Singular limit of the equations of magnetohydrodynamics in the presence of strong stratification

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Abstract: We consider a low Mach, Péclet, Froude and Alfvén number limit in the complete Navier-Stokes-Fourier system coupled with Maxwell’s equations for gases with large specific heat at constant volume. The target system is shown to be the anelastic Oberbeck-Boussinesq system coupled with Maxwell’s equations. The proof allows an intrinsic view into the process of separation of fast oscillating acoustic waves, governed by a Lighthill-type equation, from the equations describing the slow fluid flows.

keywords: anelastic approximation, magnetohydrodynamics

1 Introduction

In a number of situations, stars may be considered as compressible fluids (see [56, 57]), in which the matter behaves either as a perfect or completely degenerate gas. Moreover their dynamics is often controlled or influenced by intense magnetic fields coupled to high temperature radiation effects (see, for instance, [9, 10, 19], among others). Accordingly, their behavior at low Mach number regimes is expected to be described by the anelastic Oberbeck-Boussinesq system coupled with the Maxwell equations. The rigorous justification of this claim is the main goal of this paper.

For decades the standard Oberbeck-Boussinesq (OB) system has been the basic tool of low Mach number fluid mechanics in application to oceanography, engineering and meteorology. The anelastic version of the OB-equations introduced by Ogura & Phillips [48] and Lipps & Hemler [40] are similarly fundamental for the theoretical and numerical studies in convective and mesoscale atmospheric dynamics, see e.g. [1, 4, 16, 17, 18, 33, 37, 45, 50] as well as in the investigations of stellar convection and gaseous stars, see e.g. [31, 58].

From the point of view of mathematical modeling the anelastic OB-approximation has several advantages: firstly, it describes incompressible flow disregarding energetically insignificant fluctuations due to compressibility effects permitting at the same time variable density profiles. Secondly, they allow the pressure and the buoyancy force to be cast in a linear regime, while the advective terms remain bilinear. Thirdly, they filter out the acoustic wave propagation, which may be inconvenient in a lot of low Mach number flow simulations.

So far, in the mathematical (as well as physical) literature the OB-equations (anelastic or with constant density) have been derived from the complete Navier-Stokes-Fourier (NSF) system (or Euler if viscosity is neglected) either using various simplifying physically motivated assumptions as in [5], or by a scaling analysis and formal asymptotic low Mach number expansions [40, 48], or even from some already simplified models of compressible fluids, mostly as small thermal expansion limits in the models for isochoric motions, [20, 35]. The first rigorous limit from the complete Navier-Stokes-Fourier system to the anelastic approximation of the Oberbeck-Boussinesq equations has been obtained only recently in [46].

In the present paper we shall generalize the results of [46] taking into account the presence of the magnetic field.

Our approach leans on the concept of weak solutions to the complete NSF-system introduced in [14, 15] (see also [21, 38]). Similarly to the results [7, 13, 39, 54] (see also the survey
papers \[41, 42\] devoted to the barotropic NS-system without temperature, where the target problem has constant density, our results are based on the uniform bounds available in the framework of weak solutions defined on an arbitrarily large time interval.

To perform the singular limit, we shall benefit from various techniques introduced to treat similar problems without magnetic field, in particular \[22, 23, 25, 27, 39, 43\].

There is an alternative approach proposed in the pioneering paper by Klainerman & Majda \[36\] and elaborated e.g. in \[2, 3, 8, 11, 12, 32, 51, 52\], which is based on uniform estimates in Sobolev spaces of higher order. This approach imposes restrictions on the length of the associated existence time interval and/or on the initial data; these problems are not investigated in the present paper.

Our paper is organised as follows. The governing system, the constitutive relations and the target system are described in Section 1. After the definition of weak solutions both for the primitive and the target system in Section 2, we formulate the main theorem (Theorem 2.1).

The mathematical starting point of the paper is Section 3 introducing the dissipation inequality involving the Helmholtz function. Its fundamental coerciveness properties are formulated in Lemma 3.1. This lemma, in combination with the structural properties of the constitutive functions formulated in Paragraph 1.2, yields practically all a priori estimates available for this problem (see Lemma 3.2), except the refined pressure estimates, which are obtained later in Lemma 3.3 via the Bogovskii operator. A particular tool in the limiting process is the splitting of the phase space and of the physical domain into the residual and essential sets. The limit in the momentum equation, especially the recovering of the buoyancy force and the underlying asymptotic analysis require refined pressure estimates. It is at this point, where we shall require the constitutive law for the pressure to be “close” to a perfect gas, for small parameters of \(\rho \nu^{-3/2}\). At this stage, it is important to filter out the fast oscillating acoustic waves, which may arise due to the presence of singular coefficients in front of the pressure term, by projecting the momentum equation onto the divergence free vector fields. The convenient projection is a particular weighted Helmholtz projection introduced in equation (2.23). This projector applied to the momentum equation gives rise to a wave equation with non constant coefficients whose specific cancellation properties make it possible to pass to the limit in the convective term in a way reminiscent to \[39\]. The magnetic field does not change the wave operator in the wave equation. This is the reason why this part of the analysis is essentially the same as in \[46\].

The hypothesis of large specific heat at constant volume is exploited in Section 5 in the passage to the limit in the entropy identity. One of the particularities of this limiting process is the use of a general form of the celebrated Tartar’s div-curl-lemma \[53\] which replaces the standard Arzela-Ascoli type arguments. Due to the presence of the entropy production rate in the entropy identity, which is merely a measure, the use of such a specific compensated compactness tool seems to be inevitable.

1.1 Governing equations

We deal with a viscous compressible conducting fluid in the presence of a magnetic field. The motion of the fluid, occupying the bounded smooth domain \(\Omega \subset \mathbb{R}^3\), takes place in an arbitrarily large time interval \(I = (0, T)\), where \(T > 0\) is a given time. In such situations it is customary to use the magnetohydrodynamical approximation of Maxwell’s equations and the thermodynamic field equations, which we will describe now (for more details cf. \[34\]).

Let us start with the equations for the electromagnetic fields. The magnetic induction \(\mathbf{B}\)
is governed by Faraday’s law and the conservation of magnetic flux, which read
\[
\partial_t B + \text{curl}_x E = 0, \\
\text{div}_x B = 0.
\]
(1.1)

Neglecting the rate of change of the electric field \(E\) in Ampère’s circuital law we obtain Ampère’s law
\[
\mu_0 J = \text{curl}_x B,
\]
(1.2)

where the positive constant \(\mu_0\) is the permeability of free space\(^1\) and where the constitutive equation for the electric current \(J\) is given by Ohm’s law
\[
J = \frac{1}{\mu_0 \lambda} \mathcal{E}.
\]
(1.3)

Here \(\lambda\) denotes the magnetic diffusivity of the fluid, for which we will later specify a constitutive relation depending on the density \(\varrho \geq 0\), the temperature \(\vartheta > 0\), and the magnetic induction \(B\). The electromotive intensity \(\mathcal{E}\) is related to the electric field \(E\), the fluid velocity \(u\) and the magnetic induction \(B\) through
\[
\mathcal{E} = E + u \times B.
\]
(1.4)

Using (1.2)–(1.4) we can re-write (1.1) as
\[
\partial_t B + \text{curl}_x (B \times u) + \text{curl}_x (\lambda \text{curl}_x B) = 0.
\]
(1.5)

This equation has to be complemented by appropriate boundary and initial conditions. We assume that\(^2\)
\[
B \cdot n = 0, \quad E \times n = 0 \quad \text{on } (0, T) \times \partial \Omega, \\
B(0) = B_0 \quad \text{in } \Omega,
\]
(1.6)

where \(n\) denotes the outer normal to the boundary \(\partial \Omega\) and \(B_0\) is given satisfying \(\text{div}_x B_0 = 0\) in \(\Omega\), \(B_0 \cdot n = 0\) on \(\partial \Omega\). Note that in this situation implicitly equation (1.1) is contained in (1.5) if the solution is smooth enough.

Let us now list the thermomechanical balance laws. The conservation of mass, linear momentum and internal energy, respectively, are given by
\[
\partial_t \varrho + \text{div}_x (\varrho u) = 0, \\
\partial_t (\varrho u) + \text{div}_x (\varrho u \otimes u) - \text{div}_x S + \frac{1}{M_a^2} \nabla_x p = \frac{1}{F_t^2} \varrho \nabla_x F + \frac{1}{A_f^2} J \times B, \\
\partial_t (\varrho e) + \text{div}_x (\varrho eu) + \frac{1}{P_e^2} \text{div}_x q = M_a^2 S : \nabla_x u + \frac{M_a^2}{A_l^2} J \cdot \mathcal{E} - p \text{div}_x u,
\]
(1.8)

where \(S\) is the viscous stress tensor, \(p\) is the pressure, \(F = F(x)\) is a time independent potential generating the force \(\nabla_x F\), \(e\) is the specific internal energy, and \(q\) denotes the heat flux. We will later specify constitutive relations for \(S, p, e,\) and \(q\) depending on the density \(\varrho\), the temperature \(\vartheta\), the temperature gradient \(\nabla_x \vartheta\), the symmetric part of the velocity gradient

---

\(^1\)The constant \(\mu_0\) should not be confused with the shear viscosities \(\mu\) and \(\mu'\) defined in (1.25) and (1.44).

\(^2\)This corresponds to the assumption that the boundary \(\partial \Omega\) is a perfect conductor.
Rigorous derivation of the anelastic approximation to OBA

$\mathbf{Du} := \frac{1}{2}(\nabla_x \mathbf{u} + (\nabla_x \mathbf{u})^\top)$, and the magnetic induction $\mathbf{B}$. The non-dimensional numbers $\text{Ma}$, $\text{Fr}$, $\text{Al}$, and $\text{Pe}$ denote the Mach, Froude, Alfvén, and Péclet number, respectively. The system (1.8)–(1.10) has to be complemented by appropriate boundary and initial conditions. We assume that

$$
\begin{align*}
\mathbf{u} \cdot \mathbf{n} &= 0, & (\mathbf{Sn}) \times \mathbf{n} &= 0, & \mathbf{q} \cdot \mathbf{n} &= 0 & \text{on } (0, T) \times \partial \Omega, \\
\mathbf{u}(0) &= \mathbf{u}_0, & \varrho(0) &= \varrho_0, & \vartheta(0) &= \vartheta_0 & \text{in } \Omega,
\end{align*}
$$

where $\mathbf{u}_0$, $\varrho_0$, and $\vartheta_0$ are given.

In a standard way one can derive the total energy balance from the system (1.8)–(1.10). In fact, multiplying (1.9) by $\text{Ma}^2 \mathbf{u}$, (1.1) by $\frac{\text{Ma}^2 \text{Al}^2}{\mu_0}$, adding them to (1.10) and using (1.8) results in

$$
\begin{align*}
\partial_t \left( \frac{\text{Ma}^2}{2} \varrho |\mathbf{u}|^2 + \varrho e + \frac{1}{2\mu_0} \frac{\text{Ma}^2}{\text{Al}^2} |\mathbf{B}|^2 \right) + \\
+ \text{div}_x \left( \frac{\text{Ma}^2}{2} \varrho |\mathbf{u}|^2 \mathbf{u} + \varrho e \mathbf{u} + \frac{1}{2\mu_0} \frac{\text{Ma}^2}{\text{Al}^2} \mathbf{E} \times \mathbf{B} - \text{Ma}^2 \mathbf{S} \mathbf{u} + p \mathbf{u} + \frac{1}{\text{Pe}^2} \mathbf{q} \right) = \\
= \frac{\text{Ma}^2}{\text{Fr}^2} \varrho \mathbf{u} \cdot \nabla F.
\end{align*}
$$

From the boundary conditions and (1.8) we deduce for smooth solutions the following total energy balance of the system

$$
\frac{d}{dt} \int_\Omega \left( \frac{\text{Ma}^2}{2} \varrho |\mathbf{u}|^2 + \varrho e + \frac{1}{2\mu_0} \frac{\text{Ma}^2}{\text{Al}^2} |\mathbf{B}|^2 - \varrho F \right) dx = 0.
$$

Let us come back to the conservation of internal energy. For the variational formulation used in this paper it is more convenient to replace (1.10) by the equivalent (provided the solutions are smooth enough) entropy balance. In accordance with the second law of thermodynamics, the specific entropy $s$ is determined up to an additive constant from the constitutive laws for $p$ and $e$ through the Gibbs law

$$
\vartheta ds = de - \frac{p}{\varrho^2} d\varrho.
$$

Standard computations now yield that the entropy balance reads

$$
\begin{align*}
\partial_t (\varrho s) + \text{div}_x (\varrho s \mathbf{u}) + \frac{1}{\text{Pe}} \text{div}_x \left( \frac{\mathbf{q}}{\vartheta} \right) = \sigma,
\end{align*}
$$

where the entropy production rate $\sigma$ satisfies

$$
\sigma \geq \frac{\text{Ma}^2}{\text{Fr}^2} \frac{\nabla_x \mathbf{u}}{\vartheta} + \frac{\text{Ma}^2}{\mu_0} \frac{\lambda}{\text{Al}^2} \frac{|\text{curl}_x \mathbf{B}|^2}{\vartheta} - \frac{1}{\text{Pe}} \frac{\mathbf{q} \cdot \nabla \vartheta}{\vartheta^2}.
$$

Here we used that in view of (1.2) and (1.3) we have $\mathbf{J} \cdot \mathbf{E} = \frac{\lambda}{\mu_0} |\text{curl}_x \mathbf{B}|^2$. Note that for smooth solutions there holds equality in (1.17).

