Localized Homology

Sonntag, 7. Juni 2020 11:28

Programme

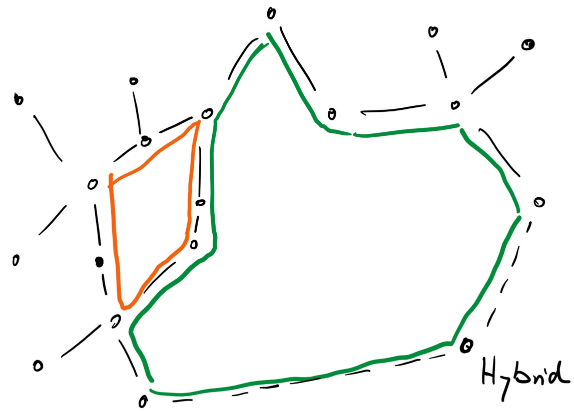
- Intro
- Measuring Cycle Sizes [Freedman-Chan 2007]
- ($\mathcal{U}$-) Localized homology [Carlsson-Zomorodian 2007]

Intro

In Data analysis we usually want to know 'where' in the data non-trivial topology arises.

- Topology of viral evolution

Recall Phylogenetic tree/network



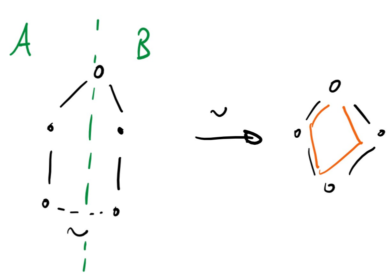
— Homoplacy — Recombination

$H_1 \neq 0 \rightsquigarrow$ obstruction for Tree

But: homology can't tell us 'where' the non-trivial topology arises!

→ For Recomb. we want to understand where Hybrids come from and which vaccines might work.

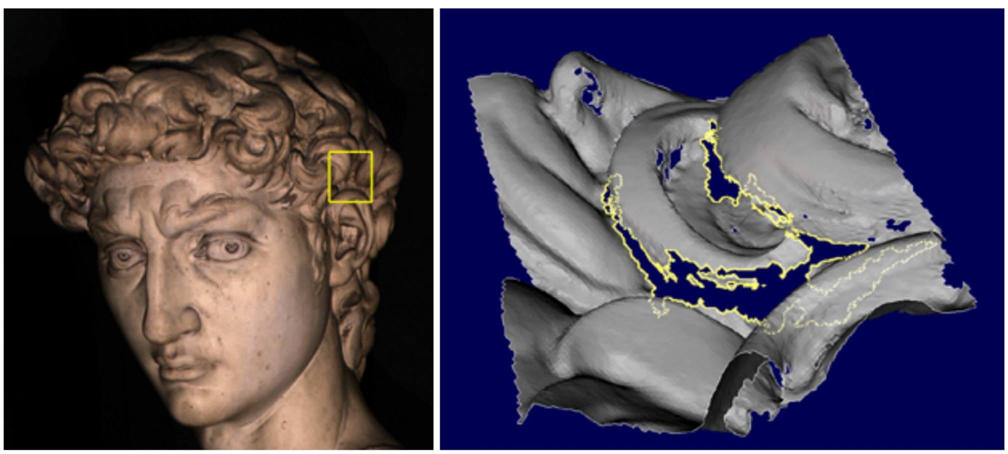
→ For homoplacies the parent / child relation still makes sense, can in principle resolve inconsistencies (e.g. w/ add. spatial information)



- Computer Vision / 3D modelling

meshes = triang. surfaces.

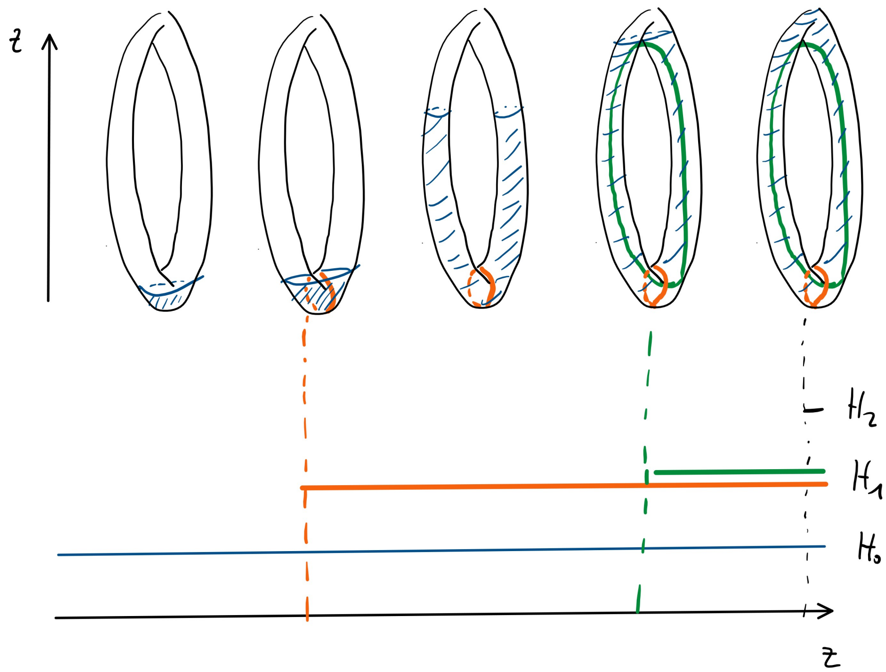
Figure 1: *Scanned meshes from Stanford 3D model repository [26]. All three meshes are valid 2-manifolds: the Buddha has genus 104, the dragon has genus 46, and David's head has genus 340. Most of these tunnels/handles are noise and can be safely removed.*



Stanford 3D modelling group

→ would like to fill in these artificial holes!
To do so we need to find cycles that sit at the boundary of a hole (or at least near one) s.t. gluing in a disk does not introduce more topology.

