Talk . Barcodes in Novikov theory

Sonntag, 28. Juni 2020 16:41

1. Novikov theory 2. Barcodes in Nov. theory 3 Stability $\oplus$ classification

Part 1 Novikov-theory

Recall Let (M,g) a riem. mfd. $f: M \to \mathbb{R}$ / df is Morse
$\Leftrightarrow$ $df \pitchfork 0$ in T^*M $\Leftrightarrow$ $\mathrm{Hess}_f(x) \neq 0$ $\forall x \in \mathrm{Crit}(f)$.
If in addition (f,g) is a Morse-Smale pair on M, then we get
$HM_*(M,f) \cong H_*(M,\mathbb{Z})$

Q: What happens if we choose instead of f a closed one form α?

Def: α closed one form is Morse $\Leftrightarrow$ $\forall x \in \mathrm{Crit}\,\alpha$ $\exists$ ngh. U of x s.t. $\exists$ Morse fct. $f: U \to \mathbb{R}$ s.t. $\alpha|_U = df$.

We can asso. to α a grad. vf X_α.

$\gamma: \mathbb{R} \to M$ neg. grad. flow line of $\alpha \Leftrightarrow \gamma$ solves $\dot{\gamma} = -X_\alpha(\gamma)$ $(*)$

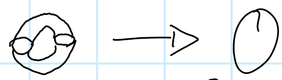
(α,g) Morse-Smale, then define for $x,y \in \mathrm{Crit}(\alpha)$:

$\widetilde{\mathcal{M}}(x,y) = \{\gamma: \mathbb{R} \to M \mid \gamma(t) \xrightarrow{t\to\infty} y,\ \gamma(t) \xrightarrow{t\to-\infty} x,\ \gamma \text{ solves } (*)\}$

$\dim \widetilde{\mathcal{M}}(x,y) = \mathrm{ind}\, x - \mathrm{ind}\, y$.

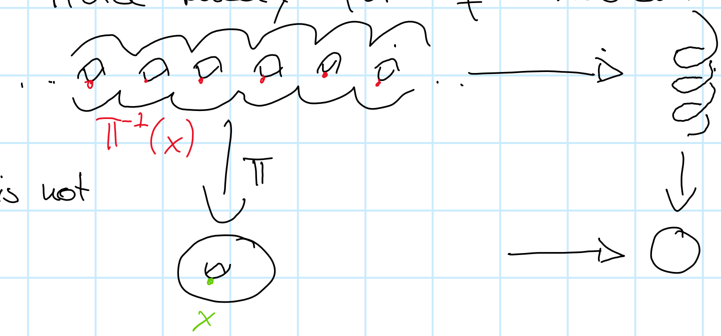
Since $\lim_{t\to\pm\infty} \gamma(t) = \lim_{t\to\pm\infty} \gamma(t+s)$ $\forall s \in \mathbb{R} \Rightarrow \mathbb{R} \overset{\text{free :)}}{\curvearrowright} \widetilde{\mathcal{M}}(x,y)$.

$\to \mathcal{M}(x,y) := \widetilde{\mathcal{M}}(x,y)/\mathbb{R}$, $\dim \mathcal{M}(x,y) = \mathrm{ind}\, x - \mathrm{ind}\, y - 1$.

Problem: α is a multivalued fct, includes f: 

$\longrightarrow$ Choose min. cover $\pi: \tilde{M} \to M$ s.t. $\pi^*\alpha = d\tilde{f}$.

So we want to do Morse theory for $\tilde{f}$. Problem $\tilde{M}$ might be non compact.



$\Rightarrow$ $\bigoplus_{\tilde{x} \in \mathrm{Crit}_k \tilde{f}} \tilde{x} \cdot \mathbb{Z}$ is not fin. gen.

$\to$ $\bigoplus_{\tilde{x} \in \mathrm{Crit}_k \tilde{f}} \tilde{x} \cdot \mathbb{Z}$ and $\prod_{\tilde{x} \in \mathrm{Crit}_k \tilde{f}} \tilde{x} \cdot \mathbb{Z}$ don't works

Since the Morse index as the Morse-Smale prop. are local

Since the Morse index as the Morse-Smale prop. are local prop. they behave nicely under π.