### 1.2 Constitutive relations and Target system

We shall study regimes where the Mach number $\text{Ma}$, the Froude number $\text{Fr}$, and the Alfvén number $\text{Al}$ are small of order $\varepsilon$, while the Péclet number is small of order $\varepsilon^2$. We consider a family of viscous compressible conducting fluids $\{\mathcal{F}_\varepsilon\}_{\varepsilon > 0}$ characterised by specific constitutive
laws for the pressure $p_\varepsilon(\varrho, \vartheta)$, the internal energy $e_\varepsilon(\varrho, \vartheta)$, and consequently the entropy $s_\varepsilon(\varrho, \vartheta)$, parametrised through a (small) positive parameter $\varepsilon$, whereas the constitutive laws for the viscous stress tensor $\mathcal{S}(\vartheta, \mathbf{Du}, |B|)$, the heat flux $\mathbf{q}(\varrho, \vartheta, \nabla \vartheta, |B|)$, and the magnetic diffusivity $\lambda(\varrho, \vartheta, |B|)$ do not depend on $\varepsilon$. Each of these fluids evolves in the fixed domain $\Omega \subset \mathbb{R}^3$ for an arbitrary large time interval $(0, T)$ according to

\begin{align}
\partial_t \mathbf{B} + \mathbf{curl}_\varrho (\mathbf{B} \times \mathbf{u}) + \mathbf{curl}_\varrho (\lambda \mathbf{curl}_\varrho \mathbf{B}) &= 0, \\
\partial_t \mathbf{g} + \text{div}_\varrho (\mathbf{g} \mathbf{u}) &= 0, \\
\partial_t (\varrho \mathbf{u}) + \text{div}_\varrho (\varrho \mathbf{u} \otimes \mathbf{u}) - \text{div}_\varrho \mathbf{S} + \frac{1}{\varepsilon^2} \text{curl}_\varrho \mathbf{p}_\varepsilon &= \frac{1}{\varepsilon^2} \varrho \nabla \vartheta F + \frac{1}{\varepsilon^2} \frac{1}{\mu_0} (\mathbf{curl}_\varrho \mathbf{B}) \times \mathbf{B}, \\
\partial_t (\varrho s_\varepsilon) + \text{div}_\varrho (\varrho s_\varepsilon \mathbf{u}) + \frac{1}{\varepsilon^2} \text{div}_\varrho \left( \frac{\mathbf{q} \cdot \nabla \vartheta}{\vartheta} \right) &= \sigma_\varepsilon, \\
\varepsilon^2 \frac{\mathbf{S} : \nabla \mathbf{u}}{\vartheta} + \frac{\lambda}{\mu_0} \frac{|\mathbf{curl}_\varrho \mathbf{B}|^2}{\vartheta} - \frac{1}{\varepsilon^2} \frac{\mathbf{q} \cdot \nabla \vartheta}{\vartheta^2} &\leq \sigma_\varepsilon, \\
\vartheta \, d s_\varepsilon &= d e_\varepsilon - \frac{p_\varepsilon}{\vartheta^2} \, d \varrho, \\
\frac{d}{dt} \int_{\Omega} \left( \frac{\varepsilon^2}{2} \varrho |\mathbf{u}|^2 + \varrho e + \frac{1}{2\mu_0} |\mathbf{B}|^2 - \varrho F \right) \, dx &= 0
\end{align}

and is subject to the boundary conditions (1.6), (1.11), as well as the initial conditions (1.7), (1.12) with given data that may depend on $\varepsilon$. The viscous stress tensor $\mathcal{S}$ shall satisfy Newton’s rheological law

\begin{equation}
\mathcal{S}(\vartheta, \mathbf{Du}, |B|) = 2\mu(\vartheta, |B|) \left( \mathbf{Du} - \frac{1}{3} \text{div}_\varrho \mathbf{u} \mathbf{I} \right) + \eta(\vartheta, |B|) \text{div}_\varrho \mathbf{u} \mathbf{I},
\end{equation}

where the bulk viscosity $\eta$ and the shear viscosity $\mu$ are $C^1([0, \infty) \times [0, \infty))$-functions which are globally Lipschitz on $[0, \infty) \times [0, \infty)$ and satisfy

\begin{equation}
c_1(1 + \vartheta^3) \leq \mu(\vartheta, |B|) \leq c_2(1 + \vartheta^3), \quad 0 \leq \eta(\vartheta, |B|) \leq c_2(1 + \vartheta^3).
\end{equation}

The heat flux $\mathbf{q}$ is given through Fourier’s law

\begin{equation}
\mathbf{q} = -\kappa(\varrho, \vartheta, |B|) \nabla \vartheta,
\end{equation}

where the heat conductivity $\kappa$ is a $C^1([0, \infty) \times [0, \infty) \times [0, \infty))$-function which obeys

\begin{equation}
c_1(1 + \vartheta^3) \leq \kappa(\varrho, \vartheta, |B|) \leq c_2(1 + \vartheta^3).
\end{equation}

The reasonability of such hypothesis is discussed, e.g., in [39]. We assume that the magnetic diffusivity $\lambda$ is a $C^1([0, \infty) \times [0, \infty) \times [0, \infty))$-function which obeys

\begin{equation}
c_1(1 + \vartheta^3) \leq \lambda(\varrho, \vartheta, |B|) \leq c_2(1 + \vartheta^3),
\end{equation}

where $\vartheta \in [1, 17/6]$. We assume that the equations of state for pressure $p_\varepsilon$, internal energy $e_\varepsilon$, and the specific entropy $s_\varepsilon$, respectively, can be split into parts referring to molecular and thermal interactions as well as radiation, which are denoted by the respective subscripts $M$, $T$, and $R$. More precisely,

\begin{align}
p_\varepsilon(\varrho, \vartheta) &= p_{M,\varepsilon}(\varrho, \vartheta) + p_{R,\varepsilon}(\vartheta), \\
e_\varepsilon(\varrho, \vartheta) &= e_{M,\varepsilon}(\varrho, \vartheta) + e_{T,\varepsilon}(\vartheta) + e_{R,\varepsilon}(\varrho, \vartheta), \\
s_\varepsilon(\varrho, \vartheta) &= s_{M,\varepsilon}(\varrho, \vartheta) + s_{T,\varepsilon}(\vartheta) + s_{R,\varepsilon}(\varrho, \vartheta),
\end{align}

\footnote{Here we dropped for simplicity the dependence of $\varrho$, $\mathbf{u}$, $\vartheta$, $\mathbf{B}$, and $\mathbf{E}$ on $\varepsilon$.}
Rigorous derivation of the anelastic approximation to OBA

where for \( \alpha \in (2, 3) \) and \( \beta \in [1, \frac{5}{2}] \)

\[
p_{M,\varepsilon} = \frac{\varepsilon}{\varepsilon^2} P\left(\varepsilon^\alpha \frac{\vartheta}{\vartheta^2}\right), \quad p_{R,\varepsilon} = \frac{\varepsilon}{3} \vartheta^4, \quad (1.33)
\]

\[
e_{M,\varepsilon} = \frac{3}{2} \frac{\varepsilon}{\varepsilon^2} P\left(\varepsilon^\alpha \frac{\vartheta}{\vartheta^2}\right), \quad e_{T,\varepsilon} = \frac{1}{\varepsilon^2} \vartheta^3, \quad e_{R,\varepsilon} = \frac{\vartheta^4}{\vartheta}. \quad (1.34)
\]

Note that

\[
p_{M,\varepsilon} = \frac{2}{3} \vartheta e_{M,\varepsilon}, \quad (1.35)
\]

which is a universal relation to be satisfied by any monoatomic gas (see [19]). The radiation pressure obeys the Boltzmann law (see [49]). In the internal energy the term \( e_{T,\varepsilon} \) keeps the specific heat at constant volume large as \( \varepsilon \to 0 \).

Each component of the internal energy and pressure satisfies the second law of thermodynamics and defines through the Gibbs relation (1.15) the specific entropy as

\[
s_{M,\varepsilon}(\vartheta, \vartheta) = S(\varepsilon^\alpha \vartheta \vartheta^{-\frac{3}{2}}) - S(\varepsilon^\alpha), \quad (1.36)
\]

\[
s_{T,\varepsilon}(\vartheta, \vartheta) = \begin{cases} \frac{\beta}{\varepsilon^2} \log \vartheta^{-1} & \text{if } \beta > 1 \\ \frac{1}{\varepsilon^2} \vartheta^{-1} & \text{if } \beta = 1 \end{cases}, \quad s_{R,\varepsilon} = \frac{4}{3} \frac{\vartheta^3}{\vartheta}, \quad (1.37)
\]

where

\[
S'(Y) = -\frac{3}{2} \frac{Y^2}{3} P(Y) - P'(Y)Y \quad \text{for any } Y > 0. \quad (1.38)
\]

The function \( P \) appearing in (1.34) and (1.38) satisfies

\[
P'(Z) > 0 \quad \text{and} \quad \frac{1}{Z} \left(\frac{5}{3} P(Z) - ZP'(Z)\right) \geq 0 \quad \text{for all } Z > 0, \quad (1.39)
\]

where \( P \in C^2[0, \infty) \) is such that

\[
P(0) = 0, \quad P'(0) =: p_0 > 0, \quad (1.40)
\]

\[
\sup_{Z > 0} \frac{1}{Z} \left(\frac{5}{3} P(Z) - ZP'(Z)\right) < \infty, \quad \lim_{Z \to \infty} P(Z)Z^{-\frac{5}{3}} =: p_\infty > 0. \quad (1.41)
\]

Assumptions (1.39) are simply reformulations of the thermodynamic stability hypotheses \( \partial_\vartheta p_{M,\varepsilon} > 0 \) and \( \partial_\vartheta e_{M,\varepsilon} > 0 \) on the phase space \((0, \infty)^2\) for any fluid \( \mathcal{F}_\varepsilon \). Hypothesis (1.41) means that \( \partial_\vartheta e_{M,\varepsilon} \) of any fluid is uniformly bounded in the phase space, while the second one says that the gas behaves like a Fermi gas for large values of the parameter \( \vartheta/\vartheta^{3/2} \). Finally, the scaling in (1.33) by \( \varepsilon^\alpha \) together with assumption (1.40) enforces the behaviour of the fluid as a perfect gas, i.e., \( p_{M,\varepsilon} = p_0 \vartheta \vartheta \), for small values of the parameter \( \vartheta/\vartheta^{3/2} \). The reader interested in more information on the physical background of (1.30)–(1.38) may consult the monographs [44, 49].

Our main aim in this paper is to perform the asymptotic limit in (1.11)–(1.15) for \( \varepsilon \to 0 \) and to identify the limit (target) problem. More specifically, we show in Theorem 2.1 that for a family \( \{B_{\varepsilon}, \vartheta_{\varepsilon}, u_{\varepsilon}, \vartheta_{\varepsilon}\}_{\varepsilon > 0} \) of (weak) solutions to the complete system endowed with so-called semi prepared initial data (see Paragraph 2.1 below) it holds

\[
B_{\varepsilon} \to 0, \quad \vartheta_{\varepsilon} \to \vartheta, \quad u_{\varepsilon} \to U, \quad \vartheta_{\varepsilon} \to \vartheta, \quad \text{and} \quad \frac{B_{\varepsilon}}{\varepsilon} \to B, \quad \frac{\vartheta_{\varepsilon} - \vartheta}{\varepsilon^2} \to \Theta \quad \text{as} \quad \varepsilon \to 0
\]
in the sense to be specified later, where \( \overline{\vartheta} > 0 \) is fixed by the initial data, and where \( \tilde{\vartheta} \) is the static state corresponding to the constitutive pressure law for the perfect gas

\[
p_0 \overline{\vartheta} \nabla_x \tilde{\vartheta} = \tilde{\vartheta} \nabla_x F \quad \text{in } \Omega, \quad \int_\Omega \tilde{\vartheta} \, dx = M_0 > 0,
\]

and \( p_0 \) is given in (1.40). The fields \( \tilde{\vartheta}, U, \Theta, \) and \( B \) solve the problem\(^4\)

\[
\partial_t B + \nabla_x (B \times U) + \nabla_x (\overline{\vartheta} \nabla_x B) = 0,
\]

\[
\partial_t (\tilde{\vartheta} U) + \nabla_x (\tilde{\vartheta} U \otimes U) + \tilde{\vartheta} \nabla_x \Pi - \overline{\mu} \Delta U - (\overline{\eta} + \frac{1}{3} \overline{\mu}) \nabla \nabla_x U = \frac{\tilde{\vartheta}}{v} \Theta \nabla_x F + \frac{1}{\mu_0} (\nabla_x B) \times B,
\]

\[
\nabla_x (\tilde{\vartheta} U) = 0,
\]

\[
A (\partial_t (\Theta U) + \nabla_x (\tilde{\vartheta} \nabla_x U)) - \frac{p_0}{\overline{\vartheta}} \nabla_x (\tilde{\vartheta} F U) - \frac{1}{\overline{\vartheta}} \nabla \nabla_x (\overline{\vartheta} \nabla_x U) = 0,
\]

where \( \overline{\lambda} = \lambda(\tilde{\vartheta}, \overline{\vartheta}, 0), \overline{\mu} = \mu(\overline{\vartheta}, 0), \overline{\eta} = \eta(\overline{\vartheta}, 0), \overline{\kappa} = \kappa(\tilde{\vartheta}, \overline{\vartheta}, 0), \) and \( A = \beta \tilde{\vartheta}^{3-2} \), with

\[
\begin{align*}
B \cdot n &= 0, & & \quad \text{on } (0, T) \times \partial \Omega, \\
E \times n &= 0, & & \text{on } (0, T) \times \partial \Omega, \\
U \cdot n &= 0, & & \text{on } (0, T) \times \partial \Omega, \\
\nabla \Theta \cdot n &= 0, & & \text{on } (0, T) \times \partial \Omega,
\end{align*}
\]

as boundary conditions. Here \( E = \overline{\lambda} \nabla_x B + B \times U \).

### 2 Constitutive relations for the governing system

#### 2.1 Weak solutions to the governing system

**Definition 2.1** Let \( \Omega \) be a bounded simply connected domain in \( \mathbb{R}^3 \) with boundary \( \partial \Omega \in C^{2,\nu} \), \( \nu \in (0, 1), T > 0, \) and \( \varepsilon > 0 \) be given. We shall say that a quadruple \( B_\varepsilon, \vartheta_\varepsilon, u_\varepsilon, \varphi_\varepsilon \) is a weak solution to the system (1.5), (1.8), (1.9), (1.15)–(1.17), supplemented with boundary conditions (1.6), (1.11) and initial conditions

\[
\begin{align*}
B_\varepsilon(0) &= B_{0,\varepsilon}, & & \quad \vartheta_\varepsilon(0) = \vartheta_{0,\varepsilon}, & & \quad u_\varepsilon(0) = u_{0,\varepsilon}, & & \quad \varphi_\varepsilon(0) = \varphi_{0,\varepsilon}, \quad \text{in } \Omega, \\
\nabla_x B_\varepsilon(0) &= 0 \quad \text{in } \Omega, & & \quad B_{0,\varepsilon} \cdot n = 0 \quad \text{on } \partial \Omega, & & \quad \text{if, firstly, the functions have the regularity}\(^5\)
\end{align*}
\]

where \( \nabla_x B_{0,\varepsilon} = 0 \) in \( \Omega \), \( B_{0,\varepsilon} \cdot n = 0 \) on \( \partial \Omega \), if, firstly, the functions have the regularity\(^5\)

\[
\begin{align*}
\begin{cases}
B_\varepsilon &\in L^\infty(0, T; L^2(\Omega)^3) \cap L^2(0, T; W^{1,2}(\Omega)^3), \\
\vartheta_\varepsilon &\in L^\infty(0, T; L^{\frac{2}{1-\varepsilon}}(\Omega)), \\
u_\varepsilon &\in L^\infty(0, T; L^3(\Omega)), \\
\varphi_\varepsilon &\geq 0, \\
\dot{\varphi}_\varepsilon &> 0 \quad \text{a.e. in } (0, T) \times \Omega,
\end{cases}
\end{align*}
\]

and secondly, the following integral identities (i)–(v) hold:

\(^4\)Recall that \( \mu_0 \) is the permeability of free space.

