Multidimensional Persistence: Theory and Applications

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Outline

- Motivation for multidimensional persistence.
- Elaboration of the algebraic objects that naturally arise from the study of multifiltrations: multigraded modules over a polynomial ring in multiple variables.
- Parameterization of these objects as a subset of a product of grassmannians together with a group action.
- By using this parameterization, show that there are no barcode-like invariants in dimension greater than one.

This talk is based on and inspired by *G. Carlsson, A. Zomorodian, The Theory of Multidimensional Persistence, 2009.*

Motivation

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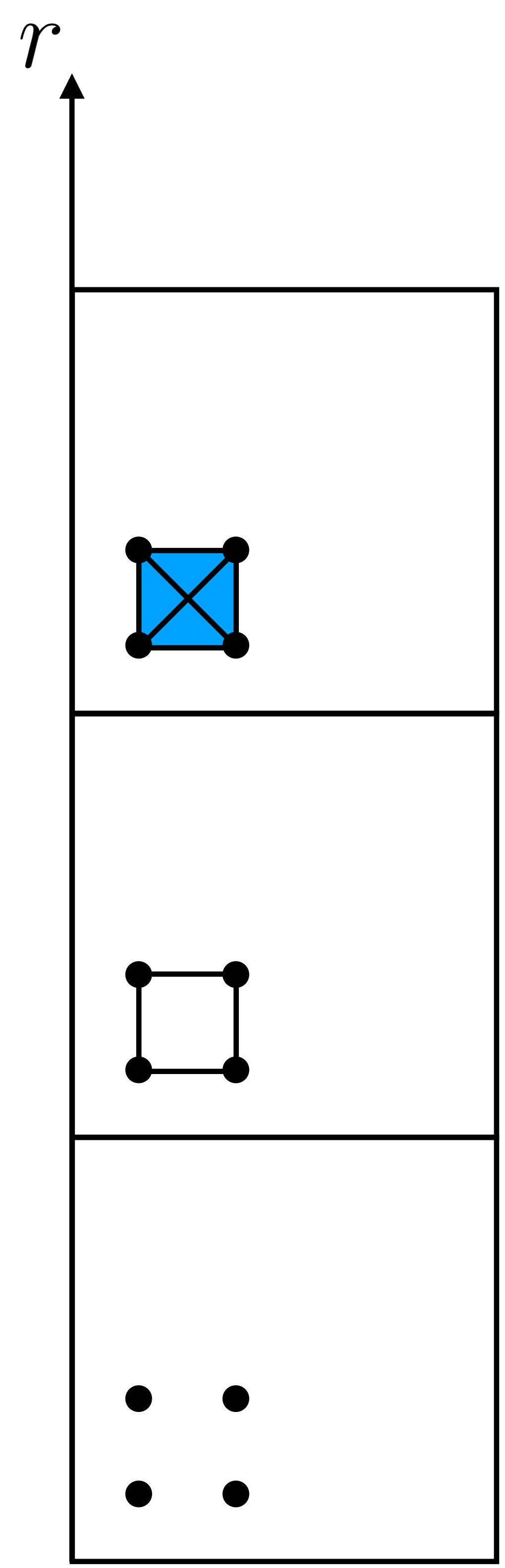
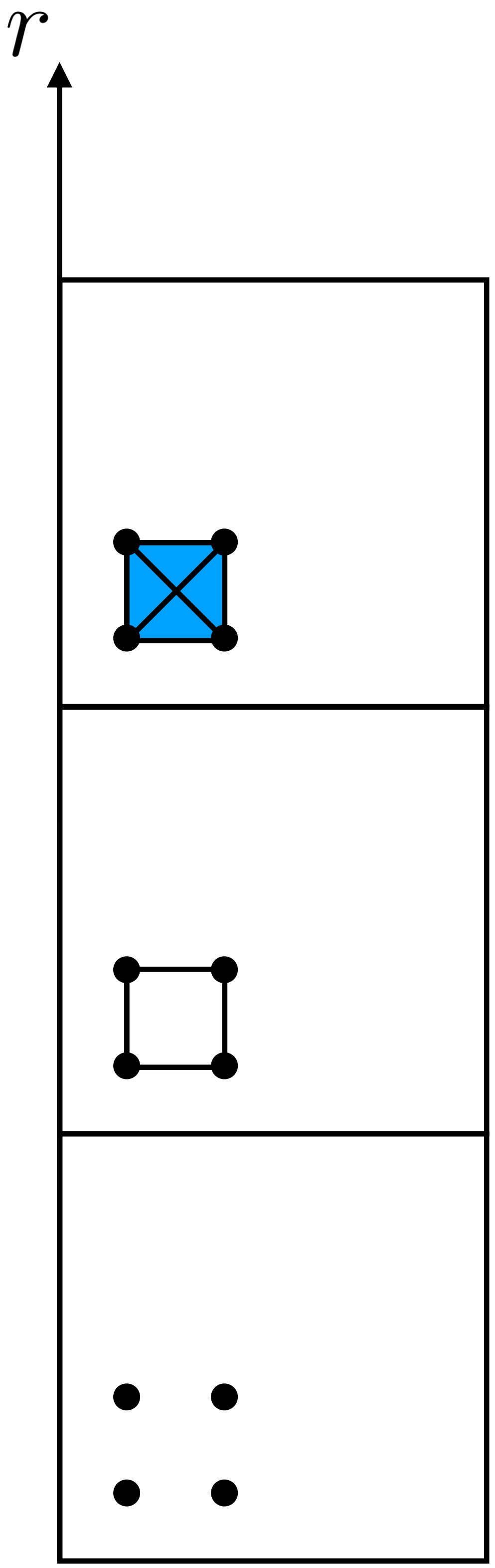


- Metric space = RNA-Sequences + Hamming-distance
- **Goal:** compute the number of reticulate events (one-dim. cycles) in which a mutation participates (detect bars of length one in barcode born in first step).
- **topological Recurrence Index (tRI)** for virus-mutations.
- A growing **tRI** of a particular mutation together with a growing prevalence of this mutation and biological evidence could serve as a heuristic for a possible **evolutionary fitness advantage** of this mutation.
- Virus data-sets grow over **time**.

M. Bleher, M. Carrière, L. Hahn, A. Ott, J. Patiño-Galindo and R. Rabadán, Topology identifies emerging adaptive mutations in SARS-CoV-2, 2021

Motivation



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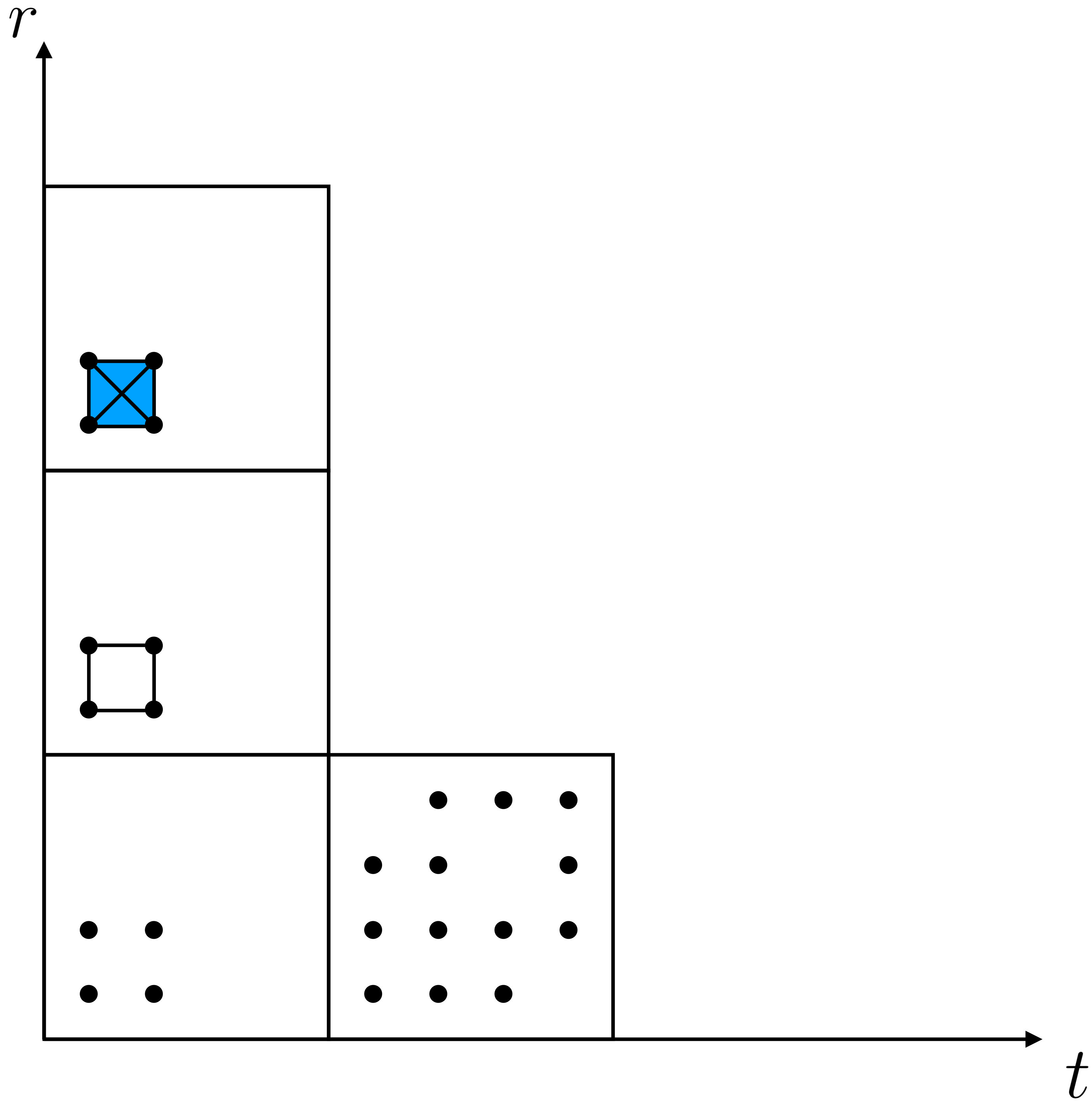

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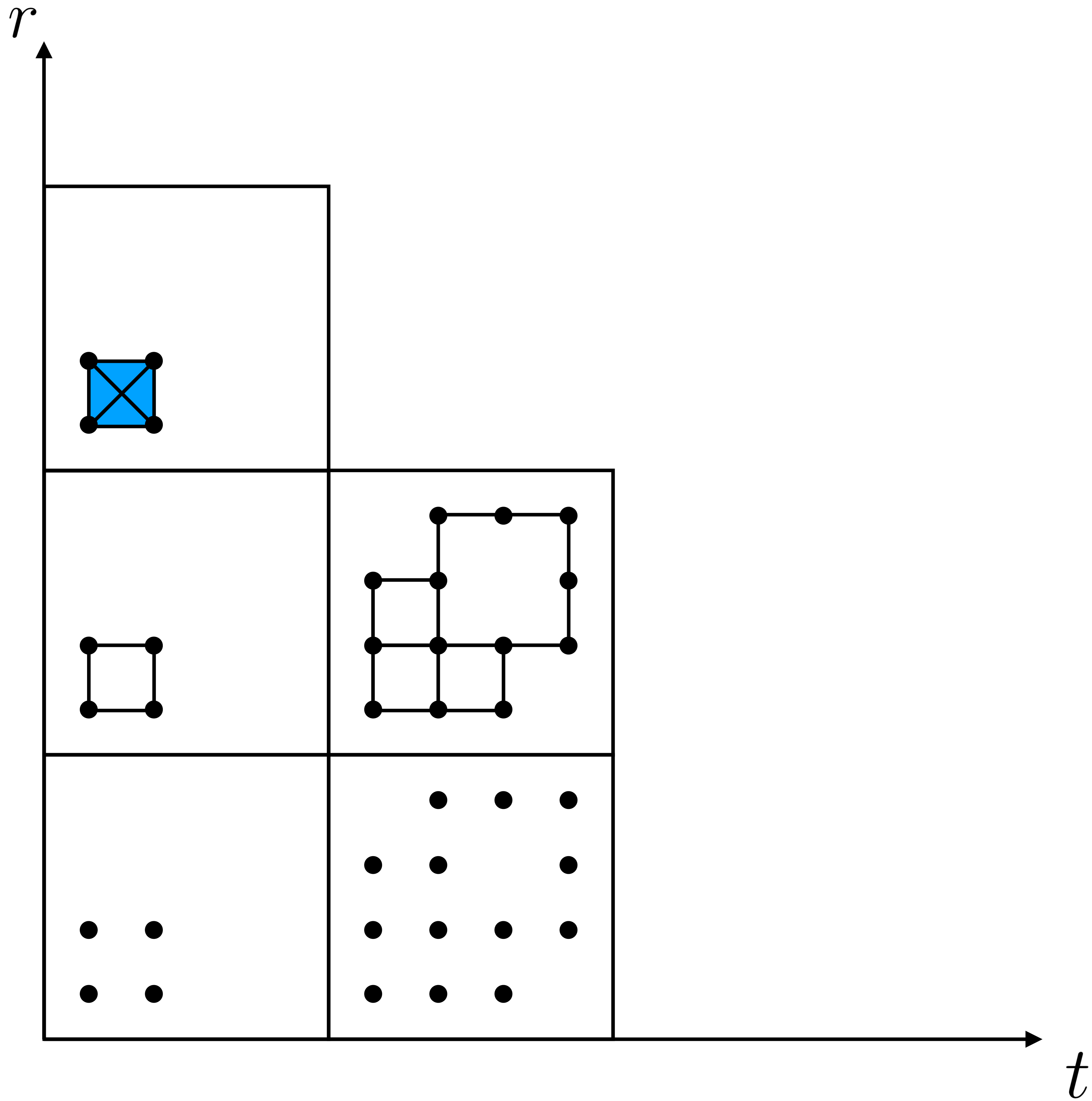