The right covering

- $\Phi_\alpha : \pi_1(M) \to \mathbb{R} : [\gamma] \mapsto \int_\gamma \alpha$ is well def.
- $\pi : \hat{M} \to M$ min. s.t. $\pi^* \alpha$ is exact. and $\pi_\# \pi_1 \hat{M} = \ker \Phi_\alpha$

$\rightsquigarrow$ Deck $\hat{M}$ is fin. gen., abelian moreover $\pi_1 M / \pi_1 \hat{M} = \text{Deck}\,\tilde{M} =: \Gamma_\alpha$

$\Rightarrow$ The map $\chi_\alpha : \Gamma_\alpha \to \mathbb{R} : A \mapsto \int_A \alpha$ is well def.

Fix such a covering.

Note that: Let $g \in \Gamma_\alpha \curvearrowright \tilde{M}$, then $\tilde{f}(g \cdot \tilde{x}) - f(\tilde{x}) = \chi_\alpha(g)$.

Define $\quad CN_k(M, \alpha) \subseteq \prod_{\tilde{x} \in \operatorname{crit}_k \tilde{f}} \tilde{x} \cdot \mathbb{Z}$ s.t.

$$\sum_{\tilde{x} \in \operatorname{Crit}_k \tilde{f}} a_{\tilde{x}} \tilde{x} \in CN_k(M, \alpha) \iff \forall c \in \mathbb{R} : \#\{\tilde{x} \mid a_{\tilde{x}} \neq 0, \tilde{f}(\tilde{x}) > c\} < \infty$$

Problem CN_k in gen. not fin. gen. as $\mathbb{Z}$-module.

$\to$ solution other coefficient ring.

Def.: The Novikov ring $\Lambda^{\Gamma_\alpha, \mathbb{Z}/K}$ over the $\mathbb{Z}$ Field K is def. as

$$\Lambda^{\Gamma_\alpha, \mathbb{Z}/K} \subseteq \prod_{A \in \Gamma_\alpha} \mathbb{Z}/K \cdot A \quad \text{s.t.}$$

$$\sum \underset{\mathbb{Z}}{\lambda_A} \cdot \underset{\Gamma_\alpha}{A} \in \Lambda^{\Gamma_\alpha, \mathbb{Z}} \iff \forall c \in \mathbb{R}\ \#\{A \mid \lambda_A \neq 0 \text{ and } \chi_\alpha(A) < c\} < \infty$$

Note 1) $\Gamma_\alpha \curvearrowright \Lambda^{\Gamma_\alpha, \mathbb{Z}} \Rightarrow \Gamma_\alpha \curvearrowright CN_k(M, \alpha, \Lambda^{\Gamma_\alpha, \mathbb{Z}}) =: CN_k(M, \alpha)$

2) $\mathbb{Z}[\Gamma^\alpha] \subseteq \Lambda^{\Gamma_\alpha, \mathbb{Z}}$

3) Identifying $\Gamma_\alpha \cong \{g = \int_A \alpha \mid A \in \Gamma_\alpha\}$, T formal var.

$$\Lambda^{\Gamma_\alpha, \mathbb{Z}} \cong \{\textstyle\sum a_g \cdot T^g \mid \forall c \in \mathbb{R} : \#\{g \mid a_g \neq 0 \text{ and } g < c\} < \infty\}$$

Thm.: $\quad CN_k(M, \alpha) \cong \bigoplus_{\tilde{x} \in \operatorname{Crit}_k \tilde{f}} \tilde{x} \cdot \Lambda^{\Gamma_\alpha, \mathbb{Z}}$, where each

$\tilde{x}$ is a repr. of a fiber $\pi^{-1}\{\pi(\tilde{x})\}$.

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Thm: $(CN_*(M,\alpha), \partial)$ is a chain cpx, where ∂ is def as

$$\partial \tilde{x} := \sum_{\substack{\tilde{y} \in \mathrm{Crit}\tilde{f} \\ \mathrm{ind}\,\tilde{y} = \mathrm{ind}\,\tilde{x} - 1}} \# \mathcal{M}(\tilde{x}, \tilde{y}) \cdot \tilde{y}$$

↑ moduli space of neg. grad flow lines of $\tilde{f}_0$

Idea: show that you're in compact window in $\tilde{M}$.

