RELATIVE ASYMPTOTIC BEHAVIOR OF PSEUDOHOLOMORPHIC HALF-CYLINDERS

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Abstract. In this article we study the asymptotic behavior of pseudoholomorphic half-cylinders which converge exponentially to a periodic orbit of a vector field defined by a framed stable Hamiltonian structure. Such maps are of central interest in symplectic field theory and its variants (symplectic Floer homology, contact homology, embedded contact homology). We prove a precise formula for the asymptotic behavior of the "difference" of two such maps, generalizing results from [15, 7, 6, 12]. Using this result with a technique from [14], we then show that a finite collection of pseudoholomorphic half-cylinders asymptotic to coverings of a single periodic orbit is smoothly equivalent to solutions to a linear equation.

Contents

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1. Introduction

Let (W, J) be an almost complex manifold. A (parametrized) pseudoholomorphic curve in W is a map $u : (S, j) \to (W, J)$, where (S, j) is a Riemann surface, whose derivative satisfies the equation

$$du \circ j = J \circ du.$$

In symplectic field theory [3] (and related theories: symplectic Floer homology, contact homology, embedded contact homology, etc.), the central objects of study are a special class of pseudoholomorphic curves. The domains of these curves are punctured Riemann surfaces, i.e. closed Riemann surfaces with a finite number of points (punctures) removed. The targets of the maps of interest are manifolds with cylindrical ends of the form $\mathbb{R}^\pm \times M^\pm$, where the $M^\pm$ are closed odd dimensional manifolds equipped with nowhere vanishing vector fields $X_\pm$ belonging to a special class of dynamical systems. Moreover, the almost complex structure on the target manifold is required to satisfy some "compatibility" conditions related to both a nondegenerate 2-form and the vector fields $X_\pm$ coming from the cylindrical ends. Requiring a finite energy condition and properness on pseudoholomorphic maps in this set-up then guarantees that near the punctures, pseudoholomorphic curves will be asymptotic to cylinders of the form $\mathbb{R} \times \gamma \subset \mathbb{R} \times M^\pm$, where γ is the image of a periodic orbit of $X_\pm$, i.e. a closed integral curve of $X_\pm$. Moreover, in the event that all periodic orbits of $X_\pm$ are nondegenerate or Morse-Bott, then the convergence near a puncture of a pseudoholomorphic curve to a cylinder over a periodic orbit is exponential.

In this work we will study the asymptotic behavior of punctured pseudoholomorphic curves. We model the behavior near a puncture by studying pseudoholomorphic half-cylinders

$$\tilde{u} : \big([R, \infty) \times S^1, j\big) \to \big(\mathbb{R} \times M, \tilde{J}\big).$$

Here we equip the half-cylinder $[R, \infty) \times S^1$ with the complex structure j that arises from viewing it as a subset of $\mathbb{R} \times S^1 \approx \mathbb{C}/i\mathbb{Z}$. The manifold M is equipped with a nowhere vanishing vector field X and a hyperplane distribution ξ everywhere transverse to X. The almost complex structure $\tilde{J}$ on $\mathbb{R} \times M$ is invariant under translations in the $\mathbb{R}$ coordinate and has some prescribed behavior with respect to X and ξ that we will describe more precisely later.

Let $P : S^1 \approx \mathbb{R}/\mathbb{Z} \to M$ be a T-periodic orbit of the vector field X, and assume that $\tilde{u}$ converges to a cylinder over P exponentially fast and in C^∞, as would be the case if P satisfies an appropriate nondegeneracy condition. In this case, we can write

$$\tilde{u}(\psi(s, t)) = \Big(Ts, \exp_{P(t)} U(s, t)\Big) \in \mathbb{R} \times M$$

where $\psi : [R_1, \infty) \times S^1 \to [R, \infty) \times S^1$ is a proper embedding, U is a map satisfying $U(s, t) \in \xi_{P(t)}$ for all $(s, t) \in [R_1, \infty) \times S^1$, and exp is the exponential map associated to a Riemannian metric on M. Previous works of Hofer, Wysocki, and Zehnder [7, 6] and Mora [15] show that the map U can be written

$$U(s, t) = e^{\lambda s} \left[e(t) + r(s, t)\right]$$

where e is an eigenvector of a self-adjoint operator on $L^2(P^*\xi)$ related to the linearized flow along P, $\lambda < 0$ is the eigenvalue of e, and r converges exponentially to zero as $s \to \infty$.

While this representation formula for the map U provides useful information about the asymptotic behavior of pseudoholomorphic half-cylinders, it does not provide any information about the relative behavior of two half-cylinders asymptotic to P in the event that the same eigenvector appears in the formula for each cylinder. The main result (Theorem 2.2) of this paper, which is a generalization and refinement of previous work of Kriener [12], addresses this issue by providing a precise description of the “difference” of two pseudoholomorphic half-cylinders asymptotic to the same periodic orbit. More precisely, assume $\tilde{v} : [R, \infty) \times S^1 \to \mathbb{R} \times M$ is another pseudoholomorphic half-cylinder asymptotic to P for which we can write

$$\tilde{v}(\phi(s,t)) = (Ts, \exp_{P(t)} V(s,t))$$

with ϕ and V satisfying the same properties as ψ and U above. Then the difference $V - U$ can be written

$$V(s,t) - U(s,t) = e^{\lambda s}\left[e(t) + r(s,t)\right]$$

where λ, e, and r satisfy the same properties as the corresponding terms in the formula for U. This result then allows one to distinguish the relative behavior of two pseudoholomorphic half-cylinders in situations where the previous results of [7, 6, 15] would not.

In the event that the periodic orbit P is multiply covered, this result can be applied to provide a refinement (Theorem 2.3) of the asymptotic description of a single half-cylinder from [7, 6, 15]. This refinement is useful for understanding the self-intersection and embeddedness properties of a single pseudoholomorphic half-cylinder. Moreover, applying a technique due to Micallef and White [14], we are able to show (Theorem 2.4) that given a finite family of pseudoholomorphic half-cylinders converging to covers of the same periodic orbit, there is a coordinate system so that the curves, after reparametrization, are given by maps of the form

$$(s,t) \in [R, \infty) \times S^1 \mapsto \left(ks, kt, \sum_{i=1}^{n} e^{\lambda_i s} e_i(t)\right) \in \mathbb{R} \times S^1 \times \mathbb{R}^{2n}$$

where k is a nonzero integer, n is a nonnegative integer, each e_i is an eigenvector of an unbounded self-adjoint operator $\mathbf{A}_k$ on $L^2(S^1, \mathbb{R}^{2n})$ arising from the linearized flow, and $\lambda_i < 0$ is the eigenvalue of e_i.

The results of this paper are most useful when the target manifold $\mathbb{R} \times M$ is 4-dimensional. In this case, these results can be used to develop an intersection theory for punctured pseudoholomorphic curves [9, 17, 16]. This intersection theory in turn is important for the foundations of a 4-dimensional variant of symplectic field theory due to Hutchings known as embedded contact homology [9, 10, 11] - a theory which exploits the fact that the embeddedness of punctured curves in dimension 4 can be controlled in terms of topological data.

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2. Background and Main Results

2.1. **Hamiltonian Structures.** Let M be a compact oriented $2n+1$ dimensional manifold. A *framed stable Hamiltonian structure* $\mathcal{H} = (\lambda, \omega)$ on M is a pair consisting of a 1-form λ and a 2-form ω satisfying

(H1) $\lambda \wedge \omega^n$ is a volume form on M.
(H2) $d\omega = 0$.
(H3) $\ker d\lambda_p \supseteq \ker \omega_p$ for every $p \in M$.

In condition (H3) we are defining the kernel of the 2-form α at $p \in M$ to be the kernel of the associated linear map $T_pM \to T_p^*M$ defined by $v \mapsto \alpha_p(v, \cdot)$. Manifolds carrying data (λ, ω) satisfying these properties were studied in [1], where the existence of this data provides sufficient "taming conditions" to prove important compactness theorems in symplectic field theory. The name "framed stable Hamiltonian structure" to describe a pair (λ, ω) satisfying (H1)-(H3) originates in [4] where it is shown that manifolds carrying this data can be identified with stable hypersurfaces in symplectic manifolds as defined in [8]. For examples of framed stable Hamiltonian structures see [1] or [4].

Assume M is equipped with a framed stable Hamiltonian structure $\mathcal{H} = (\lambda, \omega)$. Condition (H1), implies that $\ker \omega_p$ is one dimensional for all $p \in M$, and therefore ω determines a line bundle defined by $\ell_\omega = \cup_{p\in M}(p, \ker \omega_p)$. It is also clear from this condition that $\xi^{\mathcal{H}} := \ker \lambda$ is a hyperplane distribution transverse to ℓ_ω, and hence that ω restricts to a nondegenerate form on $\xi^{\mathcal{H}}$. Defining a vector field $X_{\mathcal{H}}$ to be the unique section of ℓ_ω satisfying $\lambda(X_{\mathcal{H}}) = 1$, we see that condition (H1) implies that the tangent bundle TM splits

$$TM = \mathbb{R}X_{\mathcal{H}} \oplus (\xi^{\mathcal{H}}, \omega|_{\xi^{\mathcal{H}}}) \tag{1}$$

as the direct sum of a line bundle with a preferred nowhere vanishing section, and a symplectic vector bundle.

Let ψ_t be the flow generated by $X_{\mathcal{H}}$, i.e. $\dot{\psi}_t = X_{\mathcal{H}} \circ \psi_t$. Using condition (H2) and the fact that $X_{\mathcal{H}}$ is defined to satisfy $i_{X_{\mathcal{H}}}\omega = 0$, we find that

$$\frac{d}{dt}\psi_t^*\omega = \psi_t^* L_{X_{\mathcal{H}}}\omega = \psi_t^*\left(i_{X_{\mathcal{H}}}d\omega + d(i_{X_{\mathcal{H}}}\omega)\right) = 0.$$

Similarly, using condition (H3) and the fact that $i_{X_{\mathcal{H}}}\lambda = 1$ by definition, we find that

$$\frac{d}{dt}\psi_t^*\lambda = \psi_t^* L_{X_{\mathcal{H}}}\lambda = \psi_t^*\left(i_{X_{\mathcal{H}}}d\lambda + di_{X_{\mathcal{H}}}\lambda\right) = 0.$$

We conclude that $\psi_t^*\omega = \omega$ and that $\psi_t^*\lambda = \lambda$ for all $t \in \mathbb{R}$, which together imply that the splitting (1) is preserved under the flow of $X_{\mathcal{H}}$, and that the map

$$d\psi_t|_{\xi^{\mathcal{H}}} : (\xi^{\mathcal{H}}, \omega|_{\xi^{\mathcal{H}}}) \to (\xi^{\mathcal{H}}, \omega|_{\xi^{\mathcal{H}}})$$

is symplectic.

In what follows the dynamics of the vector field $X_{\mathcal{H}}$ will play an important role, and the periodic orbits of $X_{\mathcal{H}}$ will be of particular interest. For our purposes, it will be convenient to think of periodic orbits as maps parametrized by $S^1 \approx \mathbb{R}/\mathbb{Z}$ equipped with the basepoint $0 \in \mathbb{R}/\mathbb{Z}$. More precisely, for $T \neq 0$ we define the set $\mathcal{P}_T(M, \mathcal{H})$ of T-periodic orbits of $X_{\mathcal{H}}$ by[1]

$$\mathcal{P}_T(M, \mathcal{H}) = \{P \in C^\infty(S^1, M) \,|\, dP(t)\partial_t = T \cdot X_{\mathcal{H}}(P(t)) \text{ for all } t \in S^1\}$$

[1]Note that this definition allows $T < 0$.

and we will denote the set of all periodic orbits by

$$\mathcal{P}(M,\mathcal{H}) := \cup_{T\neq 0}\mathcal{P}_T(M,\mathcal{H}).$$

We note that each set $\mathcal{P}_T(M,\mathcal{H})$ is invariant under the S^1 action on $C^\infty(S^1,M)$ defined by $c*\gamma(t)=\gamma(t+c)$ for $c\in\mathbb{R}/\mathbb{Z}$ and $\gamma\in C^\infty(S^1,M)$. A periodic orbit $P\in\mathcal{P}(M,\mathcal{H})$ will be called multiply covered if it has a nontrivial isotropy group

$$G(P)=\{c\in S^1 \,|\, c*P=P\}$$

with respect to the S^1 action, and the covering number $\operatorname{cov}(P)$ will be defined to be the order of this (necessarily finite) group. Put more simply, $\operatorname{cov}(P)$ is the largest positive integer k satisfying $\frac{1}{k}*P=P$. [2] It is straightforward to show that if $P\in\mathcal{P}_T(M,\mathcal{H})$ has covering number $k>1$, then P factors through a simply covered orbit via a k-fold covering, i.e. there exists a $p\in\mathcal{P}_{T/k}(M,\mathcal{H})$ with covering number $\operatorname{cov}(p)=1$ such that $P(t)=p(kt)$.

We will associate to any periodic orbit a differential operator related to the linearized flow. Let $P\in\mathcal{P}_T(M,\mathcal{H})$ be a T-periodic orbit and let h be a vector field along P, that is $h:S^1\to TM$ is a smooth function satisfying $h(t)\in T_{P(t)}M$ for all $t\in S^1$. Since h is defined along a flow line of $X_\mathcal{H}$ we can define the Lie derivative $L_{X_\mathcal{H}}h$ of h by

$$L_{X_\mathcal{H}}h(t)=\tfrac{d}{ds}\big|_{s=0}\,d\psi_{-s}(P(t+s/T))h(t+s/T).$$

Since the flow ψ_t of $X_\mathcal{H}$ preserves the splitting (1), so must $L_{X_\mathcal{H}}$, and we can conclude that if $h(t)\in\xi^\mathcal{H}_{P(t)}$ for all $t\in S^1$, then $L_{X_\mathcal{H}}h(t)\in\xi^\mathcal{H}_{P(t)}$ for all $t\in S^1$. Moreover, if ∇ is a symmetric connection on TM, we can use $dP(t)\partial_t=T\cdot X_\mathcal{H}(P(t))$ to write

$$T\cdot L_{X_\mathcal{H}}h=L_{T\cdot X_\mathcal{H}}h=\nabla_{T\cdot X_\mathcal{H}}h-\nabla_h(T\cdot X_\mathcal{H})=\nabla_t h-T\nabla_h X_\mathcal{H},$$

and therefore the differential operator $\nabla_t\cdot -T\nabla_\cdot X_\mathcal{H}$ maps sections of $P^*\xi^\mathcal{H}$ to sections of $P^*\xi^\mathcal{H}$.

Given any symplectic vector bundle (E,ω), we say that a complex structure J on E is compatible with ω if the bilinear form defined by

$$g_J(\cdot,\cdot)=\omega(\cdot,J\cdot)$$

is a metric on E. It is a well known fact that the space of all such J is nonempty and contractible in the C^∞ topology (see e.g. [8]). Recalling that $(\xi^\mathcal{H},\omega)$ is a symplectic vector bundle, we define the set $\mathcal{J}(M,\mathcal{H})\subset End(\xi^\mathcal{H})$ to be the set of complex structures on $\xi^\mathcal{H}$ which are compatible with ω. Choosing some $J\in\mathcal{J}(M,\mathcal{H})$, we associate to each T-periodic orbit $P\in\mathcal{P}_T(M,\mathcal{H})$ a differential operator $\mathbf{A}_{P,J}: C^\infty(P^*\xi^\mathcal{H})\to C^\infty(P^*\xi^\mathcal{H})$ acting on the space of smooth sections of $\xi^\mathcal{H}$ along P defined by

$$\mathbf{A}_{P,J}\eta=-J(\nabla_t\eta-T\nabla_\eta X_\mathcal{H}).$$

We note that the discussion of the previous paragraph implies that $\mathbf{A}_{P,J}$ does in fact map the space of sections of $\xi^\mathcal{H}$ along P to itself, and that $\mathbf{A}_{P,J}$ is independent of symmetric connection ∇ used to define it. We will refer to $\mathbf{A}_{P,J}$ as the asymptotic operator associated to the orbit P. Define an inner product on $C^\infty(P^*\xi^\mathcal{H})$ by

$$\langle h,k\rangle_J=\int_{S^1}\omega_{P(t)}\big(h(t),J(P(t))k(t)\big)\,dt.$$

[2] Throughout, we will make no notational distinction the S^1 action and the corresponding 1-periodic $\mathbb{R}$ action.

Noting that the compatibility of J with $\omega|_{\xi^{\mathcal{H}}}$ implies that $\omega(J\cdot, J\cdot) = \omega(\cdot,\cdot)$ on $\xi^{\mathcal{H}} \times \xi^{\mathcal{H}}$, and recalling that $L_{X_{\mathcal{H}}}\omega = 0$, we have that

$$\begin{aligned}\langle h, \mathbf{A}_{P,J} k\rangle_J &= \int_{S^1} \omega(h, J\mathbf{A}_{P,J}k)\, dt = \int_{S^1} \omega(h, T\cdot L_{X_{\mathcal{H}}} k)\, dt \\ &= \int_{S^1} \tfrac{\partial}{\partial t}\left(\omega(h,k)\right) - T(L_{X_{\mathcal{H}}}\omega)(h,k) - T\omega(L_{X_{\mathcal{H}}}h, k)\, dt \\ &= -\int_{S^1} \omega(T\cdot L_{X_{\mathcal{H}}}h, k)\, dt = \int_{S^1} \omega(-T\cdot JL_{X_{\mathcal{H}}}h, Jk)\, dt \\ &= \int_{S^1} \omega(\mathbf{A}_{P,J}h, Jk)\, dt = \langle \mathbf{A}_{P,J}h, k\rangle_J.\end{aligned}$$

Therefore $\mathbf{A}_{P,J}$ is formally self-adjoint, and $\mathbf{A}_{P,J}$ induces a self-adjoint operator

$$\mathbf{A}_{P,J} : D(\mathbf{A}_{P,J}) = H^1(P^*\xi^{\mathcal{H}}) \subset L^2(P^*\xi^{\mathcal{H}}) \to L^2(P^*\xi^{\mathcal{H}}).$$

In any hermitian trivialization of $(P^*\xi^{\mathcal{H}}, J, \omega(\cdot, J\cdot))$, this operator is represented by a bounded symmetric perturbation of $-i\frac{d}{dt}$. The spectrum of $\mathbf{A}_{P,J}$ thus consists of real eigenvalues of multiplicity at most $2n$ which accumulate only at $\pm\infty$.

Viewing $C^\infty(S^1, \xi^{\mathcal{H}})$ as a vector bundle over $C^\infty(S^1, M)$ with fiber $C^\infty(\gamma^*\xi^{\mathcal{H}})$ over the loop $\gamma \in C^\infty(S^1, M)$, we note that for fixed $c \in S^1$ the map $c * \cdot$ on $C^\infty(S^1, \xi^{\mathcal{H}})$ can be viewed as a bundle map covering the map $c * \cdot$ on $C^\infty(S^1, M)$. Moreover, it is clear from the definition of the asymptotic operator that $\mathbf{A}_{P,J} = (-c) * \circ \mathbf{A}_{c*P,J} \circ c*$ for any $P \in \mathcal{P}(M, \mathcal{H})$ and $c \in S^1$. Therefore $\mathbf{A}_{P,J}$ and $\mathbf{A}_{c*P,J}$ have the same spectrum, and $c * \cdot$ maps the eigenspaces of $\mathbf{A}_{P,J}$ to the eigenspaces of $\mathbf{A}_{c*P,J}$. In particular, the isotropy group $G(P) \approx \mathbb{Z}_{\mathrm{cov}(P)}$ acts on the eigenspaces of $\mathbf{A}_{P,J}$ and we define the covering number $\mathrm{cov}(e)$ of an eigenvector e of $\mathbf{A}_{P,J}$ to be the order of the isotropy group

$$G(e) = \{c \in G(P) \,|\, c * e = e\}.$$

Alternatively, we say that $\mathrm{cov}(e)$ is the largest integer m dividing $\mathrm{cov}(P)$ for which $\frac{1}{m} * e = e$. If $P \in \mathcal{P}(M, \mathcal{H})$ is any periodic orbit and we let $P_k \in \mathcal{P}(M, \mathcal{H})$ be the map defined by

$$P_k(t) = P(kt) \tag{2}$$

for any $k \in \mathbb{Z}\backslash\{0\}$, it follows from the definition of $\mathbf{A}_{P,J}$ that if e is an eigenvector of $\mathbf{A}_{P,J}$ with eigenvalue λ, then the section $e_k \in C^\infty(P_k^*\xi^{\mathcal{H}})$ defined by $e_k(t) = e(kt)$ is an eigenvector of $\mathbf{A}_{P_k,J}$ with eigenvalue $k\lambda$. If we further require that $\mathrm{cov}(P) = 1$, it is straightforward to see that any eigenvector e of $\mathbf{A}_{P_k,J}$ with covering number $m = \mathrm{cov}(e)$ is of the form $e(t) = f(mt)$ for some eigenvector f of $\mathbf{A}_{P_{k/m},J}$ with $\mathrm{cov}(f) = 1$.

We observe that if $e \in C^\infty(P^*\xi^{\mathcal{H}})$ is an eigenvector of $\mathbf{A}_{P,J}$, then $e(t) \neq 0$ for all $t \in S^1$. Indeed, assume that $e \in \ker(\mathbf{A}_{P,J} - \lambda)$. Then, in any trivialization of $P^*\xi^{\mathcal{H}}$, e satisfies a linear first order o.d.e. of the form $-J(t)\frac{d}{dt}e(t) + (S(t) - \lambda)e(t) = 0$, where S and J are smooth matrix valued functions, and $J(t)^2 = -I$. Therefore, if $e(t) = 0$ for some $t \in S^1$, $e(t) = 0$ for all $t \in S^1$. Hence nonzero vectors $e \in \ker(\mathbf{A}_{P,J} - \lambda)$ satisfy $e(t) \neq 0$ for all $t \in S^1$. Much of the utility of our main theorems derives from this fact.

2.2. **Asymptotically Cylindrical Pseudoholomorphic Half-Cylinders.** Let $(M, \mathcal{H})$ be a closed $2n+1$ dimensional manifold equipped with a framed stable Hamiltonian structure $\mathcal{H} = (\lambda, \omega)$. We will define a preferred class of almost complex structures on $\mathbb{R}\times M$ which we will then use to define a first order elliptic system. Recall that we defined the set $\mathcal{J}(M, \mathcal{H})$ to be composed of all complex structures J on $\xi^{\mathcal{H}}$ for which $\omega(\cdot, J\cdot)$ is a metric on $\xi^{\mathcal{H}}$. We extend each $J \in \mathcal{J}(M, \mathcal{H})$ to an $\mathbb{R}$-invariant almost complex structure $\tilde{J}$ on $\mathbb{R} \times M$ by requiring

$$\tilde{J}\partial_a = X_{\mathcal{H}} \quad \text{and} \quad \tilde{J}|_{\xi^{\mathcal{H}}} = J$$

where a is the parameter along $\mathbb{R}$.