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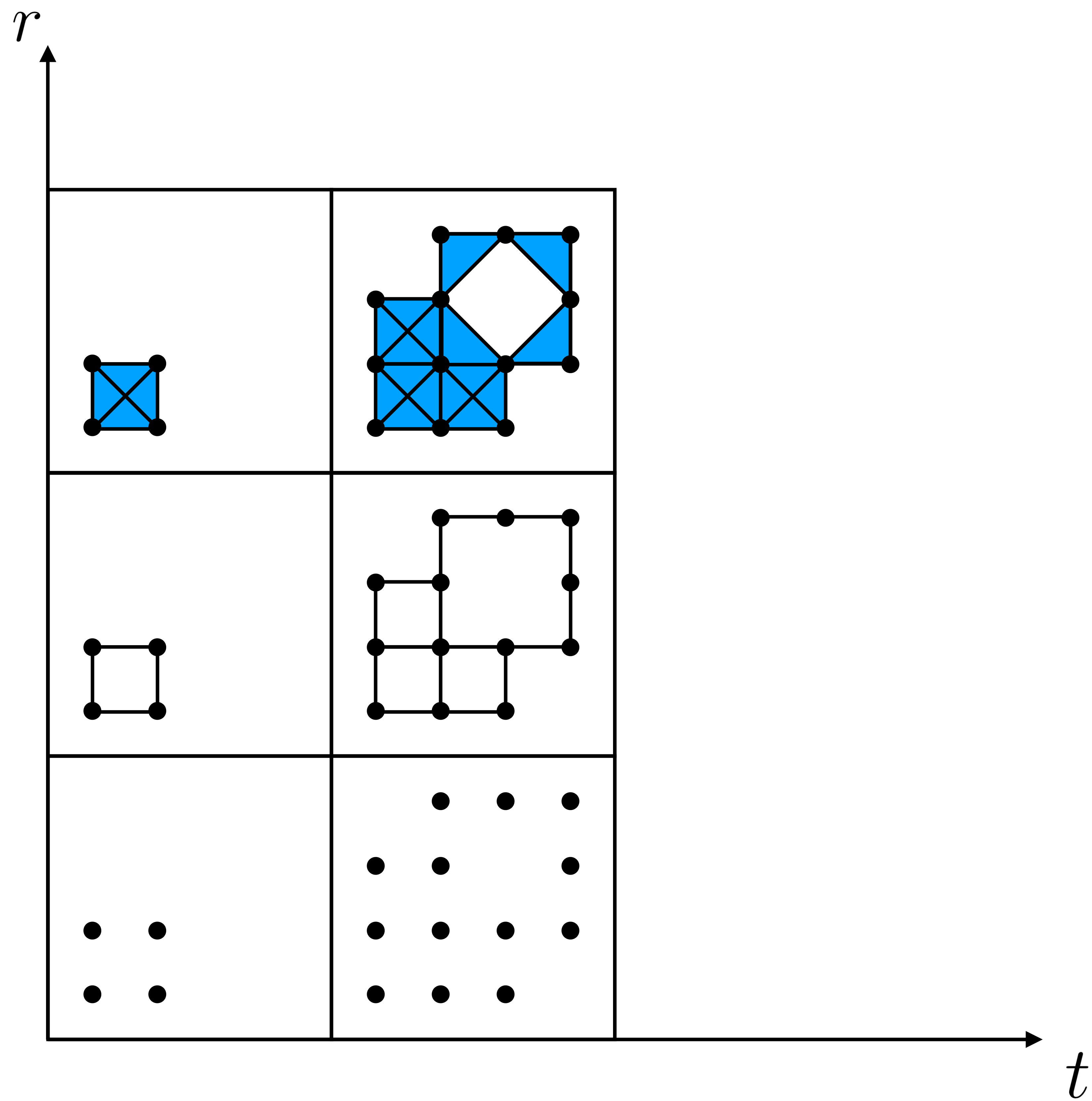

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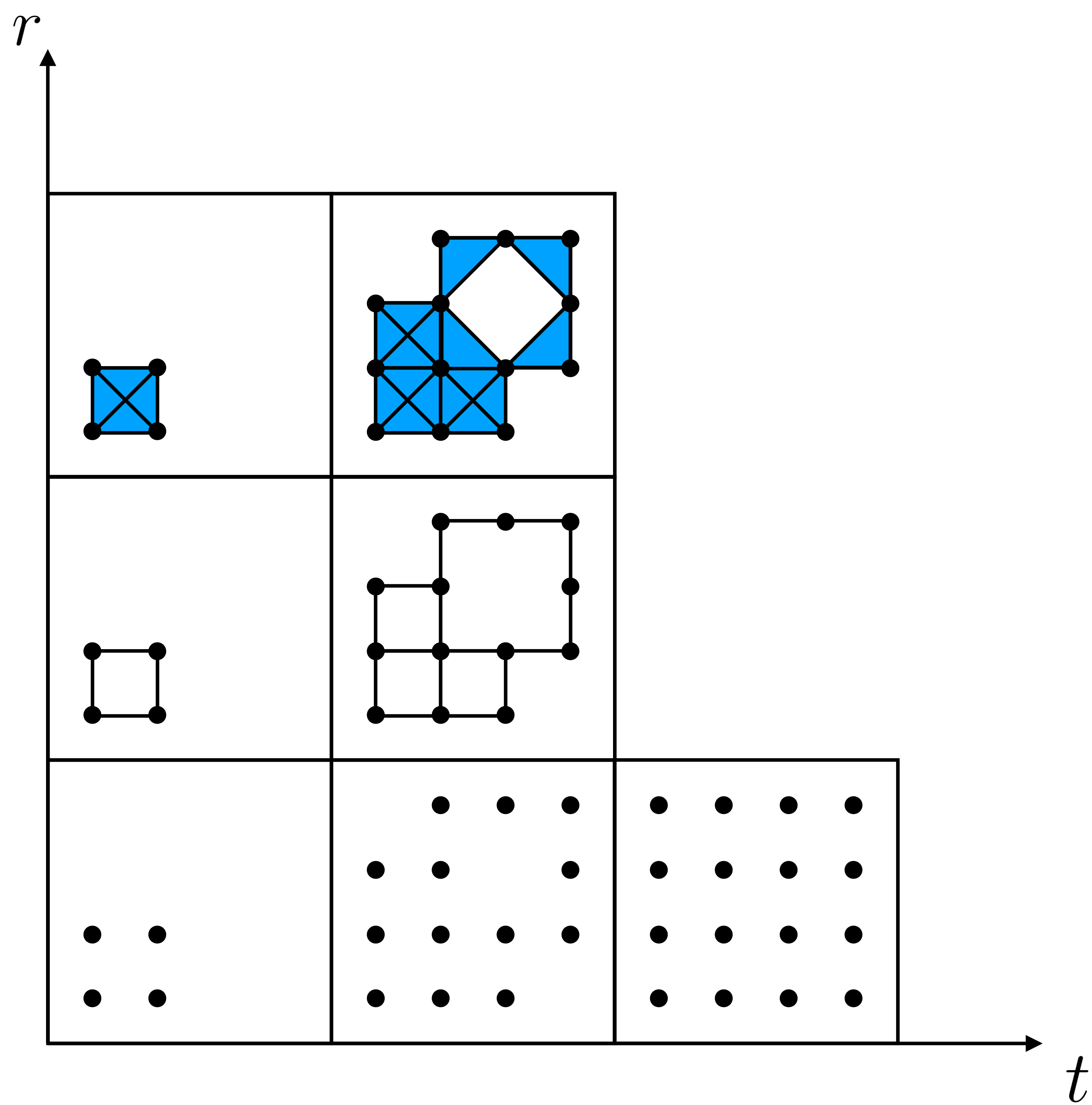

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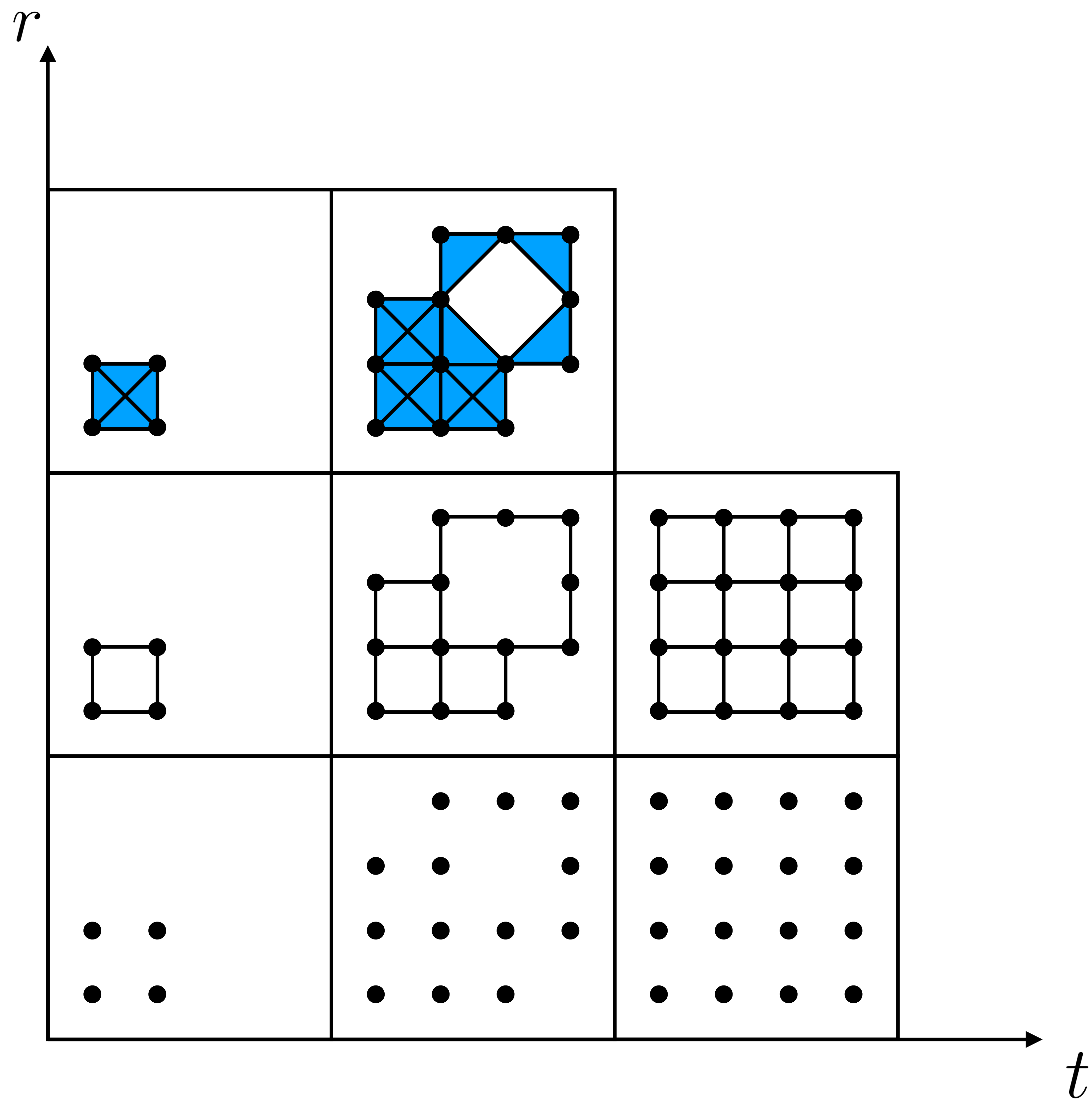

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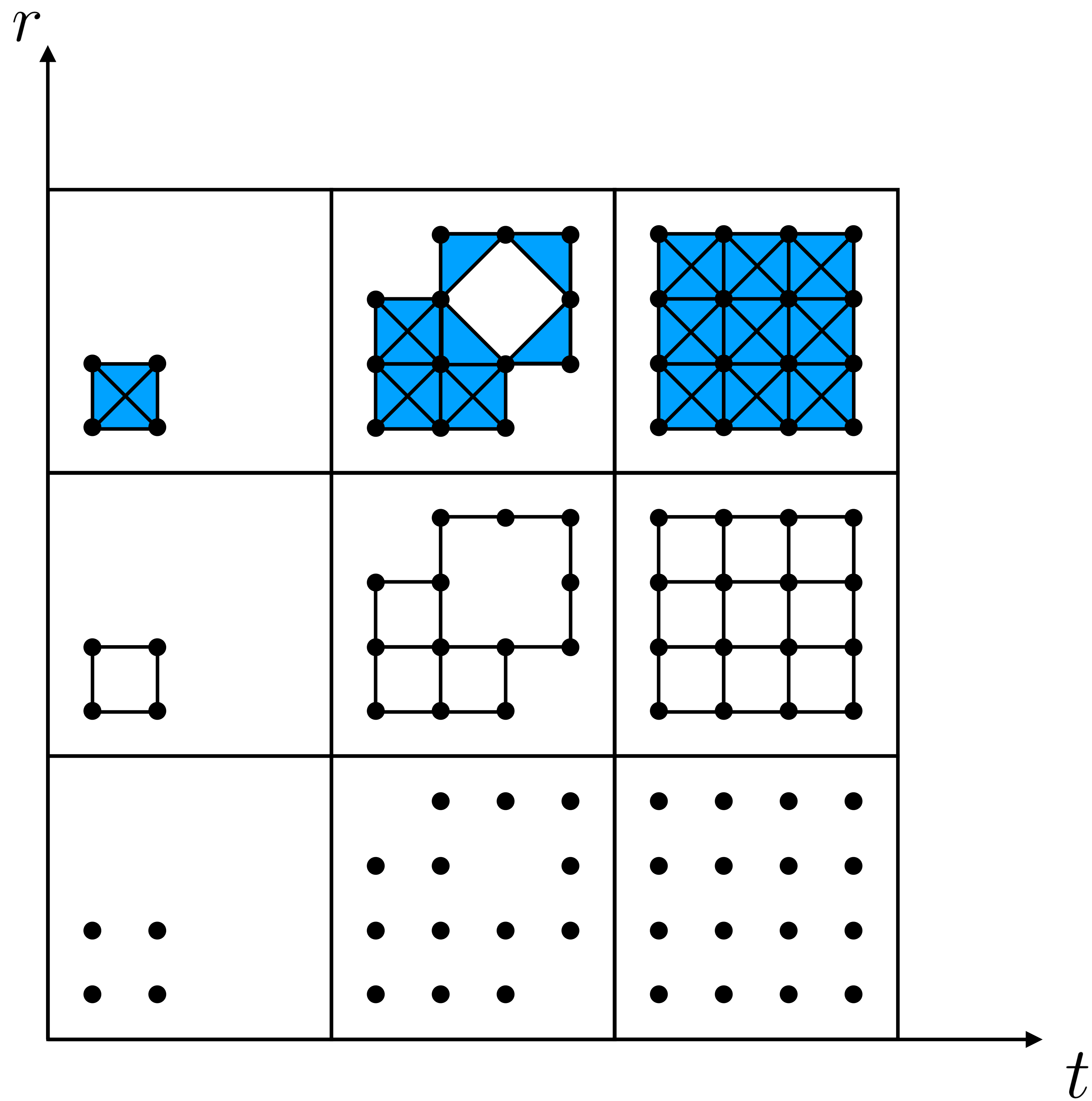

