extensive usage of connected subsets of M during this subsection and consequently in the sections 3.5 to 3.8 we may assume without loss of generality that M is connected.

3.4.0. remove this definition ?

The classical foundation for the definitions of germs and stalks in section 3.4 is the subsequent [Sheaf of holomorphic functions $\mathcal{O}_U$] The **(pre-)sheaf of holomorphic functions** $\mathcal{O}_U$ of an open subset U of M is the collection of all holomorphic functions mapping an open subset contained in U to the complex numbers, i.e. $\mathcal{O}_U = (\mathcal{O}(V))_{V \overset{\text{open}}{\subset} U}$.
We shall write $f \in \mathcal{O}_U$ if we mean $\exists\, V \overset{\text{open}}{\subset} U : f \in \mathcal{O}(V)$.

Taking direct limits on the family $\mathcal{O}_U = (\mathcal{O}(V))_{V \overset{\text{open}}{\subset} U}$ can be formulated in an easily understandable way.

[Holomorphic germ] Given a point p in $V \overset{\text{open}}{\subset} U$ and a holomorphic function $f : V \to \mathbb{C}$ in $\mathcal{O}_U$ we define f's **germ** in p as $[f]_{\mathcal{O}_{U,p}} = \left\{g \in \mathcal{O}_U : \exists\, W : p \in W \overset{\text{open}}{\subset} U \;\&\; f|_W = g|_W\right\}$. The set of these equivalence classes is the **stalk of** $\mathcal{O}_U$ **at p** denoted by $\mathcal{O}_{U,p}$.

$\mathcal{O}_{U,p}$ possesses an $\mathbb{C}$-algebra structure with addition $[f]_{\mathcal{O}_{U,p}} + [g]_{\mathcal{O}_{U,p}} := [f+g]_{\mathcal{O}_{U,p}}$ and multiplication $[f]_{\mathcal{O}_{U,p}} \cdot [g]_{\mathcal{O}_{U,p}} := [fg]_{\mathcal{O}_{U,p}}$. These binary operations are well defined, as for $f_1|_{W_f} = f_2|_{W_f}$ and $g_1|_{W_g} = g_2|_{W_g}$ we conclude $f_1|_W = f_2|_W$ and $g_1|_W = g_2|_W$ with $p \in W = W_f \cap W_g$. Therefore $(f_1+g_1)|_W = (f_2+g_2)|_W$ and

$(f_1g_1)|_W = (f_2g_2)|_W$. So all the algebra axioms for $\mathcal{O}_{U,p}$ can be deduced from the algebras $\mathcal{O}(W)$.

Furthermore we can evaluate $[f]_{\mathcal{O}_{U,p}}$ at p, as f coincides with every other representative g on an open neighbourhood of p and hence especially on p.
A consequence of the definition of a germ is the isomorphy of the restriction $\pi : \mathcal{O}_{M,p} \twoheadrightarrow \mathcal{O}_{U,p}$.

3.5 Tangent spaces

[Tangent vector] A **tangent vector** or derivation at a point p is a $\mathbb{C}$-linear map $v : \mathcal{O}_{M,p} \to \mathbb{C}$ also satisfying Leibniz's law $v([fg]_{\mathcal{O}_{U,p}}) = v([f]_{\mathcal{O}_{U,p}})\,[g]_{\mathcal{O}_{U,p}}(p) + v([g]_{\mathcal{O}_{U,p}})\,[f]_{\mathcal{O}_{U,p}}(p)$.

[Tangent space] The collection of tangent vectors at a point p is called the **holomorphic tangent space** T_pM. It is actually a vector space, if viewed with pointwise scalar multiplication and addition, i.e. $(\alpha v + \beta w)([f]_{\mathcal{O}_{U,p}}) := \alpha(v([f]_{\mathcal{O}_{U,p}})) + \beta(w([f]_{\mathcal{O}_{U,p}}))$. The interested reader may have already observed that T_pM is a subspace of the dual vector space of $\mathcal{O}_{M,p}$.

1. For a given smooth curve γ from an open interval I containing zero into M with $\gamma(0) = p$ we define v_γ by $[f]_{\mathcal{O}_{U,p}} \mapsto \frac{d}{dt}(f \circ \gamma)(0)$. This is welldefined as $f \sim g$ implies $f|_{\mathcal{D}} = g|_{\mathcal{D}}$, $f \circ \gamma|_{\gamma^{-1}(\mathcal{D})} = g \circ \gamma|_{\gamma^{-1}(\mathcal{D})}$ and so $\frac{d}{dt}(f \circ \gamma)(0) = \frac{d}{dt}(g \circ \gamma)(0)$. Furthermore

$$\begin{aligned}
v_\gamma([fg]_{\mathcal{O}_{U,p}}) &= \frac{d}{dt}((fg) \circ \gamma)(0) = \frac{d}{dt}((f \circ \gamma) \cdot (g \circ \gamma))(0) \\
&= \frac{d}{dt}(f \circ \gamma)(0) \cdot (g \circ \gamma)(0) + \frac{d}{dt}(g \circ \gamma)(0) \cdot (f \circ \gamma)(0) \\
&= v_\gamma([f]_{\mathcal{O}_{U,p}}) \cdot [g]_{\mathcal{O}_{U,p}}(p) + v_\gamma([g]_{\mathcal{O}_{U,p}}) \cdot [f]_{\mathcal{O}_{U,p}}(p)
\end{aligned}$$

proves that v_γ satisfies Leibniz's law and

$$\begin{aligned}
v_\gamma([\alpha f + \beta g]_{\mathcal{O}_{U,p}}) &= \frac{d}{dt}((\alpha f + \beta g) \circ \gamma)(0) = \frac{d}{dt}(\alpha(f \circ \gamma) + \beta(g \circ \gamma))(0) \\
&= \alpha \cdot \frac{d}{dt}(f \circ \gamma)(0) + \beta \cdot \frac{d}{dt}(g \circ \gamma)(0) \\
&= \alpha \cdot v_\gamma([f]_{\mathcal{O}_{U,p}}) + \beta \cdot v_\gamma([g]_{\mathcal{O}_{U,p}}).
\end{aligned}$$

the linearity.

2. The partial differential operators $\frac{\partial\,\cdot}{\partial z^i}\big|_{z_0}$ are elements of $T_{z_0}\mathbb{C}^n$. Let us first observe that $\frac{\partial\,\cdot}{\partial z^i}\big|_{z_0} : \mathcal{O}_{\mathbb{C}^n,z_0} \longrightarrow \mathbb{C}$ is welldefined as $f \sim g$ implies $f|_W = g|_W$ on an open subset W and hence $\frac{\partial f}{\partial z^i}\Big|_{z_0} = \frac{\partial g}{\partial z^i}\Big|_{z_0}$. The algebraic properties of a derivation are clearly satisfied.

In order to verify that the partial differential operators $\frac{\partial\,\cdot}{\partial z^i}\big|_{z_0}$ form a basis we need to state first a lemma.

It holds $v([c]_{\mathcal{O}_{U,p}}) = 0$ for every $c \in \mathbb{C}$ and $v \in T_pM$.

We start by showing that $v([1]_{\mathcal{O}_{U,p}}) = 0$, because $v([1 \cdot 1]_{\mathcal{O}_{U,p}}) = 2 \cdot v([1]_{\mathcal{O}_{U,p}}) \cdot [1]_{\mathcal{O}_{U,p}}(p) = 2 \cdot v([1]_{\mathcal{O}_{U,p}})$. $v([c]_{\mathcal{O}_{U,p}}) = c \cdot v([1]_{\mathcal{O}_{U,p}})$ completes the proof.

We prove the following lemma in a little bit bulky version so that it is easily adaptable to the real case. Furthermore the rather deep fact that every holomorphic function can be written as a power series (cf. section 3.9 on page 39) can be avoided at this stage of the thesis.

The partial differential operators $\frac{\partial\,\cdot}{\partial z^i}\big|_{z_0}$ form a basis of $T_{z_0}\mathbb{C}^n$. We use the idea presented in [Ger06, p.262] but with a deeper look into the details. In order to transform $f(z) = f(z_0) + f(z) - f(z_0)$ we define $z_t(z) := z_t := z_0 + t(z - z_0)$ and observe

$$\begin{aligned} f(z) - f(z_0) &= [f(z_t)]_0^1 = \int_0^1 \frac{d}{dt}(f(z_t))\, dt = \int_0^1 \nabla f(z_t) \frac{d}{dt}(z_t)\, dt \\ &= \int_0^1 \frac{\partial f}{\partial z^i}(z_t)(z^i - z_0^i)\, dt \end{aligned}$$

Here we used Einstein's summation convention(section 3.6 on page 33) for the first time.

$$= \int_0^1 \frac{\partial f}{\partial z^i}(z_0 + t(z - z_0))\, dt \cdot (z^i - z_0^i) \equiv S_i(z)\Delta^i(z)$$

Both $S_i(z)$ and $\Delta^i(z)$ are holomorphic functions in z by Leibniz's rule for differentiation under the integral sign. This can be found in [Ger06, lemma 9.4.3 on page 144] or derived from [FB09, lemma II.3.3., p.94]. So we deduce that $v([f]_{\mathcal{O}_{U,p}})$ equals $v(\left[f(z_0) + S_i\Delta^i\right]_{\mathcal{O}_{U,p}})$ and hence

$$v([f]_{\mathcal{O}_{U,p}}) = 0 + v([S_i]_{\mathcal{O}_{U,p}} \cdot \left[\Delta^i\right]_{\mathcal{O}_{U,p}})$$

$$\begin{aligned}
&= v([S_i]_{\mathcal{O}_{U,p}}) \cdot \left[\Delta^i\right]_{\mathcal{O}_{U,p}}(z_0) + [S_i]_{\mathcal{O}_{U,p}}(z_0) \cdot v(\left[\Delta^i\right]_{\mathcal{O}_{U,p}}) \\
&= v([S_i]_{\mathcal{O}_{U,p}})(z_0^i - z_0^i) + [S_i]_{\mathcal{O}_{U,p}}(z_0) \cdot v(\left[z^i\right]_{\mathcal{O}_{U,p}} - \left[z_0^i\right]_{\mathcal{O}_{U,p}}) \\
&= 0 + [S_i]_{\mathcal{O}_{U,p}}(z_0) \cdot v(\left[z^i\right]_{\mathcal{O}_{U,p}} - \left[z_0^i\right]_{\mathcal{O}_{U,p}}).
\end{aligned}$$

Combining

$$[S_i]_{\mathcal{O}_{U,p}}(z_0) = \int_0^1 \frac{\partial f}{\partial z^i}(z_0 + t(z_0 - z_0))\, dt = \int_0^1 \frac{\partial f}{\partial z^i}(z_0)\, dt = \left.\frac{\partial f}{\partial z^i}\right|_{z_0}$$

with $v(\left[z^i\right]_{\mathcal{O}_{U,p}} - \left[z_0^i\right]_{\mathcal{O}_{U,p}}) = v(\left[z^i\right]_{\mathcal{O}_{U,p}}) =: v^i \in \mathbb{C}$ establishes the desired formula $v([f]_{\mathcal{O}_{U,p}}) = v^i \left.\frac{\partial f}{\partial z^i}\right|_{z_0}$.

We want to transport this result to an arbitrary complex manifold. Therefore we need two further lemmas. We anticipate the notations from item 2 and section 3.6 on pages 30–32.

Given a holomorphic function $\phi : M \to N$ then there is an algebra homomorphism

$$\begin{array}{rccc}
\phi^* : & \mathcal{O}_{V,\phi(p)} & \longrightarrow & \mathcal{O}_{U,p} \\
& [f]_{\mathcal{O}_{V,\phi(p)}} & \longmapsto & [f \circ \phi]_{\mathcal{O}_{U,p}}
\end{array}$$

for U and V open sets in M and N, respectively. If ϕ is a biholomorphic function then ϕ^* is an isomorphism.

1. ϕ^* is well-defined as $f \sim g$ implies $f|_W = g|_W$ for an open subset W of V and hence $f \circ \phi = g \circ \phi$ in the open subset $\phi^{-1}(W)$.

   We have already observed in example 3.5 i) on page 27 that the algebra structure of germs is preserved by hitting the functions with one single differentiable function.

2. By evaluating $(\phi^{-1})^*$ at $[0]_{\mathcal{O}_{U,p}} = \phi^*([f]_{\mathcal{O}_{V,\phi(p)}}) = [f \circ \phi]_{\mathcal{O}_{U,p}}$ we get $[0]_{\mathcal{O}_{V,\phi(p)}} = (\phi^{-1})^*([0]_{\mathcal{O}_{U,p}}) = \left[f \circ \phi \circ \phi^{-1}\right]_{\mathcal{O}_{V,\phi(p)}} = [f]_{\mathcal{O}_{V,\phi(p)}}$ and see how to construct the preimage for an arbitrary element of $\mathcal{O}_{U,p}$.

An algebra homomorphism $\phi^* : \mathcal{O}_{N,q} \longrightarrow \mathcal{O}_{M,p}$, satisfying $\phi^*([f]_{\mathcal{O}_{N,q}})(p) = [f]_{\mathcal{O}_{N,q}}(q)$ for all f, is inducing a vector space homomorphism

$$\begin{array}{rccc}
\phi_* : & T_pM & \longrightarrow & T_qN \\
& v & \longmapsto & v \circ \phi^*.
\end{array}$$

If ϕ^* is an isomorphism, then so is ϕ_*.

1. We observe that ϕ_* maps into T_qN because ϕ^* and v are both linear and Leibniz's law is satisfied

$$\begin{aligned}\phi_*(v)([fg]_{\mathcal{O}_{N,q}}) &= v \circ \phi^* \; ([fg]_{\mathcal{O}_{N,q}}) = v(\phi^*([f]_{\mathcal{O}_{N,q}})\phi^*([g]_{\mathcal{O}_{N,q}})) \\ &= v(\phi^*([f]_{\mathcal{O}_{N,q}})) \cdot \phi^*([g]_{\mathcal{O}_{N,q}})(p) + v(\phi^*([g]_{\mathcal{O}_{N,q}})) \cdot \phi^*([f]_{\mathcal{O}_{N,q}})(p) \\ &= \phi_*(v)([f]_{\mathcal{O}_{N,q}}) \cdot [g]_{\mathcal{O}_{N,q}}(q) + \phi_*(v)([g]_{\mathcal{O}_{N,q}}) \cdot [f]_{\mathcal{O}_{N,q}}(q).\end{aligned}$$

   ϕ_* is linear as an operator concatenating ϕ^* to its input.