Given a $P \in \mathcal{P}_T(M, \mathcal{H})$, note that the map $\tilde{P} : \mathbb{R} \times S^1 = \mathbb{C}/i\mathbb{Z} \to \mathbb{R} \times M$ defined by $\tilde{P}(s, t) = (Ts, P(t))$ is $\tilde{J}$-holomorphic for any $J \in \mathcal{J}(M, \mathcal{H})$; that is $\tilde{P}$ satisfies

$$d\tilde{P} \circ j = \tilde{J} \circ d\tilde{P}.$$

We will call such a map a cylinder over the periodic orbit P. In this paper we will study $\tilde{J}$-holomorphic maps from half cylinders $[R, \infty) \times S^1 \subset \mathbb{R} \times S^1 = \mathbb{C}/i\mathbb{Z}$ to $\mathbb{R} \times M$ which are asymptotic to cylinders over periodic orbits. In particular, given real numbers $d > 0$, R, and a T-periodic orbit $P : S^1 \to M$, we define $\mathcal{M}^d_R(P, J) \subset C^\infty([R, \infty) \times S^1, \mathbb{R} \times M)$ to be the set of all smooth maps $\tilde{u} = (a, u) : [R, \infty) \times S^1 \to \mathbb{R} \times M$ satisfying:

(1) The map $\tilde{u}$ is $\tilde{J}$ holomorphic, i.e.

$$d\tilde{u} \circ j = \tilde{J} \circ d\tilde{u}.$$

(2) The loops $t \mapsto u(s, t)$ converge to P in $C^\infty(S^1, M)$ as $s \to \infty$.

(3) There exist positive constants M_β so that the $\mathbb{R}$-component a of $\tilde{u} = (a, u)$ satisfies

$$\left|\partial^\beta \left(a(s, t) - Ts\right)\right| \leq M_\beta e^{-ds}$$

for all $(s, t) \in [R, \infty) \times S^1$ and all $\beta \in \mathbb{N}^2$.

(4) Let $\phi : S^1 \times B^{2n}_\varepsilon(0) \subset S^1 \times \mathbb{R}^{2n} \to U \supset P$ be any $\operatorname{cov}(P)$-fold covering of an open neighborhood U of P satisfying $\phi(t, 0) = P(t)$. Let $R_1 \geq R$ be chosen large enough so that $u([R_1, \infty) \times S^1) \subset U$, and denote by $u_\phi : [R_1, \infty) \times S^1 \to S^1 \times \mathbb{R}^{2n}$ the unique lift of u satisfying $\lim_{s\to\infty} u_\phi(s, t) = (t, 0)$ for all $t \in S^1$. Then there exist positive constants M_β so that [3]

$$\left|\partial^\beta \left(u_\phi(s, t) - (t, 0)\right)\right| \leq M_\beta e^{-ds}$$

for all $(s, t) \in [R_1, \infty) \times S^1$ and all $\beta \in \mathbb{N}^2$.

We refer to elements of $\mathcal{M}^d_R(P, J)$ for a given $d > 0$ and R as *asymptotically cylindrical pseudoholomorphic half-cylinders*. We will abbreviate $\mathcal{M}_R(P, J) = \cup_{d>0}\mathcal{M}^d_R(P, J)$ and $\mathcal{M}(P, J) = \cup_{R\in\mathbb{R}}\mathcal{M}_R(P, J)$ when convenient.

We remark that in practice the exponential convergence to a periodic orbit that we require in the definition of asymptotically cylindrical is usually a consequence of the cylinder $\tilde{u}$ satisfying a finite energy condition and the periodic orbit P being either nondegenerate or Morse-Bott (see [7, 6, 2, 15]). However, for the purposes of the present study, only the asymptotic behavior of the cylinder is relevant, and so

[3] Here the "-" sign can be interpreted with respect to the Lie group structure on $S^1 \times \mathbb{R}^{2n}$. When $\beta = (0, 0)$ the symbol $|\cdot|$ should then be interpreted as distance from the identity element $(0, 0) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}^{2n}$.

we will neither make any assumptions about the degeneracy of the periodic orbits in question nor deal with the definition of finite energy.

For the results in this paper it will be convenient to describe elements of $\mathcal{M}(P,J)$ in terms of sections of the bundle $P^*\xi^{\mathcal{H}}$. In order to do this we associate to each $J\in\mathcal{J}(M,\mathcal{H})$, a metric g_J on M defined by

$$g_J(v,w)=\lambda(v)\lambda(w)+\omega(\pi_{\xi^{\mathcal{H}}}v,J\pi_{\xi^{\mathcal{H}}}w)$$

where $\pi_{\xi^{\mathcal{H}}}:TM=\mathbb{R}X_{\mathcal{H}}\oplus\xi^{\mathcal{H}}\to\xi^{\mathcal{H}}$ is the projection onto $\xi^{\mathcal{H}}$ along $X_{\mathcal{H}}$. We then extend g_J to an $\mathbb{R}$-invariant metric $\tilde g_J$ on $\mathbb{R}\times M$ by defining

$$\tilde g_J((h,v),(k,w))=h\cdot k+g_J(v,w)$$

where we are making the standard identification $T(\mathbb{R}\times M)\approx\mathbb{R}\oplus TM$. We will denote the exponential maps of g_J and $\tilde g_J$ by exp and $\widetilde{\exp}$ respectively and we note that these maps are related by

$$\widetilde{\exp}_{(a,p)}(h,v)=(a+h,\exp_p v)\in\mathbb{R}\times M$$

since $\tilde g$ is the sum of the standard flat metric on $\mathbb{R}$ with the metric g on M. We now make a definition which will be important for the statement of our main theorem.

Definition 2.1. Let $P\in\mathcal{P}_T(M,\mathcal{H})$ be a T-periodic orbit, let $\tilde u\in\mathcal{M}(P,J)$, and let $U:[R,\infty)\times S^1\to P^*\xi^{\mathcal{H}}$ be a smooth map satisfying $U(s,t)\in\xi^{\mathcal{H}}_{P(t)}$ for all $(s,t)\in[R,\infty)\times S^1$. We say that U is an *asymptotic representative* of $\tilde u$ if there exists a proper embedding $\psi:[R,\infty)\times S^1\to\mathbb{R}\times S^1$ asymptotic to the identity, so that

$$\tilde u(\psi(s,t))=(Ts,\exp_{P(t)}U(s,t))=\widetilde{\exp}_{(Ts,P(t))}(0,U(s,t)).$$

for all $(s,t)\in[R,\infty)\times S^1$.

Using the assumed asymptotic behavior of elements of $\mathcal{M}(P,J)$, it is straightforward to show that every asymptotically cylindrical pseudoholomorphic curve has an asymptotic representative (this also follows from a special case of Proposition 5.1). Moreover, the requirement that the embedding ψ from this definition be asymptotic to the identity implies the asymptotic representative U of a map $\tilde u\in\mathcal{M}(P,J)$ is uniquely determined up to restriction of the domain $[R,\infty)\times S^1$ of U to larger values of R. We can define an S^1 action on $C^\infty(\mathbb{R}\times S^1,\mathbb{R}\times M)$ by $c*\tilde u(s,t)=\tilde u(s,t+c)$, and note that $c*\cdot$ maps the set $\mathcal{M}^d_R(P,J)$ to the set $\mathcal{M}^d_R(c*P,J)$. In particular, if $\operatorname{cov}(P)=k$ there is a $\mathbb{Z}_k\approx G(P)$ action on $\mathcal{M}^d_R(P,J)$ generated by $\frac{1}{k}*$. Similarly, we can define an S^1 action on the space of maps $C^\infty([R,\infty)\times S^1,P^*\xi^{\mathcal{H}})$ for any $P\in\mathcal{P}(M,\mathcal{H})$, and we note that if $U\in C^\infty([R,\infty)\times S^1,P^*\xi^{\mathcal{H}})$ is an asymptotic representative for $\tilde u\in\mathcal{M}^d_R(P,J)$, then $c*U$ is an asymptotic representative for $c*\tilde u$. Indeed, let $\phi_c:\mathbb{R}\times S^1\to\mathbb{R}\times S^1$ be the map defined by $\phi(s,t)=(s,t+c)$. Then if $u(\psi(s,t))=\exp_{P(t)}U(s,t)$ with ψ converging asymptotically to the identity, it is easy to see that $c*u(\psi_c(s,t))=\exp_{c*P(t)}c*U(s,t)$ where $\psi_c:=\phi_c^{-1}\circ\psi\circ\phi_c$ converges asymptotically to the identity since ψ does.

2.3. Main Results.

In this work, we will take the existence of pseudoholomorphic half-cylinders for granted, and study their asymptotic behavior. Our main result is the following (cf. [12]).

Theorem 2.2. *Let $\tilde u\in\mathcal{M}(P,J)$ and $\tilde v\in\mathcal{M}(P,J)$, let the maps U, $V:[R,\infty)\to C^\infty(P^*\xi^{\mathcal{H}})$, be asymptotic representatives of $\tilde u$ and $\tilde v$ respectively (see Def. 2.1), and assume that $V-U$ doesn't vanish identically. Then there exists a negative* *eigenvalue λ of the asymptotic operator $\mathbf{A}_{P,J}$ and an eigenvector e with eigenvalue λ so that*

$$V(s,t) - U(s,t) = e^{\lambda s}(e(t) + r(s,t))$$

where the map r satisfies, for every $(i,j) \in \mathbb{N}^2$, a decay estimate of the form

$$\left|\nabla_s^i \nabla_t^j r(s,t)\right| \leq M_{ij} e^{-ds}$$

with M_{ij} and d positive constants.

We observe that when the periodic orbit P has covering number $k > 1$, we can apply the theorem to the reparametrized maps $\frac{i}{k} * \tilde{u}$ and $\frac{j}{k} * \tilde{v}$ to find that the maps $(s,t) \mapsto V(s, t + \frac{i}{k}) - U(s, t + \frac{j}{k})$ for any $i, j \in \mathbb{Z}$ can be represented by a formula of the same form. We can, in particular, apply this observation to the case where $\tilde{v} = \tilde{u}$ to obtain a representation formula for each of the maps $(s,t) \mapsto U(s, t + \frac{i}{k}) - U(s, t + \frac{j}{k})$.

We also observe that in the special case where $V \equiv 0$, i.e. when $\tilde{v}$ has image contained in the cylinder $\mathbb{R} \times P(S^1)$, then Theorem 2.2 just gives a formula for the asymptotic representative U of $\tilde{u}$ as in [7, 6, 15]. Combining this observation with that of the previous paragraph leads us to the following generalization of the results in [7, 6, 15].

Theorem 2.3. *Let $\tilde{u} \in \mathcal{M}(P, J)$ and let $U : [R, \infty) \times S^1 \to P^*\xi^{\mathcal{H}}$ be an asymptotic representative of $\tilde{u}$. Then U either vanishes identically or can be written*

$$U(s,t) = \sum_{i=1}^{N} e^{\lambda_i s}(e_i(t) + r_i(s,t))$$

where

- *The λ_i are a sequence of negative eigenvalues of $\mathbf{A}_{P,J}$ which is strictly decreasing in i (i.e. $\lambda_j < \lambda_i$ for $j > i$).*
- *Each $e_i (\neq 0)$ is an eigenvector of $\mathbf{A}_{P,J}$ with eigenvalue λ_i.*
- *The sequence of positive integers defined by $k_1 = \operatorname{cov}(e_1)$, $k_i = \gcd(k_{i-1}, \operatorname{cov}(e_i))$, is strictly decreasing in i.*
- *The r_i satisfy $\frac{1}{k_i} * r_i = r_i$. Moreover, each r_i satisfies decay estimates of the form*

$$\left|\nabla_s^l \nabla_t^m r_i(s,t)\right| < M_{lm} e^{-ds}$$

for some positive constants M_{lm} and d.

Theorem 2.2 can be taken a step further to give a coordinate system near a cylinder over an orbit in which a family of curves asymptotic to coverings of that orbit have a particularly simple form. In particular, we adapt an argument of Micallef and White from [14] to prove the following theorem.

Theorem 2.4. *Let $P \in \mathcal{P}_\tau(M, \mathcal{H})$ be a simply covered periodic orbit with period $\tau > 0$, let $P_k \in \mathcal{P}_{k\tau}(M, \mathcal{H})$ be as defined in (2). Let $\tilde{u}_i \in \mathcal{M}(P_{k_i}, J)$ for $i = 1 \ldots n$ be a finite collection of asymptotically cylindrical pseudoholomorphic curves, and let $\Phi : S^1 \times \mathbb{R}^{2n} \to P^*\xi^{\mathcal{H}}$ be a trivialization of $P^*\xi^{\mathcal{H}}$. Then there exist an open neighborhood U of $P(S^1)$, a smooth embedding $\tilde{\Phi} : \mathbb{R} \times U \to \mathbb{R} \times S^1 \times \mathbb{R}^{2n}$ satisfying*

$$\tilde{\Phi}(s, P(t)) = (s, t, 0) \in \mathbb{R} \times S^1 \times \mathbb{R}^{2n},$$

proper embeddings $\psi_i : [R,\infty)\times S^1 \to \mathbb{R}\times S^1$ *asymptotic to the identity, and positive integers* N_i *so that*

$$(\tilde{\Phi}\circ\tilde{u}_i\circ\psi_i)(s,t) = \left(k_i s, k_i t, \sum_{j=1}^{N_i} e^{\lambda_{i,j}s} e_{i,j}(t)\right)$$

where the $\lambda_{i,j}$ *are negative eigenvalues of* $\mathbf{A}_{P_{k_i},J}$*, and the* $e_{i,j}$ *are eigenvectors of* $\Phi^{-1}\mathbf{A}_{P_{k_i},J}\Phi$ *with eigenvalue* $\lambda_{i,j}$*.*

We remark that the precise description of curves provided by this theorem comes at the expense of the coordinate system $\tilde{\Phi}$ not respecting translations in the $\mathbb{R}$ components of $\mathbb{R}\times U$ and $\mathbb{R}\times S^1\times\mathbb{R}^{2n}$. More precisely, near an orbit it is frequently convenient to work with coordinates systems $\tilde{\Phi} : \mathbb{R}\times U \to \mathbb{R}\times S^1\times\mathbb{R}^{2n}$ of the form $\tilde{\Phi}(a,p) = (ca, \phi(p))$ where c is some nonzero constant and $\phi : U \to S^1\times\mathbb{R}^{2n}$ is an embedding. We call such a coordinate system $\mathbb{R}$-equivariant. While the coordinate system $\tilde{\Phi}$ produced by Theorem 2.4 is not $\mathbb{R}$-equivariant it is clear from the construction of $\tilde{\Phi}$ in the proof of the theorem that $\tilde{\Phi}$ is asymptotic to an $\mathbb{R}$-equivariant coordinate system in the following sense: there exist an $\mathbb{R}$-equivariant coordinate system $\tilde{\Psi}$ on $\mathbb{R}\times U$ and positive constants M_β, d so that

$$\pi_{\mathbb{R}\times S^1}\left((\tilde{\Psi}\circ\tilde{\Phi}^{-1})(s,t,w)\right) = (s,t)$$

and

$$\left|\partial^\beta\left[\pi_{\mathbb{R}^{2n}}\left((\tilde{\Psi}\circ\tilde{\Phi}^{-1})(s,t,w)\right) - w\right]\right| \le M_\beta e^{-ds}$$

for all $\beta\in\mathbb{N}^{2n+2}$ and all $(s,t,w)\in\tilde{\Phi}(\mathbb{R}\times U)$, where $\pi_{\mathbb{R}\times S^1} : \mathbb{R}\times S^1\times\mathbb{R}^{2n}\to\mathbb{R}\times S^1$ and $\pi_{\mathbb{R}^{2n}} : \mathbb{R}\times S^1\times\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ are the coordinate projections $\pi_{\mathbb{R}\times S^1}(s,t,w) = (s,t)$ and $\pi_{\mathbb{R}^{2n}}(s,t,w) = w$.

2.4. **Some Consequences of the Main Results.** In this section we provide some useful consequences of Theorems 2.2 and 2.3.

Corollary 2.5. *Let* $P\in\mathcal{P}(M,\mathcal{H})$ *be a simply covered periodic orbit, and let* $\tilde{u}\in\mathcal{M}_{R_1}(P_{k_1},J)$*,* $\tilde{v}\in\mathcal{M}_{R_2}(P_{k_2},J)$ *be asymptotically cylindrical half-cylinders, where* P_k *is as defined in* (2)*. Further, assume that neither of the sets* $\tilde{u}^{-1}(\tilde{v}([R_2,\infty)\times S^1))$ *and* $\tilde{v}^{-1}(\tilde{u}([R_1,\infty)\times S^1))$ *contain an open set. Then there exists an* $R\ge\max(R_1,R_2)$ *so that*

$$\tilde{u}([R,\infty)\times S^1)\cap\tilde{v}([R,\infty)\times S^1) = \emptyset.$$

Proof. Assuming initially that $k_1 = k_2$, we observe that all intersections of $\tilde{u}$ and $\tilde{v}$ lying outside some sufficiently large compact subset of $\mathbb{R}\times M$ correspond to zeros of a map of the form $(s,t)\mapsto U(s,t) - V(s,t+\frac{j}{k_1})$, where U and V are asymptotic representatives of $\tilde{u}$ and $\tilde{v}$ respectively. Our assumption that the sets $\tilde{u}^{-1}(\tilde{v}([R_2,\infty)\times S^1))$ and $\tilde{v}^{-1}(\tilde{u}([R_1,\infty)\times S^1))$ do not contain an open set allows us to conclude that none of the maps $U(s,t) - V(s,t+\frac{j}{k_1})$ vanish identically, so Theorem 2.2 allows us to conclude that each of the maps $U(s,t) - V(s,t+\frac{j}{k_1})$ is nonzero for all sufficiently large values of s. This completes the proof in this case.

Next observe that if k_1 and k_2 have opposite sign, the result is an easy consequence of the asymptotic behavior of the $\mathbb{R}$-components of $\tilde{u}$ and $\tilde{v}$. It therefore remains to consider the case where $k_1\ne k_2$ and k_1 and k_2 have the same sign. In this case we define maps $\tilde{u}_{|k_2|}\in\mathcal{M}_{R_1}(P_{k_1|k_2|},J)$ and $\tilde{v}_{|k_1|}\in\mathcal{M}_{R_2}(P_{k_2|k_1|},J) =$

$\mathcal{M}_{R_2}(P_{k_1|k_2|}, J)$ by $\tilde{u}_{|k_2|}(s,t) = \tilde{u}(|k_2|s, |k_2|t)$ and $\tilde{v}_{|k_1|}(s,t) = \tilde{v}(|k_1|s, |k_1|t)$. We can then apply the argument in the previous paragraph to show that $\tilde{u}_{|k_2|}$ and $\tilde{v}_{|k_1|}$ don't intersect for s sufficiently large, and hence neither do $\tilde{u}$ and $\tilde{v}$. □

Corollary 2.6. *Let $P \in \mathcal{P}(M, \mathcal{H})$ be a simply covered periodic orbit, and let P_k be as defined in (2). For any pseudoholomorphic half-cylinder $\tilde{u} \in \mathcal{M}_{R_0}(P_k, J)$, there is an $R_1 > R_0$ so that the restriction $\tilde{u}|_{[R_1,\infty)\times S^1} \in \mathcal{M}_{R_1}(P_k, J)$ factors through an embedding. More precisely, there exist a k' dividing k, an embedding $\tilde{v} \in \mathcal{M}(P_{k/k'}, J)$ and a map $p : [R_1, \infty) \times S^1 \to \mathbb{R} \times S^1$ asymptotic to the map $(s,t) \mapsto (k's, k't)$ so that $\tilde{u}|_{[R_1,\infty)\times S^1} = \tilde{v} \circ p$.*

Proof. Let $U : [R, \infty) \times S^1 \to P^*\xi^{\mathcal{H}}$ be an asymptotic representative for U, i.e.

$$\tilde{u}(\psi(s,t)) = (Ts, \exp_{P_k(t)} U(s,t))$$

for some embedding $\psi : [R, \infty) \times S^1 \to [R_0, \infty) \times S^1$ which converges asymptotically to the identity. Theorem 2.3 lets us write

$$U(s,t) = \sum_{i=1}^{N} e^{\lambda_i}(e_i(t) + r_i(s,t)) \tag{3}$$

with the λ_i, e_i and r_i satisfying the properties listed in the theorem. Assume initially that the $k_N := \gcd(\mathrm{cov}(e_1), \ldots, \mathrm{cov}(e_N)) = 1$. Then it follows that the map $(s,t) \mapsto U(s,t) - U(s, t + \frac{j}{|k|})$ does not vanish identically for any nonzero $j \in \mathbb{Z}_{|k|}$. Arguing as Corollary 2.5, this implies that there is an $R_1 \geq R$ so that $\tilde{u}|_{[R_1,\infty)\times S^1}$ is an injective immersion. Moreover, by the definition of $\mathcal{M}_R(P, J)$, $\tilde{u}$ is also a proper map, so $\tilde{u}|_{[R_1,\infty)\times S^1}$ must be an embedding.

Now assume that $k_N > 1$. We note that the e_i and r_i appearing in the formula (3) for U each satisfy $e_i(t + \frac{1}{k_N}) = e(t)$ and $r_i(s, t + \frac{1}{k_N}) = r_i(s,t)$, and hence U also satisfies $U(s, t + \frac{1}{k_N}) = U(s,t)$. This implies that there are sections $\hat{e}_i \in C^\infty(P^*_{k/k_N}\xi^{\mathcal{H}})$ and maps $\hat{r}_i : [k_N R, \infty) \times S^1 \to P^*_{k/k_N}\xi^{\mathcal{H}}$ satisfying $e_i(t) = \hat{e}_i(k_N t)$ and $r_i(s,t) = \hat{r}_i(k_N s, k_N t)$. Moreover, the $\hat{e}_i$ are easily seen to be eigenvectors of $\mathbf{A}_{P_{k/k_N}, J}$ with eigenvalues λ_i/k_N. Defining $V : [k_N R, \infty) \times S^1 \to P^*_{k/k_N}\xi^{\mathcal{H}}$ by

$$V(s,t) = \sum_{i=1}^{N} e^{(\lambda_i/k_N)s}(\hat{e}_i(t) + \hat{r}_i(s,t)) \tag{4}$$

we have that $U(s,t) = V(p_{k_N}(s,t))$ where $p_{k_N} : \mathbb{R} \times S^1 \to \mathbb{R} \times S^1$ is the covering map $(s,t) \mapsto (k_N s, k_N t)$.

