2π-GRAFTING AND COMPLEX PROJECTIVE
STRUCTURES II

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Abstract. Let \( S \) be an oriented closed surface of genus at least two. We show that, given a generic representation \( \rho: \pi_1(S) \to \text{PSL}(2, \mathbb{C}) \) in the character variety, \((2\pi)-\text{grafting}\) produces all projective structures on \( S \) with holonomy \( \rho \).

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Let \( S \) be a closed oriented surface of genus at least two throughout this paper. A (complex) projective structure on \( S \) is a \((\hat{\mathbb{C}}, \text{PSL}(2, \mathbb{C}))\)-structure (see §2.1). Then it induces a (holonomy) representation \( \rho: \pi_1(S) \to \text{PSL}(2, \mathbb{C}) \) unique up to conjugation by an element of \( \text{PSL}(2, \mathbb{C}) \).

Let \( \mathcal{P}_\rho \) be the set of all marked projective structures on \( S \) with fixed holonomy \( \rho \). Then a \((2\pi\text{-})\text{graft}\) is a surgery operation that transforms a projective structure in \( \mathcal{P}_\rho \) to another in \( \mathcal{P}_\rho \) (§2.2). In the precessing paper ([Bab]), the author showed that projective structures in \( \mathcal{P}_\rho \) are related by grafting if they are “close” in the space of geodesic laminations \( \mathfrak{G}L \) in Thurston coordinates. In this paper, we aim to relate all projective structures in \( \mathcal{P}_\rho \) by grafting.

Let \( \mathcal{P} \) be the set of all marked projective structures on \( S \). Then \( \mathcal{P} \) is naturally isomorphic to \( \mathbb{C}^{6g-6} \). Let \( \chi \) be the \( \text{PSL}(2, \mathbb{C})\)-character variety of \( S \), that is, the set of all representations \( \rho: \pi_1(S) \to \text{PSL}(2, \mathbb{C}) \), roughly, up to conjugation by an element of \( \text{PSL}(2, \mathbb{C}) \); see §2.7. Then \( \chi \) is a complex affine algebraic variety, and it consists of exactly two connected components ([Gol88]). Let \( \chi_0 \) be the canonical component of \( \chi \) consisting of representations that lifts to \( \pi_1(S) \to \text{SL}(2, \mathbb{C}) \). Let

\[
\text{Hol}: \mathcal{P} \to \chi
\]

be the holonomy map: Namely \( \text{Hol} \) takes each projective structure to its holonomy representation. Then the image of \( \text{Hol} \) is contained in \( \chi_0 \), and moreover \( \text{Hol} \) is almost onto \( \chi_0 \) ([GKM00]).

Since \( \mathcal{P}_\rho = \text{Hol}^{-1}(\rho) \), we are interested in understanding fibers of \( \text{Hol} \). This is a basic problem ([Hub81, Kap95, GKM00, Dum09]; see also [Gol80]). The holonomy map \( \text{Hol} \) is a local homeomorphism [Hej75] (moreover a locally biholomorphism [Hub81, Ear81]); however it is not a covering map onto its image. Thus \( \mathcal{P}_\rho \) is a discrete subset of \( \mathcal{P} \), but it may possibly be quite different depending on \( \rho \in \chi_0 \).

A graft of a projective surface inserts a projective cylinder along an appropriate loop (a simple closed curve), called an admissible loop. Then an ungraft is the opposite of a graft, which removes such a projective cylinder; thus it also preserves holonomy. Then

**Question 1.1** ([GKM00], Grafting Conjecture). Given two projective structures with holonomy \( \rho: \pi_1(S) \to \text{PSL}(2, \mathbb{C}) \), is there a composition of grafts and ungrafts that transforms one to the other?

The basic case of this question is when \( \rho \) is a discrete and faithful representation onto a quasifuchsian group. Then \( \mathcal{P}_\rho \) contains a unique
uniformizable projective structure (i.e. its developing map is an embedding into \( \hat{\mathbb{C}} \)); then every projective structure in \( P_\rho \) is moreover obtained by grafting the uniformizable structure along a multiloop [Gol87] (c.f. [Ito08, Bab12]). On the other hand, if \( \rho \in \chi_0 \) is a generic representation outside of the closure of the quasifuchsian space, then \( \rho \) has a dense image in \( \text{PSL}(2, \mathbb{C}) \); in particular, there is no uniformizable structure with holonomy \( \rho \). Thus Question 1.1 seems a most basic question about \( P_\rho \).