• Longevity ≠ Prominence

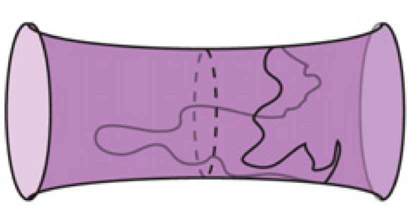


Measuring Cycle Sizes

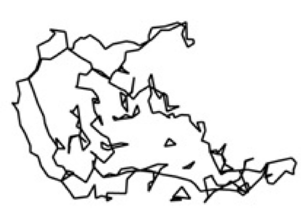
[Freedman - Chen 2007]

In this part will restrict ourselves to simpl. cplx K

Obs: algorithms yield weird representatives:

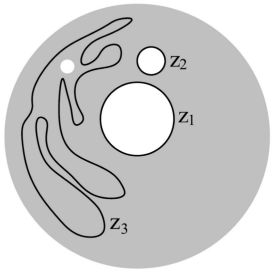


extension into irrelevant directions



multiple & irrelevant cycles

$H \ni h = \sum a_i z_i$



<u>idea:</u> Shorter paths are snugged up nicely to the 'hole' that gives rise to a homology class.

Proceed in 3 steps

(a) measure for size of hom. classes

(b) localization of representatives

(c) choice of localized basis

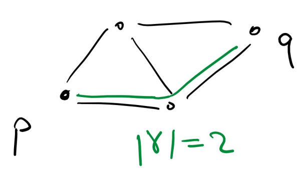
open Q: consistency of choice along pers. modules?

(a) Size of Homology Classes

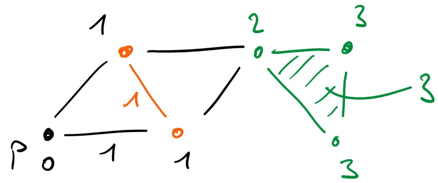
K simpl. cplx.

<u>Def.</u> <u>Geodesic distance</u>

$p, q \in \mathrm{vert}(K)$: $\quad \mathrm{dist}(p,q) = \min\limits_{\gamma} |\gamma|$ # edges

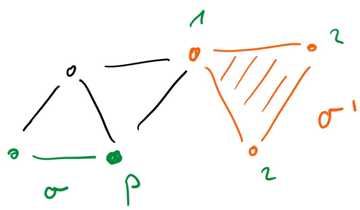


$\sigma \in K$: $\quad \mathrm{dist}(p, \sigma) = \max\limits_{q \in \mathrm{vert}\,\sigma} \mathrm{dist}(p,q)$



$$\sigma, \sigma' \in K : \quad \mathrm{dist}(\sigma, \sigma') = \min_{p \in \sigma} \mathrm{dist}(p, \sigma')$$

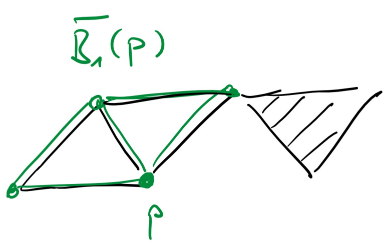
$$= \min_{p \in \sigma} \max_{q \in \sigma'} \mathrm{dist}(p, q)$$



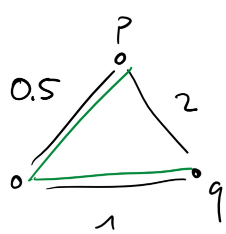
$\rightsquigarrow \mathrm{dist}(\sigma, \sigma') = 2$

Def. Geodesic Ball

$$\overline{B}_r(p) = \{ \sigma \in K \mid \mathrm{dist}(p, \sigma) \leq r \}$$



Rmks • This def. can be easily generalized to arbitrary simpl. metrics



- filtration $\mathcal{F}_t K \rightsquigarrow$ simpl. metric

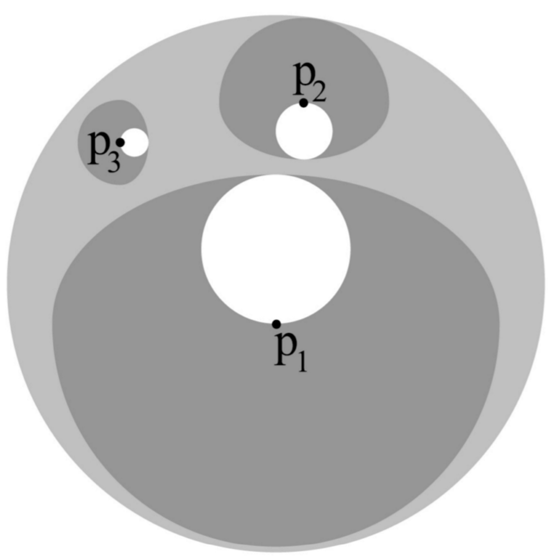
  via $|e| = \min\{t \mid e \in \mathcal{F}_t K\}$

$t = 0$ $\quad$ $t = 0.5$ $\quad$ $t = 1$ $\quad$ $t = 2$

<u>Disclaimer</u>: Didn't think this through

<u>Def.</u> $\quad \mathrm{size}\big(B_r(p)\big) := r$

Want to def. size of a homology class $h \in H_\bullet(K)$ by the size of the smallest ball that 'carries' the class.