Main thm.: 1) $HN_*(M,\alpha) := H_*(CN_*(M,\alpha), \partial)$ dep. only on the cove. and the coh. class of α.

2) $HN_*(M,[\alpha]) \cong H_*(C_*^{cell}(\tilde{M}) \otimes_{\mathbb{Z}[\Gamma_\alpha]} \Lambda^{\Gamma_\alpha, \mathbb{Z}})$

Filtration as in the Morse-case?

From now on choose $\Lambda^{\Gamma_\alpha, \Bbbk}$, $\Bbbk$ field as coeff. ring.

- $HM_*^t(M,f) := HM_*(\{f<t\}, f) \cong H_*(\{f<t\})$. $t \in \mathbb{R}$.
- $s<t \Rightarrow HM_*^s(M,f) \longrightarrow HM_*^t(M,f)$

Reasons why class. barcode theory can't applied to Novikov theory:

- Finite typ pers. modules $\mathbb{V}$ over a field $\Bbbk$ can be understood in terms of class. of fin.-gen. $\Bbbk[x]$-modules.
  If $\Gamma \neq \{0\}$ we have that $HN_*(M,\alpha)$ is inf. gen. over $\Bbbk$, leading to inf. gen. $\Bbbk[x]$-modules.
- Since we have $\Gamma_\alpha \circlearrowright CN_*(M,\alpha)$ and $\tilde{f}(g\tilde{x}) - \tilde{f}(\tilde{x}) = \chi_\alpha(g)$ $\forall g \in \Gamma_\alpha$, we get Γ_α don't preserve a filtration on CN_* and HN_*.
- Instead working over $\Lambda_{\geq 0}$, cons. of all elm. with $\chi_\alpha(g) \geq 0$, and if $\Gamma_\alpha \neq \{0\}$ and discrete $\Rightarrow \Lambda_{\geq 0} \cong \Bbbk[[t]]$ ← form. pow. series
  $\rightsquigarrow H_*^t(CN_*(M,\alpha,\Lambda_{\geq 0}))$ fin. gen. $\Lambda_{\geq 0}$-module
  $\rightarrow$ cons. of fin. gen $\Bbbk[[t]][x]$-modules ← Not PID.

Barcodes over Novikov fields $\quad \Lambda = \Lambda^{\Gamma_{\alpha},k}$

Def.: • let $\ell_\alpha : CN_k(M,\alpha) \to \mathbb{R}$; $\sum_i a_{\tilde{x}} \tilde{x} \mapsto \max\{\tilde{f}(\tilde{x})\}$.

• $CN_{k+1} \cong \bigoplus_{\substack{\operatorname{ind}\tilde{y}=k+1 \\ i=1}}^{n} \tilde{y} \cdot \Lambda$, $\quad CN_k \cong \bigoplus_{\substack{\operatorname{ind}\tilde{x}=k \\ i=1}}^{m} \tilde{x} \cdot \Lambda$

• $\partial_{k+1} : CN_{k+1} \to \ker \partial_k \subseteq CN_k$, $\quad \operatorname{rank} \partial_{k+1} =: r$.

The barcode of $(CN_*(M,\alpha), \partial, \ell_\alpha)$ is the multiset of elements of $\mathbb{R}/\Gamma_\alpha \times [0,\infty]$ cons. of

1) pair $(\ell(x_i) \bmod \Gamma_\alpha, \ell(y_i) - \ell(x_i))$ for $i = 1, \dots, r$

2) pair $(\ell(x_i) \bmod \Gamma_\alpha, \infty)$ for $i = r+1, \dots, m$

Such a elm. we denote be $([a], L) \in \mathbb{R}/\Gamma_\alpha \times [0,\infty]$

Note that the barcode is independent of the choosen lifts of $\tilde{y}, \tilde{x}$.

b/c $\quad \tilde{f}(g \cdot \tilde{x}) - \tilde{f}(\tilde{x}) = x_\alpha(g)$, $\quad \tilde{f}(g\tilde{y}) - \tilde{f}(\tilde{y}) = x_\alpha(g)$

So we get, if $\tilde{x}_1, \tilde{x}_2$ in the same fiber and $\tilde{y}_1, \tilde{y}_2$

$\rightsquigarrow \quad \tilde{f}(\tilde{x}_2) - \tilde{f}(\tilde{y}_2) = \tilde{f}(\tilde{x}_1) - \tilde{f}(\tilde{y}_1)$.