\(^5\)Due to the assumptions on \( B_{0,\varepsilon} \), one can deduce from (2.6) that \( \nabla_x B_{\varepsilon}(t) = 0 \) in \( \Omega \) and \( B_{\varepsilon}(t) \cdot n = 0 \) on \( \partial \Omega \) for a.e. \( t \in (0, T) \). This together with \( B_\varepsilon \in L^\infty(0, T; L^2(\Omega)^3) \), \( \nabla_x B_\varepsilon \in L^2(0, T; L^2(\Omega)^3) \), which can be deduced from (2.3), yields the regularity of \( B_\varepsilon \) in (2.2), since under these circumstances \( \|\cdot\|_{L^2(\Omega)^3} + \|\nabla_x \cdot\|_{L^2(\Omega)} + \|\nabla_x \nabla_x \|_{L^2(\Omega)^3} \) is an equivalent norm on \( W^{1,2}(\Omega)^3 \) (cf. [30] Section I.3.2).
(i) Magnetohydrodynamical equation:

\[
\int_0^T \int_\Omega \left( -B_\varepsilon \cdot \partial_t \varphi + (B_\varepsilon \times u_\varepsilon) \cdot \text{curl}_x \varphi + \lambda (\varrho_\varepsilon, \vartheta_\varepsilon, |B_\varepsilon|) \text{curl}_x B_\varepsilon \cdot \text{curl}_x \varphi \right) \, dx \, dt = \\
= \int_\Omega B_{0,\varepsilon} \cdot \varphi(0) \, dx,
\]

for any test function \( \varphi \in C^\infty_c([0, T) \times \overline{\Omega})^3 \);

(ii) Renormalised equation of continuity:

\[
\int_0^T \int_\Omega \varrho_\varepsilon B(\varrho_\varepsilon)(\partial_t \varphi + u_\varepsilon \cdot \nabla \varphi) \, dx \, dt = \\
= \int_0^T \int_\Omega b(\varrho_\varepsilon) \text{div}_x u_\varepsilon \varphi \, dx \, dt - \int_\Omega \varrho_{0,\varepsilon} B(\varrho_{0,\varepsilon}) \varphi(0) \, dx
\]

for any test function \( \varphi \in C^\infty_c([0, T) \times \overline{\Omega}) \), and any \( b \in C([0, \infty) \cap L^\infty([0, \infty)) \), \( B(\varrho) = B(1) + \int_1^\varrho \frac{b(s)}{s^2} \, ds \);

(iii) Momentum equation:

\[
\int_0^T \int_\Omega \varrho_\varepsilon u_\varepsilon \cdot \partial_t \varphi + \varrho_\varepsilon u_\varepsilon \times u_\varepsilon : \nabla \varphi + \frac{1}{\varepsilon^2} \varrho_\varepsilon \nabla \cdot \text{curl}_x \varphi \, dx \, dt = \\
= \int_0^T \int_\Omega \text{S}(\nabla(\varrho_\varepsilon, D u_\varepsilon, |B_\varepsilon|)) : \nabla \varphi - \frac{1}{\varepsilon^2} \varrho_\varepsilon \nabla \cdot F \cdot \varphi - \frac{1}{\mu_0 \varepsilon^2} ((\text{curl}_x B_\varepsilon) \times B_\varepsilon) \cdot \varphi \, dx \, dt
\]

for any test function \( \varphi \in C^\infty_c([0, T) \times \overline{\Omega})^3 \), \( \cdot n = 0 \) on \( \partial \Omega \), with \( \text{S} \) given by (1.25);

(iv) Entropy balance equation: For any \( \varphi \in C^\infty_c([0, T) \times \overline{\Omega}) \)

\[
\int_0^T \int_\Omega \varrho_\varepsilon s_\varepsilon(\varrho_\varepsilon, \vartheta_\varepsilon)(\partial_t \varphi + u_\varepsilon \cdot \nabla \varphi) + \frac{1}{\varepsilon^2} \varrho_\varepsilon \left( \text{curl}_x B_\varepsilon \right) \cdot \nabla \varphi \, dx \, dt + \\
+ \langle \sigma_\varepsilon; \varphi \rangle_{\mathcal{M}(\varepsilon)} | \mathcal{M}(\varepsilon) | [0, T] \times \overline{\Omega} = - \int_\Omega \varrho_{0,\varepsilon} s_\varepsilon(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) \varphi(0) \, dx,
\]

where \( \text{S} \) is given by (1.25), and \( \sigma_\varepsilon \in (C([0, T] \times \overline{\Omega}))^* \) is a non-negative measure satisfying\(^6\)

\[
\sigma_\varepsilon \geq \frac{\varepsilon^2}{2} \text{S}(\nabla(\vartheta_\varepsilon, D u_\varepsilon, |B_\varepsilon|)) : \nabla u_\varepsilon + \frac{\lambda}{\mu_0} \left| \text{curl}_x B_\varepsilon \right|^2 - \frac{1}{\varepsilon^2} \varrho_\varepsilon \left( \text{curl}_x B_\varepsilon \right) \cdot \nabla \varphi \, dx \, dt.
\]

(v) Total energy balance: For a.a. \( \tau \in (0, T) \)

\[
\int_\Omega \left( \frac{\varepsilon^2}{2} \varrho_\varepsilon |u_\varepsilon|^2 + \varrho_\varepsilon (\varrho_\varepsilon, \vartheta_\varepsilon) + \frac{1}{2 \mu_0} |B_\varepsilon|^2 - \varrho_\varepsilon F \right) (\tau) \, dx = \\
= \int_\Omega \left( \frac{\varepsilon^2}{2} \varrho_{0,\varepsilon} |u_{0,\varepsilon}|^2 + \varrho_{0,\varepsilon} (\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) + \frac{1}{2 \mu_0} |B_{0,\varepsilon}|^2 - \varrho_{0,\varepsilon} F \right) \, dx.
\]

\(^6\)We identify the space \((C([0, T] \times \overline{\Omega}))^*\) via the Riesz theorem with the space of Radon measures \(\mathcal{M}([0, T] \times \overline{\Omega})\).
This definition contains an implicit requirement that all quantities appearing in the integral formulations (i)–(v) are at least $L^1$–integrable. We also notice that due to (2.3)–(2.6) any weak solution has supplementary regularity, namely

$$\varrho \varepsilon \in \mathcal{C}^{\text{weak}}([0,T]; L^5(\Omega)) \cap \mathcal{C}([0,T]; L^1(\Omega)),$$

$$\varrho \varepsilon \mathbf{u} \in \mathcal{C}^{\text{weak}}([0,T]; L^5(\Omega)^3),$$

$$\mathbf{B} \varepsilon \in \mathcal{C}^{\text{weak}}([0,T]; L^2(\Omega)^3).$$

(2.10)

It is well known that this definition of weak solutions is reasonable in the sense that any weak solution becomes a classical one and solves (1.18)–(1.23) (with equality in (1.22)) provided it is regular enough with strictly positive temperature and density.

A slight modification of the proof of [14, Theorem 3.1] (cf. [24, Theorem 3.1] for the system without the magnetohydrodynamical equations) shows that under appropriate assumptions on the data (cf. Theorem 2.1) there exists weak solutions of the system (1.18)–(1.23). Thus our investigation makes definitely sense.

2.2 Static states

For our investigation static states play an important role. Static states are time independent solutions of system (1.18)–(1.23) with vanishing velocity field $\mathbf{u}$. One easily derives from our assumptions on the constitutive relations and (2.3) (choosing $\varphi = B \varepsilon$), (2.7) (choosing $\varphi = 1$), neglecting terms coming from the time derivatives, that for a static state we necessarily obtain $B \varepsilon = 0$, and $\varrho \varepsilon = \vartheta \varepsilon$ = const.. Moreover we assume that the constants $\vartheta \varepsilon$ do not depend on $\varepsilon$ and that their common value is denoted by $\overline{\vartheta}$. Accordingly, the density $\varrho = \tilde{\varrho} \varepsilon$ must satisfy

$$\varrho \geq 0, \quad \nabla_x p \varrho(\varrho, \overline{\vartheta}) = \varrho \nabla_x F \quad \text{in } \Omega, \quad M_0 = \int_{\Omega} \varrho \; dx.$$  

(2.11)

Note that, in general, any static solution $\varrho$ may and indeed does depend on $\varepsilon$. If one compares this solution with the solution $\tilde{\varrho} \varepsilon = \exp(F/(p_0 \overline{\vartheta}) + c M_0)$ of the linearized static problem (1.42), one obtains (see e.g. [46]):

**Lemma 2.1** Let $\Omega$ be a bounded Lipschitz domain and $F \in W^{1,\infty}(\Omega)$. Then problem (2.11), where $p \varepsilon$ satisfies hypotheses (1.30), (1.39)–(1.41), admits a unique solution $\tilde{\varrho} \varepsilon$ in the class $W^{1,\infty}(\Omega)$ with $\inf_{x \in \Omega} \tilde{\varrho} \varepsilon(x) > 0$. Besides, there exists $c > 0$ independent of $\varepsilon$ such that

$$\sup_{x \in \Omega} |\tilde{\varrho} \varepsilon(x) - \tilde{\varrho}(x)| \leq c \varepsilon^\alpha.$$ 

(2.12)

2.3 Main results

For the limiting process $\varepsilon \to 0$ we prescribe initial data in the form

$$B \varepsilon(0) = B_{0,\varepsilon} = \varepsilon B_{0,\varepsilon}^{(1)}, \quad \varrho(0) = \varrho_{0,\varepsilon} = \bar{\varrho} + \varrho \varrho_{0,\varepsilon},$$

$$\mathbf{u}(0) = \mathbf{u}_{0,\varepsilon}, \quad \vartheta(0) = \vartheta_{0,\varepsilon} = \bar{\vartheta} + \vartheta \vartheta_{0,\varepsilon},$$

(2.13)

where $\vartheta \varepsilon$ solves (1.42), $\overline{\vartheta}$ is the equilibrium temperature defined in the previous section, and

$$\int_{\Omega} \varrho_{0,\varepsilon}^{(1)} \; dx = 0, \quad \int_{\Omega} B_{0,\varepsilon}^{(1)} \; dx = 0,$$

$$\text{div}_x B_{0,\varepsilon}^{(1)} = 0 \quad \text{in } \Omega, \quad B_{0,\varepsilon}^{(1)} \cdot n \quad \text{on } \partial \Omega = 0.$$  

(2.14)
We shall also assume that
\[
\left\{ \begin{array}{l}
\{B_{0\epsilon}\}_{\epsilon>0}, \quad \{\vartheta_{0\epsilon}\}_{\epsilon>0} \quad \text{are bounded in } L^\infty(\Omega), \\
\{B_{1\epsilon}\}_{\epsilon>0}, \quad \{u_{0\epsilon}\}_{\epsilon>0} \quad \text{are bounded in } L^\infty(\Omega)^3.
\end{array} \right.
\] (2.15)

Such data are, according to the standard terminology, neither well prepared nor ill prepared.

We shall call them semi prepared initial data.

**Definition 2.2** Let \( \bar{\varrho}, A, \bar{\mu} > 0, \bar{\eta} \geq 0 \) be given constants, and \( \bar{\pi}, \bar{\lambda} > 0 \) be given \( W^{1,\infty}(\Omega) \)-functions. We say that the quadruple \( B, \bar{\varrho}, U, \Theta \) is a weak solution of the anelastic approximation of the magnetohydrodynamical OB-equations, i.e., equations (1.42)–(1.46), endowed with boundary conditions (1.47) and initial conditions

\[
B(0) = B_0, \quad U(0) = U_0, \quad \Theta(0) = \Theta_0,
\] (2.16)

where \( \text{div}_x B_0 = 0 \) in \( \Omega \), \( B_0 \cdot \mathbf{n} = 0 \) on \( \partial \Omega \), if (i)–(v) below hold and the functions belong to the regularity class:

\[
\left\{ \begin{array}{l}
B \in L^\infty(0,T;L^2(\Omega)^3) \cap L^2(0,T;W^{1,2}(\Omega))^3, \\
\bar{\varrho} \in W^{1,\infty}(\Omega), \\
U \in L^\infty(0,T;L^2(\Omega)^3) \cap L^2(0,T;W^{1,2}(\Omega))^3, \\
\Theta \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega));
\end{array} \right.
\] (2.17)

(i) Magnetohydrodynamical equation:
\[
\int_0^T \int_{\Omega} (-B \cdot \partial_t \varphi + (B \times U) \cdot \text{curl}_x \varphi + \bar{\varrho} \text{curl}_x B \cdot \text{curl}_x \varphi) \, dx \, dt = \int_{\Omega} B_0 \cdot \varphi(0) \, dx,
\] (2.18)

for any test function \( \varphi \in C^\infty_c([0,T] \times \overline{\Omega})^3 \);

(ii) Static state equation determining \( \bar{\varrho} \):
\[
p_0 \bar{\varrho} \nabla_x \bar{\varrho} = \bar{\varrho} \nabla_x F \quad \text{in } \Omega, \quad \int_{\Omega} \bar{\varrho} \, dx = M_0.
\] (2.19)

(iii) Momentum equation with buoyancy force:
\[
\begin{align*}
\int_0^T \int_{\Omega} \bar{\varrho} U \partial_t \varphi &+ \bar{\varrho} U \otimes U : \nabla_x \varphi - 2\bar{\pi} DU : \nabla_x \varphi \, dx \, dt = \\
&= \int_0^T \int_{\Omega} (\bar{\eta} - \frac{2}{3}\bar{\pi} \bar{\varrho}) \text{div}_x U \text{div}_x \varphi - \partial_t \varphi (\div_x \varphi F - \frac{1}{\bar{\mu}_0} (\text{curl}_x B) \times B) \cdot \varphi \, dx \, dt \\
&\quad - \int_{\Omega} \bar{\varrho} U_0 \cdot \varphi(0) \, dx,
\end{align*}
\] (2.20)

for all \( \varphi \in C^\infty_c([0,T] \times \overline{\Omega})^3, \varphi \cdot \mathbf{n} = 0 \) on \( \partial \Omega \), \( \text{div}_x (\bar{\varrho} \varphi) = 0 \) in \( \Omega \);

(iv) Anelastic constraint:
\[
\text{div}_x (\bar{\varrho} U) = 0 \quad \text{a.e. in } (0,T) \times \Omega.
\] (2.21)

(v) Heat equation with convection:
\[
\int_0^T \int_{\Omega} A_\vartheta \Theta \partial_t \varphi + (A_\vartheta \Theta U - \frac{p_0}{\bar{\varrho}} \bar{\varrho} F U - \frac{\bar{\pi}}{\bar{\varrho}} \nabla_x \Theta) \cdot \nabla_x \varphi \, dx = - \int_{\Omega} \Theta_0 \varphi(0) \, dx,
\] (2.22)

for all \( \varphi \in C^\infty_c([0,T] \times \overline{\Omega}) \).
In order to formulate our main result we first introduce a weighted Helmholtz projection by
\[ H_{\tilde{\varrho}}[v] := v - \tilde{\varrho} \nabla \Psi, \quad H_{\tilde{\varrho}}^\perp[v] := \tilde{\varrho} \nabla \Psi, \] (2.23)
where \( \Psi \) is the unique solution of the Neumann problem
\[ \text{div}(\tilde{\varrho} \nabla \Psi) = \text{div} v \quad \text{in} \ \Omega, \quad \tilde{\varrho} \nabla \Psi \cdot n = v \cdot n \quad \text{on} \ \partial \Omega, \quad \int_{\Omega} \Psi \, dx = 0. \] (2.24)

Introducing the Hilbert space
\[ L^2_{1/\tilde{\varrho}}(\Omega)^3 = L^2(\Omega)^3 \quad \text{with scalar product} \quad (v, w)_{1/\tilde{\varrho}} := \int_{\Omega} v \cdot w_{1/\tilde{\varrho}} \, dx \] (2.25)
one can show, using the properties of \( \tilde{\varrho} \), that \( L^2_{1/\tilde{\varrho}}(\Omega)^3 \) admits an orthogonal decomposition
\[ L^2_{1/\tilde{\varrho}}(\Omega)^3 = H_{\tilde{\varrho}}[L^2(\Omega)^3] \oplus H_{\tilde{\varrho}}^\perp[L^2(\Omega)^3], \] (2.26)
\[ H_{\tilde{\varrho}}[L^2(\Omega)^3] = \{ v \in L^2(\Omega)^3 \mid \text{div} v = 0 \}, \] (2.27)
\[ H_{\tilde{\varrho}}^\perp[L^2(\Omega)^3] = \{ \tilde{\varrho} \nabla \Psi \mid \Psi \in W^{1,2}(\Omega), \int_{\Omega} \Psi \, dx = 0 \}. \] (2.28)

Now, we are in a position to state our main result.