Formalize this via relative homology

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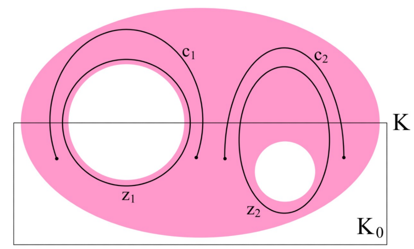
For $L \subseteq K$ subcplx. write

$$\phi_L : H_0(K) \to H_0(K, L)$$



<u>Def</u> $\quad \operatorname{size}(h) := \min_{B_r(p)} \{ r \mid \phi_{B_r(p)}(h) = 0 \}$

<u>Rmk</u> This works more generally for any choice of

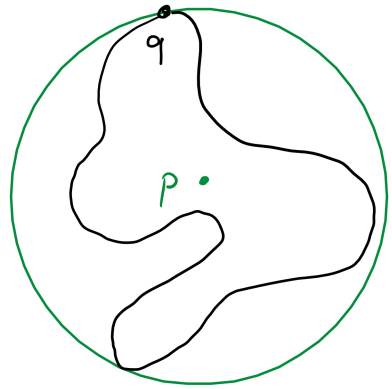
$$\mathcal{L} = \{ L \subseteq K \}, \quad \operatorname{size} : \mathcal{L} \to \mathbb{R}$$

set of subcplxs

(b) Localized Cycles

<u>Def.</u> <u>Radius</u> of a cycle $z \subseteq K$ (z repr. of $h \in H_0(K)$)

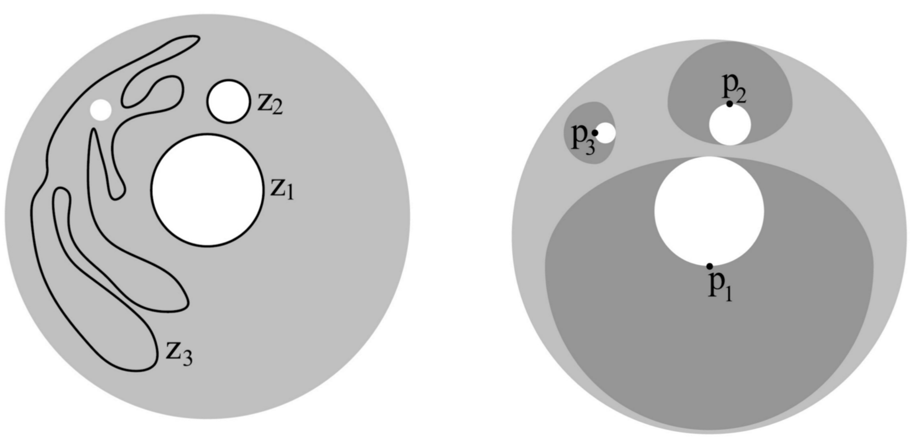
$$R(z) := \min_{p \in K} \max_{q \in z} \operatorname{dist}(p, q) \equiv \operatorname{dist}(K, z)$$



Def. A localized representative of $h \in H_0(K)$ is a cycle z_0 w/ minimal radius:

$$R(z_0) = \min_{z \in h} R(z)$$

Tracing definitions: $R(z_0) = \mathrm{size}(h)$



(c) Localized Basis

Over $\mathbb{Z}_2$ there are $2^{\beta_n} - 1$ non-triv.

Over $\mathbb{Z}_2$ there are $2^{\beta_k} - 1$ non-triv.

classes in $H_k(K)$.

A <u>localized basis</u> $\{[z_i]\}_{i=1,\dots,\beta_k}$ is one that

minimizes $\sum_{i=1}^{\beta_k} \text{size}([z_i])$. Every localized basis has

a representation by localized cycles $z_i \in [z_i]$

<u>Expl.</u> In the picture above need to choose from

$\{ [z_1], [z_2], [z_3], [z_1]+[z_2], \dots, [z_1]+[z_2]+[z_3] \}$

↳ $\{[z_1], [z_2], [z_3]\}$ is a localized basis

↳ z_1, z_2 are local. representatives. z_3 is <u>not</u>.

Freedman and Chen go on to show that there are algorithms that produce localized basis & representatives. These tend to be expensive ("high" P or even NP).

$\mathcal{O}(\beta^4 N^4)$ other measures

$\mathcal{U}$-Localized Homology [Carlsson - Zomorodian 2007]

We have seen how one can localize homology wrt. a collection of subcomplexes $\mathcal{L} = \{L \subseteq K\}$ together w/ a notion of size. The geometry / notion of locality for a given simpl. complx. is contained in this datum. It depends heavily on the real-world application.

$\to$ outsource geometry to a choice of $\mathcal{L}$. Eg. $\mathcal{L} = \{B_r(p)\}_{r \in \mathbb{R}, p \in K}$
Proceed 'topologically' from there (i.e. don't use size: $\mathcal{L} \to \mathbb{R}$)

Starting point: top. space X

cover $\mathcal{U} = \{X^i \subseteq X\}_{i \in I}$

(the cover may be provided by further insights about the system that is being analyzed, as ment. above)

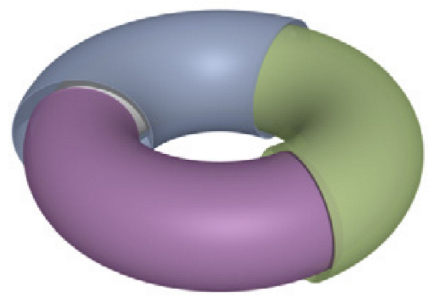
Then: $X^i \hookrightarrow X \quad \Rightarrow H_\circ(X^i) \to H_\circ(X)$

so we can call a class $h \in H_\circ(X)$ local to X^i if it is in the image of this map. More generally $\mathcal{U}$-local.