In this paper we answer Question 1.1 in the affirmative for generic representations in \( \chi_0 \), namely, of the following type: An element \( \alpha \in \text{PSL}(2, \mathbb{C}) \) is loxodromic if its trace \( \text{Tr}(\alpha) \in \mathbb{C} \), which is well-defined up to a sign, is not contained in \( [-2, 2] \subset \mathbb{R} \). A representation \( \rho : \pi_1(S) \to \text{PSL}(2, \mathbb{C}) \) is called purely loxodromic if \( \rho(\gamma) \) is loxodromic for all \( \gamma \in \pi_1(S) \). Almost all elements of \( \chi_0 \) are purely loxodromic (Proposition 2.5). Then

**Theorem 1.2.** Let \( \rho : \pi_1(S) \to \text{PSL}(2, \mathbb{C}) \) be a purely loxodromic representation in \( \chi_0 \). Then, given any \( C_\sharp, C_\flat \in P_\rho \), there is a composition of grafts and ungrafts that transforms \( C_\sharp \) to \( C_\flat \). Namely there is a finite composition of grafts \( \text{Gr}_{\ell_i} \) along loops \( \ell_i \) starting from \( C_\sharp \),

\[
C_\sharp = C_0 \xrightarrow{\text{Gr}_{\ell_1}} C_1 \xrightarrow{\text{Gr}_{\ell_2}} C_2 \rightarrow \ldots \xrightarrow{\text{Gr}_{\ell_n}} C_n, 
\]

such that the last projective structure \( C_n \) is obtained by grafting \( C_\flat \) along a multiloop \( M \),

\[
C_\flat \xrightarrow{\text{Gr}_M} C_n.
\]

(Here a multiloop is a union of disjoint loops, and “graft along a multiloop \( M \)” means simultaneous grafts along all loops of \( M \).)

In the case that \( \rho \) is a quasifuchsian representation, [Ito07, Theorem 3] implies Theorem 1.2 in a stronger form: The sequence (1) is replace by, as (2), a single graft along a multiloop. (See also [CDF].)

In order to prove Theorem 1.2, we utilize Thurston coordinates on \( \mathcal{P} \), which is given by (not necessarily \( 2\pi \))-grafting of hyperbolic surfaces. Namely there is a natural identification by a homeomorphism

\[
\mathcal{P} \cong \mathcal{T} \times \mathcal{ML},
\]

where \( \mathcal{T} \) is the space of hyperbolic structures on \( S \) and \( \mathcal{ML} \) the space of measured laminations on \( S \) (see §3). We thus denote Thurston coordinates of a projective structure \( C \) using “\( \cong \)” as \( C \cong (\tau, L) \in \mathcal{T} \times \mathcal{ML} \).

Let \( \mathcal{GL} \) be the space of geodesic laminations on \( S \). Then there is an obvious projection \( \mathcal{ML} \to \mathcal{GL} \). Given an arbitrary \( C \in \mathcal{P}_\rho \), by the preceding paper [Bab], projective structures in \( \mathcal{P}_\rho \) that are “close”
to $C$ in $\mathcal{GL}$ in Thurston coordinates are related to $C$ by grafting (see Theorem 2.6). This “local” relation will yield the graft (2). Thus our main work in this paper is to construct the sequence (1) so that $C_n$ is “close” to $C_\flat$ in $\mathcal{GL}$.

Therefore, in order to control geodesic laminations of projective structures, we observe an asymptotic, in Thurston coordinates, of projective structures given by the iteration of grafts along a fixed loop. In particular

**Theorem 1.3.** Let $C \cong (\tau, L)$ be a projective structure on $S$ in Thurston coordinates, where $(\tau, L) \in \mathcal{T} \times \mathcal{ML}$. Let $\ell$ be an admissible loop on $C$. For every $i \in \mathbb{Z}_{>0}$, let $C_i \cong (\tau_i, L_i)$ be the projective structure obtained by $i$-times grafting $C$ along $\ell$ (i.e. $2\pi i$-graft). Then $\tau_i$ converges in $\mathcal{T}$, and $L_i$ converges to a measured lamination $L_\infty$ as $i \to \infty$ such that $\ell$ is a unique leaf of $L_\infty$ of weight infinity. (See Theorem 6.5.)