**Theorem 2.1** Let \( \Omega \subset \mathbb{R}^3 \) be a bounded simply connected domain of class \( C^{2,\nu}, \nu \in (0,1], \) \( T > 0 \) and \( F \in W^{1,\infty}(\Omega) \). Let \( \{ B_\varepsilon, g_\varepsilon, u_\varepsilon, \varrho_\varepsilon \}_{\varepsilon > 0} \) be a family of weak solutions to the system (1.18)–(1.23) in the sense of Definition 2.1 with boundary conditions (1.6), (1.11), and where

- the initial data satisfy (2.13)–(2.15) with \( \bar{\varrho} > 0 \) and \( \tilde{\varrho} \) obeying (1.42),
- \( p_\varepsilon \) and \( e_\varepsilon \) are given through (1.30), (1.31) and satisfy (1.33), (1.34), (1.39)–(1.41),
- the viscous stress tensor \( S \), the heat flux \( q \), and the magnetic diffusivity \( \lambda \) are given through (1.24)–(1.29).

Then, at least for a suitable subsequence,
\[ \frac{B_\varepsilon}{\varepsilon} \rightarrow B \quad \text{in} \ L^\infty(0,T;L^2(\Omega)^3) \cap L^2(0,T;W^{1,2}(\Omega)^3), \]
\[ g_\varepsilon \rightarrow \tilde{\varrho} \quad \text{in} \ L^\infty(0,T;L^q(\Omega)) \cap C([0,T];L^q(\Omega)), \quad 1 \leq q < \frac{5}{3}, \]
\[ u_\varepsilon \rightarrow U \quad \text{in} \ L^2(0,T;W^{1,2}(\Omega)^3), \]
\[ \varrho_\varepsilon \rightarrow \bar{\varrho} \quad \text{in} \ L^2(0,T;W^{1,2}(\Omega)), \]
\[ \frac{\varrho_\varepsilon - \bar{\varrho}}{\varepsilon^2} \rightarrow \Theta \quad \text{in} \ L^2(0,T;W^{1,2}(\Omega)), \]

where \( B, \tilde{\varrho}, U, \bar{\varrho}, \Theta \) represent a weak solution to the anelastic approximation of the magnetohydrodynamical OB-equations, i.e. they satisfy (1.42)–(1.46) in the sense of Definition 2.2 with \( \bar{\varrho} = \lambda(\tilde{\varrho}, \bar{\varrho}, 0), \eta = \eta(\tilde{\varrho}, 0), \mu = \mu(\tilde{\varrho}, 0), \kappa = \kappa(\tilde{\varrho}, \bar{\varrho}, 0), \) and \( A = \beta\varrho^{3/2} \) supplemented with the initial conditions
\[ \tilde{\varrho} U_0 = w - \lim_{\varepsilon \to 0} H_{\tilde{\varrho}}[\tilde{\varrho} u_0], \quad \Theta_0 = w - \lim_{\varepsilon \to 0} \varrho_0^{(2)}, \quad B_0 = w - \lim_{\varepsilon \to 0} B_0^{(1)}. \] (2.29)
3 Uniform estimates

Now we are going to derive uniform estimates for our problem. For more details in the calculations we refer the reader to [24, Chapter 5], [46], where a similar problem is treated. We will concentrate here on the discussion of the new contributions due to the magnetic induction $B_{\varepsilon}$.

3.1 Total mass conservation and dissipation balance

Choosing $b \equiv 0$ and $B = B(1) = 1$ we get from (2.4), in accordance with hypothesis (2.14),

$$M_0 = \int_{\Omega} \varrho \, dx = \int_{\Omega} \varrho(t, \cdot) \, dx \quad \text{for all} \quad t \in [0, T].$$

(3.1)

Note that this condition makes sense due to (2.10). Combining conveniently the entropy balance (2.7), where we take $\varphi = 1$ as test function, with the energy equality (2.9), setting

$$H_{\varepsilon}(\varrho, \vartheta) := \varrho e_{\varepsilon}(\varrho, \vartheta) - \vartheta \varrho s_{\varepsilon}(\varrho, \vartheta),$$

(3.2)

we arrive at the total dissipation balance in the form

$$\int_{\Omega} \left( \frac{1}{2} \varrho |\mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2} \left( H_{\varepsilon}(\varrho_\varepsilon, \vartheta_\varepsilon) + \frac{1}{2\mu_0} |\mathbf{B}_{\varepsilon}|^2 - \varphi_\varepsilon F \right) \right)(\tau, \cdot) \, dx + \frac{\overline{\mathbf{J}}}{\varepsilon^2} \sigma_\varepsilon([0, \tau] \times \Omega) =$$

$$= \int_{\Omega} \frac{1}{2} \varrho_0 |\mathbf{u}_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^2} \left( H_{\varepsilon}(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) + \frac{1}{2\mu_0} |\mathbf{B}_{0,\varepsilon}|^2 - \varphi_{0,\varepsilon} F \right) \, dx$$

(3.3)

for a.a. $\tau \in (0, T)$, where we have used the identification of $(C([0, T] \times \Omega))^*$ with the space of Radon measures $\mathcal{M}([0, T] \times \Omega)$. In agreement with [24], the function $H_{\varepsilon}$ will be called Helmholtz function. A direct calculation employing the Gibbs’ relation (1.23) verifies that

$$\partial_{\varrho}^2 H_{\varepsilon}(\varrho, \vartheta) = \frac{1}{\varrho} \partial_{\varrho} p_{\varepsilon}(\varrho, \vartheta),$$

(3.4)

$$\partial_\vartheta H_{\varepsilon}(\varrho, \vartheta) = \varrho (\vartheta - \overline{\vartheta}) \partial_\vartheta s_{\varepsilon}(\varrho, \vartheta).$$

(3.5)

A first consequence of (3.4) in combination with (2.11) is the observation that

$$\partial_\varrho H_{\varepsilon}(\varrho_\varepsilon, \overline{\vartheta}) = F + c,$$

where $c$ is a constant. Accordingly, identity (3.3) may be rewritten as: for a.a. $\tau \in (0, T)$

$$\int_{\Omega} \left( \frac{1}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2} \frac{|\mathbf{B}_{\varepsilon}|^2}{\varepsilon} + \frac{1}{\varepsilon^2} \mathcal{H}_\varepsilon(\varrho_\varepsilon, \vartheta_\varepsilon) \right)(\tau, \cdot) \, dx + \frac{\overline{\mathbf{J}}}{\varepsilon^2} \sigma_\varepsilon([0, \tau] \times \Omega) =$$

$$= \int_{\Omega} \left( \frac{1}{2} \varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^2} \frac{|\mathbf{B}_{0,\varepsilon}|^2}{\varepsilon} + \frac{1}{\varepsilon^2} \mathcal{H}_\varepsilon(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) \right) \, dx,$$

(3.6)

where

$$\mathcal{H}_\varepsilon(\varrho, \vartheta) := H_{\varepsilon}(\varrho, \vartheta) - (\varrho - \overline{\varrho}) \partial_\varrho H_{\varepsilon}(\varrho_\varepsilon, \overline{\vartheta}) - H_{\varepsilon}(\varrho_\varepsilon, \overline{\vartheta}).$$

(3.7)

In our problem most of the available estimates will be deduced from the dissipation identity (3.6). First we shall establish the following coercivity properties of the functions $H_{\varepsilon}$ and $\mathcal{H}_\varepsilon$. The details of the classical reasoning leading to this result can be consulted in [24].
Lemma 3.1 Let the functions \( p_\varepsilon, e_\varepsilon \) satisfy the constitutive relations (1.30), (1.31), (1.33), and (1.34) with \( P \in C^2[0, \infty) \). Let \( \bar{\rho} \) be defined in (1.42) and let \( \bar{\vartheta} \) be the equilibrium temperature introduced in Section 2.2. If we set
\[
2\bar{\varrho} := \inf_{x \in \Omega} \bar{\varrho}(x) \leq \sup_{x \in \Omega} \bar{\varrho}(x) =: \overline{\varrho}/2
\]
then there exists \( c = c(\varrho, \overline{\varrho}) > 0 \) independent of \( \varepsilon \) such that the following estimates hold:
\[
\mathcal{O}_{\text{ess}} := \{ (\varrho, \vartheta) \mid \overline{\varrho} \leq \varrho \leq \vartheta, \quad \bar{\vartheta}/2 \leq \vartheta \leq 2\overline{\varrho} \} \quad \text{and} \quad \mathcal{O}_{\text{res}} := (0, \infty)^2 \backslash \mathcal{O}_{\text{ess}},
\]
where
\[
\mathcal{H}^\varepsilon(\varrho, \vartheta) \geq \left\{ \begin{array}{l}
H^\varrho_\varepsilon(\varrho, \vartheta) - H^\varrho_\varepsilon(\varrho, \overline{\varrho})
\quad \mathcal{H}^\vartheta_\varepsilon(\varrho, \vartheta) - (\varrho - \bar{\varrho}_\varepsilon)\partial_\varrho H^\varrho_\varepsilon(\bar{\varrho}_\varepsilon, \vartheta) - H^\vartheta_\varepsilon(\bar{\varrho}_\varepsilon, \overline{\varrho})
\end{array} \right\} \geq 0,
\]
\[
\forall (\varrho, \vartheta) \in \mathcal{O}_{\text{ess}}, \quad \mathcal{H}^\varepsilon(\varrho, \vartheta) \geq c(|\varrho - \bar{\varrho}|^2 + \frac{1}{\varepsilon^2}|\vartheta - \overline{\varrho}|^2),
\]
\[
\forall (\varrho, \vartheta) \in \mathcal{O}_{\text{res}}, \quad \mathcal{H}^\varepsilon(\varrho, \vartheta) \geq c(1 + \varrho + \varrho \varepsilon c(\varrho, \vartheta) + \varrho |s(\varrho, \vartheta)|).
\]

3.2 Estimates due to the total dissipation balance

In the sequel, following [24], it will be convenient to decompose a function \( h(t, x) \) on \((0, T) \times \Omega\) on its essential and residual parts as follows
\[
h(t) = [h(t)]_{\text{ess}} + [h(t)]_{\text{res}} \quad \text{with} \quad [h(t)]_{\text{ess}} = h(t)1_{\mathcal{M}_{\text{ess}}(t)}, \quad [h(t)]_{\text{res}} = h(t)1_{\mathcal{M}_{\text{res}}(t)},
\]
where
\[
\mathcal{M}_{\text{ess}}(t) = \{ x \in \Omega \mid (q_\varepsilon, \vartheta_\varepsilon)(t) \in \mathcal{O}_{\text{ess}} \}, \quad \mathcal{M}_{\text{res}}(t) = \{ x \in \Omega \mid (q_\varepsilon, \vartheta_\varepsilon)(t) \in \mathcal{O}_{\text{res}} \}.
\]

Lemma 3.2 Let the functions \( p_\varepsilon, e_\varepsilon \) satisfy the relations (1.30), (1.31), (1.33), and (1.34) with \( P \) verifying assumptions (1.39)–(1.41). Then, the following estimates hold uniformly with respect to \( \varepsilon \):
\[
\text{ess sup}_{t \in (0, T)} \| q_\varepsilon \|_{L^1(\Omega)} \leq c,
\]
\[
\text{ess sup}_{t \in (0, T)} \left\| \frac{B_\varepsilon(t)}{\varepsilon} \right\|_{L^2(\Omega)} \leq c,
\]
\[
\text{ess sup}_{t \in (0, T)} \left\| \frac{\vartheta_\varepsilon - \bar{\vartheta}(t)}{\varepsilon} \right\|_{L^2(\Omega)} \leq c,
\]
\[
\text{ess sup}_{t \in (0, T)} \left\| \frac{\vartheta_\varepsilon - \bar{\vartheta}(t)}{\varepsilon^2} \right\|_{L^2(\Omega)} \leq c,
\]
\[
\text{ess sup}_{t \in (0, T)} |\mathcal{M}_{\text{res}}| \leq c \varepsilon^2,
\]
\[
\text{ess sup}_{t \in (0, T)} \int_{\Omega} \left[ \frac{\vartheta_\varepsilon}{\varepsilon} \right]_{\text{res}}^4 \leq c \varepsilon,
\]
\[
\text{ess sup}_{t \in (0, T)} \int_{\Omega} [\vartheta_\varepsilon]_{\text{res}}^3 \leq c \varepsilon^4, \quad \beta \in [1, 5/2],
\]
\[
\text{ess sup}_{t \in (0, T)} \int_{\Omega} [\vartheta_\varepsilon]_{\text{res}}^{5/3} \leq c \varepsilon^{2-\frac{3}{2} \alpha}, \quad \alpha \in (2, 3),
\]
\[
\text{ess sup}_{t \in (0, T)} \int_{\Omega} \vartheta_\varepsilon \log \vartheta_\varepsilon \leq c \varepsilon^2,
\]
\[
\text{ess sup}_{t \in (0, T)} \int_{\Omega} \vartheta_\varepsilon \log \vartheta_\varepsilon \leq c \varepsilon^2.
\]
\[ \|\sigma_\varepsilon\|_{\mathcal{C}([0,T] \times \Omega)^*} = \sigma_\varepsilon([0,T] \times \Omega) \leq c \varepsilon^2 \] (3.24)

\[ \|u_\varepsilon\|_{L^2(0,T;W^{1,2}(\Omega)^3)} \leq c, \] (3.25)

\[ \frac{\|B_\varepsilon\|}{\varepsilon} \|B_\varepsilon\|_{L^2(0,T;W^{1,2}(\Omega)^3)} \leq c, \] (3.26)

\[ \left\| \frac{\partial^2 q - \partial^2 t}{\varepsilon^2} \right\|_{L^2(0,T;W^{1,2}(\Omega))} \leq c, \quad \gamma \in (0,3/2], \] (3.27)

\[ \left\| \log \frac{\partial q}{\partial \varepsilon} - \log \frac{\partial \tilde{q}}{\partial \varepsilon} \right\|_{L^2(0,T;W^{1,2}(\Omega))} \leq c. \] (3.28)

**Proof:** Under assumptions (2.13)–(2.15) on the initial data, the right-hand side of the dissipation identity (3.6) is uniformly bounded. This and Lemma 3.1 thus yield immediately the estimates (3.15)–(3.19), and (3.24). It follows from (1.39)–(1.41) that

\[ c_1(1 + Z^{2/3}) \leq P'(Z) \leq c_2(1 + Z^{2/3}). \] (3.29)

Integration of this inequality and (1.40) imply

\[ c_1 \left( \varepsilon^{\frac{2}{3}} q + \varepsilon \frac{\partial q}{\partial \varepsilon} \right) \leq p_{M,\varepsilon}(\varepsilon, \partial q) \leq c_2 \left( \varepsilon^{\frac{2}{3}} q + \varepsilon \frac{\partial q}{\partial \varepsilon} \right). \] (3.30)

This together with the use of the particular form of \( e_{M,\varepsilon}, e_{R,\varepsilon} \), and \( e_{T,\varepsilon} \) (cf. (1.34), (1.35)), and the lower bound in (3.12) yield (3.20)–(3.22). Coming back to (3.4) we find that

\[ \partial^2 H_{\varepsilon}(\varepsilon, \partial \varepsilon) \geq c/\varepsilon, \]

where we have used the structure of \( p_{M,\varepsilon} \) specified in (1.33) and estimate (3.29). Consequently, integrating the last expression twice from \( \tilde{\varepsilon} \) to \( \varepsilon \) and employing estimate (3.10) together with (3.12), we obtain (3.23).