Now, apply the Riemann mapping theorem to find a biholomorphic map $\rho : \psi([R,\infty) \times S^1) \to [R', \infty) \times S^1$ for some R' determined by requiring ρ to be asymptotic to the identity. Defining $\tilde{u}_1 = \tilde{u} \circ \rho^{-1} \in \mathcal{M}_{R'}(P_k, J)$ and $\psi_1 = \rho \circ \psi$ we have that

$$\tilde{u}_1(\psi_1(s,t)) = \tilde{u}(\psi(s,t)) = (Ts, \exp_{P_k(t)} V(k_N s, k_N t))$$

with $\psi_1 : [R, \infty) \times S^1 \to [R', \infty) \times S^1$ a diffeomorphism asymptotic to the identity. Letting $\phi_{k_N} : \mathbb{R} \times S^1 \to \mathbb{R} \times S^1$ be the generator $(s,t) \mapsto (s, t + \frac{1}{k_N})$ of the deck transformation group of the cover p_{k_N}, the map

$$\hat{\phi}_{k_N} = \psi_1 \circ \phi_{k_N} \circ \psi_1^{-1} : [R', \infty) \times S^1 \to [R', \infty) \times S^1$$

generates a free $\mathbb{Z}_{k_N}$-action on $[R', \infty) \times S^1$ satisfying

$$\tilde{u}_1 \circ \hat{\phi}_{k_N} = \tilde{u}_1$$

which furthermore implies that $\hat{\phi}_{k_N}$ is holomorphic. Using further that $\hat{\phi}_{k_N}$ is asymptotic to ϕ_{k_N} we must have that $\hat{\phi}_{k_N} = \phi_{k_N}$ since ϕ_{k_N} is the only biholomorphic map on $[R', \infty) \times S^1$ with this property. This in turn implies that the diffeomorphism ψ_1 commutes with ϕ_{k_N}.

Since $\frac{1}{k} * \tilde{u}_1 = \tilde{u}_1 \circ \phi_{k_N} = \tilde{u}_1 \circ \hat{\phi}_{k_N} = \tilde{u}_1$, there is a $\tilde{v} \in \mathcal{M}_{k_N R'}(P_{k/k_N}, J)$ satisfying $\tilde{u}_1(s,t) = \tilde{v}(p_{k_N}(s,t))$. Moreover, since ψ_1 commutes with the generator ϕ_{k_N} of the deck transformation group of p_{k_N}, there is an embedding $\hat{\psi} : [k_N R, \infty) \times S^1 \to [k_N R', \infty) \times S^1$ satisfying $p_{k_N}(\psi_1(s,t)) = \hat{\psi}(p_{k_N}(s,t))$. We thus find that

$$\tilde{v}(\hat{\psi}(s,t)) = (T/k_N s, \exp_{P_{k/k_N}(t)} V(s,t)).$$

Comparing the formulas (4) and (3) for V and U it is clear that $\gcd(\hat{e}_1, \ldots, \hat{e}_N) = \frac{1}{k_N} \gcd(e_1, \ldots, e_N) = 1$. We can therefore apply the argument of the first paragraph to $\tilde{v}$ and V to show that there is an R'' so that $\tilde{v}|_{[R'', \infty) \times S^1}$ is an embedding. Choosing R_1 large enough so that $(p_{k_N} \circ \rho)([R_1, \infty) \times S^1) \subset [R'', \infty) \times S^1$ we have that

$$\tilde{u}|_{[R_1, \infty) \times S^1} = \tilde{v} \circ (p_{k_N} \circ \rho)|_{[R_1, \infty) \times S^1}$$

with $\tilde{v}$ an embedding on $(p_{k_N} \circ \rho)([R_1, \infty) \times S^1)$, and $p_{k_N} \circ \rho$ asymptotic to p_{k_N} as required. □

Corollaries 2.5 and 2.6 can be combined with results of McDuff [13] or Micallef and White [14] to prove generalizations for punctured pseudoholomorphic curves of well-known results about closed pseudoholomorphic curves. Namely, two connected punctured pseudoholomorphic curves which are asymptotically cylindrical near the punctures either have identical image or intersect in at most a finite set. Also, a punctured curve which is asymptotically cylindrical near the punctures either factors through a branched cover or has a finite number of double points. (See [17] for more precise statements.)

As mentioned in the introduction, Theorems 2.2, 2.3, and 2.4 are most useful in dimension 4 where intersection theory related algebraic invariants can be computed in terms of the windings of the eigenvectors appearing in the asymptotic formulas from these theorems. These results and related topics are pursued further in [17] and the forthcoming paper [16]. In addition, Theorems 2.2 and 2.3 imply that the index inequality of Hutchings from [9] can be shown to hold for pseudoholomorphic curves in any 3-manifold equipped with a framed stable Hamiltonian structure, without needing any additional restrictions on the behavior of the almost complex structure near the periodic orbits. This index inequality is important for the definition of Hutchings' variant of symplectic field theory known as embedded contact homology [10, 11].

2.5. **Outline of Subsequent Sections.** The remainder of the paper is dedicated to the proofs of Theorems 2.2, 2.3, and 2.4. Theorems 2.3 and 2.4 are consequences of Theorem 2.2, and are proven in sections 3 and 4, respectively, assuming Theorem 2.2. Section 5 is then dedicated to the proof of Theorem 2.2. In section 5.1, we introduce some notation and outline the proof of Theorem 2.2, dividing the argument into three propositions. First, in Proposition 5.1, we show that given two half-cylinders asymptotic to a common orbit, one of the cylinders can be described as a section h of the bundle $\xi^{\mathcal{H}}$ along the other. Then, in Proposition 5.2, we use the fact that h satisfies a Cauchy-Riemann equation of a certain form to represent it by a formula involving an eigenvalue and eigenvector of the asymptotic operator.

Finally, in Proposition 5.3, we complete the proof of Theorem 2.2 by relating the formula for h to a formula for the difference of asymptotic representatives of each cylinder. The proofs of these three propositions are given in sections 5.2, 5.3, and 5.4. Finally, in the appendix, we prove a general result about a certain class of Cauchy-Riemann equations on half-cylinders which is needed in the proof of Proposition 5.2 in section 5.3.

3. Proof of Theorem 2.3

In this section we will prove Theorem 2.3 assuming Theorem 2.2. Let $P : S^1 \to M$ be a k-covered T-periodic orbit, and let $\tilde{u} \in \mathcal{M}(P, J)$. Let $U : [R, \infty) \times S^1 \to P^*\xi^{\mathcal{H}}$ be an asymptotic representative of $\tilde{u}$, that is

$$\tilde{u}(\psi(s,t)) = (Ts, \exp_{P(t)} U(s,t))$$

for some proper embedding $\psi : [R, \infty) \times S^1 \to \mathbb{R} \times S^1$ converging asymptotically to the identity. According to a special case of Theorem 2.2, we can write

$$U(s,t) = e^{\lambda_1 s} \left(e_1(t) + \rho_1(s,t)\right)$$

where λ_1 is a negative eigenvalue of $\mathbf{A}_{P,J}$, e_1 is an eigenvector with eigenvalue λ_1, and $\rho_1(s,t)$ converges exponentially to zero, as do all derivatives $\nabla_s^l \nabla_t^m \rho_1$.

If the function ρ_1 in this formula for U satisfies $\rho_1(s, t + \frac{1}{\operatorname{cov}(e_1)}) = \rho_1(s,t)$ then there is nothing more to prove. If not, we can average the functions $(s,t) \mapsto \rho_1(s, t + \frac{j}{\operatorname{cov}(e_1)})$ over $j \in \mathbb{Z}_{\operatorname{cov}(e_1)}$ to obtain a function r_1 which does satisfy $r_1(s, t + \frac{1}{\operatorname{cov}(e_1)}) = r_1(s,t)$. We will see that Theorem 2.2 allows us to write $\rho_1(s,t) - r_1(s,t) = e^{\lambda_2 s}(e_2(t) + \rho_2(s,t))$ with $\lambda_2 < \lambda_1$, $e_2 \in \ker(\mathbf{A}_{P,J} - \lambda_2) \setminus \{0\}$ satisfying $\gcd(\operatorname{cov}(e_1), \operatorname{cov}(e_2)) < \operatorname{cov}(e_1)$, and ρ_2 converging exponentially to zero. We can therefore write

$$U(s,t) = e^{\lambda_1 s} \left(e_1(t) + r_1(s,t)\right) + e^{\lambda_2 s} \left(e_2(t) + \rho_2(s,t)\right)$$

with every term satisfying the conclusions of the theorem except possibly ρ_2. Now, if ρ_2 satisfies $\rho_2(s, t + \frac{1}{k_2}) = \rho_2(s,t)$ with $k_2 = \gcd(\operatorname{cov}(e_1), \operatorname{cov}(e_2))$ there is nothing more to prove. If not, we carry out the same argument with ρ_2 to write

$$U(s,t) = \sum_{i=1}^{2} e^{\lambda_i s}(e_i(t) + r_i(s,t)) + e^{\lambda_3 s}(e_3(t) + \rho_3(s,t))$$

where every term in the sum satisfies the conclusions of the theorem except possible ρ_3, which we know decays exponentially with all derivatives. We continue this process until we have a formula for U that satisfies the conclusion of the theorem.

The details of the main step of this argument are made precise in the following lemma. The theorem is then an easy consequence of an iterative argument which applies the lemma at each step. It is clear that this argument must terminate after a finite number of steps (i.e. applications of the lemma) since each application of the lemma leads to a new term, $k_j = \gcd(\operatorname{cov}(e_1), \dots, \operatorname{cov}(e_j))$, in a strictly decreasing sequence of positive integers.

Lemma 3.1. *Let U be the map defined above, and assume that U can be written*

$$U(s,t) = \sum_{i=1}^{j-1} e^{\lambda_i s}(e_i(t) + r_i(s,t)) + e^{\lambda_j s}(e_j(t) + \rho(s,t))$$

where:

- *The* λ_i *are all negative eigenvalues of* $\mathbf{A}_{P,J}$ *satisfying* $\lambda_{i+1} < \lambda_i$.
- *The* e_i *are eigenvalues of* $\mathbf{A}_{P,J}$ *with eigenvalue* λ_i.
- *The positive integers* k_i *defined by* $k_1 = \operatorname{cov}(e_1)$ *and* $k_i = \gcd(k_{i-1}, \operatorname{cov}(e_i))$ *satisfy* $k_{i+1} < k_i$.
- *The maps* r_i *satisfy* $r_i(s, t + \frac{1}{k_i}) = r_i(s,t)$ *for all* $(s,t) \in [R, \infty) \times S^1$. *Moreover for all* $(l, m) \in \mathbb{N}^2$ *the quantities* $|\nabla_s^l \nabla_t^m r_i|$ *converge exponentially to zero.*
- *The map* ρ *converges to zero exponentially fast as do all of its derivatives* $\nabla_s^l \nabla_t^m \rho$.

Then the map ρ *either satisfies* $\rho(s, t + \frac{1}{k_j}) = \rho(s,t)$ *or there exists a map* $r_j : [R, \infty) \times S^1 \to P^*\xi^{\mathcal{H}}$ *satisfying* $r_j(s, t + \frac{1}{k_j}) = r_j(s,t)$, *an eigenvalue* λ_{j+1} *of* $\mathbf{A}_{P,J}$ *satisfying* $\lambda_{j+1} < \lambda_j$, *and an* $e_{j+1} \in \ker(\mathbf{A}_{P,J} - \lambda_{j+1}) \setminus \{0\}$ *with* $k_{j+1} := \gcd(k_j, \operatorname{cov}(e_{j+1})) < k_j$ *so that*

$$U(s,t) = \sum_{i=1}^{j} e^{\lambda_i s}(e_i(t) + r_i(s,t)) + e^{\lambda_{j+1} s}(e_{j+1}(t) + \rho_1(s,t))$$

where ρ_1 *is a function for which the maps* $\nabla_s^l \nabla_t^m \rho_1$ *converge exponentially to zero for all* $(l,m) \in \mathbb{N}^2$.

Proof. We first introduce some notation. Recall that since $k = \operatorname{cov}(P)$, there is a $\mathbb{Z}_k$ action on the space of sections $C^\infty(P^*\xi^{\mathcal{H}})$ generated by $\frac{1}{k} \in S^1 \approx \mathbb{R}/\mathbb{Z}$ and that this action fixes the eigenspaces of $\mathbf{A}_{P,J}$. In the remainder of this section we will write $(j *_k f)(t) = f(t + \frac{j}{k})$ for $j \in \mathbb{Z}_k$ to denote this action. Similarly, for any integer m which divides k, there is a $\mathbb{Z}_m$ action on $C^\infty(P^*\xi^{\mathcal{H}})$ generated by $(1 *_m f)(t) = f(t + \frac{1}{m})$. For any such m we define the map $A_m : C^\infty(P^*\xi^{\mathcal{H}}) \to C^\infty(P^*\xi^{\mathcal{H}})$ by averaging over the $\mathbb{Z}_m$ orbit of a section, i.e. we define

$$(A_m f)(t) = \frac{1}{m} \sum_{i=0}^{m-1} (i *_m f)(t).$$

We note that A_m can be thought of as the projection onto the space of sections that are fixed by the $\mathbb{Z}_m$ action; that is, any section in the image of A_m is fixed by the $\mathbb{Z}_m$ action, and A_m acts as the identity on the space of sections that are fixed by the $\mathbb{Z}_m$ action. We will make the obvious extensions of the $\mathbb{Z}_m$ action and the map A_m to the subset of the space $C^\infty([R,\infty) \times S^1, P^*\xi^{\mathcal{H}})$ defined by

$$\{u \in C^\infty([R,\infty) \times S^1, P^*\xi^{\mathcal{H}}) \,|\, u(s, \cdot) \in C^\infty(P^*\xi^{\mathcal{H}}) \text{ for all } s \in [R, \infty)\}.$$

Proceeding with the proof, we assume that ρ is not fixed by the $\mathbb{Z}_{k_j}$ action. We define $r_j := A_{k_j}\rho$ so that

$$U(s,t) = \sum_{i=1}^{j} e^{\lambda_i s}(e_i(t) + r_i(s,t)) + e^{\lambda_j s}(\rho(s,t) - r_j(s,t))$$

with r_j satisfying the required properties. Note that since all terms in the original expression for U are fixed by the $\mathbb{Z}_{k_j}$ action except for ρ, we can rewrite this as

$$U(s,t) = (A_{k_j}U)(s,t) + e^{\lambda_j s}(\rho(s,t) - r_j(s,t)).$$

Rearranging this equation gives

$$\begin{aligned}
e^{\lambda_j s}(\rho(s,t) - r_j(s,t)) &= U(s,t) - (A_{k_j}U)(s,t) \\
&= U(s,t) - \frac{1}{k_j}\sum_{i=0}^{k_j-1}(i *_{k_j} U)(s,t) \\
&= \frac{1}{k_j}\sum_{i=1}^{k_j-1}\sum_{\ell=0}^{i-1}\left(\ell *_{k_j}\left[U - 1 *_{k_j} U\right]\right)(s,t) \qquad (5)\\
&= \sum_{i=0}^{k_j-2}\frac{k_j-(i+1)}{k_j}\left(i *_{k_j}\left[U - 1 *_{k_j} U\right]\right)(s,t).
\end{aligned}$$

By Theorem 2.2 we have that $U(s,t) - 1 *_{k_j} U(s,t) = e^{\lambda_{j+1}s}(f(t) + r(s,t))$ where λ_{j+1} is a negative eigenvalue of $\mathbf{A}_{P,J}$, $f \neq 0$ is an eigenvector with eigenvalue λ_{j+1}, and $r(s,t)$ converges exponentially to zero. Using this with (5) we can write

$$e^{\lambda_j s}(\rho(s,t) - r_j(s,t)) = e^{\lambda_{j+1}s}(e_{j+1}(t) + \rho_1(s,t)) \qquad (6)$$

where

$$e_{j+1} = \sum_{i=0}^{k_j-2}\frac{k_j-(i+1)}{k_j} i *_{k_j} f$$

and

$$\rho_1 = \sum_{i=0}^{k_j-2}\frac{k_j-(i+1)}{k_j} i *_{k_j} r.$$

We therefore can write

$$U(s,t) = \sum_{i=1}^{j} e^{\lambda_i s}(e_i(t) + r_i(s,t)) + e^{\lambda_{j+1}s}(e_{j+1}(t) + \rho_1(s,t))$$

with r_j satisfying the required properties and with $e_{j+1} \in \ker(\mathbf{A}_{P,J} - \lambda_{j+1})$. Moreover, the required decay properties of ρ_1 follow easily from the definition of ρ_1 and the similar decay properties for r coming from Theorem 2.2. It remains to show that e_{j+1} is nonzero, that $\lambda_{j+1} < \lambda_j$, and that $k_{j+1} := \gcd(k_j, \operatorname{cov}(e_{j+1}))$ is strictly less than k_j.

Assuming for the moment that $e_{j+1} \neq 0$, the fact that $\lambda_{j+1} < \lambda_j$ follows easily from (6) and the exponential decay of ρ and r_j. To show both that e_{j+1} is nonzero and that $k_{j+1} < k_j$ it suffices to show that $1 *_{k_j} e_{j+1} \neq e_{j+1}$. To see that it is indeed the case that $1 *_{k_j} e_{j+1} \neq e_{j+1}$, we use the above formula for e_{j+1} to compute

$$e_{j+1} - 1 *_{k_j} e_{j+1} = f - \frac{1}{k_j}\sum_{i=0}^{k_j-1} i *_{k_j} f = f - A_{k_j} f.$$

Therefore $e_{j+1} = 1 *_{k_j} e_{j+1}$ is equivalent to $f = A_{k_j} f$ which in turn is equivalent to $f = 1 *_{k_j} f$. However, assuming $f = 1 *_{k_j} f$ leads to

$$\begin{aligned} 0 = U(s,t) - U(s,t) &= \sum_{i=0}^{k_j-1} \left(i *_{k_j} \left(U - 1 *_{k_j} U\right)\right)(s,t) \\ &= \sum_{i=0}^{k_j-1} e^{\lambda_{j+1}s} \left((i *_{k_j} f)(t) + (i *_{k_j} r)(s,t)\right) \\ &= e^{\lambda_{j+1}s} \left(k_j f(t) + \sum_{i=0}^{k_j-1} (i *_{k_j} r)(s,t) \right) \end{aligned}$$

or equivalently

$$f(t) = -\frac{1}{k_j} \sum_{i=0}^{k_j-1} (i *_{k_j} r)(s,t).$$

This is a contradiction since $|f(t)|$ is bounded away from zero while the right hand side decays exponentially in s. We have therefore shown that e_{j+1} is nonzero, that $\lambda_{j+1} < \lambda_j$, and that $k_{j+1} < k_j$. This completes the proof of the lemma. □

4. Proof of Theorem 2.4

In this section we show how Theorem 2.4 follows from Theorem 2.2. As mentioned before the statement of the theorem, the proof here adapts an argument due to Micallef and White from [14].

Let $P : S^1 \to M$ be a simply covered τ-periodic orbit with $\tau > 0$ and let P_k denote the map $t \mapsto P(kt)$ for any $k \in \mathbb{Z} \setminus \{0\}$. Let $\tilde{u}_i \in \mathcal{M}(P_{k_i}, J)$ be a family of asymptotically cylindrical pseudoholomorphic half-cylinders. Let $\Phi : S^1 \times \mathbb{R}^{2n} \to P^*\xi^{\mathcal{H}}$ be a trivialization of $P^*\xi^{\mathcal{H}}$, and define the inverse of a coordinate map $\phi : \mathbb{R} \times S^1 \times B_\varepsilon^{2n}(0) \to U \supset \mathbb{R} \times P(S^1)$ near $\mathbb{R} \times P(S^1)$ by

$$\phi(s,t,w) = (\tau s, \exp_{P(t)} \Phi(t) w).$$

The coordinate map $\tilde{\Phi}$ produced by the Theorem will given by the map ϕ^{-1} composed with a map of the form

$$(s,t,w) \mapsto (s,t,w - R(s,t,w)).$$

This will be a diffeomorphism provided $|R_w(s,t,w)| < 1$ for every (s,t,w) in the domain of R.

Before proceeding with the proof we make some simplifying assumptions. We first assume that the k_i are either all positive, or all negative. This entails no loss of generality since we are only interested in the behavior of the maps outside some set of the form $[-N,N] \times M$. Indeed, if we restrict the domains of the maps $\tilde{u}_i$ so that all of the maps lie entirely in either $[N,\infty) \times M$ or $(-\infty,-N] \times M$ we can construct the required change of coordinates separately for large positive and large negative $\mathbb{R}$ values, and then interpolate. For simplicity then we will assume that all k_i are positive. The proof in the case that the k_i are all negative is nearly identical.