As in Theorem 1.3, if stated, some closed leaves of laminations may have weight infinity in this paper (§2.3).

In the case that $C$ is a hyperbolic surface (i.e. the developing map of $C$ is an embedding onto a round disk), Theorem 1.3 is straightforward. Namely $\tau_i = \tau$ for all $i$ and $L$ is equal to $\ell$ with weight $2\pi i$ ([Gol87]). In other words, Theorem 1.3 asserts that $C_i$ asymptotically behaves similarly to grafts of a hyperbolic surface. (In contrast, the conformal structure of $C_i$ diverges as $i \to \infty$ but converges to a corresponding point in the Thurston boundary of $\mathcal{T}$ [CDR12, DK12, Hen11, Gup].)

By projectivizing transversal measures, $\mathcal{ML}$ minus the empty lamination projects onto the space of projective measured laminations, $\mathcal{PML}$. In the appendix we prove

**Theorem 1.4.** Given arbitrary $\rho \in \chi_0$, in Thurston coordinates, $\mathcal{P}_\rho$ naturally projects onto a dense subset of $\mathcal{PML}$, unless $\mathcal{P}_\rho$ is empty. (See Theorem 11.2.)

Theorem 1.4 justifies our approach of using Thurston coordinates in order to answer Question 1.1. A similar density is well-known for geodesic laminations realized by pleated surfaces in the same homotopy class in a fixed hyperbolic three-manifolds (see [CEG87]). Theorem 1.4 is obtained by carefully observing the construction of a projective structure with given holonomy in [GKM00] and applying Theorem 1.3.

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1.2. Definitions and Notations.

- Let $Y \rightarrow X$ be a covering map and let $Z$ be a subset of $X$. Then the total lift of $Z$ to $Y$ is the inverse image of $Z$ under the covering map.
- $[a, b]$ denotes the geodesic segment connecting $a$ and $b$.

2. Preliminaries

[Kap01] is a good general reference for the background materials. For hyperbolic geometry in particular, see [CEG87, EM87, Thu81]. See also the preceding paper [Bab].

2.1. Projective structures. (c.f. [Thu97].) Let $F$ be an oriented connected surface, and let $\tilde{F}$ be the universal cover of $F$. A projective structure $C$ is a $(\hat{\mathbb{C}}, \text{PSL}(2, \mathbb{C}))$-structure, i.e. it is an atlas modeled on $\hat{\mathbb{C}}$ with transition maps in $\text{PSL}(2, \mathbb{C})$. In particular $C$ is a refinement of a complex structure. In this paper all projective structures $C$ are marked by a homeomorphism $F \rightarrow C$. Then we can equivalently define a projective structure on $F$ as a pair $(f, \rho)$, where $f: \tilde{F} \rightarrow \hat{\mathbb{C}}$ is an immersion and $\rho: \pi_1(F) \rightarrow \text{PSL}(2, \mathbb{C})$ is a homomorphism such that $f$ is $\rho$-equivariant. The immersion $f$ is called the developing map, which we denote by $\text{dev}(C)$, and $\rho$ the holonomy representation of the projective structure $C$. The equivalence of projective structures on $F$ is given by the isotopies of $F$ and $(f, \rho) \sim (\gamma \circ f, \gamma \rho \gamma^{-1})$ for all $\gamma \in \text{PSL}(2, \mathbb{C})$.

2.2. Grafting. ([Gol87].) Let $C = (f, \rho)$ be a projective structure on $F$. A loop $\ell$ on $C$ is admissible if $\rho(\ell)$ is loxodromic and $f$ embeds $\tilde{\ell}$ into $\hat{\mathbb{C}}$, where $\tilde{\ell}$ is a lift of $\ell$ to $\tilde{F}$. Then $\rho(\ell)$ generates an infinite cyclic group in $\text{PSL}(2, \mathbb{C})$ and its domain of discontinuity is $\hat{\mathbb{C}}$ minus two points, the fixed points of $\rho(\ell)$. Thus its quotient is a two-dimensional torus enjoying a projective structure. Then $\ell$ is naturally embedded in $T_\ell$ as well. Therefore we can naturally combine two projective surfaces $C$ and $T_\ell$ by cutting and pasting along $\ell$, so that it results a new projective structure $\text{Gr}_\ell(C)$ on $F$. (Namely we identify boundary components $C \setminus \ell$ and $T_\ell \setminus \ell$ by the identification of $\ell$ on $C$ and on $T_\ell$ in an alternating manner.) Then it turns out $\rho$ is also the holonomy of $\text{Gr}_\ell(C)$. 
2.3. Measured laminations. Let $\tau$ be a hyperbolic surface, possibly with geodesic boundary, homeomorphic to a surface $F$. A geodesic lamination $\lambda$ is a set of disjoint geodesics whose union is a closed subset of $\tau$; then let $|\lambda|$ denote this closed subset. A geodesic lamination $\lambda$ is maximal if its complement is a union of disjoint ideal triangles. A stratum of $\lambda$ is either a leaf of $\lambda$ or the closure of a complementary region $\lambda$.