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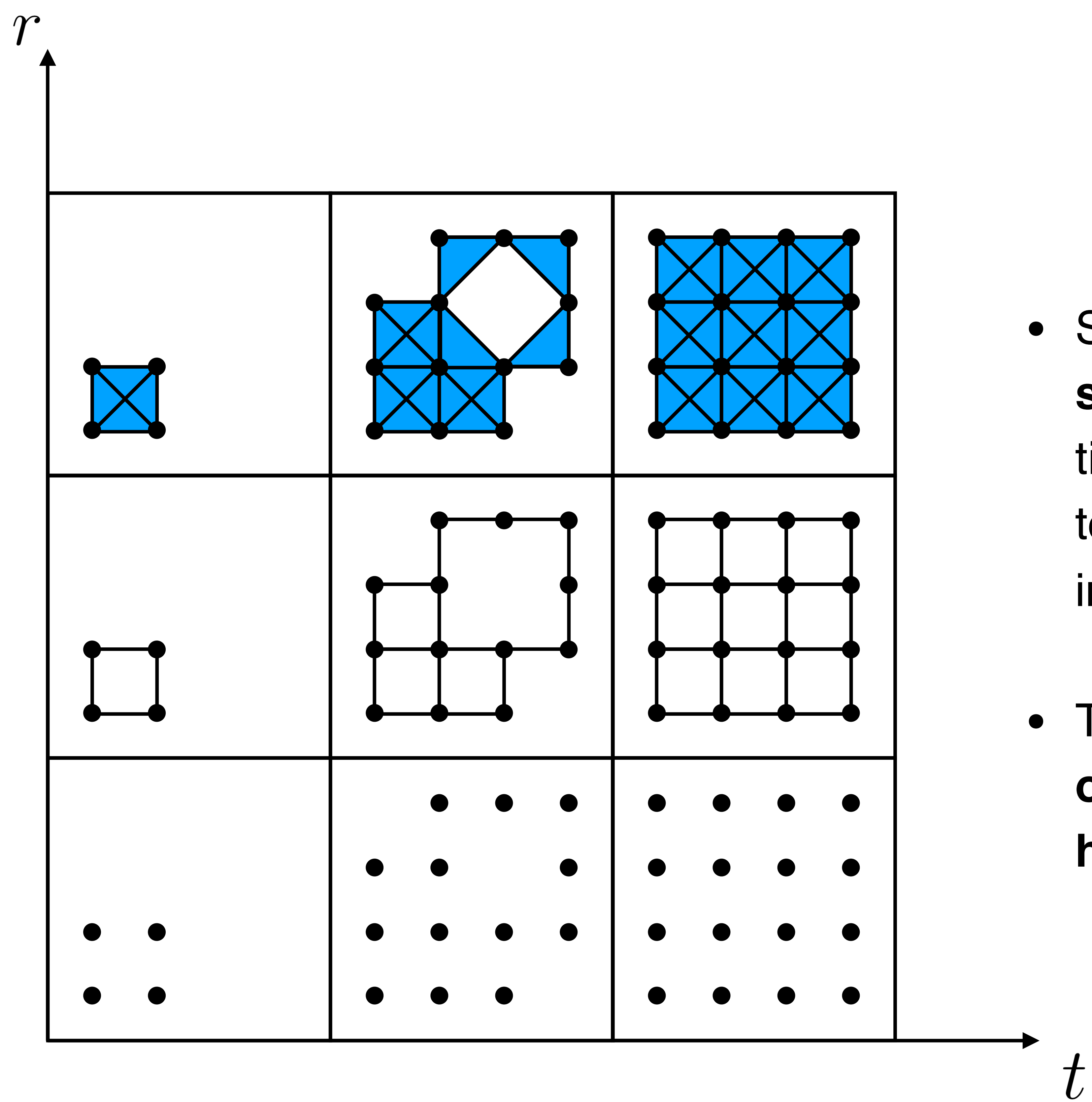

- Study of **one-dimensional subfiltrations** along the time and scaling parameter to obtain persistence information.
- This allows us to apply **one-parameter persistent homology**

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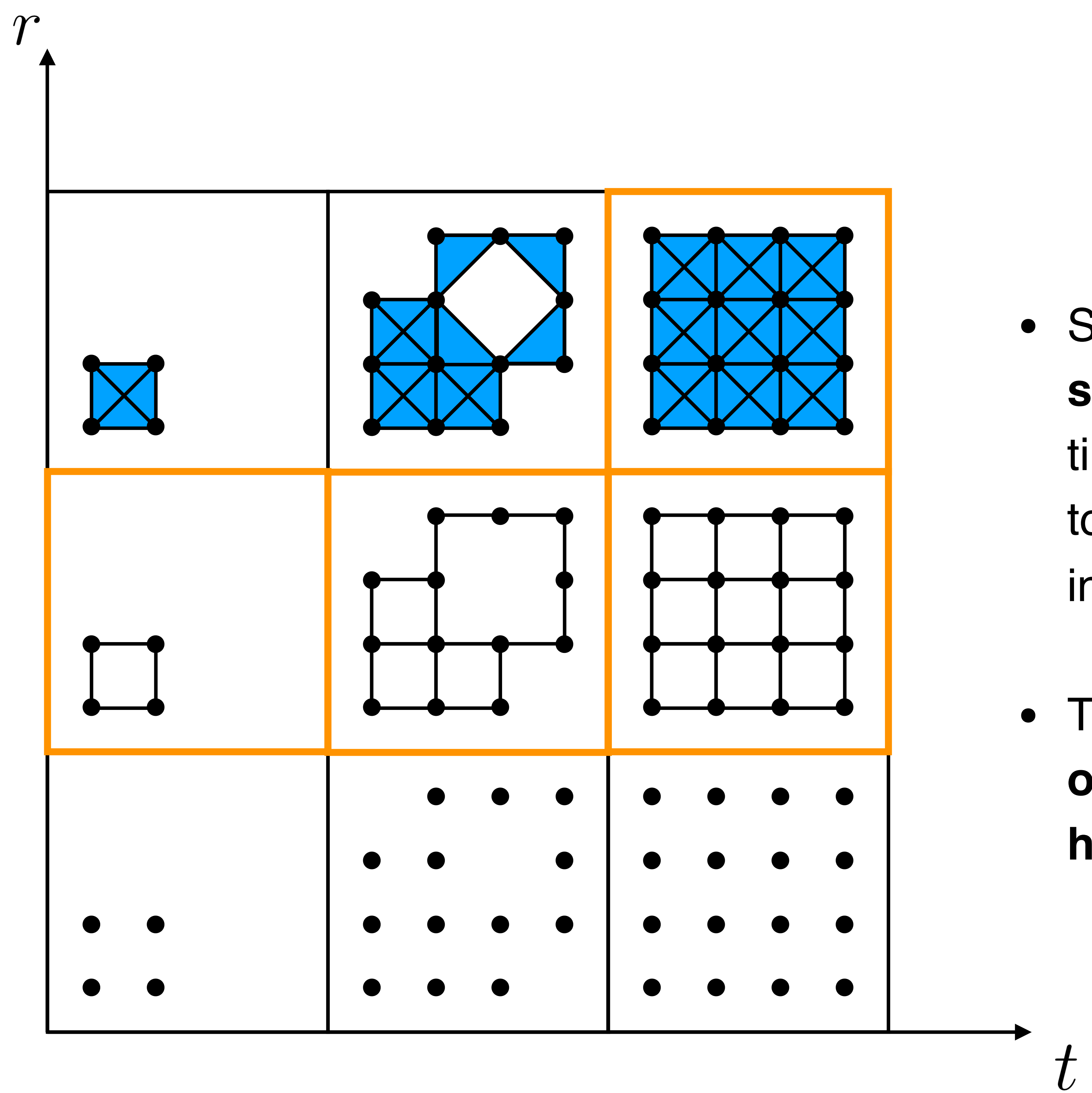

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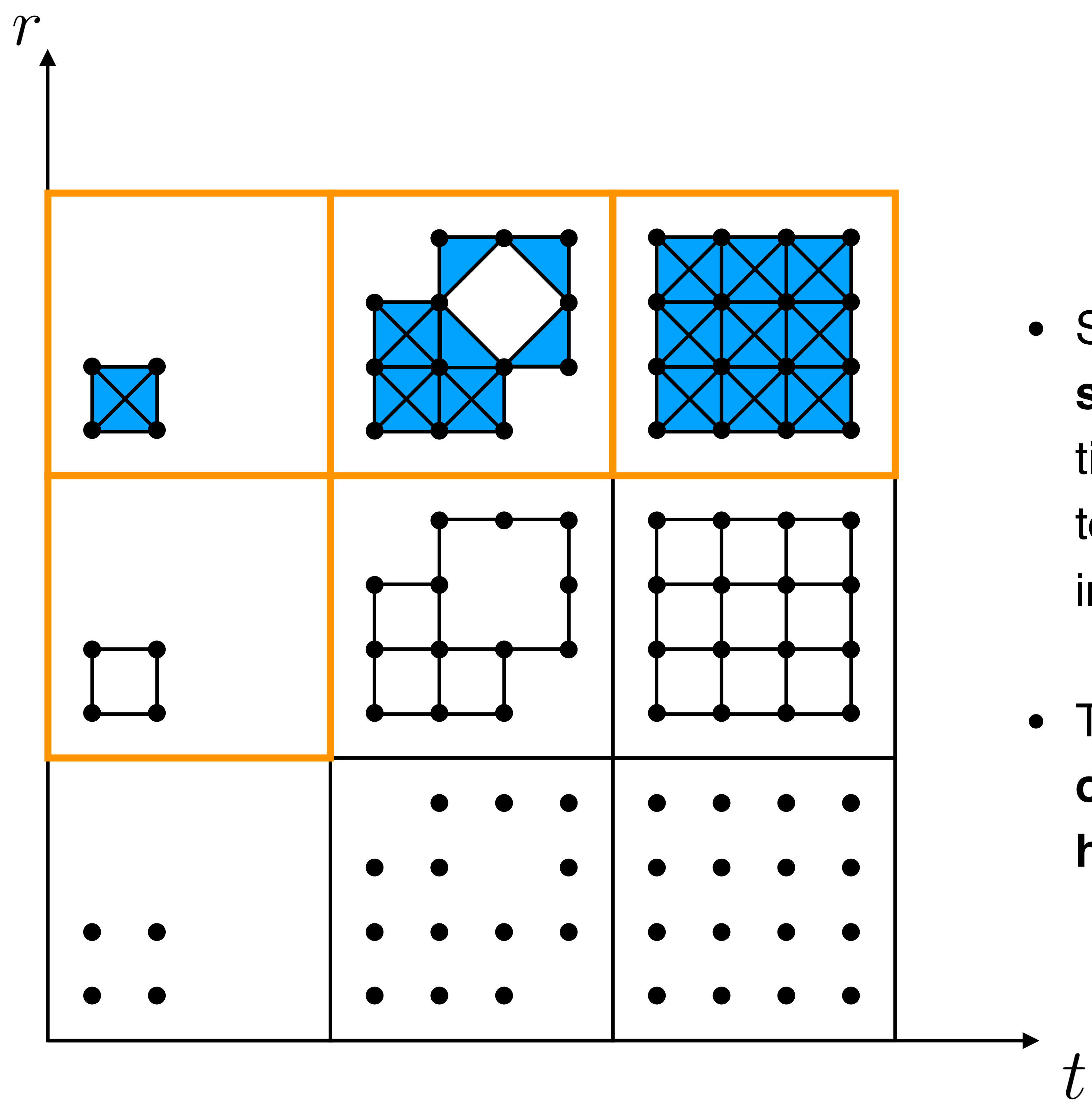

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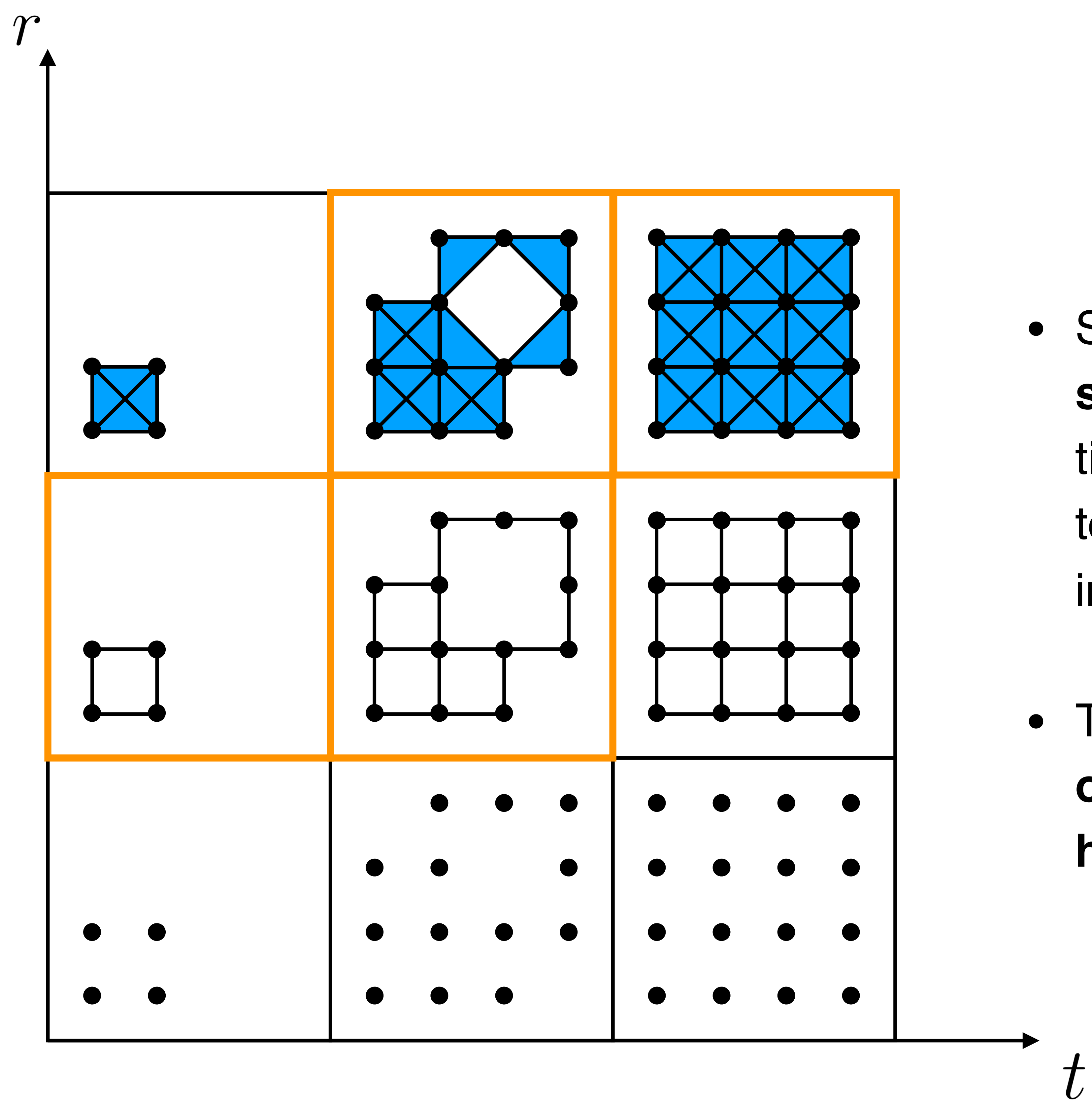