2. By hitting $0 = \phi_*(v) = v \circ \phi^*$ with $(\phi^*)^{-1}$ we get $0 = 0 \circ (\phi^*)^{-1} = v \circ \phi^* \circ (\phi^*)^{-1} = v$. The image of $w \circ (\phi^*)^{-1}$ is $w \circ (\phi^*)^{-1} \circ \phi^*$ and hence ϕ_* is surjective.

The restriction isomorphism $\Psi_U^M : \mathcal{O}_{M,p} \hookrightarrow\!\!\!\!\rightarrow \mathcal{O}_{U,p}$ induces an isomorphism $(\Psi_U^M)_* : T_pU \hookrightarrow\!\!\!\!\rightarrow T_pM$.

[Pushforward] The above constructed vector space homomorphism $\phi_* : T_pM \to T_{\phi(p)}N$ associated with a holomorphic function $\phi : M \to N$ is called the **pushforward** along ϕ.

An immediate consequence is

[Chain rule for pushforwards] For any two holomorphic functions $\phi : M \longrightarrow N$ and $\psi : N \longrightarrow P$ it holds $(\psi \circ \phi)_* = \psi_* \circ \phi_*$. Evaluating $(\psi \circ \phi)_*(v)(h)$ for arbitrary tangent vectors v and holomorphic functions $h : P \to \mathbb{C}$ leads to $v((h \circ \psi) \circ \phi) = (\phi_* v)(h \circ \psi) = (\psi_*(\phi_* v))(h)$.

We note for the interested reader that there is a functor from the category of complex manifolds with a single distinguished point to the category of vector spaces assigning to a manifold (M, p) its tangent space at the distinguished point T_pM and to a holomorphic function its pushforward between the distinguished tangent spaces T_pM and T_qN.

Let T_pM be the tangent space of a point p on a manifold $M = M^n$. Then there is a canonical isomorphism Φ_* between T_pM and $T_{z(p)}\mathbb{C}^n$ where z is a chart around p. We have already seen that there are isomorphisms $\Psi_U^M : \mathcal{O}_{M,p} \to \mathcal{O}_{U,p}$ and $\Psi_V^{\mathbb{C}^n} : \mathcal{O}_{\mathbb{C}^n,z(p)} \to \mathcal{O}_{V,z(p)}$. So we conclude from item 2 on the preceding page that we can map functions in $\mathcal{O}_{M,p}$ to functions in $\mathcal{O}_{\mathbb{C}^n,z(p)}$ by $\Phi := \left(\Psi_V^{\mathbb{C}^n}\right)^{-1} \circ (\, z^{-1})^* \circ \Psi_U^M$. Combining this result with item 2 on page 29 leads to the desired isomorphism and each element v in T_pM equals $w \circ \Phi = w^i \frac{\partial}{\partial z^i}\big|_{z(p)} \circ \Phi$ in Einstein's summation convention.

[Canonical basis of T_pM] Let Φ be the isomorphism from item 2. Then the image of $\left(\frac{\partial}{\partial z^i}\big|_{z(p)}\right)_{1\le i\le n}$ under Φ is $\left(\frac{\partial\left((\cdot)|_U \circ z^{-1}\right)}{\partial z^i}(z(p))\right)_{1\le i\le n}$ or shortened to $\left(\frac{\partial\,\cdot}{\partial z^i}\big|_p\right)_{1\le i\le n}$ and forms the canonical basis of T_pM associated to the coordinate system $(z^1,\dots,z^n)$.

We have constructed basis for the tangent spaces and defined a homomorphism between them. So one question arises naturally. What is the transformation matrix ?

Given a holomorphic function $\phi : M \to N$ and coordinates $(z^1,\dots,z^n)$ and $(w^1,\dots,w^m)$ around $p \in M$ and $\phi(p) \in N$, respectively, then the transformation matrix of the pushforward $\phi_* : T_pM \to T_{\phi(p)}N$ in terms of $\left(\frac{\partial\,\cdot}{\partial z^i}\big|_p\right)_{1\le i\le n}$ and $\left(\frac{\partial\,\cdot}{\partial w^j}\big|_{\phi(p)}\right)_{1\le j\le m}$ is the Jacobian matrix of $w \circ \phi \circ z^{-1}$. Writing $\phi_*(v)(g) = v(g \circ \phi)$ as a linear combination of the basis vectors $\frac{\partial\,\cdot}{\partial z^i}\big|_p$ gives $v^i \left.\frac{\partial g \circ \phi}{\partial z^i}\right|_p$ in Einstein's summation convention. Expanding these differential operators explicitly with coordinate functions leads to

$$\begin{aligned}
\phi_*(v)(g) &= v^i \ \frac{\partial g \circ w^{-1} \circ w \circ \phi \circ z^{-1}}{\partial z^i}(z(p)) \\
&= v^i \ \frac{\partial g \circ w^{-1}}{\partial w^j}(w(\phi(p))) \ \frac{\partial (w \circ \phi \circ z^{-1})^j}{\partial z^i}(z(p)) \\
&= v^i \ \left.\frac{\partial g}{\partial w^j}\right|_{\phi(p)} \operatorname{Jac}(w \circ \phi \circ z^{-1}, z(p))^j_i .
\end{aligned}$$

As g can be chosen arbitrarily we deduce $\phi_*\left(v^i \ \frac{\partial\,\cdot}{\partial z^i}\big|_p\right) = \operatorname{Jac}(w \circ \phi \circ z^{-1}, z(p))^j_i v^i \ \frac{\partial\,\cdot}{\partial w^j}\big|_{\phi(p)}$. Sometimes we shorten this to $v^i \frac{\partial \phi^j}{\partial z^i} \ \frac{\partial\,\cdot}{\partial w^j}\big|_{\phi(p)}$.

Taking $\phi = id$ leads to the following corollary.

[Change of the tangent space's basis] Given coordinates $(z^1,\dots,z^n)$ and $(\tilde z^1,\dots,\tilde z^n)$ around p and their associated basis $\left(\frac{\partial\,\cdot}{\partial z^i}\big|_p\right)_{1\le i\le n}$ and $\left(\frac{\partial\,\cdot}{\partial \tilde z^j}\big|_p\right)_{1\le j\le n}$ in T_pM then the change of basis matrix is the Jacobian matrix of the transition function $\tilde z \circ z^{-1}$.

3.6 Cotangent spaces

[Cotangent space] The dual space of T_pM is called the **holomorphic cotangent space** and denoted by $(T_pM)^* = T_p^*M$.
We call the elements of the cotangent space co-vectors, **1-form**s or covariant vectors.

We can associate to a holomorphic germ $[f]_{\mathcal{O}_{U,p}}$ its **total differential** at point p

$$\begin{array}{rccc} df_p : & T_pM & \longrightarrow & \mathbb{C} \\ & v & \longmapsto & v([f]_{\mathcal{O}_{U,p}}). \end{array}$$

For the advanced reader this is not totally surprising as $(\mathcal{O}_{M,p})^*$ can be written as the direct sum $T_pM \oplus V$. This implies $(\mathcal{O}_{M,p})^{**} = T_p^*M \oplus V^* \supset \mathcal{O}_{M,p}$.

The single functions z^i of a chart (z, U) are holomorphic functions on U and therefore induce total differentials $d(z^i)_p = dz_p^i$.

[Dual basis of the cotangent space] The above defined dz_p^i form a dual basis to the tangent vectors $\frac{\partial\ \cdot}{\partial z^i}\big|_p$ associated to the same chart (z, U), i.e. $dz_p^i\left(\frac{\partial\ \cdot}{\partial z^j}\big|_p\right) = \delta_j^i$. It suffices to prove $dz_p^i\left(\frac{\partial\ \cdot}{\partial z^j}\big|_p\right) = \frac{\partial z^i}{\partial z^j}\Big|_p = \frac{\partial z^i \circ z^{-1}}{\partial z^j}(z(p)) = \delta_j^i$, because T_pM is of finite dimension.

Any total differential df_p can be represented as $\frac{\partial f}{\partial z^i}\Big|_p \cdot dz_p^i$. We can easily determine the coefficients of $df_p = f_i dz_p^i$ (in ESC section 3.6 !) by evaluating it at $\frac{\partial\ \cdot}{\partial z^j}\big|_p$: $df_p(\frac{\partial\ \cdot}{\partial z^j}\big|_p) = \frac{\partial f}{\partial z^j}\Big|_p = f_i dz_p^i(\frac{\partial\ \cdot}{\partial z^j}\big|_p) = f_j$.

[Pullback] A holomorphic function $\phi : M \longrightarrow N$ induces a linear map called the **pullback**

$$\begin{array}{rccc} \phi^* : & T_{\phi(p)}^*N & \longrightarrow & T_p^*M \\ & \omega & \longmapsto & \omega \circ \phi_*. \end{array}$$

An immediate consequence is

[Chain rule for pullbacks]For any two holomorphic functions $\phi : M \longrightarrow N$ and $\psi : N \longrightarrow P$ it holds $(\psi \circ \phi)^* = \phi^* \circ \psi^*$. Evaluating $(\psi \circ \phi)^*(\omega)$ for arbitrary co-vectors ω leads to $\omega \circ \psi_* \circ \phi_* = (\psi^*\omega) \circ \phi_* = \phi^*\psi^*\omega$.

Given a holomorphic function $\phi : M \to N$ and coordinates $(z^1, \dots, z^n)$ and $(w^1, \dots, w^m)$ around $p \in M$ and $\phi(p) \in N$, respectively, then the transformation matrix of the pullback $\phi^* : T^*_{\phi(p)}N \to T^*_pM$ in terms of $\left(dz^i_p\right)_{1 \le i \le n}$ and $\left(dw^j_{\phi(p)}\right)_{1 \le j \le m}$ is the Jacobian matrix of $w \circ \phi \circ z^{-1}$. This can be seen by evaluating $\phi^*\omega = (\phi^*\omega)_i dz^i_p$ at $\frac{\partial \cdot}{\partial z^i}\big|_p$:

$$(\phi^*\omega)_i = (\phi^*\omega)\left(\frac{\partial \cdot}{\partial z^i}\bigg|_p\right) = \omega\left(\phi_*\left(\frac{\partial \cdot}{\partial z^i}\bigg|_p\right)\right) = \omega\left(\frac{\partial \phi^j}{\partial z^i}\,\frac{\partial \cdot}{\partial w^j}\bigg|_{\phi(p)}\right) = \frac{\partial \phi^j}{\partial z^i}\omega_j$$

and deducing $((\phi^*\omega)_1, \dots, (\phi^*\omega)_n) = (\omega_1, \dots, \omega_n)\,\mathrm{Jac}(w\circ\phi\circ z^{-1}, z(p))$ for the row vectors $((\phi^*\omega)_1, \dots, (\phi^*\omega)_n)$ and $(\omega_1, \dots, \omega_n)$.

Taking $\phi = id$ leads to the following corollary.

[Change of the cotangent space's basis] Given coordinates $(z^1, \dots, z^n)$ and $(\tilde{z}^1, \dots, \tilde{z}^n)$ around p and their associated basis $\left(dz^i_p\right)_{1 \le i \le n}$ and $\left(d\tilde{z}^j_p\right)_{1 \le j \le n}$ in T^*_pM then the change of basis matrix is the Jacobian matrix of the transition function $\tilde{z}\circ z^{-1}$.

[Einstein summation convention]Evaluating a co-vector $\omega = (\omega_1, \dots, \omega_n)$ at the tangent vector $v = (v^1, \dots, v^n)$ leads to $\omega(v) = \sum_{i=1}^n \omega_i dz^i_p(\sum_{j=1}^n v^j\,\frac{\partial \cdot}{\partial z^j}\big|_p) = \sum_{i=1}^n \omega_i v^i$. As the product $\omega_k v^k$ for only one single index $k \in \{1, \dots, n\}$ is extremely rarely used we stick to the Einstein summation convention(ESC) from differential geometry. This means when summing over all products of the before mentioned coefficients we omit the summation sign and write $\omega_i v^i$. The super- and subscript indices indicate when we are facing the above case.
Since under the summation sign we saw whether an index started at 0 or 1 we use from now on α and β if we want to stress that the summation index ranges from 0 to the obvious upper end say n and latin indices for ranges $\{1, \dots, n\}$.
Additionally we write component indices of entities in superscript, if the entities transform under a change of coordinates like tangent vetors (cf. item 2) and in subscript if they transform like covectors.
An example is the gradient $\nabla f = \left(\frac{\partial f}{\partial z^1}, \dots, \frac{\partial f}{\partial z^n}\right) = (g_1, \dots, g_n)$ as $\frac{\partial f}{\partial z^i} = \frac{\partial f\circ\tilde{z}^{-1}\circ\tilde{z}}{\partial z^i} = \frac{\partial f\circ\tilde{z}^{-1}}{\partial \tilde{z}^j}\,\frac{\partial \tilde{z}^j}{\partial z^i}$.

Furthermore we extend the usage of the summation convention to multiindices and even to subsets of $\{1, \ldots, n\}$, cf. section 3.8 on page 38.

3.7 Vector bundles

We define now vector bundles improving our understanding of the tangent spaces and vectors of a manifold. We generalize [GHL87, chapter I differential manifolds B Tangent bundle definition 1.32 , p. 15] and [Lee03, chapter 5 Vector Bundles, p. 103] with an eye on [GH78, Chapter 0, Sec 5 Vector bundles, pp.66].