Let $A(\lambda)$ be the set of all smooth simple arcs $\alpha$ on $\tau$ that are transversal to $\lambda$, such that $\alpha$ contains its endpoints and the endpoints may be on leaves of $\lambda$ if $\alpha$ is not tangent to $\lambda$ there. A transversal measure on $\lambda$ is a function $\mu: \mathcal{A}(\lambda) \to \mathbb{R}_{\geq 0}$ defined for almost all arcs of $\mathcal{A}(\lambda)$ so that $\mu(\alpha)$ is invariant under any isotopy of $\alpha$ preserving the way $\alpha$ transversally intersects $\lambda$. For $p, q \in \tau$, if there is a unique geodesic segment connecting $p$ to $q$ then we let $\mu(p, q)$ denote the transversal measure of the segment. The measured lamination $L$ is a pair $(\lambda, \mu)$ of a geodesic lamination $\lambda$ and the transversal measure $\mu$ supported exactly on $\lambda$.

Let $\mathcal{T}M(\lambda)$ denote the set of all transversal measure supported on $\lambda$. Let $\mathcal{ML}(F)$ be the set of all measured laminations on $(F, \tau)$. Note that we do not need to specify $\tau$, since, for different hyperbolic structures $\tau$ on $F$, the corresponding spaces $\mathcal{ML}(F)$ are naturally isomorphic.

Suppose that $\lambda$ contains a closed geodesic $\ell$. Then $\ell$ carries an atomic measure (weight), which is a positive real number.

Let $\mathcal{A}(\lambda)$ be the subset of $\mathcal{A}(\lambda)$ consisting of arcs whose endpoints are not on leaves with positive atomic measure. The transversal measure $\mu$ is well-defined on $\mathcal{A}(\lambda)$.

For $\alpha \in \mathcal{A}(\lambda) \setminus \mathcal{A}(\lambda)$, we can naturally define its transversal measure $\mu(\alpha)$ to be a closed interval in $\mathbb{R}_{\geq 0}$, so that the width of the interval is the sum of the atomic measures on leaves through the endpoints of $\alpha$.

Let $(a_i) \subset \mathcal{A}(\lambda)$ be a sequence converging to $\alpha$ with $a_i \subset \alpha$. Similarly let $b_i \subset \mathcal{A}(\lambda)$ be a sequence converging to $\alpha$ with $\alpha \subset b_i$. Then the transversal measure $\mu(\alpha)$ is the closed interval

$$[\lim_{i \to \infty} \mu(a_i), \lim_{i \to \infty} \mu(b_i)].$$

In addition, we allow closed leaves $\ell$ of $\lambda$ to have weight infinity (heavy leaves), i.e. if $\alpha \in \mathcal{A}(\lambda)$ transversally intersects $\ell$, then $\mu(\alpha) = \infty$. If a measured lamination is heavy, if it has a leaf with weight infinitely.

2.4. Pleated surfaces. A continuous map $\beta: \mathbb{H}^2 \to \mathbb{H}^3$ is a pleated plane if there exists a geodesic lamination $\lambda$ on $\mathbb{H}^2$ such that
• for each stratum $P$ of $(\mathbb{H}^2, \lambda)$, the map $\beta$ isometrically embeds $P$ into a copy of $\mathbb{H}^2$ in $\mathbb{H}^3$, and
• $\beta$ preserves the length of paths (i.e. for every smooth path $\alpha$ on $\mathbb{H}^2$ its image $\beta(\alpha)$ has the same length in $\mathbb{H}^3$).

Then if we say that the geodesic lamination $\lambda$ is realized by the pleated surface $\beta$. In this paper, we in addition assume that the realizing lamination is minimal, i.e. there is no proper sublimation of $\lambda$ satisfying the two conditions above.