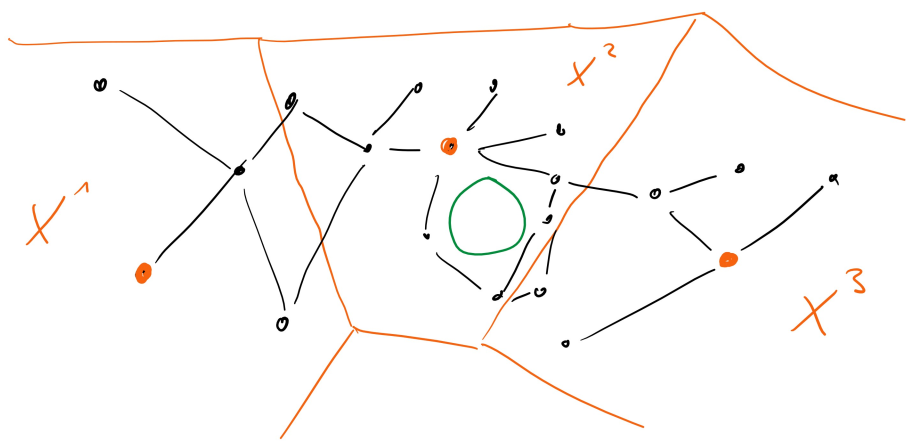
Expls

(i) $\mathcal{U}$ = hypercubes in $\mathbb{R}^n \cap X$ (refine to localize)

(ii) $\mathcal{U}$ = cylinders

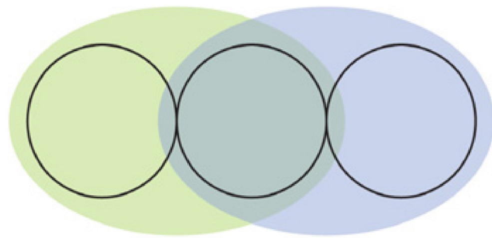


(iii) $\mathcal{U}$ = Voronoi cells around landmarks

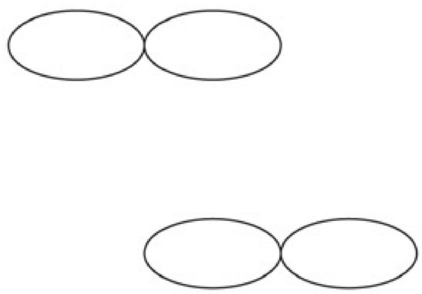


Main idea

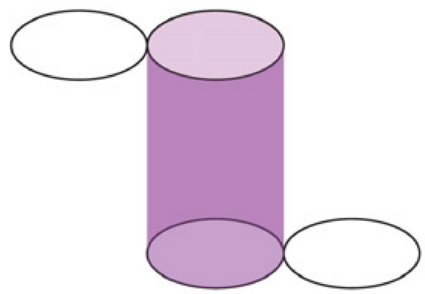
use pers. homology to "resolve $\mathcal{U}$-locality"



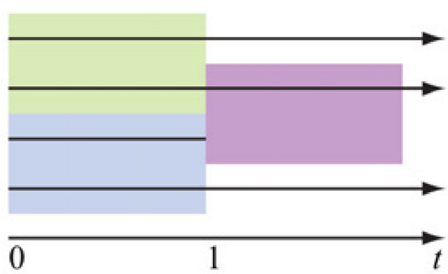
(a) Space and cover



(b) Local pieces ($t = 0$)



(c) Blowup ($t = 1$)



(d) Persistence barcode

The Mayer-Vietoris blow-up complex

X top. space, $\mathcal{U} = \{X^i \subseteq X\}_{i \in [n-1]}$

where $[k] = \{0, \dots, k\}$

$J \subseteq [n-1]$ (non-empty):

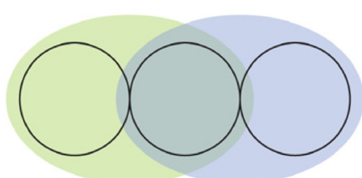
- J-face $\Delta^J \subseteq \Delta^{n-1}$ $(n-1)$-simplex

- $X^J = \bigcap_{j \in J} X^j$

<u>Def</u> (Mayer-Vietoris b↑ complex)

$$X^{\mathcal{U}} := \bigcup_{J \subseteq [n-1]} X^J \times \Delta^J \subseteq X \times \Delta^{n-1}$$

<u>Expls</u>

- 

(a) Space and cover

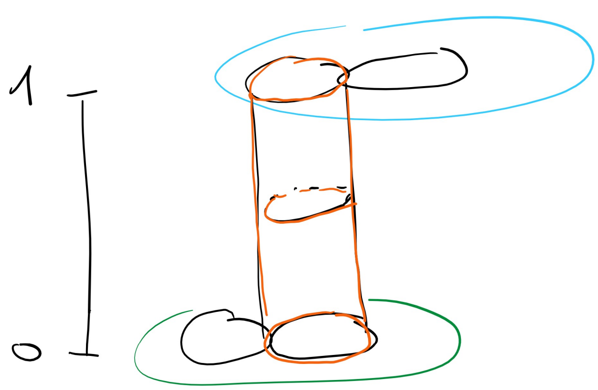
X^0 = green

X^1 = blue

$X^{\{0,1\}}$ = intersection $\equiv X^{[1]}$

$\Delta^1 = [0,1]$

$$X^{\mathcal{U}} = \underline{(X^0 \times \{0\})} \cup \underline{(X^1 \times \{1\})} \cup \underline{(X^{[1]} \times [0,1])}$$



- more generally for 2 covers

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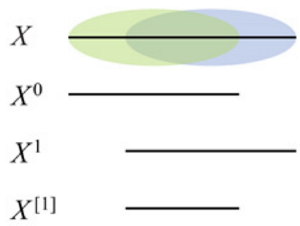
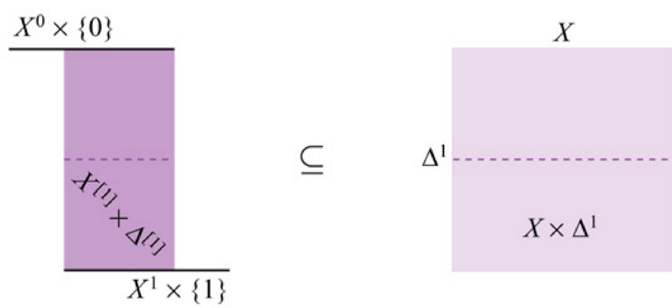
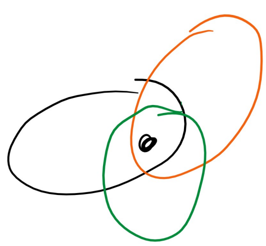