Example $\Gamma = \{0\}$: barcode above $\qquad$ class. barcode

$$(\ell(x_i), \ell(y_i) - \ell(x_i)) \longleftrightarrow [\ell(x_i), \ell(y_i))$$
$$(\ell(x_i), \infty) \longleftrightarrow [\ell(x_i), \infty)$$

Main thm.: $(CN_*(M,\alpha), \partial, \ell_\alpha)$

barcode $(\ell(x) \bmod \Gamma, \infty)$

$$\cong \bigoplus_{k\in\mathbb{Z}} \Big[\Big(\bigoplus_{\tilde{x}\in\operatorname{Crit}_k \tilde{f}} \to 0 \to \operatorname{span}_\Lambda\{\tilde{x}\} \to 0 \to \dots \bmod \Gamma \Big)$$

$$\bigoplus_{\tilde{y}\in\operatorname{Crit}_{k+1}\tilde{f}} \Big(\dots 0 \to \operatorname{span}_\Lambda\{\tilde{y}\} \to \operatorname{span}_\Lambda\{\partial\tilde{y}\} \to 0 \to \dots \Big),$$

barcode $(\ell(\partial y) \bmod \Gamma, \ell(y) - \ell(\partial y))$

Rmk.: Plug in $\Gamma = \{0\}$ we get $HN_* = HM_*$ and the barcode for the Novikov theory is the classical barcode :).

Properties of Novikov barcodes

filtered chain isomorphic

Thm A: $(CN_*(M,\alpha), \partial, \ell_\alpha) \cong (CN_*(N,\beta), \partial, \ell_\beta) \quad (\Gamma_\alpha = \Gamma_\beta)$

Thm A: $(CN_*(M,\alpha),\partial,\ell_\alpha) \overset{\text{filt.}}{\cong} (CN_*(N,\beta),\partial,\ell_\beta)$ $(\Gamma_\alpha=\Gamma_\beta)$
$\Longleftrightarrow$ $\mathcal{B}_{CN_*(M,\alpha),k} = \mathcal{B}_{CN_*(N,\beta),k}$ $\forall k\in\mathbb{Z}$. (verbose barcodes)

Rmk: $(CN_*(M,\alpha),\partial,\ell_\alpha)$ filt. chain iso to $(CN_*(N,\beta),\partial,\ell_\beta)$ $(\Gamma_\alpha=\Gamma_\beta)$ iff
$\exists\, \Phi: CN_*(M,\alpha) \longrightarrow CN_*(N,\beta)$ chain iso s.t.
$$\ell_\beta(\Phi(x)) = \ell_\alpha(x) \quad \forall x\in CN_*(M,\alpha).$$

Sketch of proof: After choosin representatives of the fibers leads to
$CN_k(M,\alpha) = \bigoplus_{\substack{\tilde{x}_i\in \mathrm{Crit}_k \tilde{f}_\alpha \\ i=1}}^{n} \tilde{x}_i \cdot \Lambda^{\Gamma_\alpha,k}$, $CN_k(N,\beta) = \bigoplus_{\substack{\tilde{y}_i\in \mathrm{Crit}_k \hat{f}_\beta \\ i=1}}^{n} \tilde{y}_i\cdot\Lambda^{\Gamma_\alpha,k}$ in each degree