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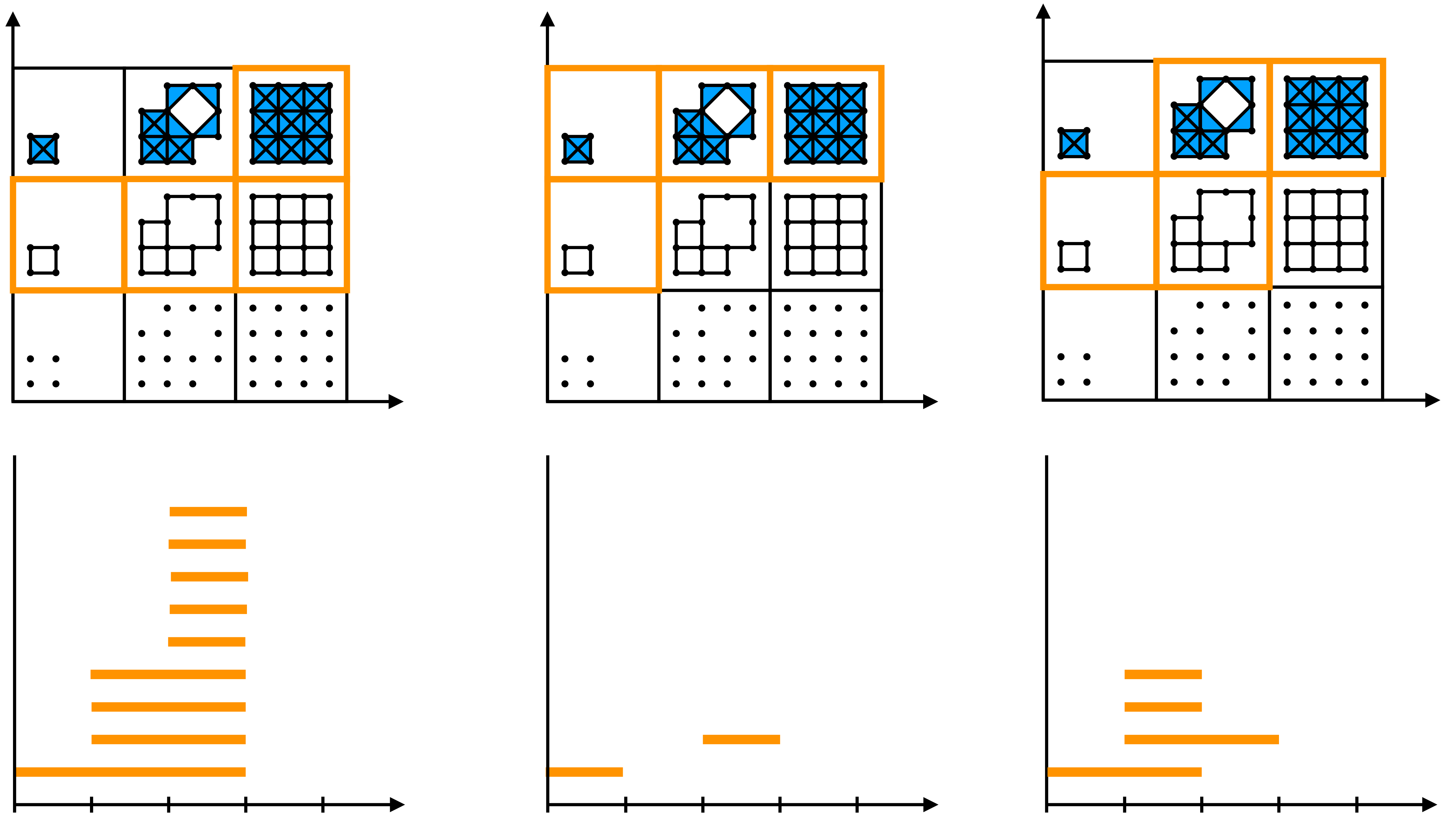

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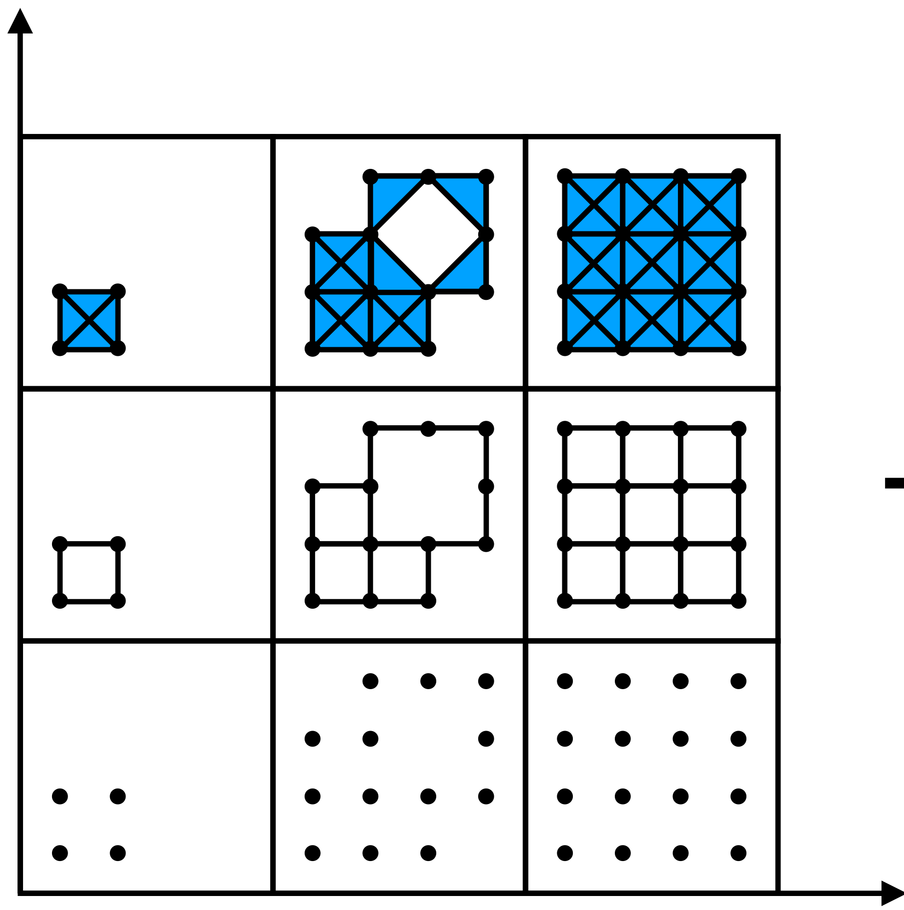

First homology

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{F} \arrow[r] & 0 \\
\mathbb{F} \arrow[r] \arrow[u] & \mathbb{F}^4 \arrow[r] \arrow[u] & \mathbb{F}^9 \arrow[u] \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}
$$

Betti numbers + ranks of transition maps

Barcodes of subfiltrations

Multifiltered Simplicial Complexes
and
Multigraded Modules

Definition (Partial order on $\mathbb{N}^n$)

For $(u_1, \ldots, u_n), (v_1, \ldots, v_n) \in \mathbb{N}^n$, we write

$$(u_1, \ldots, u_n) \preceq (v_1, \ldots, v_n)$$

if $u_i \leq v_i$ for all $i \in \{1, \ldots, n\}$.

Definition (n-filtered simplicial complex)

Let X be a finite simplicial complex and $n \geq 1$. Then X is called **$\boldsymbol{n}$-filtered** if there exists a famililiy of simplicial complexes $(X_u)_{u \in \mathbb{N}^n}$ such that

$$X = \bigcup_{u \in \mathbb{N}^n} X_u \quad \text{and} \quad X_u \subseteq X_v$$

for all $u, v \in \mathbb{N}^n$ with $u \preceq v$.

Definition (Polynomial ring)

For a field $\mathbb{F}$,

$$\mathbb{F}[\boldsymbol{x}] := \mathbb{F}[x_1, \ldots, x_n]$$

denotes the **polynomial ring in n variables**. For $v = (v_1, \ldots, v_n) \in \mathbb{N}^n$, we write

$$\boldsymbol{x}^v := x_1^{v_1} \cdot \cdots \cdot x_n^{v_n}$$

Definition (n-graded Module)

An $\mathbb{F}[\boldsymbol{x}]$-module M is called $\boldsymbol{n}$-**graded** if M admits a direct sum decomposition into $\mathbb{F}$-vector spaces

$$M = \bigoplus_{v \in \mathbb{N}^n} M_v$$

such that for all $v, w \in \mathbb{N}^n$,

$$\boldsymbol{x}^v \cdot M_w \subseteq M_{v+w}.$$

Definition (Homogeneous element)

Let M be an n-graded $\mathbb{F}[\boldsymbol{x}]$-module and $v \in \mathbb{N}^n$. An element

$$y \in M_v \setminus \{0\}$$

is called **homogenous of degree $\boldsymbol{v}$**.

Definition (Graded homomorphism)

Let N, M be n-graded $\mathbb{F}[\boldsymbol{x}]$-modules. An $\mathbb{F}[\boldsymbol{x}]$-module homomorphism

$$f : M \longrightarrow N$$

is called **graded** if for all $v \in \mathbb{N}^n$,

$$f(M_v) \subseteq N_v.$$

Example

$\mathbb{F}[\boldsymbol{x}]$ is n-graded over itself where for $u \in \mathbb{N}^n$,

$$\mathbb{F}[\boldsymbol{x}]_u := \langle \boldsymbol{x}^u \rangle_{\mathbb{F}}.$$

For $v \in \mathbb{N}^n$, $\mathbb{F}[\boldsymbol{x}](v)$ is defined by shifting the grading of $\mathbb{F}[\boldsymbol{x}]$ via

$$\mathbb{F}[\boldsymbol{x}](v)_u := \begin{cases} \langle \boldsymbol{x}^{u-v} \rangle_{\mathbb{F}}, & u \in \mathbb{N}^n_{\succeq v} \\ 0, & \text{else} \end{cases}$$

Definition (Associated n-graded module)

If X is a finite n-filtered simplicial complex, then

$$M_\ell(X) := \bigoplus_{u \in \mathbb{N}^n} H_\ell(X_u, \mathbb{F})$$

defines a f.g. n-graded $\mathbb{F}[\boldsymbol{x}]$-module where the variable action is given via the induced maps on homology

$$\boldsymbol{x}^{v-u} : H_\ell(X_u, \mathbb{F}) \longrightarrow H_\ell(X_v, \mathbb{F})$$

for $u \preceq v$.

Theorem (Realization)

Assume that $\mathbb{F} = \mathbb{F}_p$ where $p \in \mathbb{N}$ is a prime number or that $\mathbb{F} = \mathbb{Q}$. Let M be a f.g. n-graded $\mathbb{F}[\boldsymbol{x}]$-module. Then for every $\ell \geq 1$, there exists an n-filtered finite simplicial complex X such that

$$M_\ell(X) \cong M.$$

For a proof see *H. A. Harrington, N. Otter, H. Schenck, U. Tillmann, Stratifying multiparameter persistent homology, arXiv 2019*.

Conclusion

The study of finite n-filtered simplicial complexes translates into the study of f.g. n-graded $\mathbb{F}[\boldsymbol{x}]$-modules.

Invariants

Definition (Invariant)

An **invariant** is a function

$$\mathcal{I}_n^{\mathbb{F}} : \{\text{f.g. } n\text{-graded } \mathbb{F}[\boldsymbol{x}]\text{-modules}\} /_{\cong} \longrightarrow Q_n^{\mathbb{F}}$$

where $Q_n^{\mathbb{F}}$ is a set. $\mathcal{I}_n^{\mathbb{F}}$ is called

- **discrete** if $Q_n^{\mathbb{F}} = Q_n^{\mathbb{T}}$ for all fields $\mathbb{F}, \mathbb{T}$ and if $\mathcal{I}_n^{\mathbb{F}}$ has countable image.
- **complete** if $\mathcal{I}_n^{\mathbb{F}}$ is injective, i.e. each isomorphism class $[M]$ is completely determined by $\mathcal{I}_n^{\mathbb{F}}([M])$.

Theorem (Barcode)

In dimension one ($n = 1$), **the persistence barcode** defines a discrete and complete invariant.

- Are there Barcode-like invariants in dimension $n \geq 2$?

Definition (Rank invariant)

Let

$$\mathbb{D}^n := \{(u,v) \in \mathbb{N}^n \times \mathbb{N}^n \mid u \preceq v\} \subseteq \mathbb{N}^n \times \mathbb{N}^n$$

be the subset above the diagonal. The assignment

$$\rho_n^{\mathbb{F}} : \{\text{f.g. } n\text{-graded } \mathbb{F}[\boldsymbol{x}]\text{-modules}\} /_{\cong} \longrightarrow \mathrm{Hom}_{\mathrm{Sets}}(\mathbb{D}^n, \mathbb{N}^2)$$

which maps an isomorphism class $[M]$ to the function

$$\mathbb{D}^n \longrightarrow \mathbb{N}^2, \quad (u,v) \longmapsto \big(\mathrm{rank}(\boldsymbol{x}^{v-u} : M_u \to M_v), \dim_{\mathbb{F}}(M_u)\big)$$

defines a discrete invariant $\rho_n^{\mathbb{F}}$, the so-called **rank invariant**.

Theorem

In dimension one, the rank invariant $\rho_1^{\mathbb{F}}$ is equivalent to the barcode and hence complete.

Conclusion

The rank invariant $\rho_n^{\mathbb{F}}$ is a discrete generalization of the persistence barcode to dimension $n \geq 2$.

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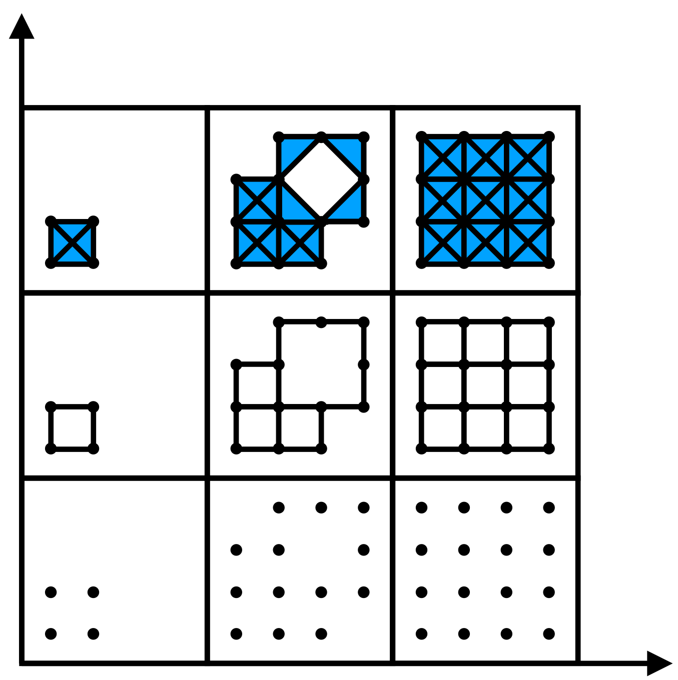

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Betti numbers + ranks of transition maps

$\downarrow\uparrow$

Barcodes of subfiltrations

Theorem

In dimension one, the rank invariant $\rho_1^{\mathbb{F}}$ is equivalent to the barcode and hence complete.