[Holomorphic vector bundle] The triple (π, E, B) of a holomorphic surjection $\pi : E \longrightarrow B$ together with its domain E and its codomain B is a **holomorphic vector bundle** of rank k, if

1. E and B are complex manifolds,

2. there is an open cover $(U_i)_{i \in I}$ of B with biholomorphic functions h_i, s.t. the diagram fig. 3.2 on the current page commutes

$$\begin{tikzcd}
\pi^{-1}(U_i) \arrow[rr, "\cong", "h_i"'] \arrow[dr, "\pi"'] & & U_i \times \mathbb{C}^k \arrow[dl, "pr_1"] \\
& U_i &
\end{tikzcd}$$

Figure 3.2: the commuting maps for the vector bundle

3. for $i, j \in I$ with $U_i \cap U_j \neq \emptyset$ there exists a holomorphic function $g_{j \to i} : U_i \cap U_j \longrightarrow \mathrm{GL}(k, \mathbb{C})$ satisfying $h_i \circ h_j^{-1}(p, v) = (p, g_{j \to i}(p)(v))$.

Let us fix some notations. E is called the **total space** and B the **base space**. We call the preimage of p in B under π the **fibre** of p, denote it by E_p . We refer to the biholomorphic maps h_i as **trivilization**s. If the base space and the surjection are obvious we identify the vector bundle with the total space.

[Manifold construction lemma] A set M with the following properties possesses the structure of a complex manifold,

1. M equals $\bigcup_{i \in I} U_i$,

2. a countable subset J of I 'covers' M, i.e. $\bigcup_{i\in J} U_i$,
3. there are bijections $\phi_i : U_i \hookrightarrow\!\!\!\!\rightarrow V_i \overset{\text{open}}{\subset} \mathbb{C}^n$,
4. furthermore it holds $\phi_i(U_i \cap U_j) \overset{\text{open}}{\subset} \mathbb{C}^n$ for all i and j in I ,
5. whenever $U_i \cap U_j \neq \emptyset$ the map $\tau_{j\to i} : \phi_j(U_i \cap U_j) \hookrightarrow\!\!\!\!\rightarrow \phi_i(U_i \cap U_j)$ is biholomorphic,
6. for 2 distinct points there exist either 2 disjoint subsets including one point each or a set containing both.

We refer to [Lee03, Lemma 1.23, p.21].

In the following example we shall investigate the prototype of a vector bundle.

[Tangent bundle] The **tangent bundle** TM over a manifold M is the collection of pairs of the form (p, v) with p a point in M and v a tangent vector in T_pM. We generalize [Lee03, p. 81] to show that the tangent bundle is a complex manifold of dimension $2n$. In order to simplify notations while checking the assumptions of manifold construction lemma(p.34) we define the following map

$$\begin{array}{rccl} \pi : & TM & \longrightarrow & M \\ & (p, v) & \longmapsto & p. \end{array}$$

For a given chart (z, U) of M we can define a chart on TM by

$$\begin{array}{rccl} \tilde{z} : & \pi^{-1}(U) & \longrightarrow & z(U) \times \mathbb{C}^n \\ & (p, v) & \longmapsto & (z^1(p), \dots, z^n(p), v^1, \dots, v^n), \end{array}$$

with $v = v^i \left.\frac{\partial}{\partial z^i}\right|_p$ in ESC.
Let us stress that $\pi^{-1}(U)$ is just $\{(p, v) \in TM : p \in U \;\&\; v \in T_pM\} =: TU$. So for the chart compatibility of $(\tilde{z}, U_1)$ and $(\tilde{w}, U_2)$ we have to observe

$$\pi^{-1}(U_1) \;\cap\; \pi^{-1}(U_2) = \{(p, v) \in TM : p \in U_1 \;\&\; p \in U_2 \;\&\; v \in T_pM\} = \pi^{-1}(U_1 \cap U_2).$$

$$\begin{array}{rccl} \tau : & \tilde{z}(\pi^{-1}(U_1 \cap U_2)) & \longrightarrow & \tilde{w}(\pi^{-1}(U_1 \cap U_2)) \\ & (z^1, \dots, z^n, v^1, \dots, v^n) & \longmapsto & \tilde{w} \circ \tilde{z}^{-1}((z^1, \dots, z^n, v^1, \dots, v^n)). \end{array}$$

With the help of item 2 on page 31 and its way of notation we can simplify the last term $\tilde{w} \circ \tilde{z}^{-1}((z^1, \ldots, z^n, v^1, \ldots, v^n))$ to

$$(w(z),\ \mathrm{Jac}(\tau_{z\to w}, z)v\ ) = \left(w^1(z), \ldots, w^n(z), v^k \frac{\partial w^1}{\partial z^k}, \ldots, v^k \frac{\partial w^n}{\partial z^k} \right).$$

Hence TM is a complex manifold. In order to get the trivialization h_z we just have to adjust the above chart $(\tilde{z}^1, \ldots, \tilde{z}^n)$ slightly to

$$\begin{array}{rccl} h_z : & \pi^{-1}(U) & \longrightarrow & U \times \mathbb{C}^n \\ & (p, v) & \longmapsto & (p, v^1, \ldots, v^n). \end{array}$$

The needed linear transformation $g_{\tilde{z}\to\tilde{w}}$ is the Jacobian of the transition function $\tau_{z\to w}$.

[Cotangent bundle] In a very similar way we can construct the **cotangent bundle** T^*M consisting of the pairs (p, ω) for p in M and ω in T^*_pM.

[Section of a vector bundle] A holomorphic map f from the base B of a vector bundle into its total space E, s.t. any point p gets mapped into its own fibre, i.e $\pi \circ f = id$, is called a **section**. We denote the set of all sections by $\Gamma(E)$.

The next definition forms the foundation of the next section.

[Tensorial operations on vector bundles] We define the **tensor product of two vector bundles** E and F over the same base space B in the following way

$$E \otimes F := \{e \otimes f := (p, v \otimes_{\mathbb{C}} u)\ :\ e = (p, v) \in E_p \text{ and } f = (p, u) \in F_p\}$$

and their **direct sum** as

$$E \oplus F := \{e \oplus f := (p, v \oplus u)\ :\ e = (p, v) \in E_p \text{ and } f = (p, u) \in F_p\}.$$

Furthermore we define the **wedge product** of E as

$$\bigwedge\nolimits^m (E) := \{e_1 \wedge \ldots \wedge e_m := (p, v_1 \wedge \ldots \wedge v_m)\ :\ e_j = (p, v_j) \in E_p\ \forall\ 1 \le j \le m\}.$$

and closing with $\mathrm{Alt}^{\bullet}(E) := \bigoplus_{m=0}^{k} \bigwedge^m (E)$ for $\mathrm{rank}(E) = k$.

Before applying manifold construction lemma(p.34) we refine the atlases used in the defin-tions of (π_E, E, B) and (π_F, F, B). This gives two trivializations

$$\begin{array}{rccl} h_{E,i}: & E|_{U_i} & \longrightarrow & U_i \times \mathbb{C}^k \\ & e & \longmapsto & (h^1_{E,i}(e), h^2_{E,i}(e)) = (p, v) \end{array}$$

and

$$\begin{array}{rccl} h_{F,i}: & F|_{U_i} & \longrightarrow & U_i \times \mathbb{C}^l \\ & f & \longmapsto & (h^1_{F,i}(f), h^2_{F,i}(f)) = (p, u). \end{array}$$

Eventually we define the new trivializations

- $$\begin{array}{rccl} h_{E,i} \otimes h_{F,i}: & (E \otimes F)|_{U_i} & \longrightarrow & U_i \times \mathbb{C}^k \otimes_{\mathbb{C}} \mathbb{C}^l = U_i \times \mathbb{C}^{kl} \\ & e \otimes f & \longmapsto & (h^1_{E,i}(e), h^2_{E,i}(e) \otimes_{\mathbb{C}} h^2_{F,i}(f)) = (p, v \otimes_{\mathbb{C}} u) \end{array}$$

  for $E \otimes F$ with associated linear transformations $g_{E\otimes F,ij}(p) = g_{E,ij}(p) \otimes_{\mathbb{C}} g_{F,ij}(p)$ and induced charts $(E \otimes F)|_{U_i} \to z(U_i) \times \mathbb{C}^{kl}$.

- $$\begin{array}{rccl} h_{E,i} \oplus h_{F,i}: & (E \oplus F)|_{U_i} & \longrightarrow & U_i \times \mathbb{C}^k \oplus \mathbb{C}^l = U_i \times \mathbb{C}^{k+l} \\ & e \oplus f & \longmapsto & (h^1_{E,i}(e), h^2_{E,i}(e) \oplus h^2_{F,i}(f)) = (p, v \oplus u) \end{array}$$

  for $E \oplus F$ with associated linear transformations $g_{E\oplus F,ij}(p) = g_{E,ij}(p) \oplus g_{F,ij}(p)$ and induced charts $(E \oplus F)|_{U_i} \to z(U_i) \times \mathbb{C}^{k+l}$.

- $$\begin{array}{rccl} \bigwedge^m (h_i): & (\bigwedge^m (E))|_{U_i} & \longrightarrow & U_i \times \left(\mathbb{C}^k \wedge \ldots \wedge \mathbb{C}^k\right) = U_i \times \mathbb{C}^{\binom{k}{m}} \\ & e_1 \wedge \ldots \wedge e_m & \longmapsto & (h^1_i(e_1), h^2_i(e_1) \wedge \ldots \wedge h^2_i(e_m)) = (p, v_1 \wedge \ldots \wedge v_m) \end{array}$$

  for $\bigwedge^m (E)$ with associated linear transformations $g_{\bigwedge^m(E),ij}(p) = \bigwedge^m (g_{E,ij}(p))$ and induced charts $(\bigwedge^m (E))|_{U_i} \to z(U_i) \times \mathbb{C}^{\binom{k}{m}}$.

3.8 Tensor bundles

We have defined in items 6 and 6 the (co-)tangent bundles of a manifold.

[Covariant tensor fields] By $T^*M^{\otimes q}$ we denote the q times tensor product of T^*M, i.e. $T^*M^{\otimes q} = \underbrace{T^*M \otimes \ldots \otimes T^*M}_{q\text{-times}}$. We call sections of $T^*M^{\otimes q}$ **covariant tensor fields**.

[q-form] The sections of $\Omega^q(M) := \bigwedge^q (T^*M)$ are called **q-forms**.

The image of p under ω is commonly denoted by $\omega_p = (p, \Psi)$, because usually the authors are far more interested in the linear operator Ψ than in holomorphic change of ω under p.
As the dz^is form a local basis of T^*M we deduce that $(dz^{\nu_1} \otimes \cdots \otimes dz^{\nu_q})_{1 \leq \nu_i \leq n}$ is a local basis of $T^*M^{\otimes q}$. We shorten $dz^{\nu_1} \otimes \cdots \otimes dz^{\nu_q}$ to dz^ν with $\nu = (\nu_1, \ldots, \nu_q)$.

[Pullback of covariant tensor fields] We define the **pullback of a covariant tensor field** ω associated with a holomorphic function $\phi : M \longrightarrow N$ between two manifolds as $(\phi^*\omega)_p(\cdot, \ldots, \cdot) = (\phi^*)_p \omega_{\phi(p)}(\cdot, \ldots, \cdot) = \omega_{\phi(p)}(\phi_*, \ldots, \phi_*)$ according to [Lee03, chapter 11 Tensors ,p. 270].

[Invariant tensors] A covariant tensor field $\omega \in \Gamma(T^*M^{\otimes q})$ is called **invariant** under a subset S of $\mathcal{O}(M, M)$ if it holds $\phi^*\omega = \omega$ for all ϕ in S. The collection of these covariant tensorfields is $\left(\Gamma(T^*M^{\otimes q})\right)^S$.

Writing the above condition more explicitely gives $\omega_p = \omega_{\phi(p)}(\phi_*, \ldots, \phi_*)$.

3.9 Holomorphic functions

In this subsection we go deeper into the theory of holomorphic functions started in section 3.4.

[Implicit function theorem] Let U be an open subset of $\mathbb{C}^n$and $f \in \mathcal{O}(U, \mathbb{C}^m)$ with $m \leq n$. Suppose the leading principal minor $(\mathrm{Jac}(f, z_0)^i_j)_{1 \leq i,j \leq m}$ is invertible for a root z_0. Then the zero set of f can be expressed as the graph of a holomorphic function g between open subsets U_1 and U_2 of $\mathbb{C}^{n-m}$ and $\mathbb{C}^m$, respectively, i.e. for z in $U_1 \times U_2$ $f(z)$ vanishes iff $(z^1, \ldots, z^m)$ equals $g(z^{m+1}, \ldots, z^n)$. The best reference here is [Huy05, Prop 1.11, p.11].

[Liouville's theorem] A bounded holomorphic function f on $\mathbb{C}^n$ is constant. We deduce from the one dimensional case that $w \mapsto f(z^1, \cdots, z^{i-1}, w, z^{i+1}, \cdots, z^n)$ is constant, hence $Df \equiv 0$. The mean value theorem leads to the desired result.

3.9.0. Identity theorem also for meromorphic functions and meromorphic tensors ?

[Identity theorem] Two holomorphic functions from a domain $\mathcal{D}$ into the complex numbers are equal if they coincide on a nonvoid open subset of $\mathcal{D}$. It is a nice exercise for the reader to prove this theorem by using the one dimensional case. A solution is presented in [Fre09, p. 309].

[Open mapping theorem] A non-constant holomorphic function from a domain $\mathcal{D}$ into the complex numbers is open. This is also a simple generalization of the one dimensional case, cf. [KK83, Theorem 6.3, p. 19].

[Power series] Every holomorphic function from an open subset $U \subset \mathbb{C}^n$ possesses a unique power series expansion $\sum_{\nu \in \mathbb{N}^n} a_\nu (z - z_0)^\nu \equiv a_\nu (z - z_0)^\nu$ around each point z_0 in U. This is proven on page 5 of [Huy05].

For certain open subsets there exist other series assigned to holomorphic functions.