Finally, estimate (3.24), the inequality for the entropy production rate (2.8) and the structure assumptions (1.25)–(1.29) yield (3.25)–(3.28). Note that we used a special Korn type inequality (cf. [24, Theorem 10.17]) to obtain (3.25), a special Poincaré type inequality (cf. [24, Th. 10.14]) to derive (3.27), (3.28). To get (3.26) we also used (3.25), (2.3) which yields \( \text{div}_{\varepsilon} B_\varepsilon(t) = 0 \) in \( \Omega \) and \( B_\varepsilon(t) \cdot n = 0 \) on \( \partial \Omega \) for a.a. \( t \in (0,T) \) and that under these circumstances \( \|\|_{L^2(\Omega)^3} + \|\|_{L^2(\Omega)} + \|\text{curl}_\varepsilon\|_{L^2(\Omega)^3} \) is an equivalent norm on \( W^{1,2}(\Omega)^3 \) (cf. [30, Section I.3.2]). For more details we refer to [24, Chapter 5] and [46]. This finishes the proof of Lemma 3.2. □

### 3.3 Refined pressure estimates

We shall see in Section 4.1 that the estimates presented in Lemma 3.2 are not enough to capture the asymptotic behaviour of \( p_\varepsilon \). To this end we introduce the linear Bogovskii operator \( B \) on \( \Omega \), see [6, 29, 24, Th. 10.11] and recall its basic properties:

If \( f \in L^p(\Omega) \), then the Bogovskii operator \( B[f] \) satisfies

\[ B[f] \in W^{1,p}(\Omega)^3, \quad \text{div}_{\varepsilon} B[f] = f - \frac{1}{|\Omega|} \int_{\Omega} f \, dx \text{ in } \Omega, \quad B[f] = 0 \text{ on } \partial \Omega, \]

\[ \|B[f]\|_{W^{1,p}(\Omega)^3} \leq c \|f\|_{L^p(\Omega)}, \quad 1 < p < \infty. \] (3.31)

If, in addition \( f = \text{div}_{\varepsilon} g, \ g \in L^q(\Omega)^3, \ g \cdot n = 0 \text{ on } \partial \Omega, \) then

\[ \|B[f]\|_{L^q(\Omega)^3} \leq c \|g\|_{L^q(\Omega)^3}, \quad 1 < q < \infty. \] (3.32)
The principal idea is to take the test functions $\varphi = \psi B[b(\varrho)]$, $\psi \in C_c^\infty(0, T)$, in the momentum equation (2.6) with the $C^\infty[0, \infty)$-functions $b$,

$$
\begin{cases}
  b(\varrho) = 0 & \text{for } 0 \leq \varrho \leq 2\vartheta, \\
  b(\varrho) \in [0, \varrho^\gamma] & \text{for } 2\vartheta < \varrho \leq 3\vartheta, \\
  b(\varrho) = \varrho^\gamma & \text{for } 3\vartheta < \varrho,
\end{cases}
$$

(3.33)

with $\gamma \in (0, 1/4)$. After a bit tedious but straightforward manipulations, we obtain

$$
\frac{1}{\varepsilon^2} \int_0^T \int_\Omega \psi \left( \rho(\varrho, D\varrho) b(\varrho) \right) dxdt = \frac{1}{\varepsilon^2} \int_\Omega \int_0^T \psi \left( \rho(\varrho, D\varrho) \right) dx \left( \int_0^T b(\varrho) \right) dxdt + \frac{1}{\varepsilon^2} \int_0^T \psi \int_\Omega \varrho \nabla F \cdot B[b(\varrho)] \left( \varrho \right) dxdt + I_\varepsilon,
$$

(3.34)

where

$$
I_\varepsilon := -\frac{1}{\mu_0} \int_0^T \int_\Omega \psi \left( \left( \text{curl}_\varepsilon \frac{\rho}{\varepsilon} \right) \times \frac{\rho}{\varepsilon} \right) \cdot B[b(\varrho)] \left( \varrho \right) dxdt + \int_0^T \int_\Omega \left( \mathcal{S}(\partial_\varepsilon, Du_\varepsilon, |B_\varepsilon|) - \varrho \varepsilon u_\varepsilon \otimes u_\varepsilon \right) : \nabla \varphi \left( \varrho \right) dxdt - \int_0^T \partial_t \psi \int_\Omega \varrho \varepsilon u_\varepsilon \cdot B[b(\varrho)] \left( \varrho \right) dxdt + \int_0^T \psi \int_\Omega \varrho \varepsilon u_\varepsilon \cdot B[\nabla_x(b(\varrho)u_\varepsilon)] \left( \varrho \right) dxdt + \int_0^T \psi \int_\Omega \varrho \varepsilon u_\varepsilon \cdot B \left[ (\varrho b'(\varrho) - b(\varrho)) \nabla_x u_\varepsilon \right] \left( \varrho \right) dxdt,
$$

(3.35)

where we have used the renormalised continuity equation (2.4) in the form

$$
\partial_t b(\varrho) = -\nabla(b(\varrho)u_\varepsilon) - (\varrho b'(\varrho) - b(\varrho)) \nabla_x u_\varepsilon.
$$

(3.36)

Taking into account the uniform estimates established in the preceding section and the properties (3.31), (3.32) of the Bogovskii operator, we can show, exactly as in [28], that all integrals contained in $I_\varepsilon$ are bounded uniformly for $\varepsilon \to 0$.

Since all terms except the first one in (3.35) have been discussed in this situation in [16] we will only estimate the new terms. First of all one easily checks that $b(\varrho) = b(\varrho)_{\text{res}}$. Thus the first integral in (3.35) can be estimated by

$$
\left\| \psi \right\|_{L^\infty(0,T)} \left\| \frac{B}{\varepsilon} \right\|_{L^2(0,T;W^{1,2}(-\Omega)^3)} \left\| \frac{B}{\varepsilon} \right\|_{L^\infty(0,T;L^2(-\Omega)^3)} \left\| B[b(\varrho)] \right\|_{L^\infty(0,T;L^\infty(-\Omega)^3)} \cdot
$$

In view of (3.16) and (3.26) we only have to estimate the last expression. Using the embedding $W^{1,2}(-\Omega)$ into $L^\infty(-\Omega)$ and the properties of the Bogovskii operator (3.31) we obtain

$$
\left\| B[b(\varrho)] \right\|_{L^\infty(-\Omega)^3} \leq c \left\| B[b(\varrho)] \right\|_{W^{1,2}(-\Omega)^3} \leq c \left\| b(\varrho) \right\|_{L^4(\Omega)}.
$$

We remark, strictly speaking, that our way of deriving (3.34) and estimating the next to the last integral in (3.35) is formal. However, nowadays these technical difficulties are well understood and various methods to overcome them are described in detail in [21] Section 2.5, [26] [17].
Due to (3.23) and (3.33) together with \( b(\varrho_\varepsilon) = b(\varrho_{\varepsilon \text{res}}) \) we have that
\[
\| b(\varrho_\varepsilon) \|_{L^\infty(0,T;L^4(\Omega))} \leq c \varepsilon^2
\]
provided \( 0 < \gamma < \frac{1}{4} \). Thus we infer
\[
\int_0^T \int_\Omega p_\varepsilon(\varrho_\varepsilon, \vartheta_\varepsilon) b(\varrho_{\varepsilon \text{res}}) \, dx \, dt \leq c \varepsilon^2 \quad \text{for} \quad \gamma \in (0, \frac{1}{4}).
\]
(3.37)
Since \( p_\varepsilon(\varrho_\varepsilon, \vartheta_\varepsilon) \geq p_{M,\varepsilon}(\varrho_\varepsilon, \vartheta_\varepsilon) \) (cf. (3.33)) we derive from (3.37), (3.30) and (3.33) that
\[
\int_0^T \int_\Omega \varepsilon^{\frac{\gamma}{2}} |\varrho_{\varepsilon \text{res}}|^{2+\gamma} 1_\geq \, dx \, dt \leq c \varepsilon^2,
\]
where \( 1_\geq \) is the characteristic function of the set \( \{ \varrho_\varepsilon \geq 3\bar{\varrho} \} \). In view of (3.39) we also get
\[
\int_0^T \int_\Omega \varepsilon^{\frac{\gamma}{2}} |\varrho_{\varepsilon \text{res}}|^{2+\gamma} 1_\leq \, dx \, dt \leq c \varepsilon^2,
\]
where \( 1_\leq \) is the characteristic function of the set \( \{ \varrho_\varepsilon < 3\bar{\varrho} \} \). Thus we conclude:

**Lemma 3.3** Let the function \( p_\varepsilon \) satisfy the constitutive relations (1.30), (1.33) with \( P \) verifying assumptions (1.39)–(1.41). Then there exists \( c = c(\gamma) > 0 \) such that
\[
\int_0^T \int_\Omega |\varrho_{\varepsilon \text{res}}|^{\frac{2}{\gamma}+\gamma} \, dx \, dt \leq c \varepsilon^{2-\frac{2}{\gamma}} \quad \text{for} \quad \gamma \in (0, \frac{1}{4}).
\]
(3.38)

### 4 Constitutive limit in the continuity and momentum equations

It follows from estimates (3.15), (3.16), (3.17), (3.19), (3.22), (3.25), (3.27) for \( \gamma = 1 \) and (3.26) that there exist \( U, B, \Theta \) and \( \varrho^{(1)} \) such that
\[
\left\{
\begin{aligned}
\varrho_\varepsilon &\to \varrho \quad \text{in} \quad L^\infty(0,T;L^\frac{2}{\gamma}(\Omega)), \\
\varepsilon \varrho_\varepsilon \to U \quad \text{in} \quad L^2(0,T;W^{1,2}(\Omega)^3), \\
\varepsilon \varrho_\varepsilon \varepsilon \varrho_\varepsilon \to \varepsilon \varrho U \quad \text{in} \quad L^2(0,T;L^{30/23}(\Omega)^3), \\
\varepsilon B_\varepsilon &\to ^* B \quad \text{in} \quad L^\infty(0,T;L^2(\Omega)^3) \cap L^2(0,T;W^{1,2}(\Omega)^3), \\
\varepsilon \varrho_\varepsilon &\to \varrho \quad \text{in} \quad L^\infty(0,T;L^q(\Omega)^3) \cap L^r((0,T) \times \Omega)^3, \quad q \in [1,6], r \in [1,10/3), \\
\varepsilon \vartheta_\varepsilon &\to \vartheta \quad \text{in} \quad L^\infty(0,T;L^2(\Omega)^3) \cap L^2(0,T;W^{1,2}(\Omega)^3), \\
\varepsilon \varrho_\varepsilon \varrho_\varepsilon &\to \varrho \varrho \quad \text{in} \quad L^2(0,T;W^{1,2}(\Omega)), \\
\varepsilon \varrho_\varepsilon \varrho_\varepsilon &\to \varrho \varrho \quad \text{in} \quad L^\infty(0,T;L^{5/3}(\Omega))
\end{aligned}\right\}
\]
(4.1)
eventually passing to suitable subsequences. In order to obtain (4.1) we used the Aubin Lions compactness theorem, where we employed (3.16), (3.25), (3.26), and (2.3) divided by \( \varepsilon \) to derive that \( \varepsilon^{-1} \vartheta_\varepsilon B_\varepsilon \) is bounded in \( L^q((0,T;W^{-1,q}(\Omega)^3) \) for some \( q > 1 \). From (3.20) with \( \gamma = 3/2 \), (3.21) and the definition of the essential set we deduce that \( \vartheta_\varepsilon \) is uniformly bounded with respect to \( \varepsilon \) in \( L^\infty(0,T;L^4(\Omega)) \cap L^3(0,T;L^{9}(\Omega)) \). Parabolic interpolation together with (4.1) yields
\[
\varrho_\varepsilon \to \varrho \quad \text{in} \quad L^q(0,T;L^q(\Omega)), \quad q \in [1,17/3).
\]
(4.2)
4.1 Anelastic constraint and asymptotics of the pressure

