Derivations

Let A and B be K-algebras and ϕ an algebra hom then $D: A \to B$ is a derivation if it satifies

- $D(\lambda f + \mu g) = \lambda D(f) + D(g)\mu \quad f, g \in A \ \lambda, \mu \in K$
- $D(fg) = \phi(f) D(g) + \phi(g) D(f)$.

We denote the vec sp of derivations by $\mathrm{Der}(A, B, \phi)$. In the literature (B, ϕ) is normally identified with the corresponding A-modul. ($\mathrm{Der}(A, B, \phi)$ is also an B-modul.)

Examples

i) $A = \mathcal{C}^\infty(U)$, $B = \mathbb{R}$ for $p \in U$ $\phi = (\cdot)|_p : \mathcal{C}^\infty(U) \to \mathbb{R}$; $f \mapsto f(p)$

$$\frac{\partial}{\partial x^i}\Big|_p (\lambda f + \mu g) = \frac{\partial f}{\partial x^i}(p)\lambda + \frac{\partial g}{\partial x^i}(p)\mu$$

$$\frac{\partial}{\partial x^i}\Big|_p (fg) = f(p)\frac{\partial g}{\partial x^i}(p) + g(p)\frac{\partial f}{\partial x^i}(p)$$
$$= \phi(f)\, D(g) + \phi(g)\, D(f)$$

ii) $A = B = \mathcal{C}^\infty(U) \quad \phi = id$

$$D = \frac{\partial}{\partial x^i} : f \mapsto \frac{\partial f}{\partial x^i}$$

$$D(fg) = \left\{ p \mapsto f(p)\frac{\partial g}{\partial x^i}(p) + g(p)\frac{\partial f}{\partial x^i}(p) = \frac{\partial}{\partial x^i}\Big|_p (fg) \right\}$$

So now we can reformulate the following

Def. (Freitag derivation)
A Freitag derivation in a point p is a family of derivations $D_U \in \mathrm{Der}(\mathcal{C}^\infty(U), \mathbb{R}, (\cdot)|_p)$ that is compatible with restrictions, i.e.
$D_V(f|_V) = D_U(f)$ for $U \subset V$ & $f \in \mathcal{C}^\infty(U)$.
Let us denote the set of such Freitag derivations by $F\text{-}\mathrm{Der}(\mathcal{C}^\infty_{X,p}, \mathbb{R})$

Thm

$$F\text{-}\mathrm{Der}(\mathcal{C}^\infty_{X,p}, \mathbb{R}) = \mathrm{Der}(\mathcal{C}^\infty_{X,p}, \mathbb{R}, (\cdot)|_p)$$

Pf.:
"$\supset$" ~~Let~~ $D \in \mathrm{Der}(\mathcal{C}^\infty_{X,p}, \mathbb{R}, (\cdot)|_p)$

Define $\mathcal{D}_U(f) := D[f]$ for $f \in \mathcal{C}^\infty(U)$
$[f] = [f|_V]$ they coincide on V !

$$\Rightarrow \mathcal{D}_V(f|_V) = D[f|_V] = D[f] = \mathcal{D}_U(f)$$

$$\mathcal{D}_U(\lambda f + g\mu) = D[\lambda f + g\mu] = \lambda D[f] + \mu D[g]$$

$$\begin{aligned}\mathcal{D}_U(fg) &= D[fg] = D([f]\cdot[g]) \\ &= D[f]\cdot([g])|_p + ([f])|_p D[g] \\ &= g(p)\cdot \mathcal{D}_U(f) + f(p)\mathcal{D}_U(g) \quad \square\end{aligned}$$

"$\subset$" Let $(\mathcal{D}_U)_U \in F\text{-}\mathrm{Der}(\mathcal{C}^\infty_{X,p}, \mathbb{R})$
define $D[f] := \mathcal{D}_U(f)$ $f \in \mathcal{C}^\infty(U)$

$$f \sim g \Rightarrow f|_W = g|_W \Rightarrow \mathcal{D}_U(f) = \mathcal{D}_W(f) = \mathcal{D}_W(g) = \mathcal{D}_V(g)$$

$$\begin{aligned}D[fg] = \mathcal{D}_W(f\cdot g) &= f(p)\mathcal{D}_W(g) + g(p)\mathcal{D}_W(f) \\ &= f(p)D[g] + g(p)D[f] \quad \square\end{aligned}$$