Assuming then that all k_i have the same sign, we can also assume that all the k_i are equal. If not, we choose some k which is a multiple of every k_i and replace the map $\tilde{u}_i$ with the map $\tilde{u}_i \circ p_{k/k_i} \in \mathcal{M}(P_k, J)$ where $p_m : \mathbb{R} \times S^1 \to \mathbb{R} \times S^1$

denotes the covering map $p(s,t) = (ms, mt)$. Suppose then that we have proven the theorem in this case, i.e. we can write

$$(\tilde{\Phi} \circ \tilde{u}_i \circ p_{k/k_i} \circ \psi_i)(s,t) = (ks, kt, U_i(s,t))$$

where $U_i(s,t) = \sum_{j=1}^N e^{\lambda_{ij}s} e_{ij}(t)$, with $\lambda_{ij} < 0$ eigenvalues of $\mathbf{A}_{P_k,J}$, and $e_{ij} \in \ker(\Phi^{-1}\mathbf{A}_{P_k,J}\Phi - \lambda_{ij})$. Letting $\phi_{k/k_i}(s,t) = (s, t + \frac{k_i}{k})$ and $\rho(s,t) := (\psi_i^{-1} \circ \phi_{k/k_i}^{-1} \circ \psi_i \circ \phi_{k/k_i})(s,t)$, we apply $\frac{k_i}{k} *$ to both sides of this equation to find

$$\begin{aligned}(ks, kt, \tfrac{k_i}{k} * U_i(s,t)) &= \tfrac{k_i}{k} * (ks, kt, U_i(s,t)) \\ &= \left(\tfrac{k_i}{k} * (\tilde{\Phi} \circ \tilde{u}_i \circ p_{k/k_i} \circ \psi_i)\right)(s,t) \\ &= \left(\tfrac{k_i}{k} * (\tilde{\Phi} \circ \tilde{u}_i \circ p_{k/k_i})\right) \circ (\phi_{k/k_i}^{-1} \circ \psi_i \circ \phi_{k/k_i})(s,t) \\ &= (\tilde{\Phi} \circ \tilde{u}_i \circ p_{k/k_i} \circ \phi_{k/k_i}^{-1} \circ \psi_i \circ \phi_{k/k_i})(s,t) \\ &= (\tilde{\Phi} \circ \tilde{u}_i \circ p_{k/k_i} \circ \psi_i) \circ \rho(s,t) \\ &= (p_k(\rho(s,t)), U_i(\rho(s,t)))\,.\end{aligned}$$

Comparing the first two components of this equation tells us that ρ is a deck transformation of the covering p_k, or more simply that $\rho(s,t) = (s, t + \frac{j}{k})$ for some $j \in \mathbb{Z}_k$. However, it is easily seen from the asymptotic behavior of ϕ_{k/k_i} and ψ_i that ρ is asymptotic to the identity, so we must have that $\rho(s,t) = (s,t)$, and hence that $\psi_i \circ \phi_{k/k_i} = \phi_{k/k_i} \circ \psi_i$. Comparing the last $2n$ components of this equation then yields $\frac{k_i}{k} * U_i = U_i$. This implies that the eigenvectors e_{ij} appearing in the formula for U_i also satisfy $\frac{k_i}{k} * e_{ij} = e_{ij}$ since the eigenspaces of $\Phi^{-1}\mathbf{A}_{P_k,J}\Phi$ are fixed by the $\mathbb{Z}_k$ action. We can therefore find maps $\hat{e}_{ij} \in \ker(\Phi^{-1}\mathbf{A}_{P_{k_i},J}\Phi - \frac{k_i}{k}\lambda_{ij})$ so that

$$(\tilde{\Phi} \circ \tilde{u}_i \circ p_{k/k_i} \circ \psi_i)(s,t) = (ks, kt, \sum_{j=1}^N e^{\lambda_{ij}s} \hat{e}_{ij}(\tfrac{k}{k_i}t)).$$

Moreover the fact that ψ_i commutes with the generator ϕ_{k/k_i} of the deck transformation group of p_{k/k_i} implies that there is an embedding $\hat{\psi} : [R,\infty) \times S^1 \to \mathbb{R} \times S^1$ satisfying $p_{k/k_i} \circ \psi = \hat{\psi} \circ p_{k/k_i}$. We therefore get that

$$(\tilde{\Phi} \circ \tilde{u}_i \circ \hat{\psi}_i \circ p_{k/k_i})(s,t) = (ks, kt, \sum_{j=1}^N e^{\lambda_{ij}s} \hat{e}_{ij}(\tfrac{k}{k_i}t)).$$

which in turn implies

$$(\tilde{\Phi} \circ \tilde{u}_i \circ \hat{\psi}_i)(s,t) = (k_i s, k_i t, \sum_{j=1}^N e^{\frac{k_i}{k}\lambda_{ij}s} \hat{e}_{ij}(t)).$$

as required.

Proceeding now with the above assumptions, we have a family of maps $\tilde{u}_i \in \mathcal{M}(P_k, J)$. It follows easily from the definition of $\mathcal{M}(P_k, J)$ (or from Proposition 5.1) that there exist proper embeddings $\psi_i : [R,\infty) \times S^1 \to \mathbb{R} \times S^1$ and maps $U_i : [R,\infty) \times S^1 \to \mathbb{R}^{2n}$ so that

$$(\phi^{-1} \circ \tilde{u}_i \circ \psi_i)(s,t) = (ks, kt, U_i(s,t)) \in \mathbb{R} \times S^1 \times \mathbb{R}^{2n}.$$

We will denote the set of functions U_i arising in this way by S. We will enlarge the set S to include each $\frac{j}{k} * U_i$ for all $j \in \mathbb{Z}_k$ and all $U_i \in S$ (alternatively, we could enlarge the original collection of maps $\tilde{u}_i$ to include all $\frac{j}{k} * \tilde{u}_i$). Let μ_1 be the largest negative eigenvalue of $\mathbf{A}_{P_k,J}$, and number the remaining negative eigenvalues of $\mathbf{A}_{P_k,J}$ by the positive integers in decreasing order, that is $\mu_j < \mu_i$ for $j > i$. In what follows we will abbreviate $\Phi^{-1}\mathbf{A}_{P_k,J}\Phi$ by $\mathbf{A}$. By Theorem 2.2, for each pair of distinct functions U, $V \in S$, we can write

$$U(s,t) - V(s,t) = e^{\lambda^{U,V}s}e^{U,V}(t) + o_\infty(\lambda^{U,V}) \tag{7}$$

where $\lambda^{U,V}$ is a negative eigenvalue of $\mathbf{A}$, $e^{U,V}$ is an eigenvector of $\mathbf{A}$ with eigenvalue $\lambda^{U,V}$, and $o_\infty(\lambda)$ represents a function $f : [R,\infty) \times S^1 \to \mathbb{R}^{2n}$ satisfying

$$\left|\partial^\beta\left(e^{-\lambda s}f(s,t)\right)\right| \leq M_\beta e^{-ds}$$

for some positive constants d, M_β.

For any $i > 0$ and $U \in S$, define $S_i(U)$ to be the set

$$S_i(U) = \left\{V \in S \,\middle|\, U - V = o_\infty(\mu_i)\right\}$$

and define $S_0(U) = S$ for all $U \in S$. Define $A_i^U : [R,\infty) \times S^1 \to \mathbb{R}^{2n}$ to be the average of all $V \in S_i(U)$, that is

$$A_i^U(z) = \frac{1}{|S_i(U)|}\sum_{V \in S_i(U)} V(z)$$

where $|S_i(U)|$ is the number of elements in $S_i(U)$. Observe that there exists an $N \in \mathbb{Z}$ so that $A_i^U \equiv U$ for all $i \geq N$ and all $U \in S$. Indeed, choose N to be the unique positive integer satisfying

$$\mu_N = \min_{U,V \in S, U \neq V} \lambda^{U,V}$$

where the $\lambda^{U,V}$ are defined by (7).

We now give some important properties of the A_i^U in a series of lemmas.

Lemma 4.1. *For any $U \in S$ and any $i \geq 1$, there exists a unique (possibly zero) $e_i^U \in \ker(\mathbf{A} - \mu_i)$ so that*

$$A_i^U(s,t) - A_{i-1}^U(s,t) = e^{\mu_i s}e_i^U(t) + o_\infty(\mu_i).$$

Moreover, $\frac{1}{k} * e_i^U = e_i^{\frac{1}{k}*U}$.

Proof. An easy computation using the definition of A_i^U leads to

$$\begin{aligned}
A_i^U - A_{i-1}^U &= \left[A_i^U - U\right] - \left[A_{i-1}^U - U\right] \\
&= \frac{-1}{|S_{i-1}(U)|}\sum_{V \in S_{i-1}(U)\setminus S_i(U)}[V - U] + \frac{|S_{i-1}(U)| - |S_i(U)|}{|S_{i-1}(U)|}\left[A_i^U - U\right] \\
&= \frac{1}{|S_{i-1}(U)|}\sum_{V \in S_{i-1}(U)\setminus S_i(U)}[U - V] + o_\infty(\mu_i).
\end{aligned}$$

By Theorem 2.2 and the definition of $S_i(U)$, we can write

$$U(s,t) - V(s,t) = e^{\mu_i s}e^{U,V}(t) + o_\infty(\mu_i)$$

for each $V \in S_{i-1}(U) \setminus S_i(U)$, where $e^{U,V} \in \ker(\mathbf{A} - \mu_i)$. With e_i^U defined by

$$e_i^U(t) = \frac{1}{|S_{i-1}(U)|} \sum_{V \in S_{i-1}(U) \setminus S_i(U)} e^{U,V}(t)$$

the first part of the result easily follows.

The fact that $\frac{1}{k} * e_i^U = e_i^{\frac{1}{k}*U}$ is a consequence of the fact that $\frac{1}{k} * A_i^U = A_i^{\frac{1}{k}*U}$, which in turn follows from the fact that $V \in S_i(U)$ if and only if $\frac{1}{k} * V \in S_i(\frac{1}{k} * U)$. □

Lemma 4.2. *Let U, $V \in S$, and assume that $\lambda^{U,V}$ (defined by (7)) satisfies $\lambda^{U,V} = \mu_j$, i.e.*

$$U(s,t) - V(s,t) = e^{\mu_j s} e^{U,V}(t) + o_\infty(\mu_j) \tag{8}$$

for some $j \in \mathbb{Z}^+$ and some nonzero $e^{U,V} \in \ker(\mathbf{A} - \mu_j)$. Then $A_i^U \equiv A_i^V$ for all $i < j$, and

$$A_i^U(s,t) - A_i^V(s,t) = e^{\mu_j s} e^{U,V}(t) + o_\infty(\mu_j)$$

for all $i \geq j$.

Proof. It is immediate from (8) that $U - V = o_\infty(\mu_{j-1})$, and therefore $S_i(U) = S_i(V)$ for all $i < j$. This of course implies that $A_i^U \equiv A_i^V$ for all $i < j$ as well. For $i \geq j$ we get

$$\begin{aligned} A_i^U(z) - A_i^V(z) &= [U(z) - V(z)] + \left[A_i^U(z) - U(z)\right] - \left[A_i^V(z) - V(z)\right] \\ &= e^{\mu_j s} e^{U,V}(t) + o_\infty(\mu_j) + o_\infty(\mu_i) + o_\infty(\mu_i) \\ &= e^{\mu_j s} e^{U,V}(t) + o_\infty(\mu_j) \end{aligned}$$

where we've used that $f = o_\infty(\mu_i) \Rightarrow f = o_\infty(\mu_j)$ since $\mu_i \leq \mu_j$ for $i \geq j$. □

Lemma 4.3. *There exists an $\varepsilon > 0$ and an $R_1 \geq R$ so that*

(1) *For all $i \geq 1$ and all $U \in S$*

$$|U(s,t) - A_i^U(s,t)| < \tfrac{1}{2}\varepsilon e^{\mu_i s}$$

for all $(s,t) \in [R_1, \infty) \times S^1$.

(2) *For $i \geq 1$ and all U, $V \in S$, either $A_i^U \equiv A_i^V$ or*

$$|A_i^U(s,t) - A_i^V(s,t)| > 2\varepsilon e^{\mu_i s}$$

for all $(s,t) \in [R_1, \infty) \times S^1$.

Proof. It is immediate from the definition of A_i^U that $U - A_i^U = o_\infty(\mu_i)$, i.e. there exist an $M_i^U > 0$ and a $d_i^U > 0$ so that

$$\left|e^{-\mu_i s}\left(U(s,t) - A_i^U(s,t)\right)\right| \leq M_i^U e^{-d_i^U s},$$

and hence

$$\left|U(s,t) - A_i^U(s,t)\right| \leq M_i^U e^{-d_i^U s} e^{\mu_i s}.$$

This shows that for every $\varepsilon > 0$, there exists an $R(i, U, \varepsilon)$ so that

$$|U(s,t) - A_i^U(s,t)| < \tfrac{1}{2}\varepsilon e^{\mu_i s}$$

for all $(s,t) \in [R(i,U,\varepsilon), \infty) \times S^1$. Moreover, since the set S_0 is assumed to be finite, and since $A_i^U \equiv U$ for all $i > N$, it follows that the choice of $R(i, U, \varepsilon)$ can be made independent of U and i.

Now we address the second claim. Recalling the definition of $e^{U,V}$ from (7), choose some ε satisfying

$$0 < \varepsilon < \frac{1}{4} \min_{U \neq V \in S_0, t \in S^1} \left| e^{U,V}(t) \right|,$$

and note that such an ε exists since the $e^{U,V}(t)$ are all nonzero for all $t \in S^1$. Assuming that $\lambda^{U,V} = \mu_j$ as in (8), it follows immediately from Lemma 4.2 that $A_i^U \equiv A_i^V$ for $i < j$ and that for $i \geq j$ there exists an $R(i, U, V)$ so that

$$\left| A_i^U(s,t) - A_i^V(s,t) \right| > 2\varepsilon e^{\mu_j s} \geq 2\varepsilon e^{\mu_i s}.$$

for all $(s,t) \in [R(i,U,V), \infty) \times S^1$. Again using that S_0 is finite and $A_i^U \equiv U$ for all $i > N$, we can choose an R_1 independent of i, U, and V so that the result holds. □

Next we define functions $R_i^U : [R_1, \infty) \times S^1 \to \mathbb{R}^{2n}$ by

$$R_i^U(s,t) = \begin{cases} A_i^U(s,t) - A_{i-1}^U(s,t) - e^{\mu_i s} e_i^U(t) & \text{for } i \geq 1 \\ A_0^U(s,t) & \text{for } i = 0 \end{cases} \tag{9}$$

and note that $R_i^U = o_\infty(\mu_i)$ for all $i \geq 1$, $U \in S$ as a result of Lemma 4.1. Moreover it follows from $\frac{1}{k} * A_i^U = A_i^{\frac{1}{k}*U}$ and $\frac{1}{k} * e_i^U = e_i^{\frac{1}{k}*U}$ that $\frac{1}{k} * R_i^U = R_i^{\frac{1}{k}*U}$. Let $\rho : \mathbb{R} \to [0,1]$ be a smooth cutoff function with $\rho(x) = 0$ for $|x| > 1$ and $\rho(x) = 1$ for $|x| < \frac{1}{2}$. Define for each $i > 0$ a function $\tilde{R}_i : [R_1, \infty) \times S^1 \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by

$$\tilde{R}_i(s,t,w) = \rho(\varepsilon^{-1} e^{-\mu_i s} |w - A_i^U(s,t)|) R_i^U(s,t)$$

if there is a $U \in S$ with $|w - A_i^U(s,t)| < \varepsilon e^{\mu_i s}$, and define

$$\tilde{R}_i(s,t,w) = 0$$

otherwise.

Lemma 4.4. *The maps $\tilde{R}_i : [R_1, \infty) \times S^1 \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ are well defined for all i. Moreover these maps are all invariant under the deck transformations of the k-fold cover $\pi_k : [R_1, \infty) \times S^1 \times \mathbb{R}^{2n} \to [kR_1, \infty) \times S^1 \times \mathbb{R}^{2n}$ defined by $\pi_k(s,t,w) = (ks, kt, w)$, and therefore there exist maps $R_i : [kR_1, \infty) \times S^1 \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ satisfying*

$$R_i(ks, kt, w) = \tilde{R}_i(s,t,w).$$

Proof. We first prove that each $\tilde{R}_i$ is well defined. Assume that there exist maps $U, V \in S_0$ and a point $(s,t,w) \in [R_1, \infty) \times S^1 \times \mathbb{R}^{2n}$ for which A_i^U and A_i^V satisfy $|w - A_i^U(s,t)| < \varepsilon e^{\mu_i s}$ and $|w - A_i^V(s,t)| < \varepsilon e^{\mu_i s}$. Then A_i^U and A_i^V satisfy

$$\left| A_i^U(s,t) - A_i^V(s,t) \right| \leq 2\varepsilon e^{\mu_i s}$$

which by Lemma 4.3 can only be true if $A_i^U \equiv A_i^V$. This in turn implies that $A_{i-1}^U \equiv A_{i-1}^V$ and hence that $R_i^U \equiv R_i^V$. Therefore the maps $\tilde{R}_i$ are well defined since they are independent of any choices made in the definition.

Next, we note that since the group of deck transformations of the cover π_k is generated by the map $(s,t,w) \mapsto (s, t + \frac{1}{k}, w)$, it suffices to show that $\tilde{R}_i(s,t,w) = \tilde{R}_i(s, t + \frac{1}{k}, w)$. Given a point $(s,t,w) \in [R, \infty) \times S^1 \times \mathbb{R}^{2n}$, assume that there is a $U \in S$ so that $\left| w - A_i^U(s, t + \frac{1}{k}) \right| < \varepsilon e^{\mu_i s}$. Then it follows from $A_i^{\frac{1}{k}*U}(s,t) =$

$\frac{1}{k} * A_i^U(s,t) = A_i^U(s, t+\frac{1}{k})$ that $\left|w - A_i^{\frac{1}{k}*U}(s,t)\right| < \varepsilon e^{\mu_i s}$. We therefore find that

$$\begin{aligned}\tilde{R}_i(s, t+\tfrac{1}{k}, w) &= \rho(\varepsilon^{-1}e^{-\mu_i s}|w - A_i^U(s,t+\tfrac{1}{k})|)R_i^U(s,t+\tfrac{1}{k}) \\ &= \rho(\varepsilon^{-1}e^{-\mu_i s}|w - \tfrac{1}{k} * A_i^U(s,t)|)\tfrac{1}{k} * R_i^U(s,t) \\ &= \rho(\varepsilon^{-1}e^{-\mu_i s}|w - A_i^{\frac{1}{k}*U}(s,t)|)R_i^{\frac{1}{k}*U}(s,t) \\ &= \tilde{R}_i(s,t,w).\end{aligned}$$

Now, assuming there is no $U \in S$ with $\left|w - A_i^U(s,t+\frac{1}{k})\right| < \varepsilon e^{\mu_i s}$, we can argue using $\frac{1}{k} * A_i^U = A_i^{\frac{1}{k}*U}$ that there is no $U \in S$ with $\left|w - A_i^U(s,t)\right| < \varepsilon e^{\mu_i s}$. Therefore, we have in this case that

$$\tilde{R}_i(s,t+\tfrac{1}{k},w) = \tilde{R}_i(s,t,w) = 0.$$

Hence, $\tilde{R}_i(s,t+\frac{1}{k},w) = \tilde{R}_i(s,t,w)$ for all $(s,t,w) \in [R_1,\infty) \times S^1 \times \mathbb{R}^{2n}$ as claimed. □

We now complete the proof of Theorem 2.4.

Proof of Theorem 2.4. Define a function $R : [kR_1, \infty) \times S^1 \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by

$$R(s,t,w) = \sum_{i=0}^{N} R_i(s,t,w)$$

where the R_i are the functions defined in Lemma 4.4. Recalling from Lemma 4.3 that $|U(s,t) - A_i^U(s,t)| < \frac{1}{2}\varepsilon e^{\mu_i s}$ for all $(s,t) \in [R_1,\infty) \times S^1$, it follows from the definition of $\tilde{R}_i$ that

$$\tilde{R}_i(s,t,U(s,t)) = R_i^U(s,t).$$

We therefore find that

$$\begin{aligned}U(s,t) - R(ks,kt,U(s,t)) &= U(s,t) - \sum_{i=0}^{N} R_i(ks,kt,U(s,t)) \\ &= U(s,t) - \sum_{i=0}^{N} \tilde{R}_i(s,t,U(s,t)) \\ &= U(s,t) - \sum_{i=0}^{N} R_i^U(s,t)\end{aligned}$$

for all $(s,t) \in [R_1,\infty) \times S^1$. Using this with the definition (9) of R_i^U and the fact that $A_N^U = U$, we find that

$$\begin{aligned}U(s,t) - R(ks,kt,U(s,t)) &= U(s,t) - \sum_{i=0}^{N} R_i^U(s,t) \\ &= A_N^U(s,t) - A_0^U(s,t) \\ &\qquad - \sum_{i=1}^{N} A_i^U(s,t) - A_{i-1}^U(s,t) - e^{\mu_i s}e_i^U(t) \\ &= \sum_{i=1}^{N} e^{\mu_i s}e_i^U(t).\end{aligned}$$

Letting $\tilde{\Phi} : \phi^{-1}(\mathbb{R} \times S^1 \times B^{2n}_{\varepsilon}(0)) \to \mathbb{R} \times S^1 \times \mathbb{R}^{2n}$ be defined by composing ϕ^{-1} with the map $(s,t,w) \mapsto (s,t,w - R(s,t,w))$, we find that

$$\tilde{\Phi} \circ \tilde{u}_j \circ \psi_j(s,t) = \big(ks, kt, U_j(s,t) - R(ks, kt, U_j(s,t))\big) = (ks, kt, \sum_{i=1}^{N} e^{\mu_i s} e_i^{U_j}(t)).$$

Thus $\tilde{\Phi} \circ \tilde{u}_j \circ \psi_j$ has the required form for all j

It remains to show that $\tilde{\Phi}$ can be chosen so that $\tilde{\Phi}(\tau s, P(t)) = (s,t,0) \in \mathbb{R} \times S^1 \times \mathbb{R}^{2n}$, and that $\tilde{\Phi}$ is an embedding for large values of the first $\mathbb{R}$ coordinate. To see that the first claim is true, we include the pseudoholomorphic map $\tilde{P}(s,t) = (\tau s, P(t))$ in the original collection. It is clear that $\phi^{-1} \circ \tilde{P}(s,t) = (s,t,0)$, so the above results show that $-R(ks,kt,0) = \sum_{j=1}^{N} e^{\mu_j s} e_j(t)$ for some appropriate $e_j \in \ker(\mathbf{A} - \mu_j)$. Thus redefining R to be the map $(s,t,w) \mapsto R(s,t,w) - R(s,t,0)$, we will have that $\tilde{\Phi} \circ \tilde{P}(s,t) = (s,t,0)$ as required, while each of the $\tilde{\Phi} \circ \tilde{u}_j \circ \psi_j$ still have the required form. Finally to see that $\tilde{\Phi}$ is an embedding, we note that the map R is easily seen to satisfy decay estimates of the form

$$\left|\partial^{\beta} R(s,t,w)\right| \leq M_{\beta} e^{(\mu_1/k)s}$$

by construction. In particular, we will have that $|\partial_w R(s,t,w)| < 1$ for all sufficiently large s. Therefore, by choosing R_1 larger if necessary, it is clear that $\tilde{\Phi}$ will be an embedding on the set $\phi([R_1, \infty) \times S^1 \times B^{2n}_{\varepsilon}(0))$. □

5. Proof of Theorem 2.2

5.1. **Overview of the Proof.** In this section, we outline the proof of Theorem 2.2 and fix some notation. In everything that follows we will identify S^1 with $\mathbb{R}/\mathbb{Z}$ unless otherwise stated. On $\mathbb{R}^{2n} = \{(x_1, \ldots, x_n, y_1, \ldots, y_n)\}$, we will denote by J_0 the standard complex structure defined by $J_0 \frac{\partial}{\partial x_j} = \frac{\partial}{\partial y_j}$ and we let the orientation of $\mathbb{R}^{2n}$ be determined by the volume form ω_0^n, where $\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$.

Through this section, we will frequently be dealing with maps of the form $f : \mathbb{R}^{m_1} \times \mathbb{T}^{n_1} \to \mathbb{R}^{m_2} \times \mathbb{T}^{n_1}$ where

$$\mathbb{T}^n = \underbrace{S^1 \times \cdots \times S^1}_{n \text{ times}}$$

represents the n-torus (in what follows we will always have n_1, $n_2 \in \{0,1\}$). It will often be convenient to work with a lifted map between the universal covers $\tilde{f} : \mathbb{R}^{m_1} \times \mathbb{R}^{n_1} \to \mathbb{R}^{m_2} \times \mathbb{R}^{n_1}$ in order to take advantage of the linear structure there. We will generally do this without saying so explicitly, and without making any notational distinction between the map and the chosen lift. It should be clear from context when the constructions we make require us to work with the lifted maps, although there will be very little harm in the reader assuming that we are always working with lifted maps.