(a) Space



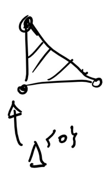
(b) Blowup

- 3-fold cover of singleton

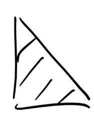


$X^0 = X^1 = X^2 = \bullet$

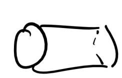
$X^{\{0,1\}} = \dots = X^{\{0,1,2\}} = \bullet$

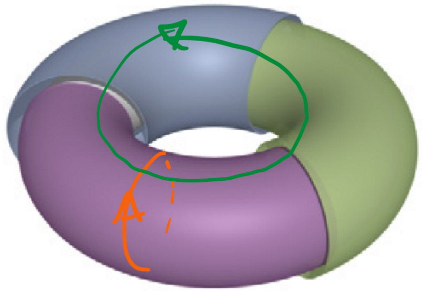
$\Delta^2 =$ 

$$\rightsquigarrow \quad X^{\mathcal{U}} = \bigcup_{J \subseteq [2]} X^J \times \Delta^J \quad =$$



- Torus w/ 3 covers

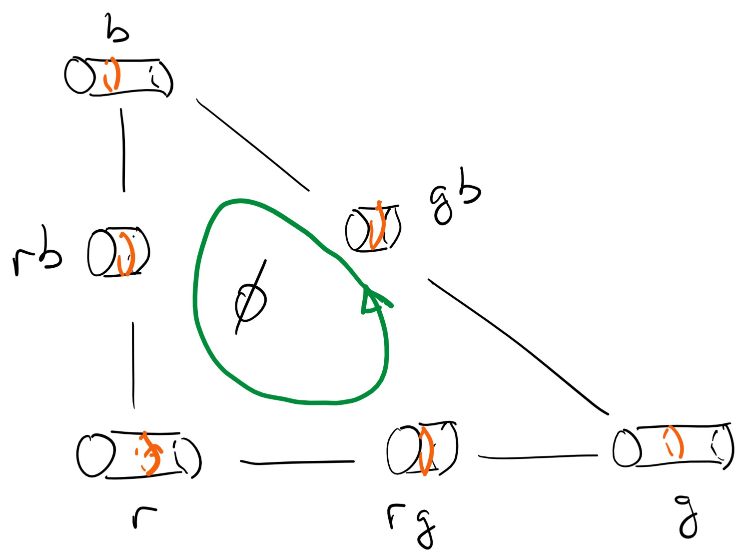
$X^i =$  $\in (r, g, b)$



$x^{\{i,j\}} =$  $\in (rg, rb, gb)$

$x^{[2]} = \emptyset$

$\Delta^2 =$ 

$X^U =$ 

<u>Lemma</u> $\pi_X : X^U \to X \times \Delta^{n-1}$

is a homotopy equivalence.

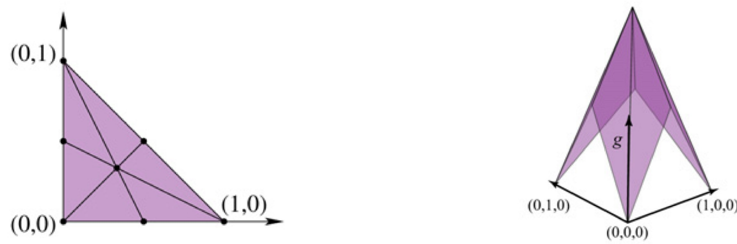
(except for "bad spaces")

<u>Cor.</u> $H_0(X^u) = H_0(X)$

In order to use persist. homology, we introduce a height function on X^u, by pulling back the following height fct. on Δ^{n-1}:

$$g(b^J) = \dim \Delta^J \quad \& \text{ linear continuation}$$

b^J: barycenter of $\Delta^J \subseteq \Delta^{n-1}$



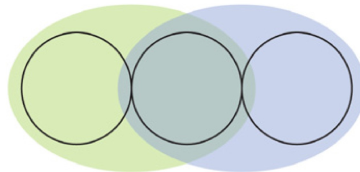
$$h: X^u \to \mathbb{R}, \; p \mapsto g \circ \pi_\Delta(p)$$