Conclusion

The rank invariant $\rho_n^{\mathbb{F}}$ is a discrete generalization of the persistence barcode to dimension $n \geq 2$.

- Is the rank invariant $\rho_n^{\mathbb{F}}$ complete for $n \geq 2$?

- Unfortunenately, the answer is no.

Goal

Show that if $n \geq 2$ there exists **no discrete and complete invariant** for

$$\{\text{f.g. } n\text{-graded } \mathbb{F}[\boldsymbol{x}]\text{-modules}\} /_{\cong}$$

Free Hulls and Graded Free Resolutions

Definition (n-dimensional multiset)

An $\boldsymbol{n}$**-dimensional multiset** is a pair $\xi = (V, \mu)$ where $V \subseteq \mathbb{N}^n$ and $\mu : V \to \mathbb{N}_{\geq 1}$ is a function such that

$$\xi = (V, \mu) = \bigcup_{v \in V} \{(v, 1), \ldots, (v, \mu(v))\}.$$

μ is also called **the mulitplicity function of** ξ.

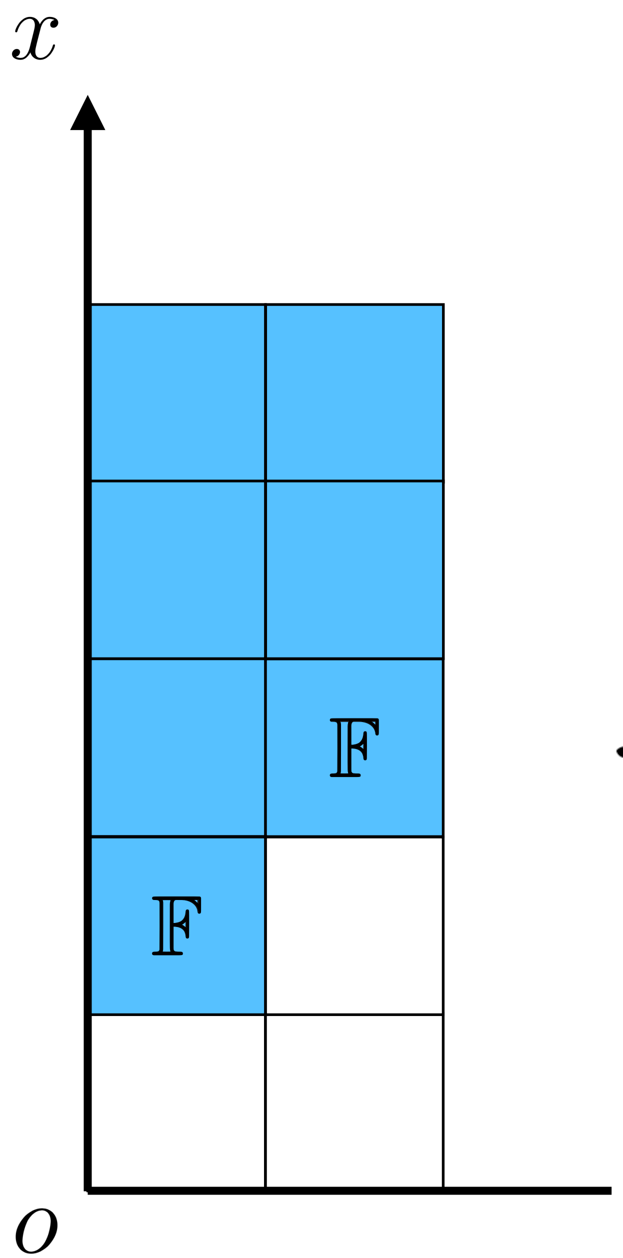
Proposition

Every f.g. n-graded free $\mathbb{F}[\boldsymbol{x}]$-module is isomorphic to

$$\mathcal{F}_n^{\mathbb{F}}(\xi) := \bigoplus_{v \in V} \mathbb{F}[\boldsymbol{x}](v)^{\mu(v)}$$

where $\xi = (V, \mu)$ is a finite n-dimensional multiset.

Example



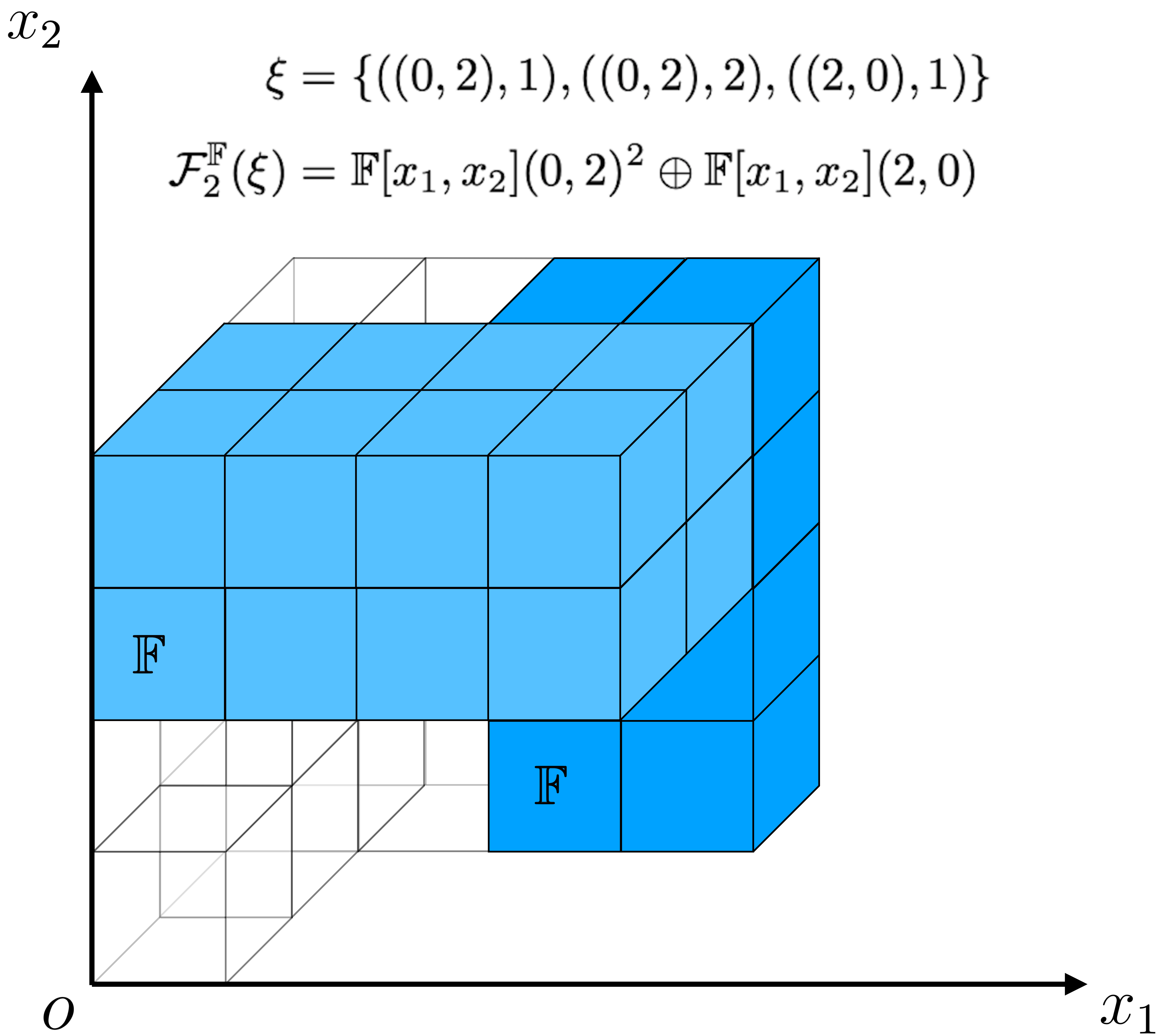
$$\xi = \{(1,1),(2,1)\}$$

$$\mathcal{F}_1^{\mathbb{F}}(\xi) = \mathbb{F}[x](1) \oplus \mathbb{F}[x](2)$$

Example

$$\xi = \{((0,2),1),((0,2),2),((2,0),1)\}$$

$$\mathcal{F}_2^{\mathbb{F}}(\xi) = \mathbb{F}[x_1,x_2](0,2)^2 \oplus \mathbb{F}[x_1,x_2](2,0)$$



- Recall that by the **structure theorem** every f.g. one-graded $\mathbb{F}[x]$-module M admits a graded isomorphism

$$M \cong \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{d_i} \mathbb{F}[x]/\langle x^{t_{i,j}} \rangle_{\mathbb{F}[x]}(v_{i,j}) \oplus \bigoplus_{i=m+1}^{r} \bigoplus_{j=1}^{d_i} \mathbb{F}[x](v_{i,j}).$$

- Thus, we obtain a commutative diagram:

$$\begin{tikzcd}
\bigoplus_{i=1}^{m} \bigoplus_{j=1}^{d_i} \langle x^{t_{i,j}} \rangle_{\mathbb{F}[x]} \arrow[r, hook] \arrow[d, "\wr"] & \bigoplus_{i=1}^{r} \bigoplus_{j=1}^{d_i} \mathbb{F}[x](v_{i,j}) \arrow[d, equal] \arrow[dr, two heads] & \\
\mathcal{F}_1^{\mathbb{F}}(\xi_1) \arrow[r, hook] & \mathcal{F}_1^{\mathbb{F}}(\xi_0) \arrow[r, two heads] & M
\end{tikzcd}$$

- ξ_0 corresponds to the start points of the **barcode** of M and ξ_1 to the endpoints.

- What about $n \geq 2$?

- We have a graded exact sequence

$$\mathcal{F}_n^{\mathbb{F}}(\xi_1) \longrightarrow \mathcal{F}_n^{\mathbb{F}}(\xi_0) \longrightarrow\!\!\!\!\rightarrow M$$

- **Problem:** in general, the left arrow is not injective. The reason is that for $n \geq 2$, submodules of free $\mathbb{F}[\boldsymbol{x}]$-modules are generally not free again.

- Nonetheless, M admits a **minimal graded free resolution**

$$\cdots \longrightarrow \mathcal{F}_n^{\mathbb{F}}(\xi_2) \longrightarrow \mathcal{F}_n^{\mathbb{F}}(\xi_1) \longrightarrow \mathcal{F}_n^{\mathbb{F}}(\xi_0) \longrightarrow\!\!\!\!\rightarrow M$$

- Here **minimal** means that each arrow maps a homogeneous basis of $\mathcal{F}_n^{\mathbb{F}}(\xi_i)$ to a minimal system of homogeneous generators of its image.

- The multisets ξ_i only depend on the isomorphism class of M. The ξ_i are also called **the i-th type of** M and we write

$$\mathrm{Type}_i(M) = \xi_i.$$

- $\mathcal{F}_n^{\mathbb{F}}(\xi_0)$ is also called **the free hull of** M. It keeps track of the degree and number of a minimal system of homogeneous generators of M.

Goal

Show that for $n \geq 2$ there exists **no discrete and complete invariant** for

$$\{\text{f.g. } n\text{-graded } \mathbb{F}[\boldsymbol{x}]\text{-modules}\} /_{\cong}$$

Definition

Let

$$I_n^{\mathbb{F}}(\xi_0, \xi_1) \subseteq \{\text{f.g. } n\text{-graded } \mathbb{F}[\boldsymbol{x}]\text{-modules}\} /_{\cong}$$

be the subset of all $[M]$ such that $\mathrm{Type}_0(M) = \xi_0$ and $\mathrm{Type}_1(M) = \xi_1$.

Goal (refined)

Show that for $n \geq 2$, there exist mulitsets ξ_0 and ξ_1 such that $I_n^{\mathbb{F}}(\xi_0, \xi_1)$ is uncountable if $\mathbb{F}$ is uncountable.

Idea

Parameterize $I_n^{\mathbb{F}}(\xi_0, \xi_1)$ as a subset of a product of **Grassmannians** together with a group action of $\mathrm{Aut}(\mathcal{F}_n^{\mathbb{F}}(\xi_0))$.

Parameterization

- As a consequence of the n-graded version of **Nakayama's Lemma** (see *E. Miller, B. Sturmfels, Combinatorial Commutative Algebra, 2005*), we have the following:

Proposition

Let M be a finitely generated n-graded $\mathbb{F}[\boldsymbol{x}]$-module. Then

$$p : \mathcal{F}_n^{\mathbb{F}}(\xi_0) \twoheadrightarrow M$$

is a free hull of M (i.e. p maps a homogeneous basis to a minimal set of homogeneous generators of M), if and only if

$$\mathrm{id}_{\mathbb{F}} \otimes_{\mathbb{F}[\boldsymbol{x}]} p : \mathbb{F} \otimes_{\mathbb{F}[\boldsymbol{x}]} \mathcal{F}_n^{\mathbb{F}}(\xi_0) \longrightarrow \mathbb{F} \otimes_{\mathbb{F}[\boldsymbol{x}]} M$$

is an isomorphism of n-graded $\mathbb{F}$-vector spaces. Here $\mathbb{F}[\boldsymbol{x}]$ acts on $\mathbb{F}$ by setting the variable action identical to zero.