[Laurent series] On the cartesian product of an $(n-1)$-dimensional domain $\mathcal{D}$ and an annulus $\mathcal{A} := \{z \in C : r < |z| < R\}$ every holomorphic function $f : \mathcal{D} \times \mathcal{A} \to \mathbb{C}$ coincides with the converging series $\sum_{k=-\infty}^{\infty} a_k(z^1, \ldots, z^{n-1}) \cdot (z^n)^k \equiv a_k(z^1, \ldots, z^{n-1}) \cdot (z^n)^k$. The uniquely determined a_ks are elements of $\mathcal{O}(\mathcal{D})$. For a proof we refer to [KK83, p.25].

3.9.0. rephrase this.

The above section 3.9 gives rise to the next theorems which cannot be deduced in the case of smooth functions on $\mathbb{R}^n$!

Let be p a point in the open set U, then $\mathcal{O}_{U,p}$ is a unique factorization domain. A proof can be found in [Huy05, Prop 1.1.15, p.14].

The germ $[z^n]_{\mathcal{O}_{U,0}}$ is irreducible in $\mathcal{O}_{U,p}$. Assume that $[z^n]_{\mathcal{O}_{U,0}}$ equals $[f]_{\mathcal{O}_{U,0}} \cdot [g]_{\mathcal{O}_{U,0}}$. Comparing the coefficients gives that one of them is a unit.

Hilbert's Nullstellensatz section 2.1 on page 11 can be generalized to germs as done in [Huy05, Proposition 1.1.29, p. 19]. [Rückert's Nullstellensatz] Let be p a point in the open set U and $[f_1]_{\mathcal{O}_{U,p}}, \ldots, [f_k]_{\mathcal{O}_{U,p}} \in \mathcal{O}_{U,p}$. Suppose $\mathfrak{a}$ is the ideal of germs "vanishing on $Z(f_1, \ldots, f_k)$", i.e.

$$\mathfrak{a} = \left\{ [f]_{\mathcal{O}_{U,p}} \in \mathcal{O}_{U,p} : \exists\ D_r^n(p) : f, f_1, \ldots, f_k \in \mathcal{O}(D_r^n(p)) : f(z) = 0\ \forall z \in Z(f_1, \ldots, f_k) \right\}.$$

Then $\mathfrak{a}$ equals the radical ideal $rad([f_1]_{\mathcal{O}_{U,p}}, \ldots, [f_k]_{\mathcal{O}_{U,p}})$ being defined by

$$\left\{[f]_{\mathcal{O}_{U,p}} \in \mathcal{O}_{U,p} : \exists\; m > 0 : [f]^m_{\mathcal{O}_{U,p}} \in ([f_1]_{\mathcal{O}_{U,p}}, \ldots, [f_k]_{\mathcal{O}_{U,p}})\right\}.$$

The above mentioned generalization implies that there is also a corollary analogous to section 2.1 on page 11, which we shall call Rückert's Corollary. [Rückert's Corollary] Given a square-free germ $[f]_{\mathcal{O}_{U,p}}$ and another germ $[g]_{\mathcal{O}_{U,p}}$ in $\mathcal{O}_{U,p}$ such that their representatives satisfy $Z(f) \subset Z(g)$ in a polydisc $D^n_r(p)$, then $[f]_{\mathcal{O}_{U,p}}$ divides $[g]_{\mathcal{O}_{U,p}}$ in $\mathcal{O}_{U,p}$.

If $[f]_{\mathcal{O}_{U,p}}$ and $[g]_{\mathcal{O}_{U,p}}$ are coprime in $\mathcal{O}_{U,p}$ then also in $\mathcal{O}_{U,q}$ for q in a small neighbourhood of p. This fact is shown in [Huy05, Prop 1.1.35, p.21].

Any bijective holomorphic function is biholomorphic. See for instance [GR65, Prop, p.19].

[Riemann extension theorem] If we denote $U \backslash Z(f)$ by V for a given holomorphic function $f : U \to \mathbb{C}$ then $g \in \mathcal{O}(V, \mathbb{C})$ is holomorphic on the whole of U if g is locally bounded around $Z(f)$. A proof can be retrieved from [Huy05, Prop 1.1.7, p.9].

3.10 Meromorphic functions and tensors

We follow [Fre09, p. 427]. [m-property of holomorphic functions] Let f be a holomorphic function on an open and dense subset $\mathcal{D}$ of a manifold M. f has got the **m-property** on M if for any $p \in M$ there are 2 holomorphic functions g and h on an open neighbourhood $U \subset M$ of p satisfying $f(z) = \frac{g(z)}{h(z)}$ for every z in $U \cap Z(h)^C$.

The functions $f : U = \mathbb{C}^2 \backslash \{0\} \to \mathbb{C}$; $(z^1, z^2) \mapsto \frac{z^1}{z^2}$
and $f : U = \mathbb{C}^2 \backslash \{0, 1\} \to \mathbb{C}$; $(z^1, z^2) \mapsto \frac{z^1}{z^2}$
have got the m-property on $\mathbb{C}^2$. The both above mentioned holomorphic functions describe (intuitively) the same 'function' on the whole plane $\mathbb{C}^2$. This impels us to make the following definition.

[Meromorphic function] We call two holomorphic functions with the m-property on M (m-)equivalent if they coincide on a dense subset of M. Such an equivalence class is a

meromorphic function.

Meromorphic functions have the subsequent properties.

1. The set of meromorphic functions on a manifold M $\mathcal{M}(M)$ is a ring. If M is connected then $\mathcal{M}(M)$ is a field.

2. If f equals $\frac{g}{h}$ on $U \cap Z(h)^C$ and $\frac{p}{q}$ on $V \cap Z(q)^C$. Suppose that U and V intersect then $g \cdot q$ coincides with $p \cdot h$ on $U \cap V \cap Z(q)^C \cap Z(h)^C$.

3. Section 3.9 on the preceding page allows us to choose U, g and h in section 3.10 such that $[g]_{\mathcal{O}_{U,q}}$ and $[h]_{\mathcal{O}_{U,q}}$ are coprime in every $\mathcal{O}_{U,q}$ with q in U.

4. As the manifold M is second countable there is a countable open cover $(U_i)_{i \in I}$ of M consisting of sets having the above mentioned properties.

3.10.0. Warum ist das folgende unzulaessig ?

So we can characterize the equivalence class of f by $(U_i, g_i, h_i)_{i \in I}$.

[Zero and pole locus of a meromorphic function] The **zero locus** of a meromorphic function $f = (U_i, g_i, h_i)_{i \in I}$ is the union $\bigcup_{i \in I} Z(g_i)$. This definition is independent of the local representations of f. Indeed given an other representation $\frac{r}{q}$ we have $g \cdot q = r \cdot h$ impliying $Z(g) \subset Z(r \cdot h)$. As $[g]_{\mathcal{O}_{U,p}}$ and $[h]_{\mathcal{O}_{U,p}}$ are coprime Rückert's Corollary(p.40) yields that all prime factors of $[g]_{\mathcal{O}_{U,p}}$ divide $[r]_{\mathcal{O}_{U,p}}$. Therefore the zero sets of r and g locally coincide. Similarly the **pole locus** is $\bigcup_{i \in I} Z(h_i)$. The sets are denoted by $Z(f)$ and $P(f)$ respectively.

Now we extend the definitions from above to arbitrary tensors.

[m-property of holomorphic tensors] Let ω be a holomorphic tensor on an open and dense subset $\mathcal{D}$ of a manifold M. ω has got the **m-property** on M if for any $p \in M$ there are a holomorphic tensor η and a holomorphic function h on an open neighbourhood $U \subset M$ of p satisfying $\omega_z = \frac{1}{h(z)} \eta_z$ for every z in $U \cap Z(h)^C$.

[$d\mathfrak{z}^i$ has got m-property on $\mathbb{P}^n\mathbb{C}$] Firstly $d\mathfrak{z}^i$ is holomorphic on $\mathbb{A}_0$ which is a dense and open subset of $\mathbb{P}^n\mathbb{C}$. Secondly for an arbitrary point outside $\mathbb{A}_0$, say $p \in \mathbb{A}_1$, we can find a holomorphic tensor η and a holomorphic function h on $\mathbb{A}_1$, s.t. $\omega = \frac{\eta}{h}$. Indeed denoting the coordinates in $\mathbb{A}_1$ by $(\mathfrak{w}^1, \dots, \mathfrak{w}^n)$ leads to $d\mathfrak{z}^i = \sum_j \frac{\partial \mathfrak{z}^i}{\partial \mathfrak{w}^j} d\mathfrak{w}^j$ in $\mathbb{A}_0 \cap \mathbb{A}_1$.

Section 3.3 on page 25 yields that $\frac{\partial \mathfrak{z}^i}{\partial \mathfrak{w}^j}$ equals $\frac{P_j(\mathfrak{w})}{(\mathfrak{w}^0)^2}$ with P_j in $\mathbb{C}[X^1, \dots, X^n]$ therefore it holds

$$d\mathfrak{z}^i_q = \sum_j \frac{1}{(\mathfrak{w}^0(q))^2}(P_j(\mathfrak{w}(q))d\mathfrak{w}^j_q) \equiv \frac{1}{(\mathfrak{w}^0(q))^2}\eta_q$$

for every q in $\mathbb{A}_1 \cap Z\left(\mathfrak{w}^0\right)^C$.

[Meromorphic tensor]We call two holomorphic tensors with the m-property on M equivalent if they coincide on a dense subset of M. Such an equivalence class is a **meromorphic tensor**.

[Vector space of meromorphic tensors $\mathcal{M}(M) \otimes_{\mathcal{O}} \Gamma(T^*M^{\otimes q})$] We denote the $\mathbb{C}$-vector space of meromorphic tensors such that their associated holomorphic tensors belong to $\Gamma(T^*M^{\otimes q})$ by $\mathcal{M}(M) \otimes_{\mathcal{O}} \Gamma(T^*M^{\otimes q})$. This awkward seeming notation is derived from a more elaborate definition of meromorphic tensors, cf. [GH78][p. 135]. The holomorphic tensor occurring in the definitions of a meromorphic tensor can be pulled back along a holomorphic function. So the question arises when is this also possible forn the meromorphic tensor.

[Pullback of covariant meromorphic tensor fields] A covariant meromorphic tensor field ω on a manifold N being holomorphic on $\mathcal{D}_\omega$ can be transported to another manifold M by a holomorphic function $\phi : M \to N$ if $\phi^{-1}(\mathcal{D}_\omega)$ becomes a dense subset in M. Indeed the equation $\omega_z = \frac{1}{h(z)}\eta_z$ is preserved.

A holomorphic function $\phi : M \to N$ pulls back any meromorphic tensor on N to M if $\phi^{-1}(\mathcal{D})$ is dense for each open and dense subset of N. Examples of such maps are open functions.

i. Obviously $\phi^{-1}(\mathcal{D}_\omega)$ is dense for each meromorphic tensor ω.

ii. Take ϕ to be an open function and let p be a point with an arbitrary neighbourhood U in M. Then $\phi(U)$ possesses a point d lying in $\mathcal{D}_\omega$. As $\phi^{-1}(\{d\}) \cap U$ is non void $\phi^{-1}(\mathcal{D}_\omega)$ is dense in M.

As meromorphic tensors are hardly described in the literature we prove some lemmas quite explicitly.

Given a meromorphic tensor ω on $\mathbb{P}^n\mathbb{C}$ with representation $\omega = \sum_\nu \omega_\nu d\mathfrak{z}^{\nu_1} \otimes \cdots \otimes d\mathfrak{z}^{\nu_q}$ on $\mathbb{A}_\alpha$ then the coefficient functions ω_ν are meromorphic functions on $\mathbb{A}_\alpha$ and consequently on $\mathbb{P}^n\mathbb{C}$.

i. ω is holomorphic on $\mathcal{D} \overset{\text{dense}}{\subset} \mathbb{P}^n\mathbb{C}$ and hence on $\mathcal{D} \cap \mathbb{A}_\alpha \overset{\text{dense}}{\subset} \mathbb{A}_\alpha \cong \mathbb{C}^n$. On $\mathcal{D} \cap \mathbb{A}_\alpha$ holomorphic sections coincide with holomorphic functions $\mathcal{D}\cap\mathbb{A}_\alpha \to (\mathbb{C}^n)^{\otimes q} = \mathbb{C}^{n \cdot q}$. Analogously η can be considered as $\eta : U \cap \mathbb{A}_\alpha \to \mathbb{C}^{n \cdot q}$. So ω_ν equals $(U \cap \mathbb{A}_\alpha, \eta_\nu, h)$.

ii. $\omega_\nu(\mathfrak{z})$ transforms to $\sum_\mu \frac{\partial \mathfrak{z}^{\mu_1}}{\partial \mathfrak{w}^{\nu_1}} \cdots \frac{\partial \mathfrak{z}^{\mu_q}}{\partial \mathfrak{w}^{\nu_q}} \omega_\mu(\mathfrak{w}) \equiv \sum_\mu \frac{P_\mu(\mathfrak{w})}{(\mathfrak{w}^0)^{2q}} \omega_\mu(\mathfrak{w}) \equiv f_\omega(\mathfrak{w})$ on $\mathbb{A}_0 \cap \mathbb{A}_1$ with $P_\mu \in \mathbb{C}[X^1, \ldots, X^n]$. $f_\omega(\mathfrak{w})$ can be considered as a meromorphic function on $\mathbb{A}_1$, because the sum's components are of the form $\frac{P_\mu(\mathfrak{w})}{(\mathfrak{w}^0)^{2q}}$ and $\omega_\mu(\mathfrak{w})$ lie in $\mathcal{M}(\mathbb{A}_1)$. The holomorphic functions ω_ν and f_ω coincide on $\mathcal{D}_{\omega_\nu} \cap \mathcal{D}_{f_\omega} \overset{\text{dense}}{\subset} \mathbb{A}_0 \cap \mathbb{A}_1 \overset{\text{dense}}{\subset} \mathbb{P}^n\mathbb{C}$ and hence represent the same equivalence class, i.e. meromorphic function, in $\mathcal{M}(\mathbb{P}^n\mathbb{C})$.