Let $\Phi: CN_*(M,\alpha) \longrightarrow CN_*(N,\beta)$ a filtered chain iso.
$\leadsto$ Φ maps generators of $CN_k(M,\alpha)$ to gen. of $CN_k(N,\beta)$ $\forall k$.
$\leadsto$ $\Phi(\tilde{x}_i) = \tilde{y}_i$ and we have since Φ is filt chain iso that $\ell_\alpha(\tilde{x}_i) = \ell_\beta(\Phi(\tilde{x}_i)) = \ell_\beta(y_i)$
So we get on Barcode level:
$$(\ell_\alpha(x_i) \bmod \Gamma_\alpha, \ell_\alpha(x_i)-\ell_\alpha(\partial x_i)) = (\ell_\beta(y_i) \bmod \Gamma_\alpha, \ell_\beta(y_i)-\ell_\beta(\partial y_i))$$
$$(\ell_\alpha(x_i) \bmod \Gamma_\alpha, \infty) = (\ell_\beta(y_i) \bmod \Gamma_\beta, \infty).$$
Since the Barcode is indep. of the choosen decomp. we get
$$\mathcal{B}_{CN_*(M,\alpha),k} = \mathcal{B}_{CN_*(N,\beta),k} \quad \forall k.$$

"$\Longleftarrow$" follows direct from the main thm in the prev. chapter. $\square$

Thm B: $(CN_*(M,\alpha),\partial_\alpha,\ell_\alpha) \overset{\text{filt. hom. equiv.}}{\simeq} (CN_*(N,\beta),\partial_\beta,\ell_\beta)$
$\Longleftrightarrow$ Barcodes are equal in all degrees.
Rmk: $(C_*^\alpha,\partial_\alpha,\ell_\alpha)$ filt. chain hom. equiv to $(C_*^\beta,\partial_\beta,\ell_\beta)$
$\Longleftrightarrow$ $\exists$ chain maps $\Phi: C_*^\alpha \to C_*^\beta$, $\Psi: C_*^\beta \longrightarrow C_*^\alpha$ s.t.
$\Phi\circ\Psi \sim \mathrm{id}_{C^\beta}$, $\Psi\circ\Phi \sim \mathrm{id}_{C^\alpha}$ and $\ell_\beta(y_i) = \ell_\alpha(x_i)$.

Stability of Novikov barcodes

Goal: 1) $2\cdot d_Q\big((CN_*(M,\alpha),\partial,\ell_\alpha),(CN_*(N,\beta),\partial,\ell_\beta)\big)$

Goal: 1) $2 \cdot d_Q((CN_*(M,\alpha), \partial, \ell_\alpha), (CN_*(N,\beta), \partial, \ell_\beta))$
$\geq d_B(\mathcal{B}_{CN_*(M,\alpha)}, \mathcal{B}_{CN_*(N,\beta)})$

2) $d_Q(CN_*(M,\alpha), CN_*(N,\beta)) \leq d_B(\mathcal{B}_{CN_*(M,\alpha)}, \mathcal{B}_{CN_*(N,\beta)})$

$\Rightarrow$ first we have the define d_Q, d_B

Def: (quasiequiv. distance).

- $\delta \geq 0$, $(CN_*(M,\alpha), \partial_\alpha, \ell_\alpha)$, $(CN_*(N,\beta), \partial_\beta, \ell_\beta)$ are δ-quasiequiv iff $\exists$ quadruple $(\Phi, \Psi, K_\alpha, K_\beta)$ s.t. 1) $(\Phi, \Psi, K_\alpha, K_\beta)$ is a hom. equiv.
  2) $\forall \tilde{x} \in CN_*(M,\alpha)$, $\forall \tilde{y} \in CN_*(N,\beta)$ we have
  - $\ell_\beta(\Phi(\tilde{x})) \leq \ell_\alpha(\tilde{x}) + \delta$, $\ell_\alpha(\Psi(\tilde{y})) \leq \ell_\beta(\tilde{y}) + \delta$
  - $\ell_\beta(K_\alpha(\tilde{x})) \leq \ell_\alpha(\tilde{x}) + 2\delta$, $\ell_\alpha(K_\beta(\tilde{y})) \leq \ell_\beta(\tilde{y}) + 2\delta$
- $d_Q((CN_*(M,\alpha), \partial_\alpha, \ell_\alpha), (CN_*(N,\beta), \partial_\beta, \ell_\beta))$
  $:= \inf\{\delta \geq 0 \mid (CN_*(M,\alpha), \partial_\alpha, \ell_\alpha) \,\&\, (CN_*(N,\beta), \partial_\beta, \ell_\beta)\ \delta\text{-quasi equiv.}\}$