Let $(M, \mathcal{H})$ be a $2n+1$ dimensional manifold equipped with a framed stable hamiltonian structure $\mathcal{H} = (\lambda, \omega)$. Let $J \in \mathcal{J}(M, \mathcal{H})$ be a compatible complex structure on $\xi^{\mathcal{H}}$, and as before extend this to an $\mathbb{R}$-invariant almost complex structure $\tilde{J}$ on $\mathbb{R} \times M$. We define a metric on M by

$$g(h,k) = \lambda(h)\lambda(k) + \omega(\pi_{\xi^{\mathcal{H}}} h, J \pi_{\xi^{\mathcal{H}}} k)$$

where $\pi_{\xi^{\mathcal{H}}} : TM = \mathbb{R}X_{\mathcal{H}} \oplus \xi^{\mathcal{H}} \to \xi^{\mathcal{H}}$ is the projection onto $\xi^{\mathcal{H}}$ along $X_{\mathcal{H}}$, and we denote the exponential map of g by exp.

Let $P \in \mathcal{P}_T(M, \mathcal{H})$ be a T-periodic orbit of the vector field $X_{\mathcal{H}}$ with covering number $\operatorname{cov}(P) = k$. Let $\Phi : P(S^1) \times \mathbb{R}^{2n} \to \xi^{\mathcal{H}}|_{P(S^1)}$ be a unitary trivialization of the hyperplane distribution $\xi^{\mathcal{H}}$ over $P(S^1)$, i.e. Φ satisfies $\Phi \circ J_0 = J \circ \Phi$ and $\langle h, k\rangle = \omega(\Phi h, J\Phi k)$, where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^{2n}$. Define a map $\phi : S^1 \times B^{2n}_{\varepsilon_0}(0) \subset S^1 \times \mathbb{R}^{2n} \to M$ by

$$\phi(t, w) = \exp_{P(t)} \Phi(P(t))w,$$

and note that if ε_0 is sufficiently small, ϕ is a k-fold covering map of some neighborhood U of $P(S^1)$. Extend Φ to a unitary trivialization (still denoted Φ) of $\xi^{\mathcal{H}}|_U$.

Pulling back the trivialization Φ to a unitary trivialization $\tilde{\Phi}$ of $(\phi^*\xi^{\mathcal{H}}, \phi^*J, \phi^*\omega)$, we note that $\tilde{\Phi}$ will be given by an $(2n+1) \times 2n$ matrix-valued function on $S^1 \times B^{2n}_{\varepsilon_0}(0)$ and that

$$\tilde{\Phi}(t, 0) = \begin{bmatrix} 0^{1\times 2n} \\ I^{2n\times 2n} \end{bmatrix}.$$

where the coordinates on $S^1 \times \mathbb{R}^{2n}$ are ordered by $(t, x_1, \ldots, x_n, y_1, \ldots, y_n)$.

Let $\tilde{u} = (a, u)$, $\tilde{v} = (b, v) \in \mathcal{M}(P, J)$. Using that the loops $u(s, \cdot)$ and $v(s, \cdot)$ converge to P as $s \to \infty$, we can find for some sufficiently large R_0 lifts u_ϕ, v_ϕ : $[R_0, \infty) \times S^1 \to S^1 \times B^{2n}_{\varepsilon_0}(0)$ of $u|_{[R_0,\infty)\times S^1}$ and $v|_{[R_0,\infty)\times S^1}$ respectively, chosen so that the loops $t \mapsto u_\phi(s, t)$ and $t \mapsto v_\phi(s, t)$ are asymptotic to the map $t \mapsto (t, 0) \in S^1 \times B^{2n}_{\varepsilon_0}(0)$. Define maps $\tilde{u}_\phi$, $\tilde{v}_\phi : [R_0, \infty) \times S^1 \to \mathbb{R} \times S^1 \times B^{2n}_{\varepsilon_0}(0)$ by $\tilde{u}_\phi = (\frac{1}{T}a, u_\phi)$ and $\tilde{v}_\phi = (\frac{1}{T}b, v_\phi)$ so that $\tilde{u}_\phi$ and $\tilde{v}_\phi$ are lifts of $\tilde{u}|_{[R_0,\infty)\times S^1}$ and $\tilde{v}|_{[R_0,\infty)\times S^1}$ relative to the covering map $\tilde{\phi} : \mathbb{R} \times S^1 \times B^{2n}_{\varepsilon_0}(0) \to \mathbb{R} \times U$ defined by $\tilde{\phi}(a, m) = (Ta, \phi(m))$. It follows from the definition of $\mathcal{M}(P, J)$ that the lifted maps $\tilde{u}_\phi$ and $\tilde{v}_\phi$ satisfy

$$\begin{aligned} \left|\partial^\beta \left(\tilde{u}_\phi(s, t) - (s, t, 0)\right)\right| &\leq M_\beta e^{-ds} \\ \left|\partial^\beta \left(\tilde{v}_\phi(s, t) - (s, t, 0)\right)\right| &\leq M_\beta e^{-ds} \end{aligned} \tag{10}$$

for some appropriate constants $d > 0$, $M_\beta > 0$, and all $(s, t) \in [R_0, \infty) \times S^1$.

Define a map $E : [R_0, \infty) \times S^1 \times \mathbb{R}^{2n} \to \mathbb{R} \times S^1 \times \mathbb{R}^{2n}$ by

$$E(s, t, w) = \tilde{u}_\phi(s, t) + (0, \tilde{\Phi}(u_\phi(s, t))w) = (\tfrac{1}{T}a(s, t), u_\phi(s, t) + \tilde{\Phi}(u_\phi(s, t))w)$$

and denote $E_{\varepsilon,R} = E|_{[R,\infty)\times S^1\times B^{2n}_\varepsilon(0)}$ for any $\varepsilon > 0$ and $R > R_0$. Note that by choosing ε small enough and R large enough, we can guarantee that the image of $E_{\varepsilon,R}$ lies within the domain of the covering map $\tilde{\phi}$. The first main step in the proof of Theorem 2.2 is to use the map E to represent $\tilde{v}_\phi$ as a section of $\phi^*\xi^{\mathcal{H}}$ along $\tilde{u}_\phi$. To this end, we prove the following proposition in Section 5.2.

Proposition 5.1. *There exist a real number $R' > R_0$, an embedding $\psi : [R', \infty) \times S^1 \to \mathbb{R} \times S^1$, and a map $h : [R', \infty) \times S^1 \to \mathbb{R}^{2n}$ satisfying*

$$\begin{gathered} \tilde{v}_\phi(\psi(s, t)) = E(s, t, h(s, t)) \\ \left|\partial^\beta \left(\psi(s, t) - (s, t)\right)\right| \leq M_\beta e^{-ds} \\ \left|\partial^\beta \left(h(s, t)\right)\right| \leq M_\beta e^{-ds} \end{gathered}$$

for all $(s, t) \in [R', \infty) \times S^1$ and all $\beta \in \mathbb{N}^2$, where $d > 0$ and where the $M_\beta > 0$ depend on β.

The next step in the proof is to get a precise description of the map h from this proposition. In order to do this we use the relationship between h and the pseudoholomorphic map $\tilde{v}$ to deduce that h satisfies a perturbed Cauchy-Riemann equation. Then applying a general result about Cauchy-Riemann equations on half-cylinders proved in the appendix, we obtain a precise description of the asymptotic behavior of h. This is the content of the following proposition which is proved in Section 5.3. We assume for this proposition and in the remainder of this section that h does not vanish identically.

Proposition 5.2. *The function $h : [R', \infty) \times S^1 \to \mathbb{R}^{2n}$ from Proposition 5.1 can be written*

$$h(s,t) = e^{\lambda s}\left[e(t) + r_0(s,t)\right]$$

where $e : S^1 \to \mathbb{R}^{2n}$ is an eigenvector of the operator $\Phi^{-1} \circ \mathbf{A}_{P,J} \circ \Phi$ with eigenvalue $\lambda < 0$, and where $\left|\partial^\beta r_0(s,t)\right|$ converges exponentially to zero for all $\beta \in \mathbb{N}^2$.

The final step in the proof of Theorem 2.2 is to relate the asymptotic behavior of the function h above to the asymptotic behavior of the difference of asymptotic representatives of $\tilde{u}$ and $\tilde{v}$. We first note that an easy special case of Proposition 5.1 (i.e. when $\tilde{u}(s,t) = (Ts, P(t))$) produces a map $\tilde{V} : [R, \infty) \times S^1 \to \mathbb{R}^{2n}$ and an embedding $\psi_v : [R, \infty) \times S^1 \to \mathbb{R} \times S^1$ satisfying

$$\tilde{v}_\phi(\psi_v(s,t)) = (s, t, \tilde{V}(s,t)) \tag{11}$$

and

$$\left|\partial^\beta\left(\psi_v(s,t) - (s,t)\right)\right| \leq M_\beta e^{-ds}.$$

Similarly for some perhaps still larger R we get a map $\tilde{U} : [R, \infty) \times S^1 \to \mathbb{R}^{2n}$ and an embedding $\psi_u : [R, \infty) \times S^1 \to \mathbb{R} \times S^1$ satisfying

$$\tilde{u}_\phi(\psi_u(s,t)) = (s, t, \tilde{U}(s,t)) \tag{12}$$

and

$$\left|\partial^\beta\left(\psi_u(s,t) - (s,t)\right)\right| \leq M_\beta e^{-ds}.$$

The asymptotic behavior of the difference of $\tilde{U}$ and $\tilde{V}$ is now given by the following proposition, proved in section 5.4.

Proposition 5.3. *Let $\tilde{U}$, $\tilde{V} : [R, \infty) \times S^1 \to \mathbb{R}^{2n}$ be maps satisfying* (11) *and* (12)*. Then*

$$\tilde{V}(s,t) - \tilde{U}(s,t) = e^{\lambda s}\left[e(t) + r_1(s,t)\right]$$

where $\lambda < 0$ and $e : S^1 \to \mathbb{R}^{2n}$ are the eigenvalue/eigenvector of $\Phi^{-1} \circ \mathbf{A}_{P,J} \circ \Phi$ appearing in Proposition 5.2, and where r_1 satisfies for some appropriate positive constants

$$\left|\partial^\beta r_1(s,t)\right| \leq M_\beta e^{-ds}$$

for all $\beta \in \mathbb{N}^2$ and all $(s,t) \in [R, \infty) \times S^1$.

This proposition is easily seen to complete the proof of Theorem 2.2. Indeed, applying $\tilde{\phi}$ to both sides of (11) and (12) gives

$$\tilde{v}(\psi_v(s,t)) = (Ts, \exp_{P(t)} \Phi(P(t))\tilde{V}(s,t))$$

and

$$\tilde{u}(\psi_u(s,t)) = (Ts, \exp_{P(t)} \Phi(P(t))\tilde{U}(s,t))$$

so $V(s,t) := \Phi(P(t))\tilde{V}(s,t)$ and $U(s,t) := \Phi(P(t))\tilde{U}(s,t)$ are asymptotic representatives of $\tilde{v}$ and $\tilde{u}$ respectively. Proposition 5.3 then tells us that

$$V(s,t) - U(s,t) = e^{\lambda s}\left[\Phi(P(t))e(t) + \Phi(P(t))r_1(s,t)\right]$$

where Φe is an eigenvector of $\mathbf{A}_{P,J}$ with eigenvalue $\lambda < 0$, and where $r(s,t) := \Phi(P(t))r_1(s,t)$ satisfies decay estimates of the form

$$\left|\nabla_s^i \nabla_t^j r(s,t)\right| \leq M_{ij} e^{-ds}$$

as a result of the exponential decay estimates for r_1.

5.2. **Lifting (Proof of Proposition 5.1).** Before proving Proposition 5.1 we will need some facts about the map E, and in particular we will need to understand the asymptotic behavior of E. The main result to this end is the following lemma.

Lemma 5.4. *There exists a constant $d > 0$ so that for $(s,t,w) \in [R_0,\infty) \times S^1 \times \mathbb{R}^{2n}$*

$$\left|\partial^\beta \left(E(s,t,w) - (s,t,w)\right)\right| \leq (1+|w|)M_\beta e^{-ds}$$

for all $\beta \in \mathbb{N}^{2n+2}$, where the $M_\beta > 0$ depend on $\beta \in \mathbb{N}^{2n+2}$.

Proof. Using the definitions of E and $\tilde{\Phi}$, we can write

$$\begin{aligned} E(s,t,w) - (s,t,w) &= \tilde{u}_\phi(s,t) + \tilde{\Phi}(u_\phi(s,t))w - (s,t,w) \\ &= \tilde{u}_\phi(s,t) - (s,t,0) + (0, \tilde{\Phi}(u_\phi(s,t))w) - (0,0,w) \\ &= [\tilde{u}_\phi(s,t) - (s,t,0)] + (0, \left[\tilde{\Phi}(u_\phi(s,t)) - \tilde{\Phi}(t,0)\right] w). \end{aligned}$$

The first term on the left satisfies exponential decay estimates given by (10), so it remains to show that the quantity $\left[\tilde{\Phi}(u_\phi(s,t)) - \tilde{\Phi}(t,0)\right] w$ also satisfies appropriate decay estimates. To see this observe that we can write

$$\begin{aligned} \tilde{\Phi}(u_\phi(s,t)) - \tilde{\Phi}(t,0) &= \int_0^1 \frac{d}{d\tau}\tilde{\Phi}\left((t,0) + \tau[u_\phi(s,t) - (t,0)]\right) d\tau \\ &= \int_0^1 D\tilde{\Phi}\left((t,0) + \tau[u_\phi(s,t) - (t,0)]\right) d\tau \cdot [u_\phi(s,t) - (t,0)] \\ &= D(s,t) \cdot [u_\phi(s,t) - (t,0)] \end{aligned}$$

where the function

$$D(s,t) := \int_0^1 D\tilde{\Phi}\left((t,0) + \tau[u_\phi(s,t) - (t,0)]\right) d\tau$$

has uniformly bounded derivatives of all orders. It then follows easily from the asymptotic behavior of u_ϕ that

$$\left|\partial^\beta \left(\left[\tilde{\Phi}(u_\phi(s,t)) - \tilde{\Phi}(t,0)\right] w\right)\right| \leq (1+|w|)M_\beta e^{-ds}$$

for all $(s,t,w) \in [R,\infty) \times S^1 \times \mathbb{R}^{2n}$ and for some constants $d > 0$ and $M_\beta > 0$ depending on $\beta \in \mathbb{N}^{2n+2}$. □

Corollary 5.5. *There exists an $\varepsilon > 0$ and an $R \geq R_0$ so that:*

- *The map $E_{\varepsilon,R} = E|_{[R,\infty)\times S^1\times\mathbb{R}^{2n}}$ is an embedding with uniformly bounded derivatives of all orders.*
- *The inverse map $E_{\varepsilon,R}^{-1} : E_{\varepsilon,R}([R,\infty)\times S^1 \times B_\varepsilon^{2n}(0)) \to [R,\infty)\times S^1\times B_\varepsilon^{2n}(0)$ has uniformly bounded derivatives of all orders.*

- *Assuming ε and R have been chosen so that the previous claims all hold, there exists an $R' > R$ and a $\delta > 0$ so that the tubular neighborhood*
$$\left\{p \in \mathbb{R}\times S^1 \times \mathbb{R}^{2n} \mid \operatorname{dist}\left(p, \tilde{u}_\phi([R',\infty)\times S^1)\right) < \delta\right\}$$
is a subset of $E([R,\infty)\times S^1 \times B^{2n}_\varepsilon(0))$. Here dist *denotes the Euclidean distance function on $\mathbb{R}\times S^1\times\mathbb{R}^{2n} = \mathbb{R}\times\mathbb{R}/\mathbb{Z}\times\mathbb{R}^{2n}$.*

Proof. We first observe that it is a straightforward consequence of Lemma 5.4 that there is an $\varepsilon > 0$ and an $R_0 > R$ so that $E_{\varepsilon,R}$ is an immersion with bounded derivatives of all orders. To prove the first claim, it therefore suffices to show that there exist constants $\varepsilon > 0$, $R > R_0$, and $c > 0$ so that
$$\operatorname{dist}(E(p), E(q)) \geq c \operatorname{dist}(p,q)$$
for all p, $q \in [R,\infty)\times S^1 \times B^{2n}_\varepsilon(0)$, where dist denotes the Euclidean distance on $\mathbb{R}\times S^1\times\mathbb{R}^{2n}$. If this is not true, then there exist sequences $p_k = (s_k, t_k, w_k)$, $q_k = (s'_k, t'_k, w'_k) \in [k,\infty)\times S^1\times B^{2n}_{\frac{1}{k}}(0)$ satisfying
$$\operatorname{dist}(E(p_k), E(q_k)) < \frac{1}{k}\operatorname{dist}(p_k, q_k). \tag{13}$$
If $\operatorname{dist}(p_k,q_k) \nrightarrow 0$, we can find a constant $\delta > 0$ and subsequences p_{k_j} and q_{k_j} so that $\operatorname{dist}(p_{k_j}, q_{k_j}) > \delta$ for all k. Using Lemma 5.4 we have
$$\begin{aligned}\operatorname{dist}(E(p_{k_j}), E(q_{k_j})) &\geq \operatorname{dist}(p_{k_j}, q_{k_j}) - \operatorname{dist}(E(p_{k_j}), p_{k_j}) - \operatorname{dist}(E(q_{k_j}), q_{k_j}) \\ &\geq \operatorname{dist}(p_{k_j}, q_{k_j}) - (1+|w_{k_j}|)Me^{-ds_{k_j}} - (1+|w'_{k_j}|)Me^{-ds'_{k_j}} \\ &\geq \operatorname{dist}(p_{k_j}, q_{k_j}) - 2(1+\tfrac{1}{k_j})Me^{-dk_j}\end{aligned}$$
so we can conclude that
$$\operatorname{dist}(E_{\varepsilon,R}(p_{k_j}), E_{\varepsilon,R}(q_{k_j})) \geq \frac{1}{2}\operatorname{dist}(p_{k_j}, q_{k_j})$$
for sufficiently large j. This contradicts (13), and we therefore assume that $\lim_{k\to\infty}\operatorname{dist}(p_k,q_k) = 0$. In this case we lift E to a map $[R_0,\infty)\times\mathbb{R}\times\mathbb{R}^{2n} \to \mathbb{R}\times\mathbb{R}\times\mathbb{R}^{2n}$ on the universal covers and use the linear structure to rewrite (13) as
$$\frac{|E(p_k) - E(q_k)|}{|p_k - q_k|} < \frac{1}{k}. \tag{14}$$
We can rewrite the left hand side as
$$\frac{|E(p_k) - E(q_k)|}{|p_k - q_k|} = \left|\int_0^1 DE(q_k + \tau(p_k - q_k))\, d\tau \cdot \tfrac{p_k - q_k}{|p_k - q_k|}\right|.$$
Passing to subsequences p_{k_j}, and q_{k_j} for which $\frac{p_{k_j} - q_{k_j}}{|p_{k_j} - q_{k_j}|} \to v \neq 0$ for some v, it follows from Lemma 5.4 that
$$\lim_{j\to\infty}\frac{|E(p_{k_j}) - E(q_{k_j})|}{|p_{k_j} - q_{k_j}|} = |v| \neq 0$$
which contradicts (14). This contradiction finishes the proof that R and ε can be chosen so that $E_{\varepsilon,R}$ is an embedding.

We next address the second claim. Abbreviating $A(s,t,w) := DE(s,t,w)$, we start by showing that all derivatives of the inverse matrix A^{-1} are bounded on $[R,\infty)\times S^1\times B^{2n}_\varepsilon(0)$ for some $\varepsilon > 0$ and R. We first show that A^{-1} is uniformly bounded on $[R,\infty)\times S^1\times B^{2n}_\varepsilon(0)$ for some $\varepsilon > 0$ and R. If not, there exists a sequence of unit vectors $v_k \in \mathbb{R}^{2n+2}$, and a sequence $(s_k, t_k, w_k) \in [k, \infty) \times S^1 \times B^{2n}_{\frac{1}{k}}(0)$ such that

$$|A^{-1}(s_k, t_k, w_k)v_k| \to \infty.$$

Defining $x_k = A^{-1}(s_k, t_k, w_k)v_k$, we get that

$$\left| A(s_k, t_k, w_k)\frac{x_k}{|x_k|} \right| = \frac{|v_k|}{|x_k|} = \frac{1}{|x_k|} \to 0.$$

Using the Lemma 5.4, we therefore find that

$$\begin{aligned} 1 = \left| I\frac{x_k}{|x_k|} \right| &\le |I - A(s_k, t_k, w_k)| + \left| A(s_k, t_k, w_k)\frac{x_k}{|x_k|} \right| \\ &\le Me^{-ds_k} + \left| A(s_k, t_k, w_k)\frac{x_k}{|x_k|} \right| \to 0. \end{aligned}$$

This contradiction implies that for some $\varepsilon > 0$ and $R > R_0$, the matrix $A^{-1}(s, t, w)$ is uniformly bounded for $(s, t, w) \in [R, \infty) \times S^1 \times B^{2n}_\varepsilon(0)$.

To see that all partial derivatives of A^{-1} are uniformly bounded, we differentiate $I = AA^{-1}$ to get

$$0 = \partial^\beta I = \partial^\beta(AA^{-1}) = \sum_{\alpha<\beta} \binom{\beta}{\alpha} \partial^\alpha A \partial^{\beta-\alpha} A^{-1}$$

where $\binom{\beta}{\alpha} = \binom{(\beta_1,\ldots,\beta_{2n+2})}{(\alpha_1,\ldots,\alpha_{2n+2})} := \binom{\beta_1}{\alpha_1}\ldots\binom{\beta_{2n+2}}{\alpha_{2n+2}}$. Solving this for $\partial^\beta A^{-1}$ gives

$$\partial^\beta A^{-1} = -A^{-1} \sum_{\alpha<\beta, |\alpha|\ge 1} \binom{\beta}{\alpha} \partial^\alpha A \partial^{\beta-\alpha} A^{-1}.$$

Therefore uniform bounds for $\partial^\beta[A(s, t, w)^{-1}]$ on $[R, \infty) \times S^1 \times B^{2n}_\varepsilon(0)$ follow from the uniform bounds for all derivatives of $A(s, t, w)$ on $[R, \infty) \times S^1 \times B^{2n}_\varepsilon(0)$ and induction on $|\beta|$.