3.10.0. rephrase this.

The product of a meromorphic function f and a meromorphic tensor ω is again a meromorphic tensor. Obviously we can multiply f with ω on their common holomorphicity locus $\mathcal{D}_f \cap \mathcal{D}_\omega$. For each p we have $f = \frac{g}{h_f}$ and $\omega = \frac{1}{h_\omega}\eta$ on $U_p \cap (Z(h_f))^C \cap (Z(h_\omega))^C$ and hence $f \cdot \omega = \frac{g}{h_f \cdot h_\omega} \cdot \eta$.

In order to push certain meromorphic functions from $\mathbb{C}^{n+1}$ to $\mathbb{P}^n\mathbb{C}$ we have to observe how to treat polynomials on $\mathbb{P}^n\mathbb{C}$.

The algebra epimorphism

$$\begin{array}{rccc} \pi_*^{\mathbb{A}_0} : & \mathbb{C}[X^0, \ldots, X^n] & \longrightarrow & \mathbb{C}[X^1, \ldots, X^n] \\ & P & \longmapsto & P(1, \cdot, \ldots, \cdot) \end{array}$$

can be restricted to a vector space isomorphism $\mathbb{C}_d[X^0, \ldots, X^n] \to \bigoplus_{i \le d} \mathbb{C}_i[X^1, \ldots, X^n]$. It should be clear that $\pi_*^{\mathbb{A}_0}$ is an algebra epimorphism.
We denote the inverse map by π^* which just homogenizes the polynomials and multiplies with an appropriate power of X^0.

The just defined map $\pi_*^{\mathbb{A}_0}$ commutes with the differential operators $\frac{\partial}{\partial z^i}$ for $0 < i \le n$. We may assume without loss of generality that $i = n$. After writing P as $\sum_{k=0}^d (z^n)^k P_k(z^0, \ldots, z^{n-1})$ the proof is trivial.

3.10.0. rephrase this.

Consider P and $Q \neq 0$ in $\mathbb{C}_d[X^1, \dots, X^{n+1}]$. Then the rational function $f = \frac{P}{Q}$ is meromorphic on $\mathbb{C}^{n+1}$ and induces the meromorphic function $\pi_* f$ on $\mathbb{P}^n\mathbb{C}$. Obviously $f = \frac{P}{Q}$ is meromorphic on $\mathbb{C}^{n+1}$. The map $\pi_* f$ completing the following commutative diagram is well defined because P and Q have got the same degree.

$$\begin{tikzcd}
\mathbb{C}^{n+1}\backslash Z(Q) \arrow[rr, two heads, "\pi"] \arrow[dr, "f = \frac{P}{Q}"'] & & \mathbb{P}^n\mathbb{C}\backslash \pi(Z(Q)) \arrow[dl, dashed, "\pi_* f"] \\
& \mathbb{C} &
\end{tikzcd}$$

In order to show that $\pi_* f$ is meromorphic around $p \in \mathbb{A}_0$ we observe the rational function $g(\mathfrak{z}^1, \dots, \mathfrak{z}^n) = \frac{\pi_*^{\mathbb{A}_0} P(\mathfrak{z}^1, \dots, \mathfrak{z}^n)}{\pi_*^{\mathbb{A}_0} Q(\mathfrak{z}^1, \dots, \mathfrak{z}^n)} = \frac{P(1, \mathfrak{z}^1, \dots, \mathfrak{z}^n)}{Q(1, \mathfrak{z}^1, \dots, \mathfrak{z}^n)}$ on $\mathbb{A}_0$. It coincides with $\pi_* f$ because it holds

$$f \circ \pi^{-1}(\mathfrak{z}) = f(z) = \frac{P(z^0, \dots, z^n)}{Q(z^0, \dots, z^n)} = \frac{(z^0)^d \cdot P\left(1, \frac{z^1}{z^0}, \dots, \frac{z^n}{z^0}\right)}{(z^0)^d \cdot Q\left(1, \frac{z^1}{z^0}, \dots, \frac{z^n}{z^0}\right)} = g(\mathfrak{z}^1, \dots, \mathfrak{z}^n)$$

for all $\mathfrak{z}$ in $\mathbb{A}_0 \cap \pi(Z(Q))^C$.

The holomorphic tensor $d\mathfrak{z}^I \otimes (d\mathfrak{z}^1 \wedge \dots \wedge d\mathfrak{z}^n)^{\otimes k}$ in $\Omega^q \otimes_{\mathcal{O}} (\Omega^n)^{\otimes k}(\mathbb{A}_0)$ is a meromorphic tensor on $\mathbb{P}^n\mathbb{C}$ with a pole locus in $\pi(\{z^0 = 0\})$. We argue analougsly to item 4 on page 41. $d\mathfrak{z}^I \otimes (d\mathfrak{z}^1 \wedge \dots \wedge d\mathfrak{z}^n)^{\otimes k}$ is holomorphic on $\mathbb{A}_0$ and changing to $\mathbb{A}_1$ gives for its different parts

- $\bigotimes_{l=1}^k \det\left(\frac{\partial \mathfrak{z}^i}{\partial \mathfrak{w}^j}\right)(d\mathfrak{w}^1 \wedge \dots \wedge d\mathfrak{w}^n) = \left(\frac{1}{\mathfrak{w}^0}\right)^{k(n+1)} \bigotimes_{l=1}^k (d\mathfrak{w}^1 \wedge \dots \wedge d\mathfrak{w}^n)$
- $d\mathfrak{z}^1 = \frac{-1}{(\mathfrak{w}^0)^2} d\mathfrak{w}^0$
- $d\mathfrak{z}^i = \frac{1}{(\mathfrak{w}^0)^2}(\mathfrak{w}^0 d\mathfrak{w}^i - \mathfrak{w}^i d\mathfrak{w}^0)$ for $i \neq 1$
- $d\mathfrak{z}^I = \frac{-1}{(\mathfrak{w}^0)^{q+1}} d\mathfrak{w}^{I\backslash\{1\}\cup\{0\}}$ for $1 \in I$, cf. item 3 on page 14, $|I| = q$,
- $d\mathfrak{z}^I = \frac{1}{(\mathfrak{w}^0)^q} d\mathfrak{w}^I - \sum_{i \in I} (-1)^{n+i} \frac{\mathfrak{w}^i}{(\mathfrak{w}^0)^{q+1}} d\mathfrak{w}^{I\backslash\{i\}\cup\{0\}}$ for $1 \notin I$

and for the whole tensor

- $d\mathfrak{z}^1 \otimes \left(d\mathfrak{z}^1 \wedge \ldots \wedge d\mathfrak{z}^n\right)^{\otimes k} = -\left(\frac{1}{\mathfrak{w}^0}\right)^{k(n+1)+2} d\mathfrak{w}^0 \otimes \bigotimes_{l=1}^{k} \left(d\mathfrak{w}^1 \wedge \ldots \wedge d\mathfrak{w}^n\right)$
- $d\mathfrak{z}^i \otimes \left(d\mathfrak{z}^1 \wedge \ldots \wedge d\mathfrak{z}^n\right)^{\otimes k} = \left(\frac{1}{\mathfrak{w}^0}\right)^{k(n+1)+2} (\mathfrak{w}^0 d\mathfrak{w}^i - \mathfrak{w}^i d\mathfrak{w}^0) \otimes \bigotimes_{l=1}^{k} \left(d\mathfrak{w}^1 \wedge \ldots \wedge d\mathfrak{w}^n\right)$ for $i \neq 1$.

[Chow's corollary] Every meromorphic function on $\mathbb{P}^n\mathbb{C}$ is rational, i.e. $\pi^* f = f \circ \pi = \frac{P}{Q}$ with $P, Q \neq 0$ in $\mathbb{C}_d[X^1, \ldots, X^{n+1}]$. For a proof please have a look in [GH78, p. 168].

3.10.0. Add reference.

Any holomorphic rational function $f = \frac{P}{Q}$ is polynomial.

3.11 Complex submanifolds and analytic subvarieties

[Complex submanifold] A subset N of a complex manifold M is called a **complex submanifold** if for every point $p \in N$ there exist natural numbers k and n with $k \leq n$ and a chart $\phi : U \to V \subset \mathbb{C}^n$ of M around p, such that

$$N \cap U \cong \phi(U) \cap \left\{ z \in \mathbb{C}^n : z^j = 0,\ k+1 \leq j \leq n \right\}.$$

Each complex submanifold possesses the structure of a complex manifold.

We can generalize the concept of a submanifold.

[Analytic subvariety] Suppose Y is a subset of a complex manifold M. If for every point $p \in Y$ there is a neighbourhood U and finitely many holomorphic functions in $\mathcal{O}(U)$ satisfying

$$U \cap Y = \bigcap_{1 \leq i \leq m_p} Z\left(f_i\right),$$

then Y is an **analytic subvariety**. The functions $f_1, \ldots, f_{m_p}$ are called **local defining functions** for Y.

The union and intersection of the closed subvarieties Y_1 and Y_2 are again analytic subvarieties.

[Regular and singular points of an analytic subvariety] A point of an analytic subvariety $Y \subset M$ is a **regular point** if there is an open neighbourhood $U \subset M$ such that $Y \cap U$ is a complex submanifold of U. A point that is not regular is **singular**.

We denote the regular and singular points of an analytic subvariety by Y_{reg} and Y_{sing}, respectively.

[Irreducible analytic subvariety] An analytic subvariety Y is **irreducible** if there are no analytic subvarieties Y_1 and Y_2 such that

$$Y_i \overset{closed}{\subsetneq} Y \quad \text{and} \quad Y = Y_1 \cup Y_2.$$

Irreducible polynomials produce irreducible varieties in the following manner.

Let P be an irreducible polynomial then its zero set $Z(P)$ is an irreducible analytic variety.

We need the three subsequent deep theorems. Their proofs can be found in [GR65] on pages 116 and 141, respectively.

The regular locus Y_{reg} is an open dense subset of Y and Y_{sing} is an analytic subvariety.

If Y is irreducible, then Y_{reg} is connected and vice versa.

Let Y be an analytic subvariety. Then, the closures of Y_{reg}'s connected components are irreducible analytic subvarieties. The above mentioned irreducible subvarieties are called the **irreducible components** of Y. It is also possible to characterize the components as maximal closed subvarieties of Y.

The preimage of an analytic subvariety $Y \subset N$ under a holomorphic function $f : M \to N$ is an analytic subvariety in M.

[Dimension of irreducible analytic varieties] The **dimension of an irreducible analytic variety** is the dimension of its regular locus.

An **analytic subvariety**'s **dimension** is the supremum of its irreducible components' dimensions. An analytic subvariety is **pure dimensional** if all irreducible components have got the same dimension.

Let Y and M be an analytic variety and a complex manifold, respectively, of pure dimensions. If Y is a subvariety of M, then its **codimension** is the natural number $\operatorname{codim} Y = \dim M - \dim Y$.

[Hypersurface] A **hypersurface** is an analytic subvariety of codimension 1.

[Negligible set] We call a analytic subvariety $A \subset M$ of a connected complex manifold **analytically negligible** if all irreducible components have codimension greater or equal 2.

Given a chain $Y_1 \subsetneq Y_2 \subsetneq M$ of irreducible subvarieties of a connected manifold M, then Y_1 is negligible. Hence, the common zero set of two holomorphic functions $f, g : M \to \mathbb{C}$ is negligible iff the germs of f and g are coprime at each point $p \in M$. A consequence is the subsequent lemma.

Suppose Q is an irreducible polynomial, then the analytic variety $Z\left(Q, \frac{\partial Q}{\partial z^i}\right)$ is negligible. Since $\frac{\partial Q}{\partial z^i}$ has a smaller degree than Q, the derivative $\frac{\partial Q}{\partial z^i}$ cannot divide Q.

[Levi's extension theorem] Given a negligible set A, any meromorphic function $f : M \backslash A \to \mathbb{C}$ extends to a meromorphic function on M.

Consequently, we can also continue meromorphic tensors over a negligible set.

[Weil-divisor] A **Weil-divisor** D on a connected complex manifold M is a mapping from the collection of irreducible hypersurfaces into the integers. Furthermore, we require it to be locally finite, i.e. every point has got an open neighbourhood U such that only finitely many hypersurfaces with $D(Y) \neq 0$ intersect U. $D(Y)$ is called **multiplicity** of Y. Sometimes, the divisor is denoted by the formal sum

$$\sum_Y D(Y) \cdot Y.$$

The **support** of D is the analytic subvariety

$$\operatorname{supp} D = \bigcup_{D(Y) \neq 0} Y.$$

Therefore, we can define the singular locus of $\operatorname{supp} D$ and denote it by D_{sing}.

3.12 Covering maps and spaces

We start this subsection with the most general definition of a covering as seen in [For77, Definition 4.1, p.18] (not to mix with [For99] !) differing from the topological and hence stricter ones in [Fre09], [For99],[Jos06] or [Bre05].

[Covering] We call a map $p : Y \to X$ between two topological spaces **covering map**, if it is open, continous and discrete, i.e. $p^{-1}(x)$ is discrete in Y for every x in X. The reason for this unconventional nomenclature of domain and codomain is, that we want to classify functions $f : X \to Z$ with the help of p. Obviously the properties of a covering are local ones.

Let us illustrate this definition by the classical example which lead to the development of the theory of Riemann surfaces.

The function sending a complex number of modulus less than 1 to its square

$$\begin{array}{rccl} p: & \mathbb{E} & \longrightarrow & \mathbb{E} = B_1(0) \\ & z & \longmapsto & z^2 \end{array}$$

is obviously holomorphic and so by the Open Mapping Theorem open [FB09, p.128]. The preimage for a given $w = \rho e^{i\theta}$ is just $\left(\sqrt{\rho}\; e^{i\left(\frac{1}{2}\theta + m\pi\right)}\right)_{0 \leq m \leq 1} = \pm\sqrt{\rho}\, e^{\frac{1}{2}\theta i}$.

Please have a look at the visualisation of this example fig. 3.3 on the current page.