Theorem (Parameterization, Part 1)

Let

$$S_n^{\mathbb{F}}(\xi_0,\xi_1) := \left\{ \begin{array}{ll} L \subseteq \mathcal{F}_n^{\mathbb{F}}(\xi_0) \text{ graded submodule} : & \\ 1. \quad \mathrm{Type}_0(L) = \xi_1 & \\ 2. \quad \mathrm{im}\left(\mathbb{F} \otimes_{\mathbb{F}[\boldsymbol{x}]} L \to \mathbb{F} \otimes_{\mathbb{F}[\boldsymbol{x}]} \mathcal{F}_n^{\mathbb{F}}(\xi_0)\right) = 0 & \end{array} \right\}.$$

We have a bijection of sets

$$\begin{array}{ccc} S_n^{\mathbb{F}}(\xi_0,\xi_1)/\mathrm{Aut}(\mathcal{F}_n^{\mathbb{F}}(\xi_0)) & & \mathrm{Aut}(\mathcal{F}_n^{\mathbb{F}}(\xi_0)) \cdot L \\ \Big\downarrow \wr & & \Big\downarrow \\ I_n^{\mathbb{F}}(\xi_0,\xi_1) & & \left[\mathcal{F}_n^{\mathbb{F}}(\xi_0)/L\right] \end{array}$$

Theorem (Parameterization, Part 1)

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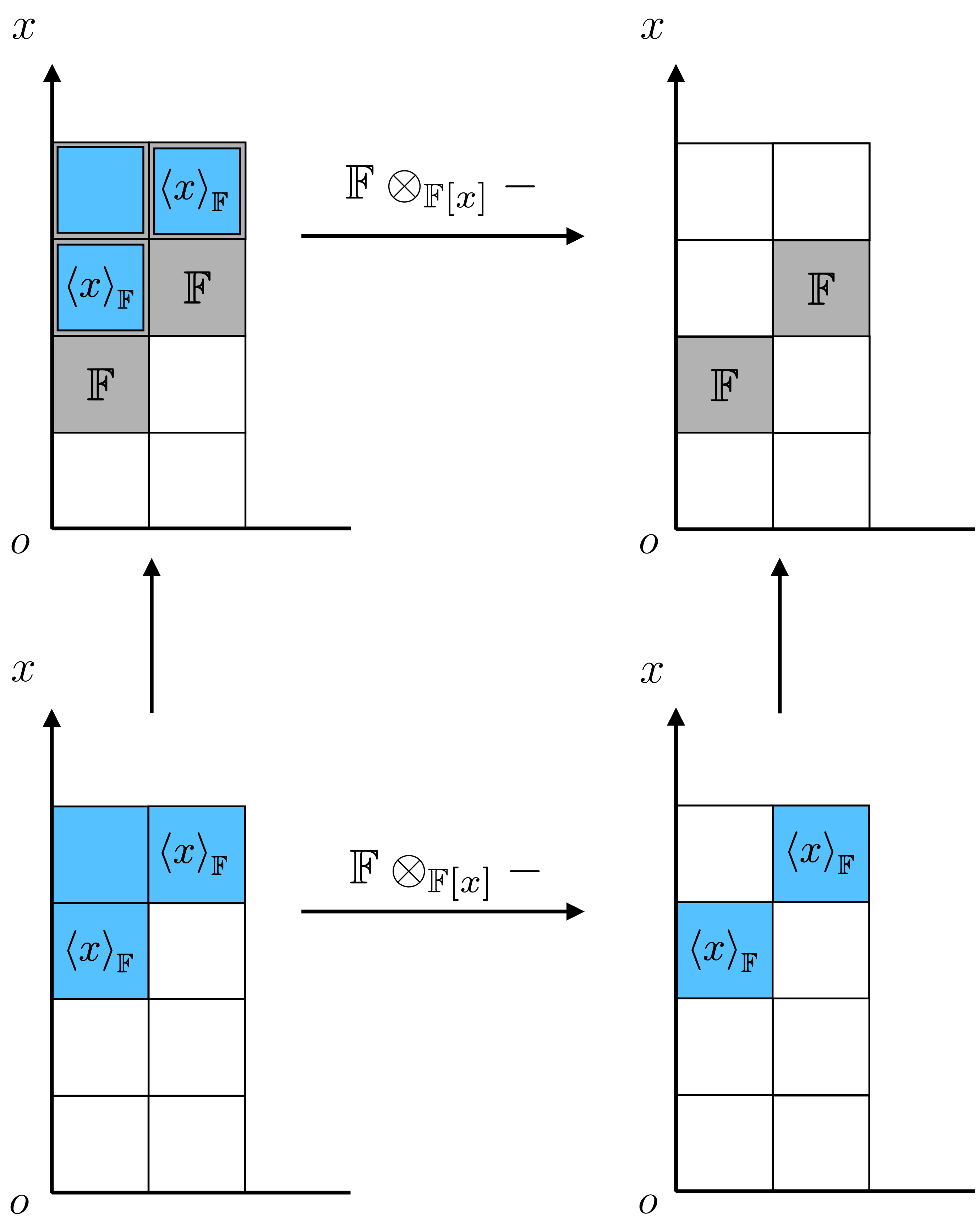
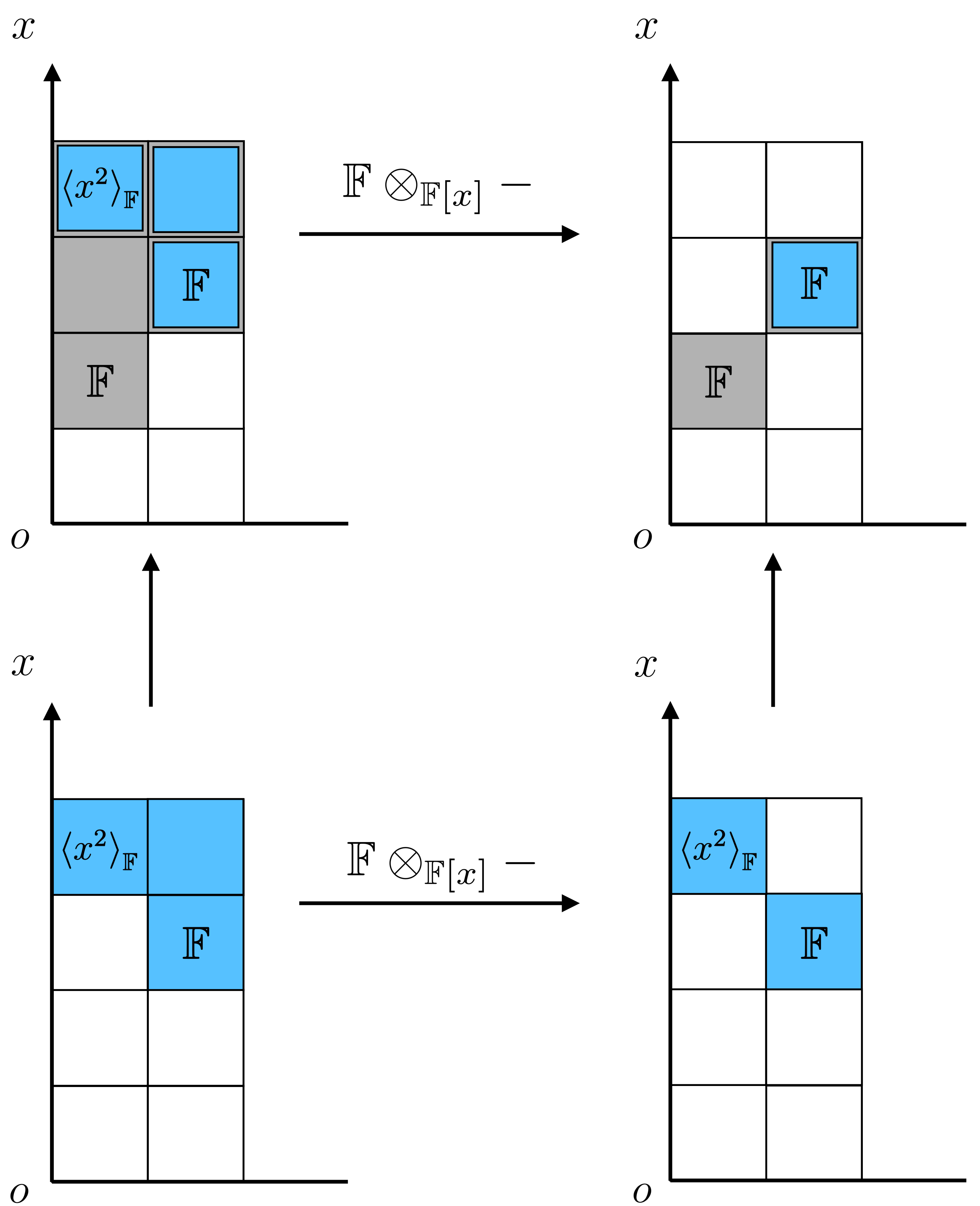
- Condition 2, which we also call **tensor-condition**, ensures that $\mathcal{F}_n^{\mathbb{F}}(\xi_0)$ is a free hull of $\mathcal{F}_n^{\mathbb{F}}(\xi_0)/L$ or equivalently that $\mathrm{Type}_0(\mathcal{F}_n^{\mathbb{F}}(\xi_0)/L) = \xi_0$.

Example



x
o
$\langle x \rangle_{\mathbb{F}}$
$\mathbb{F}$
$\mathbb{F} \otimes_{\mathbb{F}[x]} -$

Example



x
$\langle x^2 \rangle_{\mathbb{F}}$
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$\mathbb{F} \otimes_{\mathbb{F}[x]} -$
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We have a bijection of sets

$$\begin{tikzcd} S_n^{\mathbb{F}}(\xi_0,\xi_1)/\mathrm{Aut}(\mathcal{F}_n^{\mathbb{F}}(\xi_0)) \arrow[d, "\wr"] & \mathrm{Aut}(\mathcal{F}_n^{\mathbb{F}}(\xi_0)) \cdot L \arrow[d, mapsto] \\ I_n^{\mathbb{F}}(\xi_0,\xi_1) & \left[\mathcal{F}_n^{\mathbb{F}}(\xi_0)/L\right] \end{tikzcd}$$

Proposition (Automorphism group)

Recall that $\xi_0 = (V_0, \mu_0)$. Let

$$\mathrm{Aut}^{\mathbb{F}}_{\preceq}(\xi_0) := \left\{ \begin{array}{l} f \in \mathrm{Aut}\left(\bigoplus_{v\in V_0} \mathbb{F}^{\mu_0(v)}\right) : \\ \\ f\left(\mathbb{F}^{\mu_0(v)}\right) \subseteq \bigoplus_{w\preceq v} \mathbb{F}^{\mu_0(w)} \text{ for all } v \in V_0 \end{array} \right\}$$

Then

$$\mathrm{Aut}(\mathcal{F}^{\mathbb{F}}_n(\xi_0)) \cong \mathrm{Aut}^{\mathbb{F}}_{\preceq}(\xi_0)$$

Example

Let

$$\xi_0 = \{(v, 1), \ldots, (v, \mu_0(v))\}.$$

Then

$$\mathrm{Aut}^{\mathbb{F}}_{\preceq}(\xi_0) = \mathrm{GL}_{\mu_0(v)}(\mathbb{F}).$$

Example

The formula

$$f\left(\mathbb{F}^{\mu_0(v)}\right) \subseteq \bigoplus_{w \preceq v} \mathbb{F}^{\mu_0(w)}$$

can be interpreted as a vanishing condition on the entries of the transformation matrix of $f \in \mathrm{Aut}^{\mathbb{F}}_{\preceq}(\xi_0)$:

$$\xi_0 = \{(1,1),(1,2),(2,1),(3,1)\}$$

$$\begin{pmatrix} a_{21} & a_{22} & a_{23} & a_{24} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix} \quad \begin{pmatrix} 1 \leq 1 & 1 \leq 1 & 1 \leq 2 & 1 \leq 3 \\ 1 \leq 1 & 1 \leq 1 & 1 \leq 2 & 1 \leq 3 \\ 2 \not\leq 1 & 2 \not\leq 1 & 2 \leq 2 & 2 \leq 3 \\ 3 \not\leq 1 & 3 \not\leq 1 & 3 \not\leq 2 & 3 \leq 3 \end{pmatrix}$$

Example

The formula

$$f\left(\mathbb{F}^{\mu_0(v)}\right) \subseteq \bigoplus_{w \preceq v} \mathbb{F}^{\mu_0(w)}$$

can be interpreted as a vanishing condition on the entries of the transformation matrix of $f \in \mathrm{Aut}^{\mathbb{F}}_{\preceq}(\xi_0)$:

$$\xi_0 = \{((0,3),1),((1,3),1),((0,4),1)\}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \quad \begin{pmatrix} (0,3) \preceq (0,3) & (0,3) \preceq (1,3) & (0,3) \preceq (0,4) \\ (1,3) \not\preceq (0,3) & (1,3) \preceq (1,3) & (1,3) \not\preceq (0,4) \\ (0,4) \not\preceq (0,3) & (0,4) \not\preceq (1,3) & (0,4) \preceq (0,4) \end{pmatrix}$$

Theorem (Parameterization, Part 1)

Let

$$S_n^{\mathbb{F}}(\xi_0,\xi_1) := \left\{ \begin{array}{ll} L \subseteq \mathcal{F}_n^{\mathbb{F}}(\xi_0) \text{ graded submodule :} & \\ 1. \quad \mathrm{Type}_0(L) = \xi_1 & \\ 2. \quad \mathrm{im}\left(\mathbb{F} \otimes_{\mathbb{F}[\boldsymbol{x}]} L \to \mathbb{F} \otimes_{\mathbb{F}[\boldsymbol{x}]} \mathcal{F}_n^{\mathbb{F}}(\xi_0)\right) = 0 & \end{array} \right\}.$$

We have a bijection of sets

$$\begin{tikzcd} S_n^{\mathbb{F}}(\xi_0,\xi_1)/\mathrm{Aut}(\mathcal{F}_n^{\mathbb{F}}(\xi_0)) \arrow[d, "\wr"] & \mathrm{Aut}(\mathcal{F}_n^{\mathbb{F}}(\xi_0)) \cdot L \arrow[d, mapsto] \\ I_n^{\mathbb{F}}(\xi_0,\xi_1) & \left[\mathcal{F}_n^{\mathbb{F}}(\xi_0)/L\right] \end{tikzcd}$$

- Recall that $\xi_1 = (V_1, \mu_1)$. Let $L \in S_n^{\mathbb{F}}(\xi_0, \xi_1)$. Now map L to the familiy of $\mathbb{F}$-vector spaces

$$(L_w)_{w \in V_1} \in \prod_{w \in V_1} \mathrm{Grass}_{\mathbb{F}}(\dim_{\mathbb{F}}(\mathcal{F}_n^{\mathbb{F}}(\xi_1)_w), |\xi_0|)$$

- $(L_w)_{w \in V_1}$ defines a so-called **relation family over** (ξ_0, ξ_1).