With $b = 0$ and $B = B(1) = 1$ while letting $\varepsilon \to 0$ in the renormalised continuity equation (2.4), we infer
\[
\text{div}_t (\hat{\varrho} \mathbf{U}) = 0 .
\]
Moreover we recover the boundary conditions $\mathbf{U} \cdot \mathbf{n} = 0$ on $\partial \Omega$ since $\hat{\varrho} > 0$. Our aim now is to show that $\frac{1}{\varepsilon^2} p_\varepsilon \approx \frac{p_0}{\varepsilon^2} \varrho_\varepsilon \vartheta_\varepsilon$ where $p_0$ is defined in (1.40), in the limit $\varepsilon \to 0$, more precisely
\[
\frac{1}{\varepsilon^2} p_\varepsilon (\varrho_\varepsilon, \vartheta_\varepsilon) = \frac{1}{3 \varepsilon} \bar{\vartheta}_1 + \frac{p_0}{\varepsilon^2} \varrho_\varepsilon \vartheta_\varepsilon + h_\varepsilon (\varrho_\varepsilon, \vartheta_\varepsilon) , \quad \text{where } \| h_\varepsilon (\varrho_\varepsilon, \vartheta_\varepsilon) \|_{L^1((0,T) \times \Omega)} \to 0 . \tag{4.3}
\]
Looking at (1.30), we may write
\[
p_\varepsilon (\varrho_\varepsilon, \vartheta_\varepsilon) = \frac{\varepsilon}{3} \bar{\vartheta}_1 + p_0 \varrho_\varepsilon \vartheta_\varepsilon + (p_{M,\varepsilon} (\varrho_\varepsilon, \vartheta_\varepsilon) - p_0 (\varrho_\varepsilon, \vartheta_\varepsilon)) + \frac{\varepsilon}{3} \left( \vartheta_\varepsilon^4 - \bar{\vartheta}_1^4 \right) . \tag{4.4}
\]
Using (3.19), (3.20) and (3.27) on the residual set, and (3.18) together with the Taylor formula on the essential set we obtain
\[
\frac{1}{\varepsilon} \| \vartheta_\varepsilon^4 - \bar{\vartheta}_1^4 \|_{L^1((0,T) \times \Omega)} \to 0 . \tag{4.5}
\]
The remaining part of this section therefore will be devoted to showing that
\[
\frac{1}{\varepsilon^2} \left( \frac{\varepsilon^3}{3} p \left( \varepsilon^\alpha \frac{\varrho_\varepsilon}{\vartheta_\varepsilon^3} \right) - p_0 \varrho_\varepsilon \vartheta_\varepsilon \right) \to 0 \quad \text{in } L^1((0,T) \times \Omega) \quad \text{for } \varepsilon \to 0 . \tag{4.6}
\]
In agreement with the form of $p_{M,\varepsilon}$ (cf. (1.33)) and (1.40) we can write
\[
\frac{1}{\varepsilon^2} \left( \frac{\varepsilon^3}{3} p \left( \varepsilon^\alpha \frac{\varrho_\varepsilon}{\vartheta_\varepsilon^3} \right) - p_0 \varrho_\varepsilon \vartheta_\varepsilon \right) = \frac{1}{\varepsilon^2} \left( \frac{\varepsilon^3}{3} P \left( \varepsilon^\alpha \frac{\varrho_\varepsilon}{\vartheta_\varepsilon^3} \right) - \frac{\vartheta_\varepsilon^5}{\varepsilon^3} \right) . \tag{4.7}
\]
Since $P$ is twice continuously differentiable, we deduce that the essential part of (4.7) is bounded from above by $c \varepsilon^{\alpha - 2} \left[ \frac{\varepsilon^2}{\vartheta_\varepsilon^3} \right]_{\text{ess}}^{5/3}$ and thus tends to zero for $\varepsilon \to 0$ uniformly on $(0,T) \times \Omega$ provided $\alpha > 2$.

Next, considering the residual part of expression (4.7) separately on the sets $\{ \varrho_\varepsilon > \vartheta_\varepsilon^{3/2} \}$ and $\{ \varrho_\varepsilon \leq \vartheta_\varepsilon^{3/2} \}$, respectively, we get, by virtue of hypothesis (1.41) on the one hand, and by virtue of the Taylor formula on the other hand, that the residual part of (4.7) is bounded from above by $c \varepsilon^{2\alpha/3 - 2} [ [\varrho_\varepsilon]_{\text{res}}^{5/3}]$. Next we write
\[
\frac{\varepsilon^{2\alpha/3}}{\varepsilon^2} \int_0^T \int_{\Omega} \left[ [\varrho_\varepsilon]_{\text{res}}^{5/3} \right] dx dt = \frac{\varepsilon^{2\alpha/3}}{\varepsilon^2} \int_0^\varepsilon \int_{\{ \varrho_\varepsilon \leq \varepsilon \}} \left[ [\varrho_\varepsilon]_{\text{res}}^{5/3} \right] dx dt + \frac{\varepsilon^{2\alpha/3}}{\varepsilon^2} \int_{\{ \varrho_\varepsilon > \varepsilon \}} \left[ [\varrho_\varepsilon]_{\text{res}}^{5/3} \right] dx dt
\]
where, due to (3.19), the first expression is bounded by $c \varepsilon^{2\alpha/3} K^{5/3}$ and the second one by $c K^{-\gamma}$, cf. Lemma 3.3. Summing up these observations, we get
\[
\frac{\varepsilon^{2\alpha/3}}{\varepsilon^2} [ [\varrho_\varepsilon]_{\text{res}}^{5/3} ] \to 0 \quad \text{in } L^1((0,T) \times \Omega) \quad \text{for } \varepsilon \to 0 . \tag{4.8}
\]
Thus, formula (4.6) is proved.
4.2 Buoyancy force and limit in the momentum equation

Our next goal is to identify the driving force in \( (2.20) \). To this end, we rewrite the corresponding term in \( (2.6) \) as follows:

\[
\frac{1}{\varepsilon^2} \int_0^T \int_\Omega \left( p_0 \varrho_\varepsilon \varrho_\varepsilon \nabla \varphi - \varrho_\varepsilon \nabla \varphi \cdot \varphi \right) dxdt = -\frac{1}{\varepsilon^2} \int_0^T \int_\Omega p_0 (\vartheta_\varepsilon - \vartheta) \nabla \vartheta \cdot \varphi dxdt + \frac{1}{\varepsilon^2} \int_0^T \int_\Omega p_0 (\vartheta_\varepsilon - \vartheta)(\vartheta_\varepsilon - \vartheta) \nabla \varphi dxdt + \frac{1}{\varepsilon^2} \int_0^T \int_\Omega p_0 \varrho_\varepsilon \nabla \varphi dxdt + \frac{1}{\varepsilon^2} \int_0^T \int_\Omega p_0 \vartheta_\varepsilon \varphi dxdt \tag{4.9}
\]

for all \( \varphi \in C_c^\infty((0, T) \times \Omega)^3 \), \( \varphi \cdot n = 0 \) on \( \partial\Omega \), where several times we have used the static equilibrium equation \( (1.42) \) and the product rule for \( \nabla \varphi \cdot \varphi \). Notice that formula \( (4.9) \) is nothing but a rigorous weak formulation of

\[
p_0 \nabla \varphi (\vartheta_\varepsilon - \vartheta) = p_0 \nabla \varphi F = \]

\[
= p_0 (\vartheta_\varepsilon - \vartheta) \nabla \varphi + p_0 \varrho_\varepsilon \nabla \varphi \left( \frac{\varrho_\varepsilon - \varrho_\varepsilon}{\vartheta} + \frac{\varrho_\varepsilon - \varrho_\varepsilon}{\vartheta} \right) + p_0 \nabla \left( (\varrho_\varepsilon - \vartheta)(\vartheta_\varepsilon - \vartheta) \right),
\]

thanks to the static equilibrium equation \( (1.42) \). Then, with \( (4.1)_1 \) and \( (3.15) \) we see

\[
\varrho_\varepsilon u_\varepsilon \to \varrho U \quad \text{in} \quad L^\infty(0, T; L^2(\Omega)^3).
\]

This and \( (4.1)_2 \) implies that \( \varrho_\varepsilon u_\varepsilon \otimes u_\varepsilon \) is bounded in \( L^2(0, T; L^{30/29}(\Omega)^{3\times 3}) \), and therefore converges weakly in that space and in particular weakly in \( L^1(0, T; L^1(\Omega)^{3\times 3}) \). But then,

\[
\varrho_\varepsilon u_\varepsilon \otimes u_\varepsilon = (\varrho_\varepsilon - \varrho) u_\varepsilon \otimes u_\varepsilon + \varrho u_\varepsilon \otimes u_\varepsilon.
\]

The left-hand side admits a weak limit in \( L^1(0, T; L^1(\Omega)^{3\times 3}) \), while the first term at the right-hand side converges strongly in \( L^1(0, T; L^1(\Omega)^{3\times 3}) \) to 0. In view of \( (3.8) \) thus also the weak limit of \( u_\varepsilon \otimes u_\varepsilon \) exists in \( L^1(0, T; L^1(\Omega)^{3\times 3}) \). Consequently,

\[
\varrho_\varepsilon u_\varepsilon \otimes u_\varepsilon \to \varrho \cdot \mathbf{u} \otimes \mathbf{u} \quad \text{in} \quad L^2(0, T; L^{30/29}(\Omega)^{3\times 3}). \tag{4.10}
\]

Here and in the sequel - if not stated explicitly otherwise - we denote by \( \overline{f} \) a weak limit of the sequence \( f_\varepsilon \) in \( L^1((0, T) \times \Omega) \). Last but not least, noticing \( (1.25) \), \( (1.26) \), \( (4.1)_{2,6,7} \), and \( (4.2) \), we obtain using the generalized dominated convergence theorem that

\[
\mathbb{S}(\vartheta_\varepsilon, Du_\varepsilon, |B_\varepsilon|) \to 2\pi DU + (\eta - \frac{2}{3}\pi) \nabla \varphi U \quad \text{in} \quad L^q((0, T) \times \Omega)^{3\times 3}
\]

for some \( q > 1 \), where \( 2\pi DU = \nabla U + (\nabla U)^\top \). We also notice that in view of estimates \( (3.17) \), \( (3.19) \), \( (3.22) \), and \( (3.27) \), the second term on the right-hand side of \( (4.9) \) tends to 0. Using the static equation \( (1.42) \) to evaluate the first integral at the right-hand side of \( (4.9) \) and \( (4.1)_{8} \) we conclude,

\[
\frac{1}{\varepsilon^2} \int_0^T \int_\Omega \left( p_0 \varrho_\varepsilon \varrho_\varepsilon \nabla \varphi - \varrho_\varepsilon \nabla \varphi \cdot \varphi \right) dxdt \to -\int_0^T \int_\Omega \frac{\vartheta}{\varrho} \Theta \nabla \varphi \cdot \varphi dxdt \tag{4.11}
\]

for all

\[
\varphi \in C_c^\infty((0, T) \times \Omega)^3, \quad \varphi \cdot n = 0 \text{ on } \partial\Omega, \quad \nabla \varphi(\varrho \varphi) = 0 \text{ in } \Omega. \tag{4.12}
\]
From (4.1) we immediately deduce
\[
\frac{1}{\mu_0} \int_0^T \int_\Omega \left( \left( \nabla \times \frac{B_\varepsilon}{\varepsilon} \right) \cdot \varphi \right) dx dt \to \frac{1}{\mu_0} \int_0^T \int_\Omega \left( \nabla \times B \right) \cdot \varphi dx dt
\]
for all \( \varphi \in C^\infty_c((0, T) \times \Omega)^3 \). Summing up these results we may pass to the limit in the momentum equation (2.6) to obtain for any test function \( \varphi \) specified in (4.12)
\[
\int_0^T \int_\Omega \left( \varrho \nabla \cdot U \partial_t \varphi + \varrho \varphi \cdot u \otimes u : \nabla \varphi \right) dx dt = - \int_\Omega \varrho U_0 \cdot \varphi(0) dx.
\]
Note that the integral identity (4.13) represents a weak formulation of the momentum equation (2.20) in the target system as soon as we can show that the weak limit \( u \otimes u \) can be replaced by \( U \otimes U \). This step will be fully justified in Section 6.

4.3 Limit in the magnetohydrodynamical equation
From (4.1) and the assumption (1.29) we immediately obtain, using the generalized dominated convergence theorem, that
\[
\lambda(\varrho_\varepsilon, \vartheta_\varepsilon, |B_\varepsilon|) \frac{B_\varepsilon}{\varepsilon} \to \lambda \nabla \times B
\]
in \( L^q((0, T) \times \Omega)^3 \) for some \( q > 1 \). Using (4.2) we also see that
\[
\frac{B_\varepsilon}{\varepsilon} \cdot u_\varepsilon \to B \cdot U
\]
in \( L^q((0, T) \times \Omega)^3 \) for some \( q > 1 \). Hence the limit \( \varepsilon \to 0 \) of (2.3) divided by \( \varepsilon \) yields
\[
\int_0^T \int_\Omega \left( -B \cdot \partial_t \varphi + (B \times U) \cdot \nabla \times \varphi + \lambda \nabla \times B \cdot \nabla \times \varphi \right) dx dt = \int_\Omega \varrho_0 \cdot \varphi(0) dx,
\]
for any test function \( \varphi \in C^\infty_c((0, T) \times \Omega)^3 \), where we used again (4.1) for the first term. This last equation is exactly (2.18).

5 Asymptotic limit in the entropy equation
In this section, we shall pass to the limit in the entropy balance equation (2.7) in order to obtain the heat equation (2.22) in the target system. First of all by virtue of (3.24),
\[
< \sigma_\varepsilon; \varphi >_{[M, C]((0, T) \times \Omega)} \to 0
\]
for all \( \varphi \in C^\infty_c((0, T) \times \Omega) \). Using (4.2) together with (4.1) and the properties (1.27), (1.28) of the heat flux and the generalized dominated convergence theorem, we arrive at
\[
\frac{\kappa(\varrho_\varepsilon, \vartheta_\varepsilon, |B_\varepsilon|)}{\vartheta_\varepsilon} \frac{\vartheta_\varepsilon - \bar{\vartheta}}{\varepsilon^2} \to \frac{\kappa}{\vartheta} \nabla \Theta \quad \text{in} \quad L^q((0, T) \times \Omega) \quad \text{for some} \quad q > 1.
\]
We check, by means of (3.20) and (4.1), that
\[\varrho \varepsilon R,\varepsilon(\varrho \varepsilon,\vartheta \varepsilon) = \varepsilon^4 \vartheta^3 \rightarrow 0 \quad \text{in } L^\infty(0,T;L^{4/3}(\Omega)), \tag{5.3}\]
\[\varrho \varepsilon R,\varepsilon(\varrho \varepsilon,\vartheta \varepsilon)u_\varepsilon \rightarrow 0 \quad \text{in } L^2(0,T;L^{12/11}(\Omega)^3). \tag{5.4}\]

Coming back to (1.38), we may write
\[S(Z) = -p_0 \log(Z) + \tilde{S}(Z), \tag{5.5}\]
with
\[\tilde{S}'(Z) = -\frac{3}{2} \frac{1}{Z^2} \left(\frac{5}{3} (P(Z) - p_0 Z) - (P'(Z) - p_0 Z)\right), \tag{5.6}\]
where, in view of (1.39)–(1.41),
\[|\tilde{S}'(Z)| \leq c \quad \text{for all } Z > 0. \tag{5.7}\]