Figure 3.3: Here we tried to visualise the classical section 3.12 on the current page in only three dimensions. In polar coordinates p has the nice appearance $p : [0,1) \times \mathbb{R}/2\pi\mathbb{Z} \to [0,1) \times \mathbb{R}/2\pi\mathbb{Z}$, $(r,\phi) \mapsto (r^2, 2\phi) = (p^1(r), p^2(\phi))$. As p^1 is bijective on $[0,1)$, only p^2 distributes to the special character of a covering map. Hence we use the set $\left\{(\rho,\theta,\phi) \in [0,1) \times \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z} : \theta = p^2(\phi)\right\} = pr_{(\rho,\theta,\phi)} \left\{(w,z) = (\rho e^{i\theta}, re^{i\phi}) \in \mathbb{E} \times \mathbb{E} : w = z^2\right\}$.

[Ramification point] Let $p : Y \to X$ be a covering then we call a point in Y having no open neighbourhood on which p is injective a **ramification** or **branch point**.

[Ramification locus] There are three closely related sets we call the **ramification locus** of a covering p

1. the collection of all branch points $\operatorname{Ram}(p)$,

2. its image $p(\operatorname{Ram}(p))$

> 3.12.0. $p(\operatorname{Ram}(p))$ koennte man auch double point set nennen, vgl http://en.wikipedia.org/wiki/Ramification

3. and $p(\operatorname{Ram}(p))$'s preimage $\widetilde{\operatorname{Ram}}(p) := p^{-1}(p(\operatorname{Ram}(p)))$

From the context it should be clear which one we are currently using.

[Unbranched covering] We call a covering without any branch points **unbranched covering** or **local homeomorphism**.

For a given covering $p : Y \to X$ there is an associated local homeomorphism $\hat{p} := p|_{Y\setminus\operatorname{Ram}(p)}$. Later when properness of covering maps shall become important it can be preserved by restricting $\hat{p}$ further to $\tilde{p} : Y\setminus\widetilde{\operatorname{Ram}}(p) \to X\setminus p(\operatorname{Ram}(p))$. The following counterexample illustrates this.

Given the proper covering

$$\begin{array}{rccc} p: & B_1(0)\cup B_1(2) & \longrightarrow & B_1(0)=\mathbb{E} \\ & z & \longmapsto & p(z)=\begin{cases} z^2 & z\in B_1(0) \\ z-2 & z\in B_1(2) \end{cases} \end{array}$$

then its ramification loci are $\operatorname{Ram}(p) = \{0\}$ and $\widetilde{\operatorname{Ram}}(p) = \{0, 2\}$. So the preimage of $\overline{B_{\frac{1}{4}}(0)}$ under $\hat{p}$ is the non-compact set $(\overline{B_{\frac{1}{2}}(0)}\setminus\{0\})\cup\overline{B_{\frac{1}{4}}(2)}$ contradicting the properness of $\hat{p}$.

As we want to obtain good results about $p(\operatorname{Ram}(p))$ and $\widetilde{\operatorname{Ram}}(p)$, we compound a stricter definition from them. [Standard covering] We call a covering map $f : M \to N$ between two n dimensional complex manifolds **standard covering map**, if

- f is holomorphic, proper and surjective,

- M is connected,
- $\widetilde{\mathrm{Ram}}\,(p) = \mathrm{Ram}\,(p)$,
- $p\,(\mathrm{Ram}\,(p))$ and $\widetilde{\mathrm{Ram}}\,(p)$ are connected smooth hypersurfaces.

For example [Standard element p_n^k] We call the map

$$\begin{array}{cccc} p_n^k : & \mathbb{E}^n & \longrightarrow & \mathbb{E}^n \\ & (z^1, \ldots, z^n) & \longmapsto & (z^1, \ldots, z^{n-1}, (z^n)^k) \end{array}$$

the **k-th n-dimensional standard element**, with $n, k > 0$.

The standard elements are standard coverings. Clearly each standard element p is holomorphic and surjective. The ramification loci are just the zero set of the projection onto the n-th coordinate. Decomposing a polydisc $D_r^n(z_0)$ and p into their *open* components $B_{r^i}(z_0^i)$ and p^i, respectively, implies that $p(D_r(z_0)) = (p^i(B_{r^i}(z_0^i)))_i$ is open. As the preimage of $\overline{D^n_{(r^1,\ldots,r^n)}}\,(0)$ under p is $\overline{D^{n-1}_{(r^1,\ldots,r^{n-1})}}\,(0) \times \overline{B_{\sqrt[k]{r^n}}}\,(0)$ the preimages of bounded sets are bounded . The equation $p^{-1}(z^1, \ldots, z^{n-1}, \rho e^{i\theta}) = \left(z^1, \ldots, z^{n-1}, \sqrt[k]{\rho}\, \exp\left(\frac{i}{k}\,(\theta + 2m\pi)\right)\right)_{0 \le m \le k-1}$ shows that p is discrete.

A basic fact about standard coverings is the following lemma

Biholomorphic functions ϕ and ψ preserve the properties of an standard covering$p$, i.e. $p \circ \phi$, $\psi \circ p$ and $\psi \circ p \circ \phi$ are standard coverings. The main argument was given in **??** on page ??.

This paves the way for a generalization of the standard element

3.12.0. correct definition of Q-standard element ?

[Q-standard element $p^k_{n,Q}$] Let U_1 and U_2 be open in $\mathbb{C}^{n-1}$ and $\mathbb{C}$, respectively. For the holomorphic function $Q : U_1 \times U_2 \to \mathbb{C}$ and its holomorphic auxillarily function $\varphi : U_1 \to \mathbb{C}$ uniquely determined by $Q(z) = 0 \iff z^n = \varphi(z^1, \ldots, z^{n-1})$ we define

$$\begin{array}{cccc} p^k_{n,Q} : & U_1 \times U_2 & \longrightarrow & \mathbb{C} \\ & (z^1, \ldots, z^n) & \longmapsto & (z^1, \ldots, z^{n-1}, (z^n)^k + \varphi(z^1, \ldots, z^{n-1})). \end{array}$$

implicit function theorem(p.38)

Q-standard elements $p^k_{n,Q}$ satisfy

1. if $Q = z^n$ then $p^k_{n,z^n} = p^k_n$,
2. each $p^k_{n,Q}$ can decomposed $p^k_{n,Q} = p^1_{n,Q} \circ p^k_n$,
3. the pure straighenting $p^1_{n,Q}$ is biholomorphic,
4. any Q-standard element $p^k_{n,Q}$ is a standard covering,
5. for a Q-standard element $p^k_{n,Q}$ $Z\,(Q)$ is irreducible.

We start with showing that the holomorphic and injective function $p^1_{n,Q}$ is biholomorphic on its image. It is open because $p^1_{n,Q}(\Omega_1 \times \Omega_2) = \Omega_1 \times (\Omega_2 + \varphi(\Omega_1))$ and pr_n, φ are open functions by section 3.9 on page 39.
The last two properties can be deduced from item 3 and **??** on pages 50 and **??**, respectively.

A more interesting fact is Let $f : M \to N$ be a standard covering then we can deduce from A being negligible in N, that $f^{-1}(A) \subset M$ is also negligible. The fact that $f^{-1}(A)$ is an analytic subvariety was shown in **??** on page ??. Assuming that $f^{-1}(A)$ would have an irreducible component of codimension 1, say Y, would lead to a contradiction. Firstly take Y to be a subset of the analytic hypersurface $\widetilde{\mathrm{Ram}}\,(f)$. Hence f's image $f(Y)$ has to coincide with the analytic hypersurface $f\,(\mathrm{Ram}\,(f))$, a contradiction. Secondly Y is not a subset of $\widetilde{\mathrm{Ram}}\,(f)$ and possesses a point p outside of $\widetilde{\mathrm{Ram}}\,(f)$ that is without loss of generality regular due to section 3.11 on page 46. Therefore f is a local homeomorphism around p implying that Y_{sing} has got the same codimension as $(f(Y))_{sing}$, i.e greater or equal than 2.

[Isomorphic functions] Two holomorphic functions $f : M \to N$ and $f' : M' \to N'$ are **isomorphic** (in the category of holomorphic mappings between complex manifolds), if there are biholomorphic functions ϕ and ψ, such that we get a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\phi} & M' \\ \Big\downarrow f & & \Big\downarrow f' \\ N & \xrightarrow{\psi} & N'. \end{array}$$

[Ramification element]By a **ramification element** we mean a standard covering $f : M \to \mathbb{E}^n$ such that the ramification locus in $\mathbb{E}^n$ equals $z^n = 0$.

[Uniqueness of the ramification element] Let f be a ramification element, then there exists a standard element p_n^k and a biholomorphic mapping ϕ completing the following diagram

$$\begin{array}{ccc} M & \xrightarrow{\phi} & \mathbb{E}^n \\ \Big\downarrow f & & \Big\downarrow p_n^k \\ \mathbb{E}^n & \xrightarrow{id} & \mathbb{E}^n. \end{array}$$

So f is determined uniquely up to isomorphy.

[Sketch] We shall sketch the proof here by referring to parts of [For99] and generalising them if needed. We denote by $\mathcal{H}$ the left half plane $\{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$ yielding that

$$\begin{array}{rccc} \mathrm{Exp}: & \mathbb{E}^{n-1} \times \mathcal{H} & \longrightarrow & \mathbb{E}^{n-1} \times \mathbb{E}^* \\ & (z^1, \dots, z^n) & \longmapsto & (z^1, \dots, z^{n-1}, \exp(z^n)) \end{array}$$

is the universal covering of $\mathbb{E}^{n-1} \times \mathbb{E}^*$ with deck transformations $\tau_m(z^1, \dots, z^n) = (z^1, \dots, z^n + 2\pi i m)$ analogous to Exs. 4.12, 5.7 and Thm. 5.2 . There exists a biholomorphic function ϕ completing the commutative diagram

$$\begin{tikzcd} M \backslash \mathrm{Ram}\,(f) \arrow[rr, "\cong", "\phi"'] \arrow[dr, "f"'] & & \mathbb{E}^{n-1} \times \mathbb{E}^* \arrow[dl, "p_n^k"] \\ & \mathbb{E}^{n-1} \times \mathbb{E}^* & \end{tikzcd}$$

because of Thms. 5.9 and 5.10 (altered by section 3.9 on page 40 of this thesis). Following Thm. 5.11 we can show that the preimage of each point in $f(\mathrm{Ram}(f)) = \mathbb{E}^{n-1} \times \{0\}$ has just one element consequently ϕ can be extended to a bijection $\phi : M \to \mathbb{E}^n$. Even more ϕ is continuous because a sequence $b_m \to b \in \mathrm{Ram}(f)$ is getting mapped to $f(b_m) = (a_m^1, \ldots, a_m^n) \to (c^1, \ldots, c^{n-1}, 0)$ and further to $\phi(b_m) = (a_m^1, \ldots, a_m^{n-1}, \xi_m)$ with $|\xi_m| = \sqrt[k]{|a_m^n|} \to 0$. As ϕ is continuous the Riemann extension theorem(p.40) and then section 3.9 on page 40 can be applied to prove that ϕ is biholomorphic.

3.13 Orders of singularities

We generalize the order of a singularity known for one dimension in one single point from [FB09] the way it is done in [GH78, pp.130].

[Order of a holomorphic function along a hypersurface] Let p be a regular point on an irreducible analytic hypersurface Y of a complex manifold M. Suppose U is a neighbourhood of p, f a holomorphic function on U and ψ locally defines Y on U then we define the **order of f along Y at p** as

$$\sup\left\{k \in \mathbb{N} : [\psi]^k_{\mathcal{O}_{U,p}} \mid [f]_{\mathcal{O}_{U,p}} \text{ in } \mathcal{O}_{U,p}\right\} =: \mathrm{ord}\,(f, Y, p) \in \mathbb{N} \cup \{\infty\}.$$

The order of holomorphic functions has got the following properties

1. $\mathrm{ord}\,(f, Y, \cdot)$ is a locally constant function on $Y_{reg} \cap U$,
2. if f is defined on the whole manifold M then $\mathrm{ord}\,(f, Y, \cdot)$ is a constant function on M's irreducible subvariety Y,
3. $\mathrm{ord}\,(f, Y)$ equals ∞ iff f is the zero function,
4. $\mathrm{ord}\,(fg, Y) = \mathrm{ord}\,(f, Y) + \mathrm{ord}\,(g, Y)$,
5. $\mathrm{ord}\,(f, Y, p) = \mathrm{ord}\,(f \circ \phi, \phi(Y), \phi(p))$ for a biholomorphism ϕ,

i. If k is the order of f along Y in a point p then there is a decomposition of $[f]_{\mathcal{O}_{U,p}}$ into coprime elements $[\psi]^k_{\mathcal{O}_{U,p}}$ and $[g]_{\mathcal{O}_{U,p}}$ in $\mathcal{O}_{U,p}$. Section 3.9 on page 40 implies that there is a neighbourhood V of p such that $[\psi]^k_{\mathcal{O}_{U,q}}$ and $[g]_{\mathcal{O}_{U,q}}$ are coprime in every $\mathcal{O}_{U,q}$ for q in V. Hence $[\psi]_{\mathcal{O}_{U,q}}$ does not divide $[g]_{\mathcal{O}_{U,q}}$ in these $\mathcal{O}_{U,q}$ and $\operatorname{ord}(f, Y, q) = k$ for q in $V \cap Y_{reg}$.

ii. As f lies in $\mathcal{O}(M)$ $\operatorname{ord}(f, Y, \cdot)$ is locally constant on $Y_{reg} \cap M$. Sections 3.11 and 3.11 on pages 46 and 46, respectively, yield that Y_{reg} is connected and dense in Y.

iii. Clearly units have got order zero and non-zero non-units are assigned the exponent associated to $[\psi]_{\mathcal{O}_{U,p}}$ in their prime factorization. Additionally it holds $0 = [\psi]^k_{\mathcal{O}_{U,p}} \cdot 0$ for all k in $\mathbb{N}$.

iv. Let k and l be the orders of f and g respectively. Once again we use the induced decompositions $[f]_{\mathcal{O}_{U,p}} = [\psi]^k_{\mathcal{O}_{U,p}} [f_0]_{\mathcal{O}_{U,p}}$ and $[g]_{\mathcal{O}_{U,p}} = [\psi]^l_{\mathcal{O}_{U,p}} [g_0]_{\mathcal{O}_{U,p}}$ inducing $[fg]_{\mathcal{O}_{U,p}} = [\psi]^{k+l}_{\mathcal{O}_{U,p}} [f_0]_{\mathcal{O}_{U,p}} [g_0]_{\mathcal{O}_{U,p}}$. As $[\psi]_{\mathcal{O}_{U,p}}$ is irreducible this implies $\operatorname{ord}(fg, Y) = \operatorname{ord}(f, Y) + \operatorname{ord}(g, Y)$.

v. Firstly **??** on page ?? secures that $\phi(Y)$ is an irreducible analytic subvariety. Secondly we have seen in item 2 on page 29, that $\phi^* : \mathcal{O}_{U,p} \to \mathcal{O}_{V,\phi(p)}$ is an isomorphism of UFDs. Finally applying section 2.1 on page 11 establishes the desired formula.