Finally, to see that $E^{-1}_{\varepsilon,R}$ has bounded derivatives of all orders we write

$$DE^{-1}_{\varepsilon,R}(p) = B(E^{-1}_{\varepsilon,R}(p))$$

where we have abbreviated $B(s, t, w) := [A(s, t, w)]^{-1} = [DE(s, t, w)]^{-1}$. Applying ∂^β to both sides of this equation gives

$$\partial^\beta DE^{-1}_{\varepsilon,R}(p) = \partial^\beta \left[B(E^{-1}_{\varepsilon,R}(p)) \right].$$

The left hand side of this equation is a matrix containing partial derivatives of $E^{-1}_{\varepsilon,R}$ of order $|\beta| + 1$, and the right hand side can be expanded using the chain and product rules to a polynomial in the derivatives of B and $E^{-1}_{\varepsilon,R}$ of order at most $|\beta|$. Therefore uniform bounds on the derivatives of $E^{-1}_{\varepsilon,R}$ of all orders follow from the uniform bounds on the derivatives of B and induction on $|\beta|$.

We now address the final claim. Recall that we showed that $E_{\varepsilon,R}$ is an embedding by showing that it satisfies an inequality of the form

$$\operatorname{dist}(E_{\varepsilon,R}(p), E_{\varepsilon,R}(q)) \ge c \operatorname{dist}(p, q)$$

for some $c > 0$ and all p, q in its domain. It follows that for any $p \in [R, \infty) \times S^1 \times B^{2n}_\varepsilon(0)$ and any ε' with $B_{\varepsilon'}(p) \subset [R, \infty) \times S^1 \times B^{2n}_\varepsilon(0)$ the image $E_{\varepsilon,R}(B_{\varepsilon'}(p))$ contains a ball of radius $c\varepsilon'$ around $E_{\varepsilon,R}(p)$. Therefore, choosing $R' \ge R + \varepsilon$ and

$\delta < c\varepsilon$ it follows that any point within distance δ of the set $\tilde{u}_\phi([R',\infty)\times S^1)$ is contained in the image of $E_{\varepsilon,R}$ as required. $\square$

We are now ready to complete the proof of Proposition 5.1.

Proof of Proposition 5.1. Note that it follows immediately from (10) that for appropriate positive constants M_β, d that

$$\left|\partial^\beta\left(\tilde{u}_\phi(s,t)-\tilde{v}_\phi(s,t)\right)\right| \leq M_\beta e^{-ds}$$

for all $\beta\in\mathbb{N}^2$ and all $(s,t)\in[R_0,\infty)\times S^1$. Choosing an ε and R so that $E_{\varepsilon,R}$ satisfies all conclusions of Corollary 5.5, it follows that there exists some R' for which the set $\tilde{v}_\phi([R',\infty)\times S^1)$ will be in the image of the map $E_{\varepsilon,R}$. Define functions $\psi_0:[R',\infty)\times S^1\to[R,\infty)\times S^1$ and $h_0:[R',\infty)\times S^1\to\mathbb{R}^{2n}$ by

$$(\psi_0(z),h_0(z)) = E_{\varepsilon,R}^{-1}(\tilde{v}_\phi(z)).$$

Using $(z,0)=E_{\varepsilon,R}^{-1}(\tilde{u}_\phi(z))$ we get that

$$(\psi_0(z),h_0(z))-(z,0) = E_{\varepsilon,R}^{-1}(\tilde{v}_\phi(z)) - E_{\varepsilon,R}^{-1}(\tilde{u}_\phi(z)),$$

and since $E_{\varepsilon,R}^{-1}$ has bounded derivatives of all orders, it follows that

$$\left|\partial^\beta\left(\psi_0(s,t)-(s,t)\right)\right| \leq M_\beta e^{-ds}$$

and

$$\left|\partial^\beta\left(h_0(s,t)\right)\right| \leq M_\beta e^{-ds}$$

for all $\beta\in\mathbb{N}^2$.

Arguing as in Corollary 5.5, the estimate

$$\left|\partial^\beta\left(\psi_0(s,t)-(s,t)\right)\right| \leq M_\beta e^{-ds}$$

implies that $\psi_0|_{[R'',\infty)\times S^1}$ is invertible for sufficiently large R'', and that the inverse $\psi := \psi_0^{-1}$ has bounded derivatives of all orders. We can then write

$$\begin{aligned}\psi(s,t)-(s,t) &= \psi(s,t)-\psi(\psi_0(s,t))\\ &= D(s,t)\cdot[(s,t)-\psi_0(s,t)]\end{aligned}$$

where $D(s,t) := \int_0^1 D\psi(\psi_0(s,t)+\tau[(s,t)-\psi_0(s,t)])\,d\tau$ has bounded derivatives of all orders, and we conclude that for appropriate constants M_β, $d>0$ we will have

$$\left|\partial^\beta\left(\psi(s,t)-(s,t)\right)\right| \leq M_\beta e^{-ds}$$

for all $\beta\in\mathbb{N}^2$. Defining $h=h_0\circ\psi$, we get that $E(z,h(z))=\tilde{v}_\phi(\psi(z))$. Moreover, it follows from the exponential decay of h_0 and the uniform bounds for ψ that there exist positive constants d, M_β so that

$$\left|\partial^\beta h(s,t)\right| \leq M_\beta e^{-ds}$$

for all $\beta\in\mathbb{N}^2$. $\square$

5.3. **The Cauchy-Riemann Equation (Proof of Proposition 5.2).** In this section we derive the PDE satisfied by the map h from Proposition 5.1. After deriving this PDE, we apply a general result about a class of Cauchy-Riemann equations on half-cylinders to obtain a precise description of the asymptotic behavior of h. From here on we assume that h does not vanish identically.

Let $\bar{J}_\infty$ denote the almost complex structure on $\mathbb{R} \times S^1 \times B^{2n}_{\varepsilon_0}(0)$ obtained by pulling back $\tilde{J}$ via the covering $\tilde{\phi}$, i.e.

$$\bar{J}_\infty(z,w) = \tilde{\phi}_*(z,w)^{-1}\tilde{J}(\tilde{\phi}(z,w))\tilde{\phi}_*(z,w).$$

Observe that it is a straightforward consequence of the $\mathbb{R}$-invariance of $\tilde{J}$ and the definition of $\tilde{\phi}$ that $\bar{J}_\infty$ is also $\mathbb{R}$-invariant. We will therefore write $\bar{J}_\infty$ as a function of just the $t \in S^1$ and $w \in \mathbb{R}^{2n}$ variables when convenient. Choosing $\varepsilon > 0$ and R so that all the conclusions of Corollary 5.5 are satisfied, and so that $E([R,\infty) \times S^1 \times B^{2n}_{\varepsilon}(0)) \subset \mathbb{R} \times S^1 \times B^{2n}_{\varepsilon_0}(0)$, we use the map $E_{\varepsilon,R}$ to pull back the almost complex structure $\bar{J}_\infty$ on $\mathbb{R} \times S^1 \times B^{2n}_{\varepsilon_0}(0)$ to an almost complex structure

$$\bar{J}(z,w) = DE_{\varepsilon,R}(z,w)^{-1} \cdot \bar{J}_\infty(E_{\varepsilon,R}(z,w)) \cdot DE_{\varepsilon,R}(z,w).$$

on $[R,\infty) \times S^1 \times B^{2n}_{\varepsilon}(0)$. We will write

$$\bar{J}(z,w) = \begin{bmatrix} i(z,w) & \beta(z,w) \\ \gamma(z,w) & J(z,w) \end{bmatrix}$$

where $i(z,w) \in \mathrm{End}(T_z([R',\infty) \times S^1))$, $\beta(z,w) \in \mathrm{Hom}(\mathbb{R}^{2n}, T_z([R',\infty) \times S^1))$, $\gamma(z,w) \in \mathrm{Hom}(T_z([R',\infty) \times S^1), \mathbb{R}^{2n})$, and $J(z,w) \in \mathrm{End}(\mathbb{R}^{2n})$. It is immediate from the definition of $\bar{J}$ that the map

$$z \mapsto f(z) \in [R,\infty) \times S^1 \times B^{2n}_{\varepsilon}(0)$$

is $\bar{J}$-holomorphic precisely when the map $z \mapsto E(f(z))$ is $\bar{J}_\infty$-holomorphic, which in turn is true precisely when the map $z \mapsto \tilde{\phi}(E(f(z)))$ is $\tilde{J}$-holomorphic. Moreover, it follows from the definition of $E_{\varepsilon,R}$ and the fact that that $\tilde{\Phi}$ is a unitary trivialization of $(\phi^*\xi^{\mathcal{H}}, \bar{J}_\infty, \phi^*\omega)$ that

$$\bar{J}(z,0) = \begin{bmatrix} j_0 & 0 \\ 0 & J_0 \end{bmatrix}$$

where j_0 is the complex structure on $[R,\infty) \times S^1$ and J_0 is the standard complex multiplication on $\mathbb{R}^{2n}$.

Now recalling that $h : [R',\infty) \times S^1 \to \mathbb{R}^{2n}$ satisfies $E(z,h(z)) = \tilde{v}_\phi(\psi(z))$, where $\tilde{v}_\phi$ is a lift under $\tilde{\phi}$ of a $\tilde{J}$-holomorphic map, we must have that the tangent space of the graph of h is invariant under $\bar{J}$. More precisely, if we define a map $\Gamma_h : [R',\infty) \times S^1 \to [R',\infty) \times S^1 \times \mathbb{R}^{2n}$ by $\Gamma_h(z) = (z, h(z))$, then Γ_h will satisfy

$$d\Gamma_h(z)(T_z([R',\infty) \times S^1)) = \bar{J}(\Gamma_h(z))d\Gamma_h(z)(T_z([R',\infty) \times S^1)).$$

Observing Γ_h is an immersion, we can define an almost complex structure $\bar{j}$ on $C_{R'} = [R',\infty) \times S^1$ by

$$\bar{j}(z)v = (d\Gamma_h(z)|_{d\Gamma_h(z)T_zC_{R'}})^{-1}\bar{J}(\Gamma_h(z))d\Gamma_h(z)v$$

so that Γ_h will satisfy the equation

$$d\Gamma_h + \bar{J} \circ d\Gamma_h \circ \bar{j} = 0. \tag{15}$$

Considering the $T([R',\infty)\times S^1)$ components of this equation, and using $\bar{j}^2=-id$ gives that

$$\bar{j}(z)=i(z,h(z))+\beta(z,h(z))dh(z) \tag{16}$$

while the $\mathbb{R}^{2n}$ components give

$$dh(z)+J(z,h(z))\circ dh(z)\circ\bar{j}(z)+\gamma(z,h(z))\circ\bar{j}(z)=0. \tag{17}$$

We would like to apply Theorem A.1 to obtain a formula for h. In order to do this, we must understand the asymptotic behavior of $\bar{J}$, and then rewrite (17) in a form that allows us to apply the theorem. The main result to this end is the following.

Lemma 5.6. *With $\bar{J}$, $\bar{J}_\infty$ defined as above, there exist constants d, $M_\beta>0$ so that*

$$\left|\partial^\beta\left(\bar{J}(s,t,w)-\bar{J}_\infty(t,w)\right)\right|\le M_\beta e^{-ds}$$

for all $\beta\in\mathbb{N}^{2n+2}$ and all $(s,t,w)\in[R,\infty)\times S^1\times B^{2n}_\varepsilon(0)$.

Proof. First observe that it follows from Lemma 5.4 and the $\mathbb{R}$-invariance of $\bar{J}_\infty$ that after potentially choosing larger R, and smaller ε we will have

$$\left|\partial^\beta\left(\bar{J}_\infty(E_\varepsilon(s,t,w))-\bar{J}_\infty(s,t,w)\right)\right|\le M_\beta e^{-ds}$$

for all $\beta\in\mathbb{N}^{2+2n}$ and all $(s,t,w)\in[R,\infty)\times S^1\times B^{2n}_\varepsilon(0)$. Given this, it is a straightforward consequence of Lemma 5.4, Corollary 5.5, and the definitions of $\bar{J}$ and $\bar{J}_\infty$ that

$$\left|\partial^\beta\left(\bar{J}(s,t,w)-\bar{J}_\infty(t,w)\right)\right|\le M_\beta e^{-ds}$$

for all $(s,t,w)\in[R,\infty)\times S^1\times B^{2n}_\varepsilon(0)$ and $\beta\in\mathbb{N}^{2+2n}$, as claimed. ☐

For the following corollary, we write

$$\bar{J}_\infty(t,w)=\begin{bmatrix} i_\infty(t,w) & \beta_\infty(t,w)\\ \gamma_\infty(t,w) & J_\infty(t,w)\end{bmatrix}$$

to represent how $\bar{J}_\infty$ decomposes with respect to the splitting $T([R,\infty)\times S^1\times B^{2n}_\varepsilon(0))\approx T([R,\infty)\times S^1)\oplus\mathbb{R}^{2n}$.

Corollary 5.7. *There exists functions*

$$J^*:[R,\infty)\times S^1\to\mathrm{Hom}(\mathbb{R}^{2n},\mathrm{End}(\mathbb{R}^{2n}))$$
$$j^*:[R,\infty)\times S^1\to\mathrm{Hom}(\mathbb{R}^{2n},\mathrm{End}(T([R,\infty)\times S^1)))$$
$$\gamma^*:[R,\infty)\times S^1\to\mathrm{Hom}(\mathbb{R}^{2n},\mathrm{Hom}(T([R,\infty)\times S^1),\mathbb{R}^{2n}))$$

satisfying

$$\bar{J}(s,t,h(s,t))=J_0+J^*(s,t)h(s,t)$$
$$\bar{j}(s,t)=j_0+j^*(s,t)h(s,t)$$
$$\gamma(s,t,h(s,t))=D_2\gamma_\infty(t,0)h(s,t)+\gamma^*(s,t)h(s,t),$$

where D_2 represents the partial derivative with respect to the $\mathbb{R}^{2n}$ variable of $S^1\times\mathbb{R}^{2n}$. Moreover, J^* and j^* have uniformly bounded derivatives of all orders, and there exist constants d, $M_\beta>0$ so that

$$\left|\partial^\beta\gamma^*(s,t)\right|\le M_\beta e^{-ds}$$

for all $\beta\in\mathbb{N}^2$.

Proof. Using that $\bar{J}(z,0) = J_0$, we write

$$\begin{aligned}\bar{J}(z,h(z)) &= J_0 + \big(\bar{J}(z,h(z)) - \bar{J}(z,0)\big) \\ &= J_0 + \int_0^1 \frac{d}{d\tau}\bar{J}(z,\tau h(z))\,d\tau \\ &= J_0 + \int_0^1 D_2\bar{J}(z,\tau h(z))(h(z))\,d\tau.\end{aligned}$$

The asymptotic behavior of $\bar{J}$ from the lemma tells us that $D_2\bar{J}$ has bounded derivatives of all orders on $[R,\infty)\times S^1\times B_\varepsilon^{2n}(0)$. Defining $J^*(s,t)w = \int_0^1 D_2\bar{J}(z,\tau h(z))(w)\,d\tau$ we get

$$\bar{J}(s,t,h(s,t)) = J_0 + J^*(s,t)h(s,t)$$

where J^* has bounded derivatives of all orders as required. Similarly, using (16) with $i(z,0) = j_0$ and $\beta(z,0) = 0$, we get

$$\bar{j}(s,t) = j_0 + j^*(s,t)h(s,t)$$

where j^* is defined by

$$j^*(z)w = \int_0^1 D_2 i(z,\tau h(z))(w)\,d\tau + \int_0^1 [D_2\beta(z,\tau h(z))(w)]\circ dh(z)\,d\tau.$$

It then follows from the asymptotic behavior of $\bar{J}$ that j^* has bounded derivatives of all orders.

Finally, using $\gamma(z,0) = 0$ we write

$$\gamma(z,h(z)) = D_2\gamma(z,0)h(z) + \int_0^1\int_0^1 \tau D_2^2\gamma(z,\sigma\tau h(z))(h(z),h(z))\,d\tau\,d\sigma,$$

so that defining γ^* by

$$\begin{aligned}\gamma^*(s,t)w &= [D_2\gamma(s,t,0) - D_2\gamma_\infty(t,0)]\,w \\ &\quad + \int_0^1\int_0^1 \tau D_2^2\gamma(s,t,\sigma\tau h(s,t))(h(s,t),w)\,d\tau\,d\sigma\end{aligned}$$

we get

$$\gamma(s,t,h(s,t)) = D_2\gamma_\infty(t,0)h(s,t) + \gamma^*(s,t)h(s,t).$$

Once again, the asymptotic behavior of $\bar{J}$ implies that $D_2^2\gamma$ has bounded derivatives of all orders on $[R,\infty)\times S^1\times B_\varepsilon^{2n}(0)$, which, with the exponential decay of h, implies that γ^* decays exponentially with all derivatives. □

We now give the proof of Proposition 5.2.

Proof of Proposition 5.2. The formula for h will follow from Theorem A.1 once we have shown that h satisfies a PDE of the form

$$\partial_s h - (\Phi^{-1}\mathbf{A}_{P,J}\Phi)h + \tilde{\Delta}h = 0$$

for some $\tilde{\Delta} : [R',\infty)\times S^1 \to End(\mathbb{R}^{2n})$ satisfying the exponential decay estimate

$$\left|\partial^\beta\tilde{\Delta}(s,t)\right| \le M_\beta e^{-ds}$$

for all $\beta \in \mathbb{N}^2$ and for some positive constants d, M_β.

Evaluating (17) on ∂_s and using Corollary 5.7 and $j_0\partial_s = \partial_t$, we get that h satisfies the equation

$$\partial_s h(s,t) + J_0\partial_t h(s,t) + (D_2\gamma_\infty(t,0)h(s,t))\,\partial_t + \tilde{\Delta}(s,t)h(s,t) = 0$$

where

$$\tilde{\Delta}(z)w = \tilde{\Delta}(z, h, dh)w = ([J^*(z)w] \circ dh(z) \circ \bar{j}(z) + J(z, h(z)) \circ dh(z) \circ [j^*(z)w] + [\gamma^*(z)w] \circ \bar{j}(z))\partial_s.$$

It follows from the uniform bounds on the derivatives of $\bar{J}$, J^*, and j^*, and the exponential decay of all the derivatives of h and γ^* that $\tilde{\Delta}$ decays exponentially together will all of its derivatives. It therefore remains to show that $\Phi^{-1}\mathbf{A}_{P,J}\Phi = -J_0\partial_t - D_2\gamma_\infty(t,0)(\cdot)\partial_t$.

Let ∇ be some symmetric connection on M and let $\tilde{\nabla}$ be the extension of ∇ to $\mathbb{R} \times M$ obtained by requiring ∂_a to be a parallel field, where a again denotes the parameter along $\mathbb{R}$. Viewing $\xi^{\mathcal{H}}_{P(t)}$ as lying in $T_{(0,P(t))}(\mathbb{R} \times M)$, we rewrite $\mathbf{A}_{P,J}$ as

$$\begin{aligned}\mathbf{A}_{P,J}\eta &= -J\nabla_t\eta + TJ\nabla_\eta X_{\mathcal{H}} \\ &= \pi_{\xi^{\mathcal{H}}}\left(-\tilde{J}\tilde{\nabla}_t\eta + T\tilde{J}\tilde{\nabla}_\eta X_{\mathcal{H}}\right)\end{aligned}$$

where $\pi_{\xi^{\mathcal{H}}}$ again denotes the projection onto $\xi^{\mathcal{H}}$ along $X_{\mathcal{H}}$. Using the definition of $\tilde{J}$ and $\tilde{\nabla}$ we find that

$$\begin{aligned}\tilde{J}\tilde{\nabla}X_{\mathcal{H}} &= \tilde{\nabla}(\tilde{J}X_{\mathcal{H}}) - (\tilde{\nabla}\tilde{J})X_{\mathcal{H}} \\ &= \tilde{\nabla}\partial_a - (\tilde{\nabla}\tilde{J})X_{\mathcal{H}} = -(\tilde{\nabla}\tilde{J})X_{\mathcal{H}},\end{aligned}$$

and so we find that $\mathbf{A}_{P,J}$ can be written

$$\mathbf{A}_{P,J}\eta = \pi_{\xi^{\mathcal{H}}}\left(-\tilde{J}\tilde{\nabla}_t\eta - T(\tilde{\nabla}_\eta\tilde{J})X_{\mathcal{H}}\right) = -J\nabla_t\eta - \pi_{\xi^{\mathcal{H}}}T(\tilde{\nabla}_\eta\tilde{J})X_{\mathcal{H}}.$$

Using that the coordinate fields ∂_t, ∂_{x_i}, ∂_{y_i} are invariant under the deck transformations of the covering ϕ, we can choose the connection ∇ to be a flat connection near $P(S^1)$ which pulls back via ϕ to a connection which is just the standard derivative on $S^1 \times \mathbb{R}^{2n}$. Moreover, the extension $\tilde{\nabla}$ to $\mathbb{R} \times M$ will pull back to the standard derivative on $\mathbb{R} \times S^1 \times \mathbb{R}^{2n}$. Now, observing that ϕ was constructed to satisfy

$$\begin{aligned}d\phi(t,0)\partial_t &= T \cdot X_{\mathcal{H}}(P(t)) \\ d\phi(t,0)(0,w) &= \Phi(P(t))w\end{aligned}$$

and that

$$\phi^*\pi_{\xi^{\mathcal{H}}}(t,0) = d\phi(t,0)^{-1}\pi_{\xi^{\mathcal{H}}}(P(t))d\phi(t,0) = \pi_{\mathbb{R}^{2n}}$$

where $\pi_{\mathbb{R}^{2n}}$ is the projection onto the $\mathbb{R}^{2n}$ coordinates of $S^1 \times \mathbb{R}^{2n}$, it is straightforward to deduce that term $\pi_{\xi^{\mathcal{H}}}T(\tilde{\nabla}_\eta\tilde{J})X_{\mathcal{H}}$ at $P(t)$ lifts via $\tilde{\phi}$ to $D_2\gamma_\infty(t,0)(\Phi^{-1}\eta(t))\partial_t$, and that $J\nabla_t\eta$ lifts via ϕ to $J_0\frac{d}{dt}\left(\Phi^{-1}\eta(t)\right)$. We therefore find that

$$\Phi^{-1}\mathbf{A}_{P,J}\Phi = -J_0\partial_t - D_2\gamma_\infty(t,0)(\cdot)\partial_t$$

as claimed, and this completes the proof. □

5.4. **Completing the Proof (Proof of Proposition 5.3).** In this section we prove Proposition 5.3 and, as explained in Section 5.1, this completes the proof of Theorem 2.2. Recall that we have defined maps $\tilde{U}$, $\tilde{V}$, $h : [R', \infty) \times S^1 \to \mathbb{R}^{2n}$ and embeddings ψ_u, ψ_v, $\psi : [R', \infty) \times S^1 \to \mathbb{R} \times S^1$ satisfying

$$(z, \tilde{U}(z)) = \tilde{u}_\phi(\psi_u(z)) \tag{18}$$

$$(z, \tilde{V}(z)) = \tilde{v}_\phi(\psi_v(z)) \tag{19}$$

$$E_{\varepsilon,R}(z, h(z)) = \tilde{v}_\phi(\psi(z)) \tag{20}$$

$$E_{\varepsilon,R}(z, 0) = \tilde{u}_\phi(z) \tag{21}$$

and

$$\begin{aligned} \left|\partial^\beta \left(\psi_u(s,t) - (s,t)\right)\right| &\leq M_\beta e^{-ds} \\ \left|\partial^\beta \left(\psi_v(s,t) - (s,t)\right)\right| &\leq M_\beta e^{-ds} \\ \left|\partial^\beta \left(\psi(s,t) - (s,t)\right)\right| &\leq M_\beta e^{-ds}. \end{aligned} \tag{22}$$

We know that h can be written

$$h(s,t) = e^{\lambda s}\left[e(t) + r(s,t)\right]$$

where $\lambda < 0$, $\Phi e \in \ker(\mathbf{A}_{P,J} - \lambda) \setminus \{0\}$, and where r and all its derivatives exhibit exponential decay. We would like to prove a similar formula for the function $z \mapsto \tilde{V}(z) - \tilde{U}(z)$ with the same eigenvalue λ and eigenvector e. The following lemma will be important in the proof of Proposition 5.3. In this section, if $f : [R, \infty) \times S^1 \to \mathbb{R}^k$, we will use the shorthand $f(s,t) = o_\infty(c)$ for $c \in \mathbb{R}$ if there are positive constants d, M_β so that

$$\left|\partial^\beta \left(e^{-cs} f(s,t)\right)\right| \leq M_\beta e^{-ds}$$

for all $(s,t) \in [R, \infty) \times S^1$ and all $\beta \in \mathbb{N}^2$.

