## 5. The de-Rham complex

For a differentiable manifold the de-Rham complex is defined to be the sequence of maps

$$\cdots \longrightarrow A^{p-1}(X) \longrightarrow A^p(X) \longrightarrow A^{p+1}(X) \longrightarrow \cdots$$

We use the notation

$$\begin{aligned} C^p(X) &:= \text{kernel}(d : A^p(X) \longrightarrow A^{p+1}(X)), \\ B^p(X) &:= \text{image}(d : A^p(X) \longrightarrow A^{p+1}(X)). \end{aligned}$$

The elements of $C^p(X)$ are called *closed* ($d\omega = 0$) and the elements of $B^p(X)$ are called *exact*. They are of the form $d\omega'$. Because of $d \circ d = 0$ exact forms are closed. The converse is not always true and it is important to understand this. To measure the difference between exact and closed forms on introduces the de-Rham cohomology groups. (One should better say "de-Rham cohomology vector spaces", but this is unusual.) They are defined as factor space of $C^p(X)$ by the subspace $B^p(X)$.

$$H^p(X, \mathbb{R}) = C^p(X)/B^p(X).$$

The elements of $H^p(X, \mathbb{R})$ are classes of elements from $C^p(X)$. The class $[\omega]$ of an element $\omega \in C^p(X)$ consists of all elements of the form $\omega + d\alpha$, $\alpha \in A^{p-1}(X)$. The set of all classes can be made to a vector space in a natural way. The vector space structure is defined through the fact that the natural projection

$$C^p(X) \longrightarrow H^p(X), \quad \omega \longmapsto [\omega],$$

is a linear map. This linear map is surjective and its kernel is $B^p(X)$. Hence we see: The group $H^p(X, \mathbb{R})$ vanishes if and only of each closed $p$-form is exact.

Let $X$ be a differentiable manifold of dimension $n$. (This means that all charts land in $\mathbb{R}^n$.) Then of course $H^p(X, \mathbb{R}) = 0$ for $p > n$. Of course $H^p(X, \mathbb{R}) = 0$ for $p < 0$ is always true. Let's consider the case $p = 0$. Clearly $B^0(X) = 0$ since every form of degree $-1$ is zero. Hence $H^0(X, \mathbb{R}) = C^0(X)$. The space $C^0(X)$ consists of all functions with $df = 0$. Such functions are locally constant. If we assume that $X$ is connected then the are constant. Hence $H^0(X, \mathbb{R})$ for a connected differentiable manifold can be identified with $\mathbb{R}$.

**5.1 Remark.** *For each connected differentiable manifold one has*

$$H^0(X, \mathbb{R}) = \mathbb{R}.$$

A basic result is the

**5.2 Lemma of Poincaré.** *Let $U \subset \mathbb{R}^n$ be an open convex subset. Then*

$$H^p(U, \mathbb{R}) = 0 \quad \textit{for} \quad p > 0.$$

*Proof.* Let $\omega$ be a closed form. We decompose it as

$$\omega = \alpha + \beta \wedge dx_n,$$

where $\alpha$ doesn't contain any term with $dx_n$. We write

$$\beta = \sum f_a d_a$$

where $a$ are subsets of $\{1, \dots, n-1\}$ that do nor contain $n$. Integrating with respect to the last variable we find differentiable functions $F_a$ such that $\partial_n F_a = f_a$. Now the difference $\omega - d \sum_a F_a dx_a$ doesn't contain any term in which $dx_n$ occurs. Hence we can assume that in $\omega$ no term with $dx_n$ occurs. We write

$$\omega = \sum_a g_a dx_a,$$

where all $a$ are subsets of $\{1, \dots, n-1\}$. Now we use $d\omega = 0$. We obtain $\partial_n g_a = 0$. Hence $g_a$ do not depend on $x_n$. But now $\omega$ can be considered as differential form in one dimension less (on the image of $U$ with respect to the projection map that cancels the last variable) and an induction argument completes the proof. □

Next we want to give an important class of examples for a non-vanishing de-Rham cohomology groups. It rests on the theorem of Stokes and hence on integration of differential forms. We just recall the basic concept.

First one has to introduce the concept of orientation: A differentiable manifold $X$ is called *orientable*, if there exists a defining atlas $\mathcal{A}$ such that all chart transformations inside $\mathcal{A}$ have positive functional determinant everywhere. Two such atlases are called oriented equivalent if their union is oriented as well. An orientation of a differentiable manifold is given by an equivalence class of oriented equivalent atlases (consisting of differentiable charts with respect to the given differentiable structure on $X$). In this equivalence class there exists a unique maximal (oriented) atlas $\mathcal{A}^+$. The elements of this atlas are called the oriented differentiable charts on $X$.

Let now $X$ be of dimension $n$ and $\omega$ a top form $\omega \in A^n(X)$. We assume that $\omega$ has compact support. Then one can define

$$\int_X \omega.$$