Consequently, using the uniform bounds established in (3.15), (3.19), (3.22), (3.28), we obtain
\[\left[\varrho \varepsilon s_{M,\varepsilon}(\varrho \varepsilon,\vartheta \varepsilon)\right]_{\text{res}} \rightarrow 0 \quad \text{in } L^q((0,T) \times \Omega) \quad \text{for a certain } q > 1, \tag{5.8}\]
\[\left[\varrho \varepsilon s_{M,\varepsilon}(\varrho \varepsilon,\vartheta \varepsilon)u_\varepsilon\right]_{\text{res}} \rightarrow 0 \quad \text{in } L^q((0,T) \times \Omega)^3 \quad \text{for a certain } q > 1. \tag{5.9}\]

Next, recalling the form (1.37) of \(s_{T,\varepsilon}\) we easily find that
\[\left[\varrho \varepsilon(s_{T,\varepsilon}(\vartheta \varepsilon) - s_{T,\varepsilon}(\bar{\vartheta}))\right]_{\text{res}} \rightarrow 0 \quad \text{in } L^2((0,T;L^{30/23}(\Omega)), \tag{5.10}\]
\[\left[\varrho \varepsilon(s_{T,\varepsilon}(\vartheta \varepsilon) - s_{T,\varepsilon}(\bar{\vartheta}))u_\varepsilon\right]_{\text{res}} \rightarrow 0 \quad \text{in } L^2(0,T;L^{30/29}(\Omega)^3), \tag{5.11}\]
where we have used (3.19), (3.22), (3.27), (3.28) in the first case and (3.15), (3.19), (3.22), (3.27), (3.28) in the second one. Interpolating with estimates (3.18) and (3.27), (3.28) we readily obtain the bound
\[\|s_{T,\varepsilon}(\vartheta \varepsilon) - s_{T,\varepsilon}(\bar{\vartheta})\|_{L^q((0,T) \times \Omega)} \leq c, \tag{5.12}\]
whence
\[\left[\varrho \varepsilon(s_{T,\varepsilon}(\vartheta \varepsilon) - s_{T,\varepsilon}(\bar{\vartheta}))\right]_{\text{res}} \rightarrow A\tilde{\vartheta} \Theta \quad \text{in } L^p((0,T) \times \Omega)) \quad \text{with any } 1 \leq p < 10/3, \quad (5.13)\]
In order to do it, we recall the celebrated Tartar’s div-curl lemma [53] in its $L^p$-version, see e.g. [24] Theorem 10.21, stating that if $Q \subset \mathbb{R}^N$ is a domain, then

\[
\left\{ \begin{array}{l}
\text{div}_x V_\varepsilon \text{ precompact in } W^{-1,s}(Q), \\
\text{curl}_x W_\varepsilon \text{ precompact in } W^{-1,s}(Q)^{N\times N}
\end{array} \right\}, \quad s > 1
\]

implies

\[
V_\varepsilon \cdot W_\varepsilon \rightharpoonup V \cdot W \quad \text{in } L^r(Q).
\] (5.13)

We shall apply this statement to

\[
V_\varepsilon := \left( [s_{T,\varepsilon}(\vartheta_\varepsilon) - s_{T,\varepsilon}(\overline{\vartheta})]_{\text{ess}}, \varrho_\varepsilon s_\varepsilon(\varrho_\varepsilon, \vartheta_\varepsilon) u_\varepsilon - \kappa(\vartheta_\varepsilon) \nabla \vartheta_\varepsilon - \varrho_\varepsilon \varepsilon^2 \right) \subset \mathbb{R}^4,
\]

\[
W_\varepsilon := (G((u_\varepsilon)), 0, 0, 0) \subset \mathbb{R}^4,
\]

where $i = 1, 2, 3$ and $G \in W^{1,\infty}(\mathbb{R})$. In accordance with (1.32), (2.7), we notice

\[
\text{Div}_{t,x} V_\varepsilon = -\partial_t \left( \varrho_\varepsilon s_{M,\varepsilon}(\varrho_\varepsilon, \vartheta_\varepsilon) + \varrho_\varepsilon s_{R,\varepsilon,p}(\varrho_\varepsilon, \vartheta_\varepsilon) - \varrho_\varepsilon [s_{T,\varepsilon}(\vartheta_\varepsilon) - s_{T,\varepsilon}(\overline{\vartheta})]_{\text{res}} \right) + \sigma_\varepsilon,
\]

where the latter expression is precompact in $W^{-1,s}((0,T) \times \Omega)$, $1 \leq s < 4/3$ by virtue of (5.3), (5.6), (5.8), (5.9) and (3.24) together with the compact embedding $(C([0,T] \times \overline{\Omega}))^* \hookrightarrow W^{-1,s}((0,T) \times \Omega)$, $s < 4/3$. Moreover, Curl$_{t,x} W_\varepsilon$ being bounded in $L^2((0,T) \times \Omega)^{4 \times 4}$, is compact in $W^{-1,s}((0,T) \times \Omega)$. The boundedness of $V_\varepsilon$ in $L^q((0,T) \times \Omega)^3$ for some $q > 1$ is ensured by Lemma 3.2 in the way described through (5.2), (5.4), (5.7), (5.8), (5.9), (5.11). Thus, the div-curl lemma provides, e.g.,

\[
[s_{T,\varepsilon}(\vartheta_\varepsilon) - s_{T,\varepsilon}(\overline{\vartheta})]_{\text{ess}} T_k((u_\varepsilon)_i) \rightharpoonup A\varrho \Theta T_k(u_i) \quad \text{in } L^1((0,T) \times \Omega),
\] (5.14)

where $k > 0$ and

\[
T_k(s) = \begin{cases} 
\min\{s,k\} & \text{if } s \geq 0, \\
-\min\{-s,k\} & \text{if } s < 0.
\end{cases}
\]

Evidently, due to (3.25)

\[
\sup_{\varepsilon > 0} \|T_k((u_\varepsilon)_i) - (u_\varepsilon)_i\|_{L^q((0,T) \times \Omega)} \to 0 \quad \text{for } 1 \leq q < 2,
\]

which in combination with (5.10) and (5.14) yields (5.12). Now, everything is ready to pass to the limit in the entropy balance equation (2.7). First of all, we realise that

\[
\int_{\Omega} \varrho_{0,\varepsilon} \left( s_{M,\varepsilon}(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) + (s_{T,\varepsilon}(\vartheta_{0,\varepsilon}) - s_{T,\varepsilon}(\overline{\vartheta})) + s_{R,\varepsilon}(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) \right) \varphi(0,\cdot) \, dx
\]

\[
\rightharpoonup p_0 \int_{\Omega} \varrho \left( \frac{3}{2} \log \vartheta - \log \varrho \right) \, dx + A \int_{\Omega} \varrho \vartheta_0^2 \varphi(0,\cdot) \, dx,
\]
Rigorous derivation of the anelastic approximation to OBA

where we have used formulae (5.5), (1.36) and (1.37) together with the structure of the initial data specified in (2.13) and (2.15). Next we subtract from equation (2.7) the identity

\[ s_{T,\varepsilon}(\vec{v}) \int_0^T \int_\Omega \left( \varrho_\varepsilon \partial_t \varphi + \varrho_\varepsilon u_\varepsilon \cdot \nabla \varphi \right) d\Omega d\tau = -s_{T,\varepsilon}(\vec{v}) \int_\Omega \varrho_0 \varphi(0, \cdot) d\Omega. \]

Finally, we employ in the new identity the limits established in this section, namely (5.1), (5.2), (5.4), (5.6), (5.7), (5.8), (5.9), (5.12). The resulting limiting equation is precisely the heat equation (2.22).

6 Constitutive limit in the convective term

In this technical part, we closely follow [22]; the method exploits ideas from Lions, Masmoudi [39] adapting them from the case of the constant equilibrium density to the space variable case. The reader can consult Masmoudi [43] for another approach.

6.1 Lighthill’s equations and Spectral analysis of the wave operator

The continuity equation (2.4) with \( b = 0 \) and \( B = B(1) = 1 \) can be written in the form

\[ \int_0^T \int_\Omega \varepsilon r_\varepsilon \partial_t \varphi + \nabla \varphi \cdot \varepsilon V_\varepsilon dx dt = 0 \]  

for any \( \varphi \in C^\infty_c((0, T) \times \Omega) \), where

\[ r_\varepsilon := \frac{\varrho_\varepsilon - \tilde{\varrho}}{\varepsilon}, \quad V_\varepsilon := \varrho_\varepsilon u_\varepsilon. \]

Similarly, taking into account (4.3), (4.9), we can rewrite the momentum equation (2.6) multiplied by \( \varepsilon \) as

\[ \int_0^T \int_\Omega \varepsilon \dot{V}_\varepsilon \cdot \partial \varphi \varepsilon + p_0 \frac{\partial \varepsilon}{\partial \varphi} \text{div}_x(\varepsilon \varphi) dx dt = \int_0^T \int_\Omega \varepsilon g_\varepsilon \text{div}_x \varphi + \varepsilon \mathbb{G}_\varepsilon \cdot \varphi + \varepsilon h_\varepsilon \cdot \varphi dx dt \]

for any \( \varphi \in C^\infty_c((0, T) \times \Omega)^3 \), \( \varphi \cdot \mathbf{n} = 0 \) on \( \partial \Omega \), where we have denoted

\[ \mathbb{G}_\varepsilon := S(\partial_\varepsilon, \mathbf{D}u_\varepsilon, |B_\varepsilon|) - \varrho_\varepsilon u_\varepsilon \otimes u_\varepsilon, \quad h_\varepsilon := -\frac{1}{\mu_0} \left( \text{curl}_x \frac{B_\varepsilon}{\varepsilon} \right) \times \frac{B_\varepsilon}{\varepsilon}, \]

\[ g_\varepsilon := \frac{1}{\varepsilon^2} \left( p_0 \varrho_\varepsilon \partial_\varepsilon + \frac{\varepsilon}{3} \frac{\partial \varepsilon}{\partial \varphi} - \rho_\varepsilon \partial_\varepsilon \rho_\varepsilon \right) - p_0 \varepsilon \frac{\partial \varepsilon}{\varrho_\varepsilon} \frac{\partial \varepsilon}{\varepsilon^2} - p_0 \tilde{\varrho} \left( \partial_\varepsilon - \tilde{\varrho} \right). \]

We observe that by virtue of (4.3) and Lemma 3.2

\[ g_\varepsilon \to 0 \quad \text{in} \quad L^1((0, T) \times \Omega), \]

\[ \{ \mathbb{G}_\varepsilon \}_{\varepsilon > 0} \quad \text{is bounded in} \quad L^q((0, T) \times \Omega)^{3 \times 3}, \]

\[ \{ h_\varepsilon \}_{\varepsilon > 0} \quad \text{is bounded in} \quad L^q((0, T) \times \Omega)^3 \]

for a certain \( q > 1 \). Notice that system (6.1), (6.2) corresponds to the weak formulation of a wave equation whose strong formulation reads

\[ \varepsilon \partial_t r_\varepsilon + \text{div}_x \varepsilon V_\varepsilon = 0, \]

\[ \varepsilon \partial_t V_\varepsilon + p_0 \frac{\partial \varepsilon}{\partial \varphi} \text{div}_x \frac{r_\varepsilon}{\varepsilon} = \varepsilon \left( \nabla \varphi \varepsilon + \text{div}_x \mathbb{G}_\varepsilon - h_\varepsilon \right). \]
We consider an eigenvalue problem associated to (6.1), (6.2), namely,
\[ \tilde{\varrho} \nabla_x \left( \omega \tilde{\varrho} \right) = \lambda Q, \quad p_0 \vartheta \text{div}_x Q = \omega \quad \text{in } \Omega, \quad Q \cdot n = 0 \quad \text{on } \partial \Omega, \] (6.4)
or, equivalently,
\[ - \text{div}_x \left( \tilde{\varrho} \nabla_x \left( \omega \tilde{\varrho} \right) \right) = \Lambda \omega \quad \text{in } \Omega, \quad \lambda^2 = - \Lambda p_0 \vartheta \] (6.5)
supplemented with the Neumann boundary condition
\[ \nabla_x \left( \frac{\omega}{\varrho} \right) \cdot n = 0 \quad \text{on } \partial \Omega. \] (6.6)

Similarly to Chapter 3 in [55] and in agreement with the general theory of compact symmetric operators on Hilbert spaces, it is a routine matter to check that problem (6.5), (6.6) possesses a complete system of real eigenfunctions \( E := \{ \omega_{j,m} \}_{j=0, m=1}^{\infty} \), together with the associated real eigenvalues \( \Lambda_{j,m} \) such that
\[ m_0 = 1, \quad \Lambda_{0,1} = 0, \quad \omega_{0,1} = \left( \int_\Omega \tilde{\varrho} \, dx \right)^{-1/2} \tilde{\varrho}, \quad 0 < \Lambda_{1,1} = \cdots = \Lambda_{1,m_1} (:= \Lambda_1) < \Lambda_{2,1} = \cdots = \Lambda_{2,m_2} (:= \Lambda_2) < \cdots, \] (6.7)
where \( m_j \) stands for the multiplicity of the eigenvalue \( \Lambda_j \). System \( \mathbb{E} := \{ \omega_{j,m} \}_{j=0, m=1}^{\infty} \), together with the associated real eigenvalues \( \Lambda_{j,m} \) such that
\[ \lim_{j \to \infty} \Lambda_j = \infty. \]

Consequently, we easily check by using (6.4)–(6.6) and (2.24), (2.28) that
\[ \{ Q_{j,m} \}_{j=1,m=1}^{\infty} \quad \text{with} \quad Q_{j,m} = \frac{i}{\sqrt{\Lambda_j}} \tilde{\varrho} \nabla_x \left( \frac{\omega_{j,m}}{\varrho} \right) \] (6.8)
forms an orthonormal basis in \( \mathbb{H}^1_{\tilde{\varrho}} [L^2_{1/\tilde{\varrho}}(\Omega)]^3 \), where the orthogonal weighted Helmholtz projector is defined in (2.23). Now, we take \( \varphi = \psi(t)\varpi_{j,m}/\tilde{\varrho} \) in (6.1), and \( \varphi = \psi(t)\mathbf{Q}_{j,m}/\tilde{\varrho} \) in (6.2) in order to obtain a system of equations:
\[ \varepsilon \partial_t [r_{\varepsilon}]_{j,m} - i \sqrt{\Lambda_j} [V_{\varepsilon}]_{j,m} = 0, \quad \varepsilon \partial_t [V_{\varepsilon}]_{j,m} - i p_0 \vartheta \sqrt{\Lambda_j} [r_{\varepsilon}]_{j,m} = \varepsilon H_{\varepsilon}^{j,m}, \] (6.9, 6.10)
\[ j = 1, 2, \ldots, m = 1, \ldots, m_j, \] where we have set
\[ [r_{\varepsilon}]_{j,m} := \int_\Omega r_{\tilde{\varrho}} \varpi_{j,m} \frac{1}{\varrho} \, dx, \quad [V_{\varepsilon}]_{j,m} := \int_\Omega V_{\varepsilon} \cdot \mathbf{Q}_{j,m} \frac{1}{\varrho} \, dx. \] (6.11)

Here, in accordance with (6.3)
\[ \{ H_{\varepsilon}^{j,m} \}_{j=1}^{\infty} \quad \text{is bounded in } L^1(0, T). \] (6.12)
6.2 Analysis of the convective term