We generalize the above definition and lemma to

[Order of a meromorphic function along a hypersurface] Suppose p is a regular point of an irreducible analytic hypersurface Y. Then the **order of a meromorphic function** f with local -not necessarily coprime-representation (U, g, h) **in** p is well defined by

$$\operatorname{ord}(f, Y, p) := \operatorname{ord}(g, Y, p) - \operatorname{ord}(h, Y, p) \in \mathbb{Z} \cup \{\infty\}$$

and has got the following properties

1. if f is meromorphic on M then $\operatorname{ord}(f, Y)$ is also well defined,

2. the orders of a meromorphic function f with coprime representation (U, g, h) are related by $\operatorname{ord}(f, Y_i, p) \geq 0 \iff \operatorname{ord}(h, Y_i, p) = 0$ and $\operatorname{ord}(f, Y_i, p) \leq 0 \iff \operatorname{ord}(g, Y_i, p) = 0$,

3. the properties stated in section 3.13 can be generalized,

4. the order of a sum of functions is greater than the smallest single order, i.e. $\operatorname{ord}\left(\sum_{i=1}^{m} f_i, Y\right) \geq \min_{1\leq i\leq m}\{\operatorname{ord}(f_i, Y)\}$, with equality if there is an index j with $\operatorname{ord}(f_j, Y) < \operatorname{ord}(f_i, Y)$ for all $i \neq j$.

0. We have to show that it holds

$$\operatorname{ord}(g_i, Y, p) - \operatorname{ord}(h_i, Y, p) = \operatorname{ord}(g_j, Y, p) - \operatorname{ord}(h_j, Y, p)$$

for any pair of local representations (U_i, g_i, h_i) and (U_j, g_j, h_j) with $p \in U_i \cap U_j$. Indeed $g_i h_j = g_j h_i$ implies by section 3.13,4 $\operatorname{ord}(g_i, Y, p) + \operatorname{ord}(h_j, Y, p) = \operatorname{ord}(g_j, Y, p) + \operatorname{ord}(h_i, Y, p)$.

i. $\operatorname{ord}(f, Y, \cdot)$ assigns to each p in Y_{reg} an integer or ∞. As $\operatorname{ord}(g_i, Y, \cdot)$ and $\operatorname{ord}(h_i, Y, \cdot)$ are locally constant so is $\operatorname{ord}(f, Y, \cdot)$. As seen in section 3.13 this makes $\operatorname{ord}(f, Y, \cdot)$ constant on Y.

ii. $\operatorname{ord}(f, Y_i, p) \geq 0$ implies $\operatorname{ord}(g, Y_i, p) \geq \operatorname{ord}(h, Y_i, p)$. But $\operatorname{ord}(h_j, Y_i, p) \geq 1$ would contradict that $[g_j]_{\mathcal{O}_{U,p}}$ and $[h_j]_{\mathcal{O}_{U,p}}$ are coprime, hence $\operatorname{ord}(h_j, Y, p) = 0$.

iv. Clearly if $[\psi]^k_{\mathcal{O}_{U,p}}$ divides each holomorphic $[f_i]_{\mathcal{O}_{U,p}}$ then also $\left[\sum_i f_i\right]_{\mathcal{O}_{U,p}}$. Conversely if $[\psi]^k_{\mathcal{O}_{U,p}}$ divides all $[f_i]_{\mathcal{O}_{U,p}}$ s except for $[f_j]_{\mathcal{O}_{U,p}}$ then neither $\left[\sum_i f_i\right]_{\mathcal{O}_{U,p}}$. Inded assuming $[\psi]^k_{\mathcal{O}_{U,p}} [h]_{\mathcal{O}_{U,p}} = \left[\sum_i f_i\right]_{\mathcal{O}_{U,p}}$ leads to $[f_j]_{\mathcal{O}_{U,p}} = \left[\sum_i f_i\right]_{\mathcal{O}_{U,p}} - \left[\sum_{i\neq j} f_i\right]_{\mathcal{O}_{U,p}} = [\psi]^k_{\mathcal{O}_{U,p}} \left([h]_{\mathcal{O}_{U,p}} - \left[\sum_{i\neq j} h_i\right]_{\mathcal{O}_{U,p}}\right)$, a contradiction!
For meromorphic functions $f_i = \frac{g_i}{\psi^k h_i}$ their sum is $\frac{\sum_i g_i \prod_{j\neq i} h_j}{\psi^k \prod_i h_i}$ the divisor of which has got order $k = -\min_{1\leq i\leq m}\{\operatorname{ord}(f_i, Y)\}$. So the total order can only be altered by the nominator's positive order $\operatorname{ord}\left(\sum_i g_i \prod_{j\neq i} h_j, Y\right)$, i.e. more than one summand with order zero.

Let P be a polynomial and Q a prime polynomial then Q^k divides P in $\mathbb{C}[X^1, \ldots, X^n]$ iff $\operatorname{ord}(P, Z(Q)) \geq k$. In fact the maximal of these ks is just $\operatorname{ord}(P, Z(Q))$. If $P = Q^k \cdot R$ in $\mathbb{C}[X^1, \ldots, X^n]$ then obviously also in $\mathcal{O}_{U,p}$ and so $\operatorname{ord}(P, Z(Q)) \geq k$. Conversely we define for $m = \operatorname{ord}(P, Z(Q))$ the rational function $\frac{P}{Q^m}$ that is everywhere holomorphic. Indeed it locally coincides with $[g]_{\mathcal{O}_{U,p}}$ for $[Q^m]_{\mathcal{O}_{U,p}} \cdot [g]_{\mathcal{O}_{U,p}} = [P]_{\mathcal{O}_{U,p}}$. We conclude from item ii on page 45 that f is a polynomial.

3.13.0. prove this.

The Laurent series of a holomorphic function f on $\mathcal{D} \times \mathbb{E}^*$ has got a lower bound on the index set of non-zero coefficients iff f is meromorphic on $\mathcal{D} \times \mathbb{E}$. It turns out that this lower bound coincides with the order of f along $\{z^n = 0\}$. Additionally f is holomorphic iff the lower bound is 0. If f has got such a lower bound, say $-m$, then f equals

$$\sum_{k=-m}^{\infty} a_k(z^1, \ldots, z^{n-1}) \cdot (z^n)^k = (z^n)^{-m} \cdot \sum_{l=0}^{\infty} a_{-m+l}(z^1, \ldots, z^{n-1}) \cdot (z^n)^l \equiv (z^n)^{-m} \cdot g(z).$$

3.13.0. Rephrase the proof of $-m = \operatorname{ord}(f, p, \{z^n = 0\}) = \operatorname{ord}(f, \{z^n = 0\})$

The Riemann extension theorem(p.40) allows us to analytically continue $g \in \mathcal{O}(\mathcal{D} \times \mathbb{E}^*)$ onto $\mathcal{D} \times \mathbb{E}$ leading to the equality $f = (\mathcal{D} \times \mathbb{E}, g, (z^n)^m)$. We conclude from the identity theorem(p.39) that $a_{-m}(z^1, \ldots, z^{n-1})$ does not vanish in most points of $\{z^n = 0\}$. It holds $-m = \operatorname{ord}(f, p, \{z^n = 0\}) = \operatorname{ord}(f, \{z^n = 0\})$ for all such points. Analogously $m = 0$ implies $f \in \mathcal{O}(\mathcal{D} \times \mathbb{E})$.
Conversely for a local representation (U, g, h) around a point $p \in \{z^n = 0\}$ we decompose h's germ in $\mathcal{O}_{U,p}$ into its prime factors $[\psi_1]_{\mathcal{O}_{U,p}}, \ldots, [\psi_{r_p}]_{\mathcal{O}_{U,p}}$. As f is holomorphic outside of $\{z^n = 0\}$ it holds $Z(\psi_i) \subset Z(h) \subset \{z^n = 0\}$ for all i. We deduce from Rückert's Corollary(p.40) that $[\psi_i]_{\mathcal{O}_{U,p}}$ divides $[z^n]_{\mathcal{O}_{U,p}}$ in $\mathcal{O}_{U,p}$. Due to section 3.9 on page 39 $[z^n]_{\mathcal{O}_{U,p}}$ is irreducible in $\mathcal{O}_{U,p}$ hence it equals $[\epsilon_i]_{\mathcal{O}_{U,p}} \cdot [\psi_i]_{\mathcal{O}_{U,p}}$ for all i. Therefore $[h]_{\mathcal{O}_{U,p}}$ equals $[\epsilon]_{\mathcal{O}_{U,p}} \cdot [z^n]^{r_p}_{\mathcal{O}_{U,p}}$ leading to $r_p = \operatorname{ord}(h, p, \{z^n = 0\})$, $h = \epsilon \cdot (z^n)^{r_p}$ and eventually

$$f = g/h = \epsilon^{-1} \cdot g \cdot (z^n)^{-r_p} = \sum_{k=-r_p}^{\infty} a_k(z^1, \ldots, z^{n-1}) \cdot (z^n)^k.$$

r_p is a global lower bound because the order $\operatorname{ord}(h, p, \{z^n = 0\})$ is constant on $\{z^n = 0\}$.

Let Y be a hypersurface of M with irreducible components Y_i and fa meromorphic function on M that is holomorphic on $M \backslash Y$. Then f is holomorphically extendable on M iff it holds $\operatorname{ord}(f, Y_i) \geq 0$ for all i. By Levi's extension theorem and section 3.11 on pages 47 and 46, respectively, holomorphic functions can be analytically continued over Y_{sing}. Consequently it is sufficient to observe the holomorphicity in Y_{reg}. As f's order is invariant under charts, cf. item v, the proof is completed by applying item iv.

Let $p^k_{n,Q} : \mathbb{E}^n \to \mathbb{E}^n$ be the Q-standard element and $f : \mathbb{E}^n \to \mathbb{C}$ a non-zero meromorphic function. Then the order of f along the ramification locus $p^k_{n,Q}\left(\operatorname{Ram}\left(p^k_{n,Q}\right)\right)$ and the order of $f \circ p^k_{n,Q}$ along the ramification locus $\widetilde{\operatorname{Ram}}\left(p^k_{n,Q}\right)$ vary directly, i.e.

(3.2) $\operatorname{ord}\left(f \circ p_{n,Q}^k, \widetilde{\operatorname{Ram}}\left(p_{n,Q}^k\right)\right) = k \cdot \operatorname{ord}\left(f, p_{n,Q}^k\left(\operatorname{Ram}\left(p_{n,Q}^k\right)\right)\right)$.

i. For $Q = z^n$ and f being holomorphic we prove the following auxiliary inequalities

(3.3)
$$\operatorname{ord}\left(f, p_n^k\left(\operatorname{Ram}\left(p_n^k\right)\right)\right) = l \qquad \Longrightarrow \operatorname{ord}\left(f \circ p_n^k, \widetilde{\operatorname{Ram}}\left(p_n^k\right)\right) = l \cdot k,$$
(3.4)
$$\operatorname{ord}\left(f \circ p_n^k, \widetilde{\operatorname{Ram}}\left(p_n^k\right)\right) = m \geq 0 \quad \Longrightarrow \operatorname{ord}\left(f, p_n^k\left(\operatorname{Ram}\left(p_n^k\right)\right)\right) = \left\lceil \frac{m}{k} \right\rceil.$$

This is done by writing down the explicit definitions of the orders

$$\operatorname{ord}\left(f \circ p_n^k, \widetilde{\operatorname{Ram}}\left(p_n^k\right)\right) = \max\left\{l \in \mathbb{N} : (z^n)^l \mid f(z^1, \ldots, (z^n)^k)\right\}$$
$$\operatorname{ord}\left(f, p_n^k\left(\operatorname{Ram}\left(p_n^k\right)\right)\right) = \max\left\{l \in \mathbb{N} : (z^n)^l \mid f(z^1, \ldots, z^n)\right\}.$$

As f is holomorphic we can extend it and $f \circ p_n^k$ in a power series $f = a_\nu z^\nu \equiv \alpha_m (z^n)^m$ and $f \circ p_n^k = \alpha_m (z^n)^{km}$, respectively.

a) The first step to prove eq. (3.3) is concluding that $\operatorname{ord}\left(f \circ p_{n,Q}^k, \widetilde{\operatorname{Ram}}\left(p_{n,Q}^k\right)\right)$ lies in $k\mathbb{N}$. Furthermore let l_{max} be the order of f along $p_n^k\left(\operatorname{Ram}\left(p_n^k\right)\right)$ then $f(z)$ equals $(z^n)^{l_{max}} g(z)$ and hence $\alpha_{l_{max}}$ is the first coefficient not to vanish. Therefore $(z^n)^{k \cdot l_{max}} \mid f \circ p_n^k$ and consequently $\operatorname{ord}\left(f \circ p_{n,Q}^k, \widetilde{\operatorname{Ram}}\left(p_{n,Q}^k\right)\right) = l \cdot k$.

b) Equation (3.4) is proven indirectly. Assume that $l = \operatorname{ord}\left(f, p_n^k\left(\operatorname{Ram}\left(p_n^k\right)\right)\right)$ lies in $\mathbb{N} \backslash \left\{\left\lceil \frac{m}{k} \right\rceil, \left\lfloor \frac{m}{k} \right\rfloor\right\}$, then eq. (3.3) implies

$$\operatorname{ord}\left(f \circ p_n^k, \widetilde{\operatorname{Ram}}\left(p_n^k\right)\right) = l \cdot k > \left\lceil \frac{m}{k} \right\rceil \cdot k \geq m$$

or

$$\operatorname{ord}\left(f \circ p_n^k, \widetilde{\operatorname{Ram}}\left(p_n^k\right)\right) = l \cdot k < \left\lfloor \frac{m}{k} \right\rfloor \cdot k \leq m, \text{ a contradiction !}$$

The case $\operatorname{ord}\left(f, p_n^k\left(\operatorname{Ram}\left(p_n^k\right)\right)\right) = \left\lfloor \frac{m}{k} \right\rfloor < \left\lceil \frac{m}{k} \right\rceil$ also returns $\operatorname{ord}\left(f \circ p_n^k, \widetilde{\operatorname{Ram}}\left(p_n^k\right)\right) < m$, because the difference of $\left\lfloor \frac{m}{k} \right\rfloor$ and $\left\lceil \frac{m}{k} \right\rceil$ shows that $\left\lfloor \frac{m}{k} \right\rfloor < \frac{m}{k}$ and furthermore by eq. (3.3) $\operatorname{ord}\left(f \circ p_{n,Q}^k, \widetilde{\operatorname{Ram}}\left(p_{n,Q}^k\right)\right) = \left\lfloor \frac{m}{k} \right\rfloor \cdot k < \frac{m}{k} \cdot k = m$.