Lemma 5.8. *The maps ψ_u, ψ_v, and ψ satisfy*

$$(\psi^{-1} \circ \psi_v)(s,t) - \psi_u(s,t) = o_\infty(\lambda)$$

where $\lambda < 0$ is the negative eigenvalue of $\mathbf{A}_{P,J}$ appearing in Proposition 5.2.

Proof. Using (18) and (21), we can write

$$E(z, 0) = \tilde{u}_\phi(z) = \left(\psi_u^{-1}(z), \tilde{U}(\psi_u^{-1}(z))\right)$$

and similarly using (19) and (20) gives

$$E(z, h(z)) = \tilde{v}_\phi(\psi(z)) = \left((\psi_v^{-1} \circ \psi)(z), \tilde{V}((\psi_v^{-1} \circ \psi)(z))\right).$$

Letting $\pi_{\mathbb{R}\times S^1}$ denote the projection of $\mathbb{R} \times S^1 \times \mathbb{R}^{2n}$ onto the first two coordinates, we find that

$$\begin{aligned} (\psi_v^{-1} \circ \psi)(z) - \psi_u^{-1}(z) &= \pi_{\mathbb{R}\times S^1}\left(E_{\varepsilon,R}(z, h(z)) - E_{\varepsilon,R}(z, 0)\right) \\ &= \pi_{\mathbb{R}\times S^1}\left(\int_0^1 DE_{\varepsilon,R}(z, \tau h(z))\, d\tau \cdot (0, h(z))\right) \\ &= \pi_{\mathbb{R}\times S^1}\left(\int_0^1 \left[DE_{\varepsilon,R}(z, \tau h(z)) - I\right]\, d\tau \cdot (0, h(z))\right) \end{aligned}$$

where in the third line we have subtracted $0 = \pi_{\mathbb{R}\times S^1}(0, h(z))$. If follows from Lemma 5.4 that the function

$$D(z) := \int_0^1 \left[DE_{\varepsilon,R}(z, \tau h(z)) - I\right]\, d\tau$$

satisfies

$$\left|\partial^\beta D(s,t)\right| \leq M_\beta e^{-ds}$$

for some appropriate positive constants M_β and d, and it follows from the formula for h in Proposition 5.2 that h satisfies

$$\left|\partial^\beta h(s,t)\right| \leq M_\beta e^{\lambda s} \tag{23}$$

where $\lambda < 0$ is the eigenvalue of $\mathbf{A}_{P,J}$ appearing in the proposition and M_β are some positive constants. Therefore

$$(\psi_v^{-1} \circ \psi)(z) - \psi_u^{-1}(z) = o_\infty(\lambda)$$

and, as in the proof of Proposition 5.1, this implies that

$$(\psi^{-1} \circ \psi_v)(z) - \psi_u(z) = o_\infty(\lambda)$$

as well. □

We now proceed with the proof of Proposition 5.3.

Proof of Proposition 5.3. Using (19), (20), Lemma 5.8 and uniform bounds on the derivatives of $E_{\varepsilon,R}$ (from Corollary 5.5) and h (from either Proposition 5.1 or 5.2) we get that

$$\begin{aligned}\tilde{V}(z) &= \pi_{\mathbb{R}^{2n}} \tilde{v}_\phi(\psi_v(z)) \\ &= \pi_{\mathbb{R}^{2n}} E_{\varepsilon,R}\left((\psi^{-1} \circ \psi_v)(z), h((\psi^{-1} \circ \psi_v)(z))\right) \\ &= \pi_{\mathbb{R}^{2n}} E_{\varepsilon,R}\left(\psi_u(z), h(\psi_u(z))\right) + o_\infty(\lambda)\end{aligned}$$

and using (18) and (21) gives

$$\tilde{U}(z) = \pi_{\mathbb{R}^{2n}} \tilde{u}_\phi(\psi_u(z)) = \pi_{\mathbb{R}^{2n}} E_{\varepsilon,R}\left(\psi_u(z), 0\right).$$

Here $\pi_{\mathbb{R}^{2n}} : \mathbb{R} \times S^1 \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ denotes the projection onto the last $2n$ coordinates of $\mathbb{R} \times S^1 \times \mathbb{R}^{2n}$. Subtracting these expressions gives

$$\begin{aligned}\tilde{V}(s,t) - \tilde{U}(s,t) &= \pi_{\mathbb{R}^{2n}}\left[E_{\varepsilon,R}\left(\psi_u(z), h(\psi_u(z))\right) - E_{\varepsilon,R}\left(\psi_u(z), 0\right)\right] + o_\infty(\lambda) \\ &= \pi_{\mathbb{R}^{2n}} \int_0^1 DE_{\varepsilon,R}\left(\psi_u(z), \tau h(\psi_u(z))\right)\, d\tau \cdot (0, h(\psi_u(z))) + o_\infty(\lambda) \\ &= h(\psi_u(z)) + o_\infty(\lambda) \\ &= e^{\lambda s} e(t) + o_\infty(\lambda).\end{aligned}$$

Going from the second to third line, we have used the exponential decay estimates (23) for h resulting from Propositions 5.2, and that the function

$$z \mapsto \int_0^1 \left[DE_{\varepsilon,R}\left(\psi_u(z), \tau h(\psi_u(z))\right) - I\right]\, d\tau$$

exhibits exponential decay as a result of Lemma 5.4. Going from the third to final line we have used (22) with the formula for h from Proposition 5.2. This completes the proof of Proposition 5.3. □

Appendix A. The Asymptotic Formula

In this appendix, we give the proof of the asymptotic formula from Proposition 5.2. This result is implicit in [15] and our arguments are adapted from those in [7], [6], [5], and [15].

Throughout we will use the notation $\bar{\mathbb{R}}^+$ to denote the closed half-line $[0, \infty)$. We will equip $\mathbb{R}^{2n}$ with its standard norm and inner product, and all Banach or Hilbert spaces of $\mathbb{R}^{2n}$-valued functions will be equipped with the standard norms and inner products arising from those on $\mathbb{R}^{2n}$. We will consider $S^1 = \mathbb{R}/\mathbb{Z}$ with coordinate $t \in \mathbb{R}/\mathbb{Z}$, and we will equip S^1 with the measure dt.

Theorem A.1. *Let $w : \bar{\mathbb{R}}^+ \times S^1 \to \mathbb{R}^{2n}$ satisfy the equation*

$$\partial_s w + J_0 \partial_t w + (S(t) - \tilde{\Delta}(s,t))w = 0 \tag{24}$$

where

$$J_0 = \begin{pmatrix} 0 & -I^{n\times n} \\ I^{n\times n} & 0 \end{pmatrix},$$

$S : S^1 \to End(\mathbb{R}^{2n})$ is a smooth family of symmetric matrices, and $\tilde{\Delta} : \bar{\mathbb{R}}^+ \times S^1 \to End(\mathbb{R}^{2n})$ is a smooth map. Assume there exist positive constants M_β and d such that

$$|\partial^\beta w(s,t)| \le M_\beta e^{-ds} \tag{25}$$

$$|\partial^\beta \tilde{\Delta}(s,t)| \le M_\beta e^{-ds} \tag{26}$$

for all $\beta \in \mathbb{N}^2$. Then either w vanishes identically or we have

$$w(s,t) = e^{\lambda s}(e(t) + r(s,t)) \tag{27}$$

where λ is a negative eigenvalue of the self-adjoint operator

$$\mathbf{A} : H^1(S^1, \mathbb{R}^{2n}) \subset L^2(S^1, \mathbb{R}^{2n}) \to L^2(S^1, \mathbb{R}^{2n})$$

defined by

$$(\mathbf{A}h)(t) = -J_0 \partial_t h(t) - S(t)h(t),$$

$e : S^1 \to \mathbb{R}^{2n}$ is an eigenvector of $\mathbf{A}$ with eigenvalue λ, and $|\partial^\beta r(s,t)| \le M'_\beta e^{-cs}$ for some positive constants constant c and M'_β and all $\beta \in \mathbb{N}^2$.

To prove this we assume that w does not vanish identically and we study the functions

$$v(s,t) := \frac{w(s,t)}{\|w(s)\|} \tag{28}$$

$$\mu(s) := \langle v(s), (\mathbf{A} + \tilde{\Delta}(s))v(s)\rangle \tag{29}$$

where $w(s) : S^1 \to \mathbb{R}^{2n}$ denotes the smooth loop $t \mapsto w(s,t)$, $v(s)$ is defined similarly to $w(s)$, $\tilde{\Delta}(s) \in \mathcal{L}(L^2(S^1, \mathbb{R}^{2n}))$ is the bounded linear operator defined by $(\tilde{\Delta}(s)x)(t) = \tilde{\Delta}(s,t)x(t)$ for $t \in S^1$, and where $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ denote respectively the standard norm and inner product on $L^2(S^1, \mathbb{R}^{2n})$. We note that the similarity principle (see Appendix A.6 in [8]) implies that w either vanishes identically or has only isolated zeroes. In the latter case, we must have $\|w(s)\| > 0$ for all $s \ge 0$ so $v(s,t)$ is well-defined and smooth. Theorem A.1 will follow from the following proposition.

Proposition A.2. *With v, μ defined as above there exist positive constants c, C_β, C_k and a smooth eigenvector $\hat{e}(t) : S^1 \to \mathbb{R}^{2n}$ of $\mathbf{A}$ with eigenvalue $\lambda < 0$ such that*

$$\left|\partial^\beta[v(s,t) - \hat{e}(t)]\right| \leq C_\beta e^{-cs} \quad \text{and} \quad \left|\tfrac{d^k}{ds^k}[\mu(s) - \lambda]\right| \leq C_k e^{-cs}$$

for all $\beta \in \mathbb{N}^2$ and all $k \in \mathbb{N}$.

We prove this proposition by proceeding in a series of smaller results.

Lemma A.3. *Assume that w does not vanish identically. Then w satisfies*

$$\|w(s)\| = e^{\int_0^s \mu(\tau)\,d\tau}\|w(0)\|. \tag{30}$$

Proof. As stated in the discussion preceding Proposition A.2, we can assume by the similarity principle that w has only isolated zeroes, and hence $\|w(s)\| > 0$ for all $s \geq 0$. Using (24) and (29) we compute

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\|w(s)\|^2 &= \langle w_s(s), w(s)\rangle \\ &= \langle(\mathbf{A} + \tilde{\Delta}(s))w(s), w(s)\rangle \\ &= \|w(s)\|^2\mu(s).\end{aligned}$$

Integrating this equation gives (30). □

Lemma A.4. *The map $\mu : [0,\infty) \to \mathbb{R}$ satisfies*

$$\lim_{s\to\infty} \mu(s) = \lambda \in \sigma(\mathbf{A})$$

for some $\lambda < 0$.

Proof. We start by deriving an inequality for the derivative μ' of μ. Differentiating (28) and using (24) gives

$$\partial_s v = (\mathbf{A} + \tilde{\Delta}(s,t) - \mu(s))v, \tag{31}$$

which along with $\|v(s)\| = 1$ gives

$$\langle(\mathbf{A} + \tilde{\Delta} - \mu)v, v\rangle = \langle v_s, v\rangle = 0. \tag{32}$$

Differentiating (29), and using (31),(32), and self-adjointness of $\mathbf{A}$ gives

$$\mu'(s) = 2\|v_s\|^2 + \langle\tilde{\Delta}_s v, v\rangle + \langle\tilde{\Delta}v_s, v\rangle - \langle\tilde{\Delta}v, v_s\rangle. \tag{33}$$

Using $\|v(s)\| = 1$ we get

$$\langle\tilde{\Delta}_s v, v\rangle \geq -\|\tilde{\Delta}_s\|\|v\|^2 = -\|\tilde{\Delta}_s\|$$

$$\langle\tilde{\Delta}v_s, v\rangle - \langle\tilde{\Delta}v, v_s\rangle \geq -2\|\tilde{\Delta}\|\|v_s\|\|v\| = -2\|\tilde{\Delta}\|\|v_s\| \geq -\|v_s\|^2 - \|\tilde{\Delta}\|^2$$

which gives us

$$\mu'(s) \geq \|v_s\|^2 - \|\tilde{\Delta}_s\| - \|\tilde{\Delta}\|^2. \tag{34}$$

Using (31) and (26), this is equivalent to

$$\mu'(s) \geq [\|(\mathbf{A} - \mu)v\| - \varepsilon(s)]^2 - \varepsilon(s) \tag{35}$$

where $\varepsilon(s)$ is used to denote quantities that approach 0 as $s \to \infty$.

We now show that μ is bounded from above. If there is a number $K > 0$ such that $\mu(s) > K$ for all $s > s_0$ then it follows from (30) that $\|w(s)\| \to \infty$ as $s \to \infty$ in contradiction to the decay estimates (25). Therefore if μ is not bounded from above it must oscillate. In particular there exists a positive real number $\eta \notin \sigma(\mathbf{A})$ and a sequence $s_n \to \infty$ such that $\mu(s_n) = \eta$ and $\mu'(s_n) \leq 0$. However, the assumption $\eta \notin \sigma(\mathbf{A})$ implies that

$$\|(\mathbf{A} - \eta)v\| \geq \|(\mathbf{A} - \eta)^{-1}\|^{-1}\|v\| = \operatorname{dist}(\eta, \sigma(\mathbf{A})) > 0,$$

where we have used that $\|(\mathbf{A} - \eta)^{-1}\| = [\operatorname{dist}(\eta, \sigma(\mathbf{A}))]^{-1}$ since $\mathbf{A}$ is a self-adjoint operator with a compact resolvent. This together with (35) and $\mu(s_n) = \eta$ implies that for large n we will have

$$\mu'(s_n) \geq \frac{1}{2} \operatorname{dist}(\eta, \sigma(\mathbf{A}))^2 > 0$$

which contradicts $\mu'(s_n) \leq 0$. Therefore μ must be bounded from above.

Define

$$g(s) = \int_s^\infty \|\tilde{\Delta}_s(\rho)\| + \|\tilde{\Delta}(\rho)\|^2 \, d\rho$$

and observe that by (26), $g(s)$ is finite and satisfies $\lim_{s\to\infty} g(s) = 0$. Since μ is bounded from above, the function $\mu_1 : \bar{\mathbb{R}}^+ \to \mathbb{R}$ defined by

$$\mu_1(s) = \mu(s) - g(s)$$

is also bounded from above. Moreover using (34) we have that

$$\begin{aligned} \mu_1'(s) &= \mu'(s) + \|\tilde{\Delta}_s(s)\| + \|\tilde{\Delta}(s)\|^2 \\ &\geq \|v_s(s)\|^2 \geq 0 \end{aligned}$$

so μ_1 is nondecreasing, and therefore $\lim_{s\to\infty} \mu_1(s)$ exists. Since $\lim_{s\to\infty} g(s) = 0$, the limit $\lim_{s\to\infty} \mu(s)$ must also exist and satisfy

$$\lim_{s\to\infty} \mu(s) = \lim_{s\to\infty} \mu_1(s).$$

Defining

$$\lambda = \lim_{s\to\infty} \mu(s)$$

we now show that $\lambda \in \sigma(\mathbf{A})$. We have that

$$\begin{aligned} \|(\mathbf{A} - \mu)v\| &\geq \|(\mathbf{A} - \lambda)v\| - \|(\lambda - \mu)v\| \\ &\geq \operatorname{dist}(\lambda, \sigma(\mathbf{A})) - \varepsilon(s) \end{aligned} \tag{36}$$

where again we use $\varepsilon(s)$ to denote a quantity that goes to 0 as $s \to \infty$. If $\lambda \notin \sigma(\mathbf{A})$, it follows from (35) and (36) that for sufficiently large s we will have

$$\mu'(s) \geq \frac{1}{2} \operatorname{dist}(\lambda, \sigma(\mathbf{A}))$$

which contradicts the upper bound for μ. We therefore conclude $\lambda \in \sigma(\mathbf{A})$.

It remains to show that $\lambda < 0$. First observe that it follows from (25) that

$$\|w(s)\| \leq Me^{-ds} \tag{37}$$

for some M, $d > 0$. If $\lambda \geq 0$ we can find an s_0 such that for $s > s_0$ we will have

$$\mu(s) \geq -\frac{d}{2}.$$

This with (30) gives us

$$\|w(s)\| \geq e^{-\frac{d}{2}(s-s_0)}\|w(s_0)\| = M'e^{-\frac{d}{2}s}$$

with $M' > 0$, which contradicts (37). We therefore conclude $\lambda < 0$. □

Lemma A.5. *The functions v and μ satisfy*

$$\sup_{\bar{\mathbb{R}}^+\times S^1} |\partial^\beta v(s,t)| < \infty$$
$$\sup_{\bar{\mathbb{R}}^+} |\tfrac{d^k}{ds^k}\mu(s)| < \infty$$

for all multi-indices β and nonnegative integers k.

Proof. For any $s'\in\mathbb{R}$ define a sequence of nested intervals

$$I_j(s') = [s'-(1+2^{-j}), s'+(1+2^{-j})]$$

and set $Q_j(s')\subset\mathbb{R}\times S^1$ by $Q_j(s') = I_j(s')\times S^1$. Choose a sequence of smooth cut-off functions $\beta_j:\mathbb{R}\to[0,1]$ satisfying

$$\beta_j|_{\mathbb{R}\setminus I_{j-1}(0)} \equiv 0 \quad\text{ and }\quad \beta_j|_{I_j(0)}\equiv 1$$

and we observe that the translated functions $\beta_j^{s'}:\mathbb{R}\to[0,1]$ defined by $\beta_j^{s'}(s) = \beta_j(s-s')$ satisfy

$$\beta_j^{s'}|_{\mathbb{R}\setminus I_{j-1}(s')} \equiv 0 \quad\text{ and }\quad \beta_j^{s'}|_{I_j(s')}\equiv 1.$$

We will use elliptic estimates for the $\bar\partial$-operator to show inductively that

$$\|v\|_{W^{k,p}(Q_k(s'))} \le C_{k,p} \tag{38a-k,p}$$
$$\|\mu\|_{W^{k,p}(I_k(s'))} \le C_{k,p} \tag{38b-k,p}$$

for all $k\in\mathbb{N}$ and $p>2$, where $C_{k,p}$ are constants that are independent of s'. The result will then follow from the Sobolev embedding theorem.

Recalling the well known a priori estimate for the $\bar\partial$-operator, we have

$$\|v\|_{W^{k,p}(Q_k(s'))} \le M_{k,p}\|\bar\partial(\beta_k^{s'}v)\|_{W^{k-1,p}(Q_{k-1}(s'))} \tag{39}$$

for $p>1$ where $M_{k,p}$ is independent of s'. Rewriting (31) as

$$\bar\partial v = -(S-\tilde\Delta+\mu)v \tag{40}$$

we use C^0 bounds on S, $\tilde\Delta$, and μ to estimate

(41)

$$\begin{aligned}\|v\|_{W^{1,p}(Q_1(s'))} &\le M_{1,p}\|\bar\partial(\beta_1^{s'}v)\|_{L^p(Q_0(s'))} = M_{1,p}\|\bar\partial(\beta_1^{s'})v + \beta_1^{s'}\bar\partial v\|_{L^p(Q_0(s'))}\\ &= M_{1,p}\|\bar\partial(\beta_1^{s'})v - \beta_1^{s'}(S-\tilde\Delta+\mu)v)\|_{L^p(Q_0(s'))}\\ &\le c\|v\|_{L^p(Q_0(s'))}.\end{aligned}$$

Here c depends on $p>1$, the C^1 norm of β_1, and the C^0 norms of S, $\tilde\Delta$, and μ, but not on s'. Recalling $\|v(s)\|=1$, we have $\|v\|_{L^2(Q_0(s'))}=\sqrt{2}$ so we conclude $\|v\|_{W^{1,2}(Q_1(s'))} < C_{1,2}$ for $C_{1,2}$ independent of s'. Therefore, by the Sobolev embedding theorem, we have for all $p>1$

$$\|v\|_{L^p(Q_1(s'))} \le C_p\|v\|_{W^{1,2}(Q_1(s'))} \le C'_p,$$

from which we can conclude (38a-0,p) for all $p>1$. Using (41) again, we conclude that

$$\|v\|_{W^{1,p}(Q_1(s'))} \le c\|v\|_{L^p(Q_0(s'))} \le C_{1,p}$$

thus establishing (38a-1,p) for all $p>1$.

We observe that (38b-0,p) follows directly from Lemma A.4. For $k \geq 1$ and $p > 1$ we differentiate formula (33) and use Hölder's inequality to find

$$\begin{aligned} |\partial_s^k \mu(s)|^p &\leq c(\sum_{i=0}^k \|\partial_s^i v(s)\|^2)^p \leq c_{k,p} \sum_{i=0}^k \|\partial_s^i v(s)\|^{2p} \\ &= c_{k,p} \sum_{i=0}^k \|1 \cdot |\partial_s^i v(s)|^2\|^p_{L^1(S^1)} \\ &\leq c_{k,p} \sum_{i=0}^k \||\partial_s^i v(s)|^2\|^p_{L^p(S^1)} \|1\|^p_{L^{p/(p-1)}(S^1)} \\ &= c_{k,p} \sum_{i=0}^k \|\partial_s^i v(s)\|^{2p}_{L^{2p}(S^1,\mathbb{R}^{2n})}. \end{aligned}$$

Integrating this inequality over $I_k(s')$ and using the uniform bound on μ, we conclude that

$$\|\mu\|^p_{W^{k,p}(I_k(s'))} \leq c_1 \|v\|^{2p}_{W^{k,2p}(Q_k(s'))} + c_2. \tag{42}$$

This establishes (38b-1,p) for all $p > 1$.