Having established all the necessary preliminary material, we are ready to show that the term $\mathbf{u} \times \mathbf{u}$ can be replaced by $U \times U$ in the momentum equation (4.13), i.e. we shall show that

$$\int_0^T \int_\Omega \varrho \mathbf{e} (\mathbf{u}_e \otimes \mathbf{u}_e) : \nabla \varphi \, dx dt \to \int_0^T \int_\Omega \varrho \mathbf{u} \otimes \mathbf{U} : \nabla \varphi \, dx dt$$

for any test function $\varphi$ such that

$$\varphi \in C_c^\infty ([0,T) \times \overline{\Omega})^3, \quad \varphi \cdot \mathbf{n} = 0 \text{ on } \Omega, \quad \nabla_x (\varrho \varphi) = 0 \text{ in } \Omega.$$ (6.14)

To begin, we split the expression in the convective term as follows

$$\varrho \mathbf{e} \mathbf{u}_e \otimes \mathbf{u}_e = \mathbf{H}_\varrho [\varrho \mathbf{e} \mathbf{u}_e \otimes \mathbf{u}_e] + \frac{1}{\varrho} \mathbf{H}_{\varrho}^2 [\varrho \mathbf{e} \mathbf{u}_e] \otimes \mathbf{H}_\varrho [\varrho \mathbf{e} \mathbf{u}_e] + \frac{1}{\varrho} \mathbf{H}_{\varrho}^3 [\varrho \mathbf{e} \mathbf{u}_e] \otimes \mathbf{H}_{\varrho}^2 [\varrho \mathbf{e} \mathbf{u}_e]$$

and investigate each of terms in this decomposition separately. Employing (4.1) and continuity properties of $\mathbf{H}_\varrho$ and $\mathbf{H}_{\varrho}^2$ we infer

$$\mathbf{H}_\varrho [\varrho \mathbf{e} \mathbf{u}_e] \to \mathbf{H}_\varrho [\varrho \mathbf{e} \mathbf{U}] = \varrho \mathbf{U} \text{ in } L^\infty (0,T; L^{5/4} (\Omega)^3),$$

$$\mathbf{H}_{\varrho}^2 [\varrho \mathbf{e} \mathbf{u}_e] \to 0 \text{ in } L^2 (0,T; W^{1,2} (\Omega)^3),$$

$$\mathbf{H}_{\varrho}^3 [\varrho \mathbf{e} \mathbf{u}_e] \to 0 \text{ in } L^\infty (0,T; L^2 (\Omega)^3).$$

Uniform estimates established in Lemma 3.2 together with estimate (4.3) implemented in the momentum equation (2.6) with the pressure and driving terms written according to (4.3) and (4.9) imply that the mapping

$$t \in [0,T] \mapsto \int_\Omega \mathbf{H}_\varrho [\varrho \mathbf{e} \mathbf{u}_e (t)] \cdot \varphi \, dx = \int_\Omega \varrho \mathbf{e} \mathbf{u}_e (t) \cdot \mathbf{H}_\varrho [\varrho \mathbf{e} \varphi] \frac{dx}{\varrho}$$

is equi-uniformly continuous in $C[0,T]$ if $\varphi \in C_c^\infty (\overline{\Omega})^3$, $\varphi \cdot \mathbf{n} = 0$ on $\partial \Omega$. Thus, with the Arzela-Ascoli Theorem, the separability of $L^5 (\Omega)$ and the diagonalisation procedure, we arrive at

$$\mathbf{H}_\varrho [\varrho \mathbf{e} \mathbf{u}_e] \to \mathbf{H}_\varrho [\varrho \mathbf{e} \mathbf{U}] = \varrho \mathbf{U} \text{ in } C_{\text{weak}} ([0,T]; L^2 (\Omega)^3)$$

in agreement with (4.1). Finally, the compact embedding $L^{5/4} (\Omega) \hookrightarrow W^{-1,2} (\Omega)$ provides

$$\mathbf{H}_\varrho [\varrho \mathbf{e} \mathbf{u}_e] \to \varrho \mathbf{U} \text{ in } L^2 (0,T; W^{-1,2} (\Omega)^3).$$

(6.20)

Now, given (4.1), we readily check that

$$\mathbf{H}_\varrho [\varrho \mathbf{e} \mathbf{u}_e] \otimes \mathbf{u}_e \to \varrho \mathbf{U} \otimes \mathbf{U} \text{ in } L^2 (0,T; L^{30/29} (\Omega)^3 \times \Omega^3).$$

Moreover, as $\mathbf{H}_\varrho$ and $\mathbf{H}_{\varrho}^2$ are orthogonal projections in the weighted space $L^2_{1/\varrho} (\Omega)^3$, we find with the help of (4.1) and (6.20) that

$$\mathbf{H}_\varrho [\varrho \mathbf{e} \mathbf{u}_e] \to \varrho \mathbf{U} \text{ in } L^2 (0,T; L^2 (\Omega)^3),$$

which in combination with (6.18) and (4.1) yields

$$\mathbf{H}_{\varrho}^2 [\varrho \mathbf{e} \mathbf{u}_e] \otimes \mathbf{H}_\varrho [\varrho \mathbf{e} \mathbf{u}_e] \to 0 \text{ in } L^2 (0,T; L^{20/29} (\Omega)^3 \times \Omega^3).$$
Coming back to (6.13)–(6.15) we easily see that the problem reduces to showing
\[
\int_0^T \int_\Omega H_\varphi^1 [\varrho_\varepsilon u_\varepsilon] \otimes H_\varphi^1 [\varrho u_\varepsilon] : \nabla_x \varphi \frac{1}{\varrho} \, dx \, dt \to 0 \quad \text{as } \varepsilon \to 0
\] (6.21)
for any test function \( \varphi \) satisfying (6.14). Now we set, for any \( Z \in L^1(\Omega)^3 \),
\[
\left\{ H_\varphi^1 [Z] \right\}_M := \sum_{\{j:0<\Lambda_j\leq M\}} \sum_{m=1}^{m_j} [Z]_{j,m} Q_{j,m}, \quad \text{where} \quad [Z]_{j,m} = \int_\Omega Z \cdot \tilde{Q}_{j,m} \frac{1}{\varrho} \, dx,
\] cf. (6.11). A straightforward computation yields
\[
H_\varphi^1 [\varrho_\varepsilon u_\varepsilon] \otimes H_\varphi^1 [\varrho u_\varepsilon] = \left( \left\{ H_\varphi^1 [\varrho_\varepsilon u_\varepsilon] \right\}_M + \left\{ H_\varphi^1 [\varrho_\varepsilon u_\varepsilon] - \left\{ H_\varphi^1 [\varrho_\varepsilon u_\varepsilon] \right\}_M \right\} \otimes \left( \left\{ H_\varphi^1 [\varrho u_\varepsilon] \right\}_M + \left\{ H_\varphi^1 [\varrho u_\varepsilon] - \left\{ H_\varphi^1 [\varrho u_\varepsilon] \right\}_M \right\} \right) \right),
\] (6.22)
where
\[
H_\varphi^1 [\varrho_\varepsilon u_\varepsilon] - \left\{ H_\varphi^1 [\varrho_\varepsilon u_\varepsilon] \right\}_M =
= H_\varphi^1 [(\varrho_\varepsilon - \varrho) u_\varepsilon] - \left\{ H_\varphi^1 [(\varrho_\varepsilon - \varrho) u_\varepsilon] \right\}_M + H_\varphi^1 [\varrho u_\varepsilon] - \left\{ H_\varphi^1 [\varrho u_\varepsilon] \right\}_M.
\]
Here, by virtue of (4.1),
\[
H_\varphi^1 [(\varrho_\varepsilon - \varrho) u_\varepsilon] - \left\{ H_\varphi^1 [(\varrho_\varepsilon - \varrho) u_\varepsilon] \right\}_M \to 0 \quad \text{in } L^1(0,T; L^1(\Omega)^3).
\]
On the other hand, we may use the orthogonality of functions \( \omega_{j,m} \), together with Parseval’s identity with respect to the scalar product of \( L^2_{1/\varrho}(\Omega) \), to find
\[
\| \text{div}_x(\tilde{\varrho} u_\varepsilon) \|_{L^2_{1/\varrho}(\Omega)}^2 = \sum_{j=1}^\infty \sum_{m=1}^{m_j} \Lambda_j [\tilde{\varrho} u_\varepsilon]_{j,m}^2 \quad \text{as well as}
\]
\[
\| H_\varphi^1 [\tilde{\varrho} u_\varepsilon] - \left\{ H_\varphi^1 [\tilde{\varrho} u_\varepsilon] \right\}_M \|_{L^2_{1/\varrho}(\Omega)}^2 = \sum_{\{j:1<\Lambda_j>M\}} \sum_{m=1}^{m_j} [u_\varepsilon]_{j,m}^2 \leq \frac{1}{M} \| \text{div}_x(\tilde{\varrho} u_\varepsilon) \|_{L^2_{1/\varrho}(\Omega)}^2.
\]
We are allowed to conclude that for the limit \( M \to \infty \), uniformly in \( \varepsilon \) we find
\[
H_\varphi^1 [\tilde{\varrho} u_\varepsilon] - \left\{ H_\varphi^1 [\tilde{\varrho} u_\varepsilon] \right\}_M \to 0 \quad \text{in } L^2(0,T; L^2_{1/\varrho}(\Omega)^3).
\]
In the light of the previous arguments, the proof of (6.21) reduces to showing that
\[
\int_0^T \int_\Omega \left\{ H_\varphi^1 [\varrho_\varepsilon u_\varepsilon] \right\}_M \otimes \left\{ H_\varphi^1 [\varrho u_\varepsilon] \right\}_M : \nabla_x \varphi \frac{dx}{\varrho} \, dt \to 0
\]
or, equivalently,
\[
\int_0^T \int_\Omega \left\{ H_\varphi^1 [\varrho_\varepsilon u_\varepsilon] \right\}_M \otimes \left\{ H_\varphi^1 [\varrho_\varepsilon u_\varepsilon] \right\}_M : \nabla_x \varphi \frac{dx}{\varrho} \, dt \to 0
\] (6.23)
for all \( \phi \) satisfying \((6.14)\) and for any fixed \( M \). In order to show \((6.23)\), we first observe that, by means of \((6.8)\) and \((6.11)\),

\[
\int_0^T \int_\Omega \left\{ H^\xi_0 [\varrho \varepsilon u_\varepsilon] \right\}_M : \nabla \phi \frac{d\varepsilon}{\varepsilon} \, dt = \\
= \int_0^T \int_\Omega (\varrho \nabla \psi_\varepsilon \otimes \nabla \psi_\varepsilon) : \nabla \psi_\varepsilon \, dx \, dt,
\]

where

\[
\psi_\varepsilon = i \sum_{\{j | \lambda_j \leq M \}} \sum_{m=1}^{j_m} \frac{[V_\varepsilon]_{j,m}}{\sqrt{\lambda_j}} \left( \frac{\omega_{j,m}}{\varepsilon} \right).
\]

Integrating by parts and using the fact that \( \psi \) is a solenoidal function, we notice

\[
- \int_0^T \int_\Omega \nabla_x (\varrho \nabla \psi_\varepsilon) \nabla_x \psi_\varepsilon \cdot \varphi \, dx \, dt = - \int_0^T \int_\Omega \nabla_x (\varrho \nabla \psi_\varepsilon) \nabla_x \psi_\varepsilon \cdot \varphi \, dx \, dt,
\]

where, in accordance with \((6.5)\),

\[
- \nabla_x (\varrho \nabla \psi_\varepsilon) = i \sum_{\{j | \lambda_j \leq M \}} \sum_{m=1}^{j_m} \sqrt{\lambda_j} [V_\varepsilon]_{j,m} \omega_{j,m}.
\]

It is only now when we use the fact that the quantities \([V_\varepsilon]_{j,m}\) satisfy the acoustic equation \((6.9)\), \((6.10)\) yielding

\[
- \int_0^T \int_\Omega \nabla_x (\varrho \nabla \psi_\varepsilon) \nabla_x \psi_\varepsilon \cdot \varphi \, dx \, dt = \varepsilon \int_0^T \int_\Omega \sum_{\{j | \lambda_j \leq M \}} \sum_{m=1}^{j_m} \partial_t [r_\varepsilon]_{j,m} \omega_{j,m} \nabla_x \psi_\varepsilon \cdot \varphi \, dx \, dt \\
= -\varepsilon \int_0^T \int_\Omega \sum_{\{j | \lambda_j \leq M \}} \sum_{m=1}^{j_m} \omega_{j,m} ([r_\varepsilon]_{j,m} \nabla_x \psi_\varepsilon) \cdot \partial_t \varphi \, dx \, dt - \\
- \varepsilon \int_0^T \int_\Omega \sum_{\{j | \lambda_j \leq M \}} \sum_{m=1}^{j_m} \omega_{j,m} [r_\varepsilon]_{j,m} \partial_t \nabla_x \psi_\varepsilon \cdot \varphi \, dx \, dt.
\]

Thus in order to complete the proof of \((6.13)\), it is enough to show that

\[
\varepsilon \left| \int_0^T \int_\Omega \sum_{\{j | \lambda_j \leq M \}} \sum_{m=1}^{j_m} \omega_{j,m} [r_\varepsilon]_{j,m} \partial_t \nabla_x \psi_\varepsilon \cdot \varphi \, dx \, dt \right| \rightarrow 0. \tag{6.24}
\]

To this end, we make use of equation \((6.10)\) to obtain

\[
\varepsilon \partial_t \nabla_x \psi_\varepsilon = -p_0 \overrightarrow{\nabla} \sum_{\{j | \lambda_j \leq M \}} \sum_{m=1}^{j_m} [r_\varepsilon]_{j,m} \nabla_x \left( \frac{\omega_{j,m}}{\varepsilon} \right) + \varepsilon \sum_{\{j | \lambda_j \leq M \}} \sum_{m=1}^{j_m} \frac{1}{\varepsilon} H^\xi_{j,m} Q_{j,m},
\]

where \( H^\xi_{j,m} \) satisfy \((6.12)\). Finally, as \( \psi \) belongs to the class \((6.14)\)

\[
\int_0^T \int_\Omega \left( \sum_{\{j | \lambda_j \leq M \}} \sum_{m=1}^{j_m} [r_\varepsilon]_{j,m} \left( \frac{\omega_{j,m}}{\varepsilon} \right) \right) \left( \sum_{\{j | \lambda_j \leq M \}} \sum_{m=1}^{j_m} [r_\varepsilon]_{j,m} \nabla_x \left( \frac{\omega_{j,m}}{\varepsilon} \right) \right) \cdot (\tilde{\varphi} \varphi) \, dx \, dt = 0.
\]

Hence, \((6.24)\), and, consequently \((6.23)\) as well as \((6.13)\), hold true. This completes the proof of Theorem \(2.1\).
References


Rigorous derivation of the anelastic approximation to OBA