Definition (Relation family)

Recall that $\xi_0 = (V_0, \mu_0)$ and $\xi_1 = (V_1, \mu_1)$. A **relation family** is a family

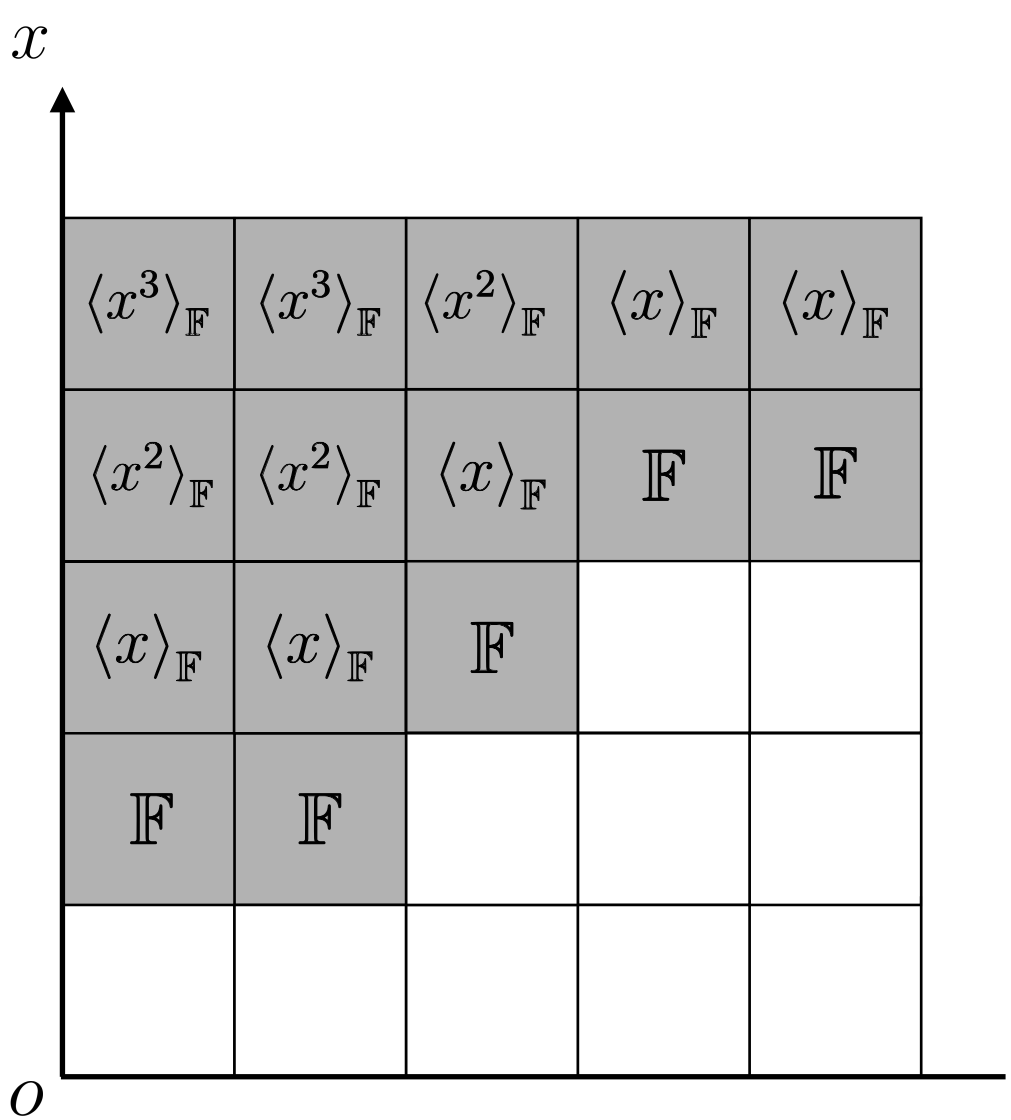
$$(L_w)_{w \in V_1} \in \prod_{w \in V_1} \mathrm{Grass}_{\mathbb{F}}(\dim_{\mathbb{F}}(\mathcal{F}_n^{\mathbb{F}}(\xi_1)_w), |\xi_0|)$$

such that for all $w \in V_1$:

- $\pi_v(L_w) = 0$ for all $v \in V_0$ with $v \not\prec w$ where $\pi_v : \mathbb{F}^{|\xi_0|} \to \mathbb{F}^{\mu_0(v)}$ denotes the canonical projection (note that $\mathbb{F}^{|\xi_0|} = \bigoplus_{v \in V_0} \mathbb{F}^{\mu_0(v)}$).
- if $v \in V_1$ with $v \prec w$, then $L_v \subseteq L_w$.
- $\dim_{\mathbb{F}} \left(L_w / \sum_{v \prec w} L_v \right) = \mu_1(w)$.

$Y_n^{\mathbb{F}}(\xi_0, \xi_1)$ denotes the set of all relations families over (ξ_0, ξ_1).

Example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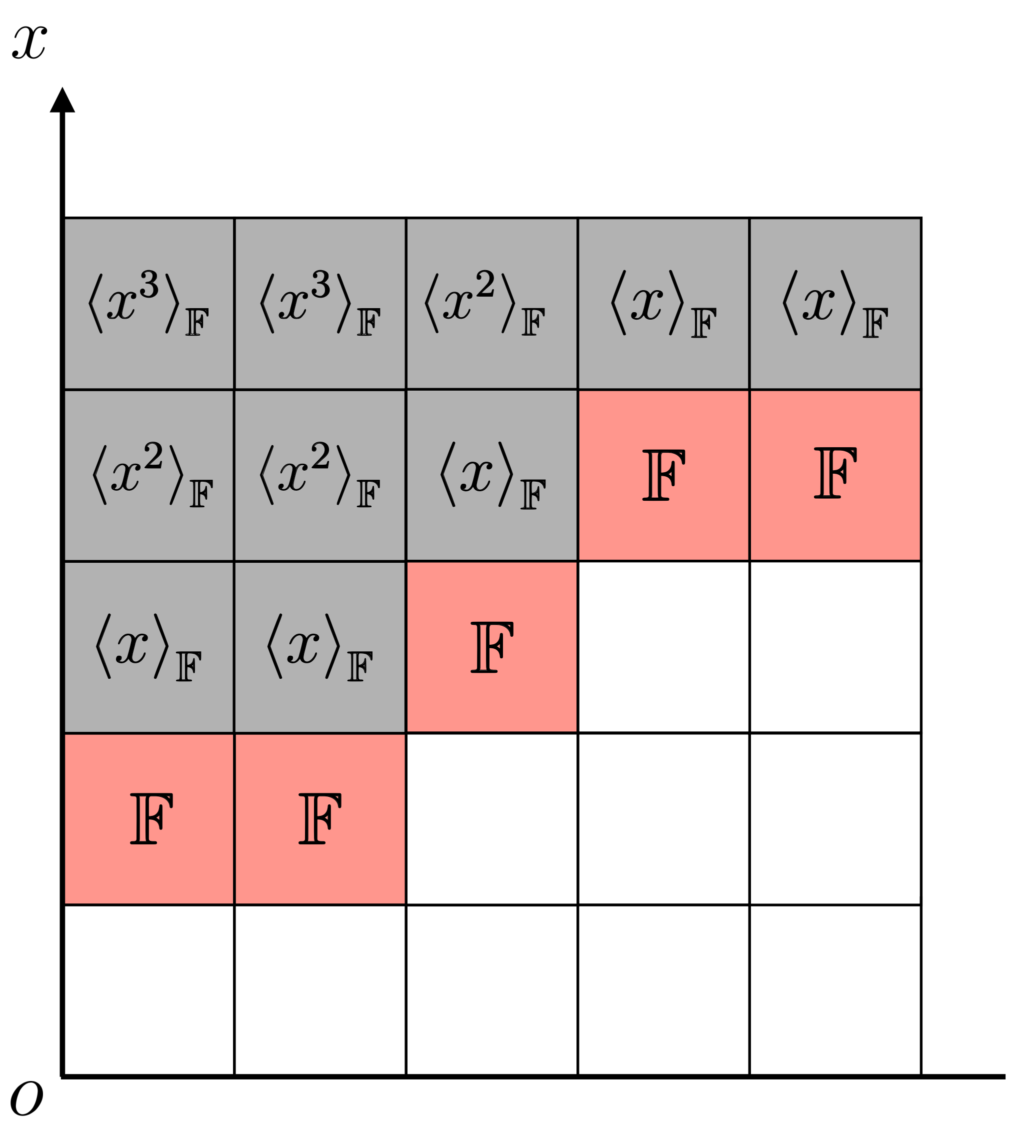
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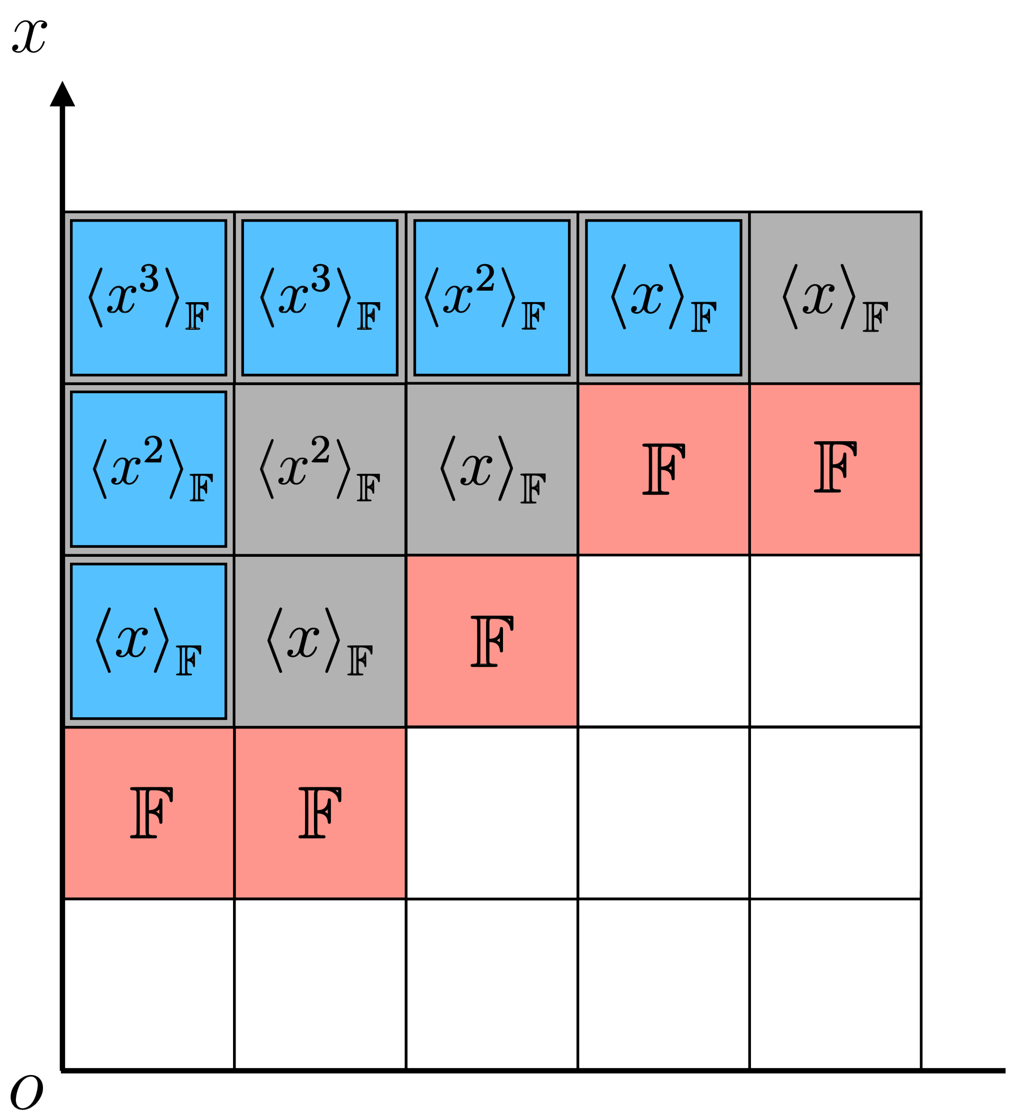

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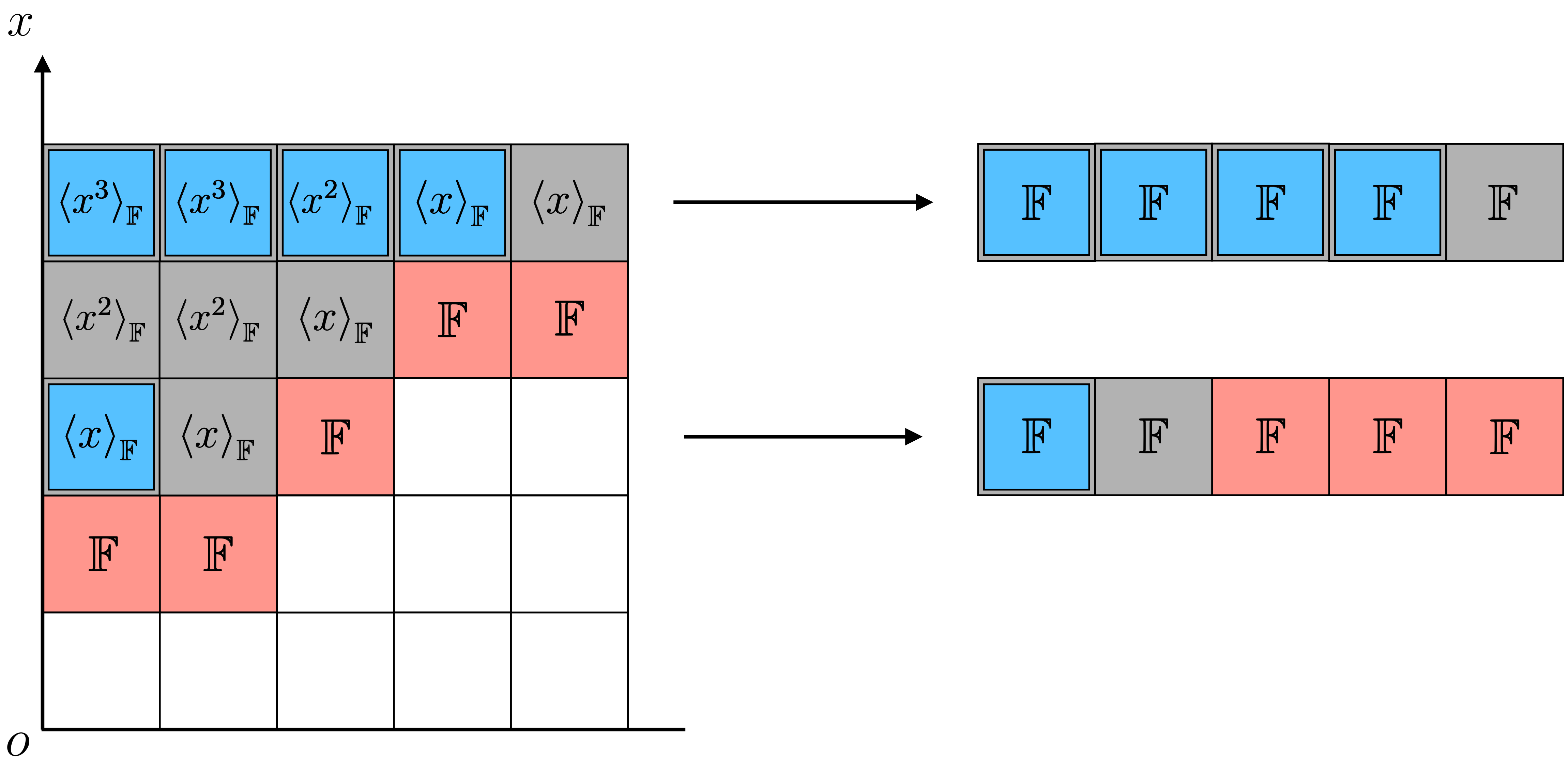