<u>Def.</u> <u>Filtered b4</u> $X^u_t = h^{-1}([0,t])$

Exp's

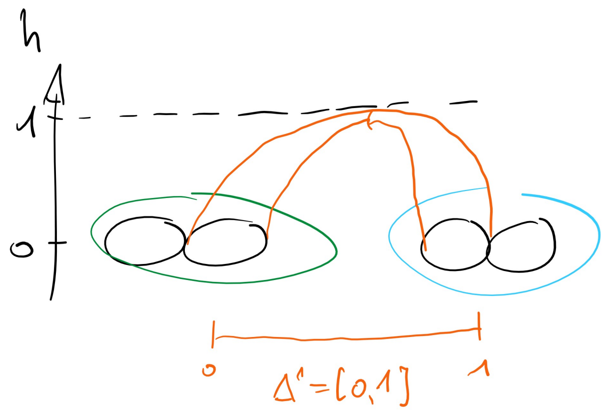
Exp's

- $(X, \mathcal{U}) =$



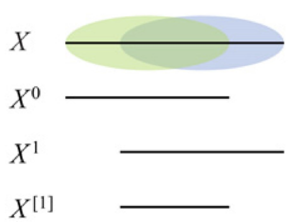
(a) Space and cover

$X^{\mathcal{U}} =$



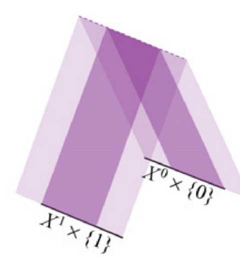
- more generally for 2-covered space

$(X, \mathcal{U}) =$



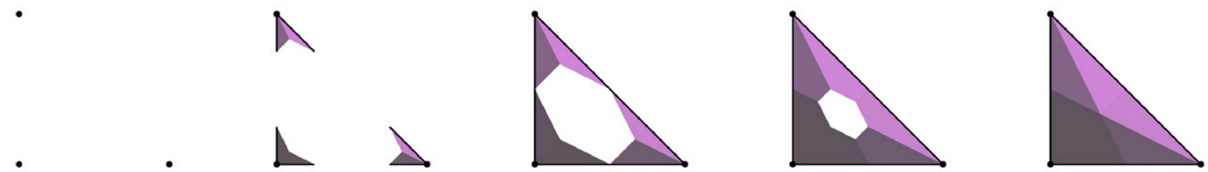
(a) Space

$X^{\mathcal{U}} =$



(c) Height function

- 3-covered singleton



(a) 0.0 (b) 0.5 (c) 1.0 (d) 1.5 (e) 2.0

Fig. 6. Singular blowup $X_t^{\mathcal{U}}$ for specified t in Example 8. The space X is a point and its cover $\mathcal{U}$ is three sets that contain it.

Observe:

$$H_\bullet(X_0^{\mathcal{U}}) = \bigoplus H_\bullet(X^i)$$

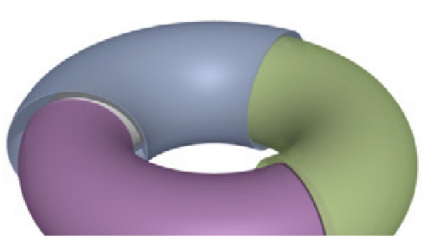
$$H_\bullet(X_\infty^{\mathcal{U}}) = H_\bullet(X)$$

Def The $\mathcal{U}$-localized homology of X are the classes $h \in H_0(X)$ that are in

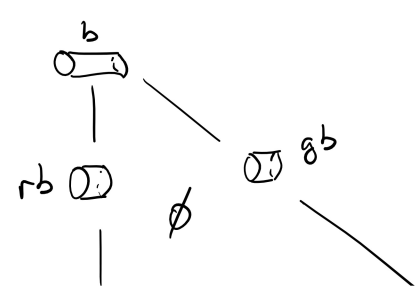
$$\operatorname{im}\left(H_0(X_0^{\mathcal{U}}) \longrightarrow H_0(X_\infty^{\mathcal{U}}) \right)$$

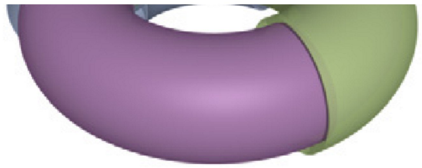
equiv.: $h \in H_0(X)$ is $\mathcal{U}$-local if h is $[0,\infty)$-persistent.

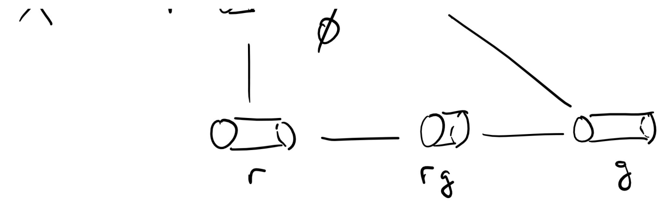
Expl.

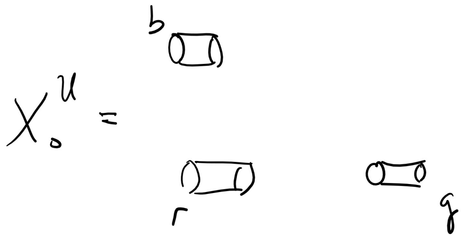


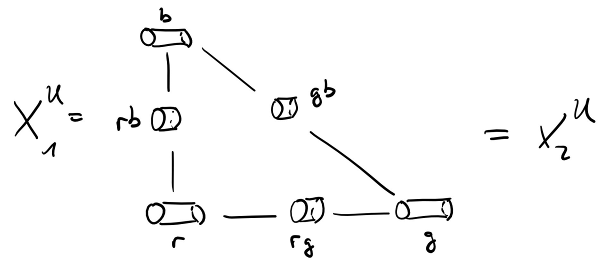
$X^{\mathcal{U}} =$

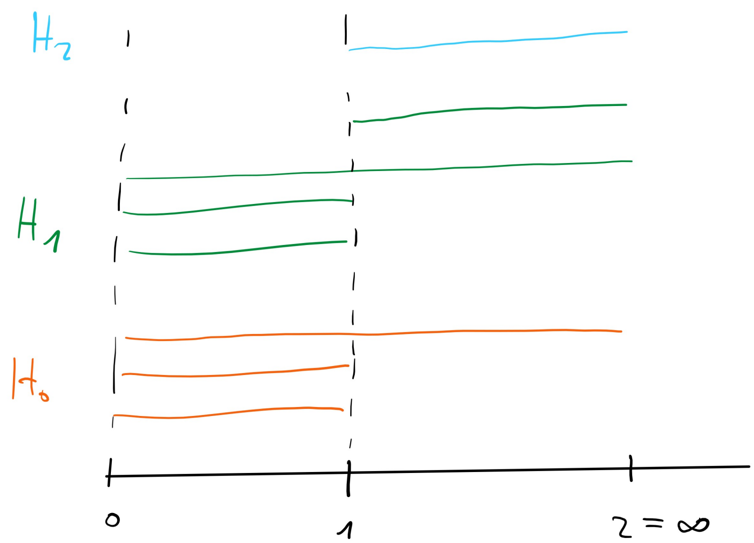












Also other persistence classes carry interesting information, as we will see momentarily.