Finally, for $k \geq 2$ and $p > 2$ we use (39), (40), the uniform C^k bounds for S and $\tilde{\Delta}$, and the uniform C^{k+1} bounds for β_k to estimate

$$\begin{aligned} \|v\|_{W^{k,p}(Q_k(s'))} &\leq c\|\bar{\partial}(\beta_k^{s'})v - \beta_k^{s'}(S - \tilde{\Delta} + \mu)v)\|_{W^{k-1,p}(Q_{k-1}(s'))} \\ &\leq c\left(1 + \|\mu\|_{W^{k-1,p}(I_{k-1}(s'))}\right)\|v\|_{W^{k-1,p}(Q_{k-1}(s'))}. \end{aligned} \tag{43}$$

Here we've used that

$$\|\mu v\|_{W^{k-1,p}(Q_{k-1}(s'))} \leq c\|\mu\|_{W^{k-1,p}(I_{k-1}(s'))}\|v\|_{W^{k-1,p}(Q_{k-1}(s'))}$$

provided $(k-1)p > 2$. The inequalities (38) now follow from (42), (43), and induction on k. □

As an immediate corollary we have the following.

Corollary A.6. *The function μ satisfies*

$$\lim_{s\to\infty} |\tfrac{d^k}{ds^k}(\mu(s) - \lambda)| = 0$$

for all $k \in \mathbb{N}$.

Proof. If not we can find a $k \geq 1$, a constant $c > 0$, and a sequence $s_n \to \infty$ such that $|\mu^{(k)}(s_n)| > c$ for all n. Defining a sequence of functions $\mu_n : [-s_n, \infty) \to \mathbb{R}$ by $\mu_n(s) = \mu(s + s_n)$, it follows from Arzelà-Ascoli and the previous lemma that a subsequence converges in C^∞_{loc} to a smooth function $\mu_\infty : \mathbb{R} \to \mathbb{R}$. Moreover, by Lemma A.4 we must have $\mu_\infty \equiv \lambda$. Indexing the subsequence still by n it follows that $\mu_n^{(k)}(0) \to \mu_\infty^{(k)}(0) = 0$, which gives the immediate contradiction

$$0 < c \leq \lim_{n\to\infty} |\mu^{(k)}(s_n)| = \lim_{n\to\infty} |\mu_n^{(k)}(0)| = 0.$$

□

Lemma A.7. *For all positive integers k we have*

$$\lim_{s\to\infty} \|\partial_s^k v(s)\| = 0.$$

Proof. From (34) we have

$$\|\partial_s v(s)\|^2 \leq \mu'(s) + \|\tilde{\Delta}_s(s)\| + \|\tilde{\Delta}(s)\|^2$$

and it follows from Corollary A.6 and (26) that the conclusion of the lemma holds for $k = 1$. For the higher order derivatives, we rewrite (31) as

$$\partial_s v = (\mathbf{A} - \lambda)v + (\tilde{\Delta}(s,t) + [\lambda - \mu(s)])v \tag{44}$$

and differentiate to get

$$\partial_s^{k+1} v = (\mathbf{A} - \lambda)\partial_s^k v + \sum_{l=0}^{k} \binom{k}{l} \partial_s^l (\tilde{\Delta}(s,t) + [\lambda - \mu(s)])\partial_s^{k-l} v. \tag{45}$$

Taking an inner product with $\partial_s^{k+1} v$ and using the self-adjointness of $\mathbf{A}$ and Lemma A.5 gives

$$\begin{aligned}
\|\partial_s^{k+1} v\|^2 &\leq \|(\mathbf{A} - \lambda)\partial_s^{k+1} v\| \|\partial_s^k v\| \\
&\qquad + \sum_{l=0}^{k} \binom{k}{l} \left(\|\partial_s^l \tilde{\Delta}(s)\| + |\partial_s^l [\lambda - \mu(s)]| \right) \|\partial_s^{k-l} v\| \|\partial_s^{k+1} v\| \\
&\leq C \left(\|\partial_s^k v\| + \sum_{l=0}^{k} \left(\|\partial_s^l \tilde{\Delta}(s)\| + |\partial_s^l [\lambda - \mu(s)]| \right) \right).
\end{aligned}$$

The lemma now follows from Corollary A.6, (26), and induction on k. □

Lemma A.8. *Let $P : L^2(S^1, \mathbb{R}^{2n}) \to E := \ker(\mathbf{A} - \lambda)$ be the orthogonal projection onto the λ-eigenspace of $\mathbf{A}$, and let $Q = (I - P) : L^2(S^1, \mathbb{R}^{2n}) \to E^{\perp}$ be the complementary projection onto the orthogonal complement to E. Then for every $k \in \mathbb{N}$, we have*

$$\|\partial_s^k Q v(s)\| \leq M_k e^{-cs}$$

for some positive constants M_k and c.

Proof. We start by observing that there is a constant $C_\lambda > 0$ so that

$$C_\lambda \|Qx\| \leq \|(\mathbf{A} - \lambda)Qx\|$$

for any $x \in H^1(S^1, \mathbb{R}^{2n})$. Using this with (44) and $\|v\| = 1$ gives us

$$\begin{aligned}
C_\lambda \|Qv\| \leq \|(\mathbf{A} - \lambda)Qv\| &= \|(\mathbf{A} - \lambda)v\| = \|v_s - (\tilde{\Delta} + [\lambda - \mu])v\| \\
&\leq \|v_s\| + \|\tilde{\Delta}\| + |\lambda - \mu|
\end{aligned}$$

so it follows from Lemma A.7, Corollary A.6 and (26) that $\|Qv(s)\| \to 0$ as $s \to \infty$. Define the map $V : \bar{\mathbb{R}}^+ \times S^1 \to \mathbb{R}^{2n(k+1)}$ by $V = (v, \partial_s v, \ldots, \partial_s^k v) \in \oplus_{k+1} \mathbb{R}^{2n}$ and let $\bar{Q}$ be the diagonal operator on $L^2(S^1, \mathbb{R}^{2n(k+1)}) \approx \oplus_{k+1} L^2(S^1, \mathbb{R}^{2n})$ associated to Q. Define $a(s) = \frac{1}{2}\|\bar{Q}V(s)\|^2$ and observe that Lemma A.7 along with the fact that $\|Qv(s)\| \to 0$ as $s \to \infty$ imply that $a(s) \to 0$ as $s \to \infty$. We will prove an estimate of the form

$$a''(s) \geq c_1^2 a(s) - c_2^2 e^{-|r|s} \tag{46}$$

for all s larger than some s_0, and this will imply that $a(s) \leq Me^{-cs}$ for some constants $M, c > 0$. Indeed, if we define $f(s) = a(s) - Me^{-cs}$ we find that

$$f''(s) \geq c_1^2 f(s) \text{ and } f(s_0) \leq 0$$

provided that we choose [4] $c < \min(|c_1|, |r|)$ and $M \geq \max\left(e^{cs_0}a(s_0), \frac{c_2^2}{c_1^2 - c^2}\right)$. This implies that f can not have any positive local maxima, and since $f(s) \to 0$ as $s \to \infty$ we must have $f(s) \leq 0$ for all $s \geq s_0$. Therefore $a(s) \leq Me^{-cs}$ for all $s \geq s_0$ and choosing M larger if necessary, we get $a(s) \leq Me^{-cs}$ for all $s \geq 0$.

We now proceed with the proof of (46). Letting $\bar{\mathbf{A}}$ be the diagonal operator on $\oplus_{k+1} L^2(S^1, \mathbb{R}^{2n})$ associated to $(\mathbf{A} - \lambda)$ we observe that (45) implies that V satisfies an equation of the form

$$\partial_s V = \bar{\mathbf{A}}V + \Gamma(s)V + \bar{\Delta}(s,t)V. \tag{47}$$

for matrix functions Γ and $\bar{\Delta}$. We observe that Γ is made up of $2n \times 2n$ blocks of the form $c_{jl}\partial_s^j(\lambda - \mu)I^{2n\times 2n}$ for some constants c_{jl}, and thus satisfies $\bar{Q}\Gamma = \Gamma\bar{Q}$ and $\partial_s^k \Gamma(s) \to 0$ as $s \to \infty$ for all $k \in \mathbb{N}$. Moreover $\bar{\Delta}$ is made up of blocks of the form $c_{jl}\partial_s^k \tilde{\Delta}$, and thus satisfies $|\partial^\beta \bar{\Delta}(s,t)| \leq M_\beta e^{-ds}$ for appropriate constants $M_\beta > 0$ and all $\beta \in \mathbb{N}^2$. Applying $\bar{Q}$ to (47) and using $\partial_s(\bar{Q}V) = \bar{Q}\partial_s V$ we get that $V^\perp := \bar{Q}V$ satisfies the equation

$$\partial_s V^\perp = \bar{\mathbf{A}}V^\perp + \Gamma(s)V^\perp + \delta(s,t) \tag{48}$$

where $\delta(s,t) = \bar{Q}\bar{\Delta}(s,t)V$ satisfies

$$|\partial^\beta \delta(s,t)| \leq M_\beta e^{-ds}$$

for all $\beta \in \mathbb{N}^2$ because of the exponential decay of $\bar{\Delta}$ and the uniform bounds for V resulting from Lemma A.5.

Differentiating the equation $a(s) = \frac{1}{2}\|\bar{Q}V(s)\|^2 = \frac{1}{2}\|V^\perp(s)\|^2$ twice gives us

$$a''(s) = \|V_s^\perp\|^2 + \langle V_{ss}^\perp, V^\perp\rangle.$$

Using (48) and the self-adjointness of $\bar{\mathbf{A}}$ gives us

$$\begin{aligned}
\langle V_{ss}^\perp, V^\perp\rangle &= \langle \partial_s(\bar{\mathbf{A}}V^\perp + \Gamma V^\perp + \delta), V^\perp\rangle \\
&= \langle V_s^\perp, \bar{\mathbf{A}}V^\perp\rangle + \langle \Gamma_s V^\perp, V^\perp\rangle + \langle \Gamma V_s^\perp, V^\perp\rangle + \langle \delta_s, V^\perp\rangle \\
&= \|V_s^\perp\|^2 - \langle V_s^\perp, \Gamma V^\perp\rangle - \langle V_s^\perp, \delta\rangle + \langle \Gamma_s V^\perp, V^\perp\rangle + \langle V_s^\perp, \Gamma^T V^\perp\rangle + \langle \delta_s, V^\perp\rangle \\
&= \|V_s^\perp\|^2 - \langle V_s^\perp, \delta\rangle + \langle \Gamma_s V^\perp, V^\perp\rangle + \langle V_s^\perp, (\Gamma^T - \Gamma)V^\perp\rangle + \langle \delta_s, V^\perp\rangle
\end{aligned}$$

so we find that

$$a''(s) = 2\|V_s^\perp\|^2 - \langle V_s^\perp, \delta\rangle + \langle \Gamma_s V^\perp, V^\perp\rangle + \langle V_s^\perp, (\Gamma^T - \Gamma)V^\perp\rangle + \langle \delta_s, V^\perp\rangle.$$

For any $\sigma > 0$ we have

$$-\langle V_s^\perp, \delta\rangle \geq -\|V_s^\perp\|\|\delta\| \geq -\|V_s^\perp\|Me^{-ds} \geq -\sigma\|V_s^\perp\|^2 - M_\sigma e^{-2ds}$$

and similarly

$$\langle \delta_s, V^\perp\rangle \geq -\sigma\|V^\perp\|^2 - M_\sigma e^{-2ds}$$

for some constant M_σ depending on σ. Furthermore, letting $\varepsilon(s)$ represent any positive quantity that converges to 0 as $s \to \infty$ we can estimate

$$\langle \Gamma_s V^\perp, V^\perp\rangle \geq -\|\Gamma_s\|\|V^\perp\|^2 = -\varepsilon(s)\|V^\perp\|^2$$

and

$$\langle V_s^\perp, (\Gamma^T - \Gamma)V^\perp\rangle \geq -\|\Gamma^T - \Gamma\|\|V_s^\perp\|\|V^\perp\| \geq -\varepsilon(s)\|V_s^\perp\|^2 - \varepsilon(s)\|V^\perp\|^2.$$

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[4] If $|r| < |c_1|$ we can take $c = |r|$, but c must be strictly smaller than $|c_1|$.

Putting these estimates together gives

$$a''(s) \geq (2-\sigma-\varepsilon(s))\|V_s^\perp\|^2 - (\sigma+\varepsilon(s))\|V^\perp\|^2 - M_\sigma e^{-2ds}.$$

Using (48) again, we find

$$\begin{aligned}\|V_s^\perp\|^2 &= \|\bar{\mathbf{A}}V^\perp\|^2 + \|\Gamma V^\perp\|^2 + \|\delta\|^2 + 2\langle \bar{\mathbf{A}}V^\perp, \Gamma V^\perp\rangle + 2\langle \bar{\mathbf{A}}V^\perp, \delta\rangle + 2\langle \Gamma V^\perp, \delta\rangle \\ &\geq \|\bar{\mathbf{A}}V^\perp\|^2 + 2\langle \bar{\mathbf{A}}V^\perp, \Gamma V^\perp\rangle + 2\langle \bar{\mathbf{A}}V^\perp, \delta\rangle + 2\langle \Gamma V^\perp, \delta\rangle \\ &\geq (1-\sigma-\varepsilon(s))\|\bar{\mathbf{A}}V^\perp\|^2 - \varepsilon(s)\|V^\perp\|^2 - M_\sigma e^{-2ds}.\end{aligned}$$

Finally using $\|\bar{\mathbf{A}}V^\perp\| \geq C_\lambda\|V^\perp\|$, the previous estimates yield

$$\begin{aligned}a''(s) &\geq (2C_\lambda^2 - \sigma - \varepsilon(s))\|V^\perp\|^2 - M_\sigma e^{-2ds} \\ &= (4C_\lambda^2 - \sigma - \varepsilon(s))a(s) - M_\sigma e^{-2ds}.\end{aligned}$$

Therefore choosing σ small, we can find for any $c \in (0, \min(C_\lambda, d))$ a constant M_c so that

$$\frac{1}{2}\|V^\perp(s)\|^2 = a(s) \leq M_c e^{-2cs}$$

for all $s \geq 0$. The lemma follows directly from this estimate, and the definition of $V^\perp$. □

Lemma A.9. *For every $k \in \mathbb{N}$ we have that*

$$\left|\tfrac{d^k}{ds^k}(\mu(s)-\lambda)\right| \leq C_k e^{-cs}$$

for some positive constants C_k and c.

Proof. Rewriting (32) as

$$\langle([\mathbf{A}-\lambda] + \tilde{\Delta} + [\lambda-\mu])v, v\rangle = 0$$

and using $\|v(s)\| = 1$ and the self-adjointness of $\mathbf{A}$ we have that

$$\begin{aligned}\mu - \lambda &= \langle(\mathbf{A}-\lambda)v, v\rangle + \langle\tilde{\Delta}v, v\rangle \\ &= \langle(\mathbf{A}-\lambda)v, Qv\rangle + \langle\tilde{\Delta}v, v\rangle.\end{aligned}$$

Differentiating this equation and using Lemma A.5 gives

$$\begin{aligned}\left|\tfrac{d^k}{ds^k}(\mu-\lambda)\right| &= \left|\sum_{j=0}^k \binom{k}{j}\left(\langle(\mathbf{A}-\lambda)\partial_s^j v, \partial_s^{k-j}Qv\rangle + \sum_{l=0}^j \binom{j}{l}\langle(\partial_s^l\tilde{\Delta})\partial_s^{j-l}v, \partial_s^{k-j}v\rangle\right)\right| \\ &\leq \sum_{j=0}^k \binom{k}{j}\left(\|(\mathbf{A}-\lambda)\partial_s^j v\|\|\partial_s^{k-j}Qv\| + \sum_{l=0}^j \binom{j}{l}\|\partial_s^l\tilde{\Delta}\|\|\partial_s^{j-l}v\|\|\partial_s^{k-j}v\|\right) \\ &\leq C\left(\sum_{j=0}^k \|\partial_s^j Qv\| + \|\partial_s^j\tilde{\Delta}\|\right).\end{aligned}$$

The lemma now follows from Lemma A.8, and (26). □

The following lemma completes the proof of Proposition A.2.

Lemma A.10. *There exists a unit length eigenvector $\hat{e} \in E$ and positive constants c, C_β so that*

$$\left|\partial^\beta[v(s,t) - \hat{e}(t)]\right| \leq C_\beta e^{-cs}$$

for all $\beta \in \mathbb{N}^2$.

Proof. Letting $P : L^2(S^1, \mathbb{R}^{2n}) \to E := \ker(\mathbf{A} - \lambda)$ still denote the orthogonal projection onto the λ-eigenspace of $\mathbf{A}$, we let $\bar{P}$ denote the diagonal operator on $L^2(S^1, \mathbb{R}^{2n(k+1)}) \approx \oplus_{k+1} L^2(S^1, \mathbb{R}^{2n})$ associated to P. Applying $\bar{P}$ to (47), we get

$$\begin{aligned}(\bar{P}V)_s &= \bar{\mathbf{A}}(\bar{P}V) + \Gamma(\bar{P}V) + \bar{P}\bar{\Delta}V \\ &= \Gamma(\bar{P}V) + \bar{P}\bar{\Delta}V.\end{aligned}$$

Recalling that the entries in Γ are constant multiples of $\partial_s^k(\lambda - \mu)$, it follows from Lemma A.9 that $\Gamma(s) \to 0$ exponentially. This along with the exponential decay of $\bar{\Delta}$ and uniform bounds on V imply that

$$|\bar{P}V_s(s,t)| \le Me^{-cs}$$

for some $M > 0$. This gives us

$$\|\bar{P}V(s_0) - \bar{P}V(s)\| \le \left| \int_s^{s_0} \|\bar{P}V_s(\rho)\| \, d\rho \right| \le M \left| \int_s^{s_0} e^{-c\rho} \, d\rho \right|$$

which implies that $\bar{P}V(s)$ must converge in L^2 as $s \to \infty$ to some $\tilde{E} \in \oplus_{k+1} E$. Moreover, it follows from Lemma A.7 and $\|v\| = 1$ that $\tilde{E}(t) = (\hat{e}(t), 0, \ldots, 0)$ for some $\hat{e} \in E$ with $\|\hat{e}\| = 1$. Taking the limit as $s_0 \to \infty$ in the above inequality then gives

$$\|\bar{P}V(s) - \tilde{E}\| \le M' e^{-cs}$$

which along with Lemma A.8 implies that

$$\|\partial_s^k(v(s) - \hat{e})\| \le M_k e^{-cs} \tag{49}$$

for all $k \in \mathbb{N}$.

Combining (44) with

$$\partial_s \hat{e} = 0 = (\mathbf{A} - \lambda)\hat{e}$$

gives us

$$\partial_s(v - \hat{e}) = (\mathbf{A} - \lambda)(v - \hat{e}) + (\lambda - \mu)v + \tilde{\Delta}v$$

which, using $\mathbf{A} = -J_0\partial_t - S$ and $J_0^2 = -I$, is equivalent to

$$\partial_t(v - \hat{e}) = J_0 \Big[\partial_s(v - \hat{e}) + (S(t) + \lambda)(v - \hat{e}) + (\mu(s) - \lambda)v - \tilde{\Delta}(s,t)v\Big].$$

Applying $\partial_s^i \partial_t^j$ to this equation gives

$$\begin{aligned}\partial_s^i \partial_t^{j+1}(v - \hat{e}) = J_0 \partial_s^{i+1}\partial_t^j(v - \hat{e}) + J_0 \sum_{k=0}^{j} \binom{j}{k} \Bigg[&\partial_t^k(S(t) + \lambda)\partial_s^i \partial_t^{j-k}(v - \hat{e}) \\ &+ \sum_{l=0}^{i} \binom{i}{l} \partial_s^l \partial_t^k \Big[(\mu(s) - \lambda) - \tilde{\Delta}(s,t)\Big] \partial_s^{i-l}\partial_t^{j-k} v \Bigg]\end{aligned}$$

which with Lemma A.5 implies

$$\|\partial_s^i \partial_t^{j+1}(v - \hat{e})\| \le C\Big(\|\partial_s^{i+1}\partial_t^j(v - \hat{e})\| + \sum_{\beta \le (i,j)} \|\partial^\beta(v - \hat{e})\| + |\partial^\beta(\mu - \lambda)| + \|\partial^\beta \tilde{\Delta}\|\Big).$$

It then follows from (26), (49), Lemma A.9, and induction on j that

$$\|\partial^\beta(v(s) - \hat{e})\| \le M_\beta e^{-cs}$$

for all $\beta \in \mathbb{N}^2$. Finally applying the Sobolev embedding theorem, we can conclude that

$$\left|\partial^\beta[v(s,t) - \hat{e}(t)]\right| \le M_\beta e^{-cs}$$

for all $\beta \in \mathbb{N}^2$. □

Having completed the proof of Proposition A.2, we now prove Theorem A.1.

Proof of Theorem A.1. We observe that the exponential decay estimates of Lemma A.9 imply that $\mu - \lambda$ is integrable on $[0, \infty)$ and that the function $\delta(s) := \int_s^\infty (\lambda - \mu(\tau))\, d\tau$ satisfies

$$\left| \tfrac{d^k}{ds^k} \delta(s) \right| \leq M_k e^{-cs}$$

for all $k \in \mathbb{N}$. Moreover, we have similar exponential decay estimates for $e^{\delta(s)} - 1$ since we can write

$$\begin{aligned} e^{\delta(s)} - 1 &= \int_0^1 \tfrac{d}{dt} e^{t\delta(s)}\, dt \\ &= \delta(s) \int_0^1 e^{t\delta(s)}\, dt \end{aligned}$$

and the function $s \mapsto \int_0^1 e^{t\delta(s)}\, dt$ has bounded derivatives of all orders. Using the definition (28) of v with (30) gives us

$$\begin{aligned} w(s,t) &= \|w(s)\| v(s,t) \\ &= e^{\int_0^s \mu(\tau)\, d\tau} \|w(0)\| v(s,t) \\ &= e^{\lambda s - \delta(0) + \delta(s)} \|w(0)\| \left(\hat{e}(t) + [v(s,t) - \hat{e}(t)]\right) \\ &= e^{\lambda s}(a\hat{e}(t) + r(s,t)) \end{aligned}$$

where $a = e^{-\delta(0)}\|w(0)\| \neq 0$ and $r(s,t) = a\left(e^{\delta(s)} v(s,t) - \hat{e}(t)\right)$. Moreover, the function $r(s,t)$ and all of its derivatives will satisfy exponential decay estimates since the functions $e^{\delta(s)} - 1$ and $v(s,t) - \hat{e}(t)$ and all their derivatives do. □

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