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Theorem (Parameterization, Part 2)

We have a bijection of sets

$$S_n^{\mathbb{F}}(\xi_0, \xi_1) \xrightarrow{\sim} Y_n^{\mathbb{F}}(\xi_0, \xi_1)$$

where L mapped to the familiy of $\mathbb{F}$-vector spaces $(L_w)_{w \in V_1}$. The inverse is given by mapping $(L_w)_{w \in V_1}$ to $\langle \bigcup_{w \in V_1} L_w \rangle_{\mathbb{F}[\boldsymbol{x}]}$. This leads to a bijection of sets on the orbit spaces

$$S_n^{\mathbb{F}}(\xi_0, \xi_1)/\mathrm{Aut}(\mathcal{F}_n^{\mathbb{F}}(\xi_0)) \xrightarrow{\sim} Y_n^{\mathbb{F}}(\xi_0, \xi_1)/\mathrm{Aut}_{\preceq}^{\mathbb{F}}(\xi_0).$$

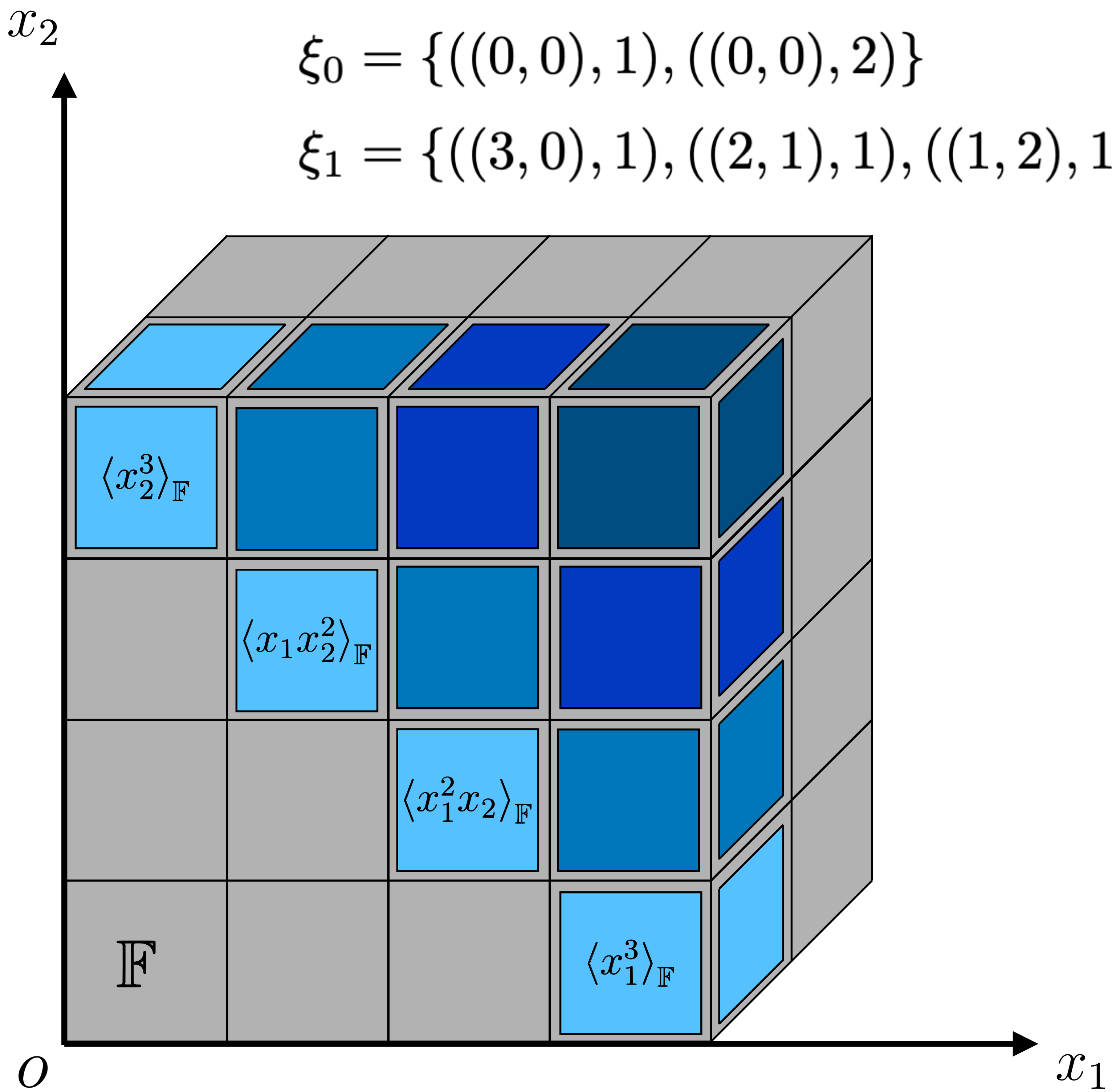
Thus,

$$I_n^{\mathbb{F}}(\xi_0, \xi_1) \cong Y_n^{\mathbb{F}}(\xi_0, \xi_1)/\mathrm{Aut}_{\preceq}^{\mathbb{F}}(\xi_0).$$

Example (Non-existence of a discrete and complete invariant)

$$\xi_0 = \{((0,0),1),((0,0),2)\}$$

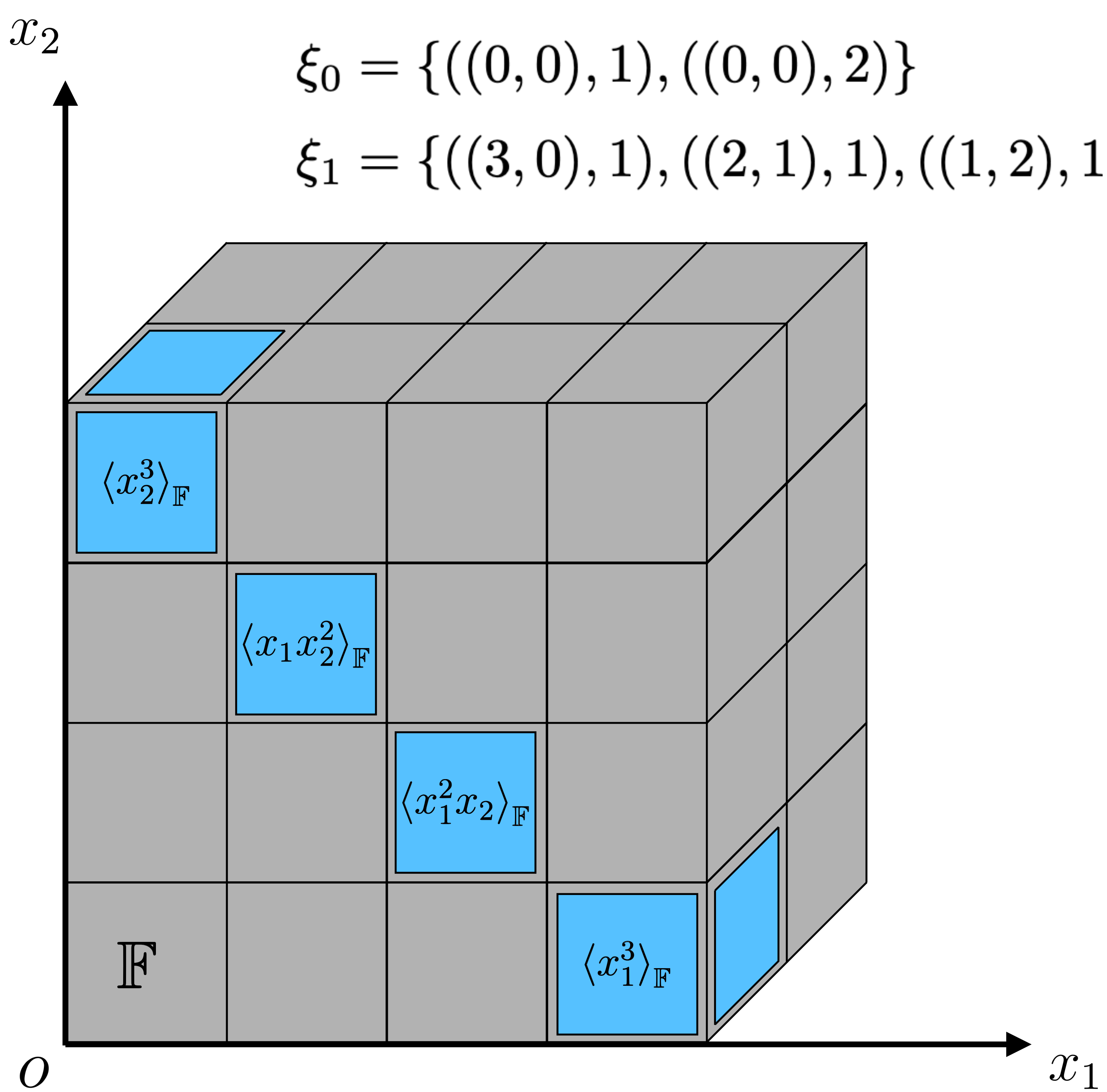
$$\xi_1 = \{((3,0),1),((2,1),1),((1,2),1),((0,3),1)\}$$



Example (Non-existence of a discrete and complete invariant)

$$\xi_0 = \{((0,0),1),((0,0),2)\}$$

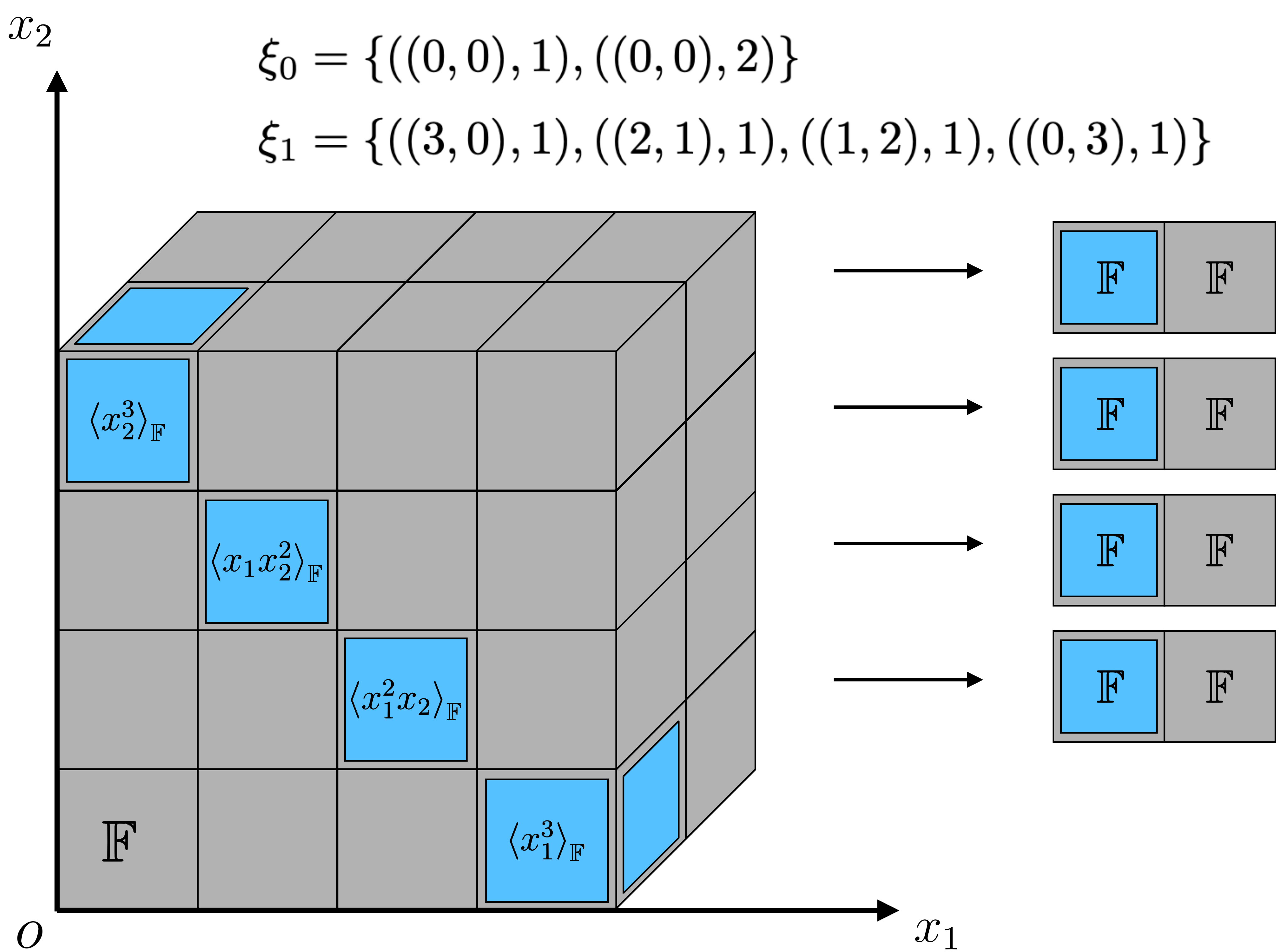
$$\xi_1 = \{((3,0),1),((2,1),1),((1,2),1),((0,3),1)\}$$



Example (Non-existence of a discrete and complete invariant)

$$\xi_0 = \{((0,0),1),((0,0),2)\}$$

$$\xi_1 = \{((3,0),1),((2,1),1),((1,2),1),((0,3),1)\}$$



Example (Non-existence of a discrete and complete invariant)

$$\xi_0 = \{((0,0),1),((0,0),2)\}$$
$$\xi_1 = \{((3,0),1),((2,1),1),((1,2),1),((0,3),1)\}$$

- We have

$$Y_2^{\mathbb{F}}(\xi_0,\xi_1)/\mathrm{Aut}_{\preceq}^{\mathbb{F}}(\xi_0) = \mathrm{Grass}_{\mathbb{F}}(1,2)^4/\mathrm{GL}_2(\mathbb{F}) = \mathbb{P}_1(\mathbb{F})^4/\mathrm{GL}_2(\mathbb{F})$$

  which is uncountable if $\mathbb{F}$ is uncountable. Thus,

$$\{\text{f.g. } n\text{-graded } \mathbb{F}[\boldsymbol{x}]\text{-modules}\} /_{\cong} \supseteq I_2^{\mathbb{F}}(\xi_0,\xi_1) \cong \mathbb{P}_1(\mathbb{F})^4/\mathrm{GL}_2(\mathbb{F})$$

  is uncountable if $\mathbb{F}$ is uncountable which shows that there is **no discrete and complete invariant.**

- For $n > 2$, append zeros to the entries of ξ_0 and ξ_1.

Thanks for your attention!