Example: ◦ (f_1, g_1), (f_2, g_2) Morse-Smale pairs on M.
◦ $\delta := \|f_1 - f_2\|_{L^\infty}$
$\Rightarrow$ $CM_*(M, f_1)$ and $CM_*(M, f_2)$ δ-quasiequiv.

Def: (Bottleneck distance)

1) S, T multisets of elm. of $\mathbb{R}/\Gamma \times [0, \infty]$.
A δ-matching between S and T cons. of the foll. data:
(i) $\exists$ submultisets S_{short}, T_{short} s.t. $\forall ([a], L) \in S_{short} \cup T_{short}$ we have $L \leq 2\delta$
(ii) $\exists$ bij. $\sigma: S / S_{short} \to T / T_{short}$ s.t. for each $([a], L) \in S / S_{short}$ we have $\sigma([a], L) = ([a'], L')$ where
$\forall \varepsilon > 0$ the repr. a' of $[a']$ can be chosen s.t $|a' - a| \leq \delta + \varepsilon$ and either $L = L' = \infty$ or $|(a' + L') - (a + L)| \leq \delta + \varepsilon$.

2) $d_B(J, S) = \inf\{\delta \geq 0 \mid \exists\ \delta\text{-matching betw. } S \text{ and } T\}$.

3) Let $J = \{J_k\}_{k \in \mathbb{Z}}$, $S = \{S_k\}_{k \in \mathbb{Z}}$ barcodes.
$\rightsquigarrow$ $d_B(J, S) := \sup_{k \in \mathbb{Z}} d_B(J_k, S_k)$.

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<u>Stab. thm.</u>: 1) $d_B(\mathcal{B}_{CN_*(M,\alpha)}, \mathcal{B}_{CN_*(N,\beta)})$
$\leq 2 d_Q((CN_*(M,\alpha), \partial_\alpha, \ell_\alpha), (CN_*(N,\beta), \partial_\beta, \ell_\beta))$.

2) $k \in \mathbb{Z}$, $\Delta_{\beta,k} :=$ small. second. coord. of $([a'], L'), ([a'], L') \in \mathcal{B}_{CN_*(N,\beta)}$

If $d_Q((CN_*(M,\alpha), \partial_\alpha, \ell_\alpha), (CN_*(N,\beta), \partial_\beta, \ell_\beta)) < \frac{\Delta_{\beta,k}}{4} \quad \forall k \in \mathbb{Z}$

$\Longrightarrow d_B(\mathcal{B}_{CN_*(M,\alpha)}, \mathcal{B}_{CN_*(M,\beta)})$
$\leq d_Q((CN_*(M,\alpha), \partial_\alpha, \ell_\alpha), (CN_*(N,\beta), \partial_\beta, \ell_\beta))$

<u>Converse stab. thm.</u>

$d_Q((CN_*(M,\alpha), \partial_\alpha, \ell_\alpha), (CN_*(N,\beta), \partial_\beta, \ell_\beta))$
$\leq d_B(\mathcal{B}_{CN_*(M,\alpha)}, \mathcal{B}_{CN_*(N,\beta)})$.

$\Longrightarrow$ If $d_Q((CN_*(M,\alpha), \dots), (CN_*(N,\beta), \dots)) < \frac{\Delta_{k,\beta}}{4} \quad \forall k$

we have $d_Q(CN_*(M,\alpha).., CN_*(N,\beta)..) = d_B(\mathcal{B}_{CN_*(M,\alpha)}, \mathcal{B}_{CN_*(N,\beta)})$.

$\square$

<u>Rmk.</u>: All the things we've done we can do in a sim. way for the Floer cpx.

$\rightsquigarrow$ We get through the barcode theory somehow controll on how the change of the Ham. changes the number of periodic orbits. (Highly non triv.)