Applying eqs. (3.3) and (3.4) to $\operatorname{ord}\left(f \circ p_n^k, \widetilde{\operatorname{Ram}}\left(p_n^k\right)\right) = m$ yields $m = \left\lceil \frac{m}{k} \right\rceil k$ or equivalently $\frac{m}{k} = \left\lceil \frac{m}{k} \right\rceil$. Consequently eqs. (3.3) and (3.4) can be merged to eq. (3.2).

ii. We have seen in items 2 and 3 on pages 29 and 51, respectively, that $(p_{n,Q}^1)^* : \mathcal{O}_{U,p} \to \mathcal{O}_{V,f(p)}$ is an isomorphism of UFDs. Applying section 2.1 on page 11 completes the proof for $p_{n,Q}^1$ and holomorphic f.

iii. Any $p_{n,Q}^k$ can be written as $p_{n,Q}^k = p_{n,Q}^1 \circ p_n^k$.

iv. For a meromorphic function$f$ with local representation (U, g, h) $\operatorname{ord}\left(f \circ p_{n,Q}^k, \widetilde{\operatorname{Ram}}\left(p_{n,Q}^k\right)\right)$ equals $\operatorname{ord}\left(g \circ p_{n,Q}^k, \widetilde{\operatorname{Ram}}\left(p_{n,Q}^k\right)\right) - \operatorname{ord}\left(h \circ p_{n,Q}^k, \widetilde{\operatorname{Ram}}\left(p_{n,Q}^k\right)\right)$ reducing the problem to the above discussed cases.

3.14 The $(\Omega^\bullet)^{\otimes k}(M, D)$ spaces

[$(\Omega^\bullet)^{\otimes k}(M, D)$ or generalised logarithmic tensors] Let D be a divisor on a n-dimensional complex manifold M, we define $(\Omega^\bullet)^{\otimes k}(M, D)$ as the space of tensors $\omega \in (\Omega^\bullet)^{\otimes k}(M \backslash \operatorname{supp} D)$ with the supplementary property : If $p : X \twoheadrightarrow U \overset{\text{open}}{\subset} M \backslash D_{sing}$ is a holomorphic and surjective covering satisfying

1. X is connected,
2. U just intersects one irreducible component of $\operatorname{supp}(D)$, say Y,
3. the ramification locus in U equals $U \cap Y$,
4. given a chart ϕ transforming $V \overset{\text{open}}{\subset} U$ or $V \cap Y$, respectively, to $W \overset{\text{open}}{\subset} \mathbb{C}^n$ or $W \cap \left(\mathbb{C}^{n-1} \times \{0\}\right)$, respectively, then $\phi \circ p$ is isomorphic to the standard element $p_n^{D(Y)+1}$, i.e. $\phi \circ p = p_n^{D(Y)+1} \circ \psi$ or equivalently $p = \phi^{-1} \circ p_n^{D(Y)+1} \circ \psi$

then ω's pullback $p^*\left(\omega|_{V \backslash Y}\right)$ is holomorphically extendable to the whole of X.

We can show that the space of generalised logarithmic tensors $(\Omega^\bullet)^{\otimes k}(M, D)$ is a subset of the vector space of meromorphic tensors $\mathcal{M}(M) \otimes_{\mathcal{O}} (\Omega^\bullet)^{\otimes k}(M)$ and in particular of

$\left(\Omega^1 \otimes_{\mathcal{O}} (\Omega^n)^{\otimes k}\right)(M\setminus \operatorname{supp} D) \cap \mathcal{M}(M) \otimes_{\mathcal{O}} (\Omega^\bullet)^{\otimes k}(M)$.

Every element ω of $(\Omega^\bullet)^{\otimes k}(M,D)$ is a meromorphic tensor on M. By Levi's extension theorem for meromorphic tensors(p.**??**) it is sufficient to show that ω is meromorphic in $\operatorname{supp} D\setminus D_{sing}$. Using the notations of section 3.14 $\phi^{-1}\circ p_n^{D(Y)+1}$ is certainly one of the observed coverings. It is sufficient to check whether $(\phi^{-1})^*\omega$ is meromorphic because we could apply ϕ's pullback to $(\phi^{-1})^*\omega$ and deduce that ω is meromorphic, cf. item 4 on page 42.
The coefficient functions of

$$\omega = \sum_{\substack{\mathcal{I}=\mathcal{I}_1^1\times\cdots\times\mathcal{I}_{l_1}^1\times\mathcal{I}_1^2\times\cdots\times\mathcal{I}_{l_n}^n \\ \mathcal{I}_j^i\subset\{1,\ldots,n\} \;\&\; \left|\mathcal{I}_j^i\right|=i}} \omega_{\mathcal{I}}\, dz^{\mathcal{I}_1^1}\otimes\cdots\otimes dz^{\mathcal{I}_{l_1}^1}\otimes dz^{\mathcal{I}_1^2}\otimes\cdots\otimes dz^{\mathcal{I}_{l_n}^n},$$

transform in the following manner

$$\left(p_n^{D(Y)+1}\right)^*\omega_{\mathcal{I}}(z) = \omega_{\mathcal{I}}(p_n^{D(Y)+1}(z))\cdot\Big((D(Y)+1)(z^n)^{D(Y)}\Big)^N$$

where N is the amount of ns in $\mathcal{I}$ due to the diagonal form of $p_n^{D(Y)+1}$'s Jacobian matrix. But they are also holomorphic on the annulus $\mathbb{E}^{n-1}\times\mathbb{E}^*$ and consequently possess a Laurent series

$$\omega_{\mathcal{I}}(z) = \sum_{m\in\mathbb{Z}} a_m\cdot(z^n)^m\,.$$

Therefore $\left(p_n^{D(Y)+1}\right)^*\omega_{\mathcal{I}}(z)$ can be rewritten to

$$\Big((D(Y)+1)(z^n)^{D(Y)}\Big)^N\sum_{m\in\mathbb{Z}} a_m\cdot(z^n)^{m(D(Y)+1)} \equiv \sum_{l\in\mathbb{Z}} b_l\,(z^n)^l$$

which is holomorphic by definition of $(\Omega^\bullet)^{\otimes k}(M,D)$. It holds $b_l\equiv 0$ for all $l<0$ by item iv on page 56, and so $a_m\equiv 0$ for all $\frac{-D(Y)}{D(Y)+1}N > m$. The desired result can be deduced from item iv on page 55.

The above proof could have given you an idea of the line of arguments in chapter 5 or **??** on page ??, respectively.

1. Given a meromorphic tensor ω and two entire coverings p and q with $p = q\circ\phi$. We conclude that if $q^*\omega$ is holomorphic, then also $p^*\omega$, because $p^*\omega$ equals $\phi^*(q^*\omega)$. Therefore the uniqueness of the ramification element, item 5 on page 52, allows us

to prove the extension property of ω's pullback in x for one chosen p with pleasant properties. In chapter 5 it is the ramification element defined in item 3 on page 51. The aforementioned property is equivalent to ω to lie in $(\Omega^\bullet)^{\otimes k}(M, D)$.

2. $D_1 \leq D_2$ implies $(\Omega^\bullet)^{\otimes k}(M, D_1) \subset (\Omega^\bullet)^{\otimes k}(M, D_2)$.

3. The space of generalised logarithmic tensors to the zero divisor coincides with the holomorphic tensors of the same type, i.e. $(\Omega^\bullet)^{\otimes k}(M, 0) = (\Omega^\bullet)^{\otimes k}(M)$.

4. We specify certain subspaces such as $\left(\Omega^i \otimes_{\mathcal{O}} (\Omega^n)^{\otimes k}\right)(M, D) := (\Omega^\bullet)^{\otimes k+1}(M, D) \cap \left(\Omega^i \otimes_{\mathcal{O}} (\Omega^n)^{\otimes k}\right)(M \setminus \operatorname{supp} D)$.

5. The space of generalised logarithmic 0-forms to any divisor D equals the vector space of holomorphic functions on M, i.e. $\Omega^0(M, D) = \mathcal{O}(M)$.

4 Modular functions

5 Existence results

6 Bibliography

[BBI01] Dmitri Burago, Yuri Burago, and Sergei Ivanov. *A Course in Metric Geometry.* American Mathematical Society, 7 2001. (cited on page 15)

[Bre05] Glen Eugene Bredon. *Topology and geometry.* Graduate texts in mathematics; 139. Springer-Verlag Berlin Heidelberg, Germany, 2005. (cited on page 48)

[Bro12] Luitzen Egbertus Jan Brouwer. Beweis der Invarianz des n-dimensionalen Gebiets. *Mathematische Annalen*, 71:305–315, 1912. (cited on page 22)

[FB09] Eberhard Freitag and Rolf Busam. *Complex analysis.* Springer-Verlag Berlin Heidelberg, Germany, Berlin; Heidelberg, 2009. (cited on pages 28, 48, and 53)

[FG02] Klaus Fritzsche and Hans Grauert. *From holomorphic functions to complex manifolds.* Number 213 in Graduate texts in mathematics. Springer, New York; Berlin; Heidelberg, 2002. (cited on page 25)

[For77] Otto Forster. *Riemannsche Flächen.* Heidelberger Taschenbücher; 184. Springer-Verlag Berlin Heidelberg, Germany, 1977. (cited on pages 23 and 48)

[For99] Otto Forster. *Lectures on Riemann surfaces.* Graduate texts in mathematics; 81. Springer-Verlag Berlin Heidelberg, Germany, 1999. (cited on pages 48 and 52)

[Fre09] Eberhard Freitag. *Funktionentheorie 2.* Springer-Lehrbuch. Springer-Verlag Berlin Heidelberg, Germany, 2009. (cited on pages 39, 40, and 48)

[Ger06] Claus Gerhardt. *Analysis.* International Press, Somerville, MA, 2006. (cited on page 28)

[GH78] Phillip A. Griffiths and Joseph Harris. *Principles of algebraic geometry.* Pure and applied mathematics; A Wiley-Interscience publication. Wiley, New York [a.o.], 1978. (cited on pages 25, 34, 42, 45, and 53)

[GHL87] Sylvestre Gallot, Dominique Hulin, and Jacques Lafontaine. *Riemannian geometry*. Universitext. Springer, Berlin; Heidelberg; New York; London; Paris; Tokyo, 1987. (cited on pages 23 and 34)

[GR65] Robert C. Gunning and Hugo Rossi. *Analytic functions of several complex variables*. Prentice-Hall series in modern analysis. Prentice-Hall, Englewood Cliffs, NJ, 1965. (cited on pages 40 and 46)

[Huy05] Daniel Huybrechts. *Complex geometry, an introduction*. Springer-Verlag Berlin Heidelberg, Germany, 2005. (cited on pages 38, 39, and 40)

[Jos06] Jürgen Jost. *Compact Riemann surfaces*. Universitext. Springer, Berlin; Heidelberg, 3. ed. edition, 2006. (cited on page 48)

[KK83] Ludger Kaup and Burchard Kaup. *Holomorphic functions of several variables*. Number 3 in De Gruyter studies in mathematics. De Gruyter, Berlin [a.o.], 1983. (cited on page 39)

[Lan02] Serge Lang. *Algebra*. Graduate texts in mathematics; 211. Springer-Verlag Berlin Heidelberg, Germany, 2002. (cited on pages 11 and 12)

[Lee03] John M. Lee. *Introduction to smooth manifolds*. Number 218 in Graduate texts in mathematics. Springer, New York; Heidelberg, 2003. Includes bibliographical references and index. (cited on pages 34, 35, and 38)

[Ser65] Jean-Pierre Serre. *Lie algebras and Lie groups*. Benjamin, New York, 1965. (cited on page 26)

[Shi71] Goro Shimura. *Introduction to the arithmetic theory of automorphic functions*. Number 11 in Publications of the Mathematical Society of Japan. Princeton University Press, Princeton, New Jersey, Tokyo, 1971. (cited on page 21)

[Spr77] Tonny A. Springer. *Invariant theory*. Number 585 in Lecture notes in mathematics. Springer, Berlin, 1977. (cited on page 16)

[Wie10] Thomas Wieber. Symplectic geometry and hamiltonian group actions. Master's thesis, University of Leeds, United Kingdom, 2010. (cited on page 22)

Todo list