Refining & Coarsening

Refining & Coarsening

Can always refine a cover st. $H_0(X^i) = 0$.

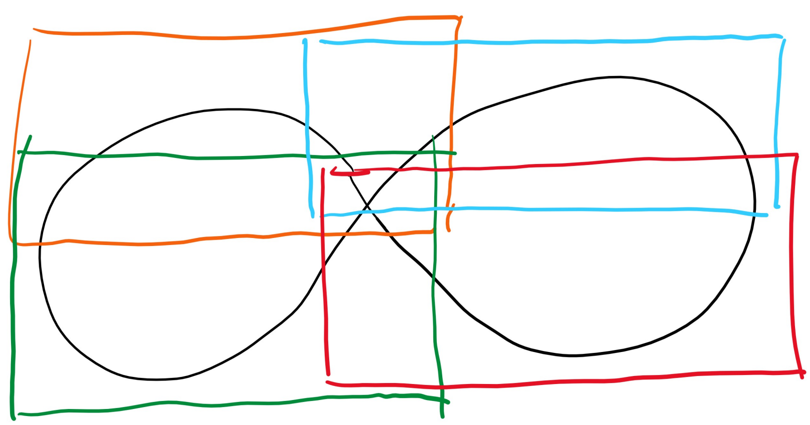
Start from there and coarsen the cover by unions of X^i:

$$\mathcal{U}[\ell] = \Big\{ \bigcup_{j \in J} X^j \;\Big|\; \operatorname{card} J = \ell \Big\}$$

Thm $h \in H_\bullet(X)$ $\mathcal{U}[\ell]$-local $\Rightarrow$ h is $[\ell-1, \infty)$ persistent

proof omitted

Expl. 4-covered figure 8



This does not have non-trivial $\mathcal{U}$-local classes in $H_1(\infty)$,

because $X_0^{\mathcal{U}} \simeq 4 \times$ "$\propto$"

The barcode of the MV-bt cplx is

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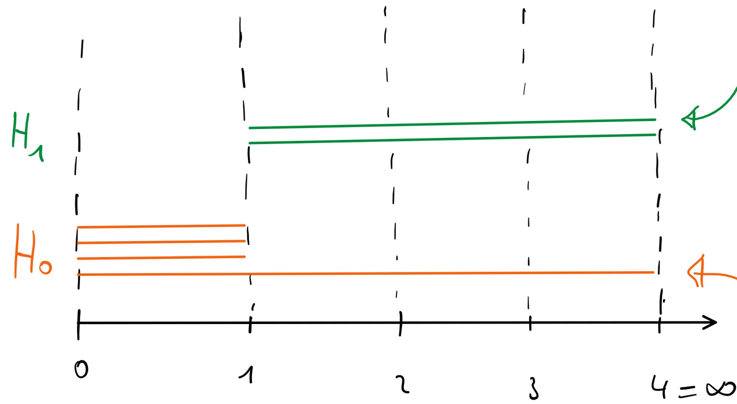
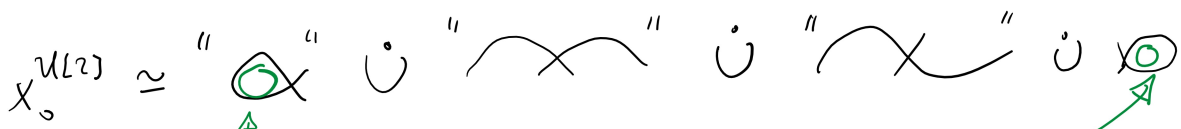


appear once we can "move vertically" btw. X^i e.g. green $\to$ orange

only $\mathcal{U}$-local class of X corrp. to X being connected.

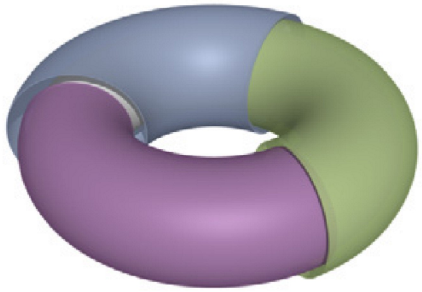
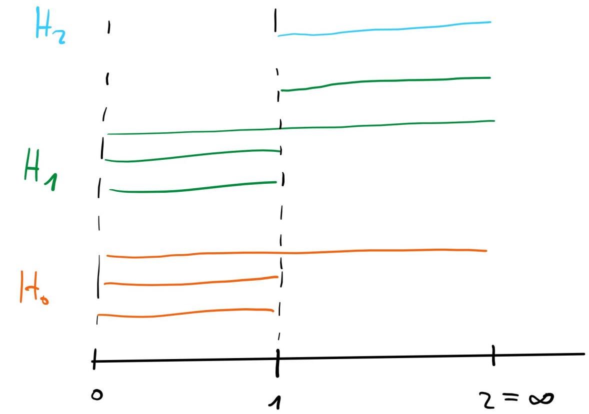
<u>Indeed</u> the $[1,\infty)$-persist. classes in H_1 are local to the union (green $\cup$ orange) and (red $\cup$ blue):



$\mathcal{U}[2]$-local homology class

corresp. to $[1,\infty)$-pers. class

<u>However</u>, the converse statement of the Thm is false!

E.g. the class in H_2 is $[1,\infty]$ persistent, but <u>not</u> $\mathcal{U}[2]$-local, bc. each set in $\mathcal{U}[2]$ is a cylinder w/ $H_2(\text{cylinder}) = 0$.

A note on Ethymology

Let $\mathcal{U} = \{X^0, X^1\}$ be a two-fold cover.

Then we have the <u>long-exact seq. of pair</u> $(X_1^{\mathcal{U}}, X_0^{\mathcal{U}})$

$$\cdots \to H_i(X_0^{\mathcal{U}}) \to H_i(X_1^{\mathcal{U}}) \to H_i(X_1^{\mathcal{U}}, X_0^{\mathcal{U}}) \to \cdots$$

$\cong$ – see above – $\cong$ $\qquad\qquad$ $\cong$

$$\cdots \to H_i(X^0) \oplus H_i(X^1) \to H_i(X) \to H_{i-1}(X^0 \cap X^1) \to \cdots$$

<u>Mayer-Vietoris sequence</u>

where "$\cong$" holds due to:

$$H_i(X_1^u, X_0^u) \overset{\text{excision}}{=} H_i(X^0 \cap X^1 \times [0,1],\ X^0 \cap X^1 \times \{0,1\})$$

$$\underset{\text{Künneth}}{=} H_{i-1}(X^0 \cap X^1)$$

Examples (proof of concept pictures)

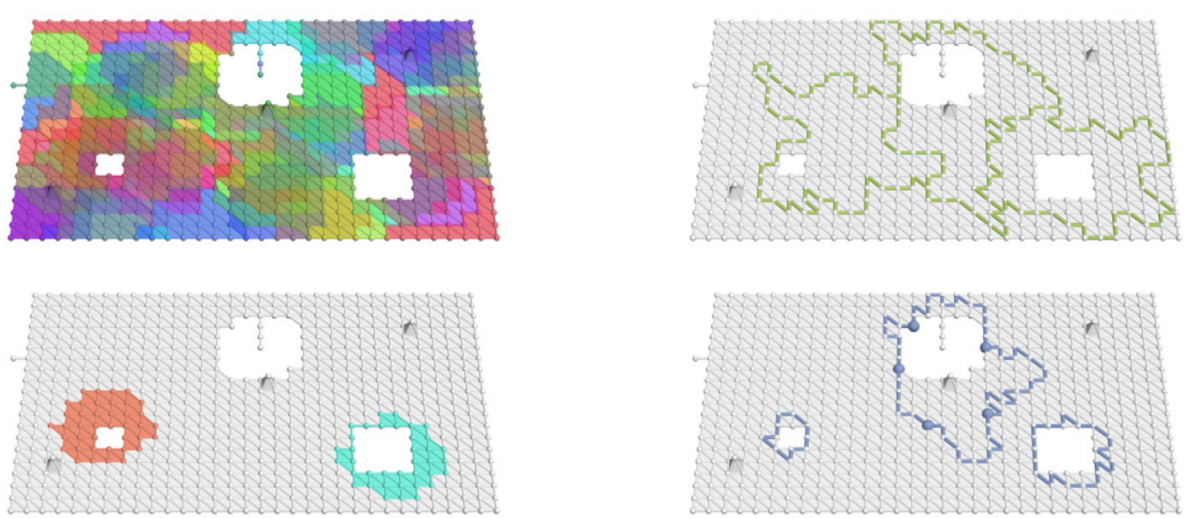


Fig. 12. Top left: A defective surface with extra edges and tetrahedra, covered by transparently colored sets based on ϵ-balls. Top right: The 1-cycles computed with homology are non-local and one goes around two holes. Bottom left: The complex highlighting the first sets in the cover that contain the small and medium holes, respectively. Bottom right: The projection (π_X) of the 1-cycles of the blowup complex localizes the two smaller holes. The large 1-cycle has portions that project onto vertices, as indicated.

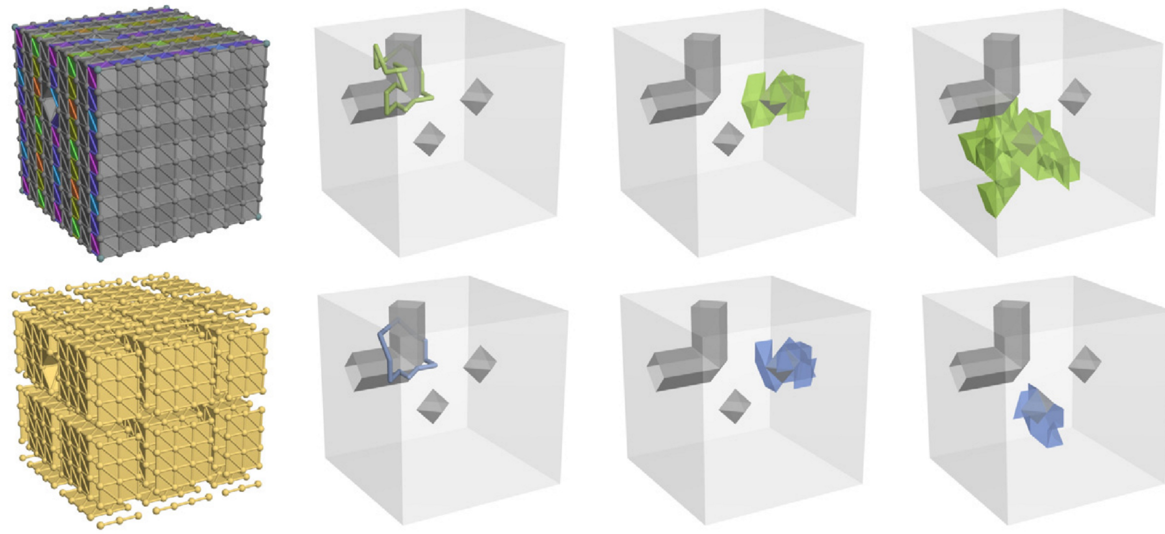


Fig. 13. We carve a tunnel and two voids from a solid cubical block. Top left: the simplicial complex representing our space, colored by the 8 sets in the cover. (For colors see the web version of this article.) Bottom left: a set in the cover has components of dimensions 1, 2, and 3. On the right, we show the descriptions on a transparent rendering of the volume. The top row renders the descriptions found by homology for the 1-cycle (tunnel) and two-cycles (voids), and the bottom row shows our localized descriptions.