Suppose in addition that $\lambda$ is the total lift of a measured lamination $\nu$ on a complete hyperbolic surface $\tau$. Let $F$ be the underlying topological surface of $\tau$. Then the action of $\pi_1(F)$ on $\mathbb{H}^2$ preserves $\lambda$. Let $\rho: \pi_1(F) \to \text{PSL}(2,\mathbb{C})$ be a homomorphism. The pleated surface $\beta: \mathbb{H}^2 \to \mathbb{H}^3$ is $\rho$-equivariant if $\beta \circ \gamma = \rho(\gamma) \circ \beta$ for all $\gamma \in \pi_1(F)$. Then we say that the pair $(\tau, \nu)$ is realized by the $\rho$-equivariant pleated surface $\beta$.

**Definition 2.1.** Let $\psi: X \to Y$ be a map between metric spaces $(X,d_X)$ and $(Y,d_Y)$. Then, for $\epsilon > 0$, the map $\psi$ is an $\epsilon$-rough isometric embedding if
\[
d_Y(\psi(a), \psi(b)) - \epsilon < d_X(a,b) < d_Y(\psi(a), \psi(b)) + \epsilon,
\]
for all $a, b \in X$. Then $\psi$ is an $\epsilon$-rough isometry if, in addition, $Y$ is the $\epsilon$-neighborhood of the image of $\psi$.

Then, in the preceding paper, we proved

**Theorem 2.2 ([Bab], Theorem C).** Suppose that there are a representation $\rho: \pi_1(S) \to \text{PSL}(2,\mathbb{C})$ and a $\rho$-equivariant pleated surface $\beta_0: \mathbb{H}^2 \to \mathbb{H}^3$ realizing $(\sigma_0, \nu_0) \in \mathcal{T} \times \mathcal{G}$. Then, for every $\epsilon > 0$, there exists $\delta > 0$, such that if there is another $\rho$-equivariant pleated surface $\beta: \mathbb{H}^2 \to \mathbb{H}^3$ realizing $(\sigma, \nu) \in \mathcal{T} \times \mathcal{G}$ with $\angle_{\sigma_0}(\nu_0, \nu) < \delta$, then $\beta_0$ and $\beta$ are $\epsilon$-close: Namely there is an $\epsilon$-rough isometry $\psi: \sigma_0 \to \sigma$ such that $\beta_0$ and $\beta \circ \tilde{\psi}: \mathbb{H}^2 \to \mathbb{H}^3$ are $\epsilon$-close in the $C^0$-topology, where $\tilde{\psi}: \mathbb{H}^2 \to \mathbb{H}^2$ is the lift of $\psi$, and moreover in the $C^\infty$-topology in the complement of the $\epsilon$-neighborhood $|\nu| \cup |\nu_0|$ in $\sigma$.

2.5. **Traintracks.** (See [Kap01]. Also [PH92]) Given a rectangle $R$, pick a pair of opposite edges as horizontal edges and the other pair vertical edges. A (fat) traintrack $T$ is a collection $\{R_i\}_i$ of rectangles, called branches, embedded in a surface $F$ so that $R_i$ are disjoint except overlaps of their vertical edges in a particular manner: After dividing each vertical edge, if necessarily, into finitely many (sub-)edges, there is a unique bijective pairing of the vertical edges and homeomorphisms between paired edges that are induced by the overlaps of $R_i$ on $F$. 
The points dividing vertical edges are called branch points of $T$. Let $|T| \subset F$ denote the union of the branches $R_i$ over all $i$. Then the boundary of $|T|$ is the union of the horizontal edges of $R_i$.

In this paper we assume that the traintracks are at most trivalent. To be more precise, their corresponding traintrack graphs have degree two or three at each vertex: Each rectangle is foliated by vertical lines parallel to its vertical edges, and the rectangle projects onto a single edge by collapsing each vertical line to a point. Then accordingly a traintrack $T$ collapses to a unique graph by applying this projection to each branch of $T$.

A lamination $\lambda$ on $F$ is carried by a traintrack $T$ if

- $|\lambda| \subset |T|$,
- leaves of $\lambda$ are transversal to the vertical edges of the branches of $T$, and
- if $R$ is a branch of $T$, then $R \cap \lambda$ is a lamination on $R$ consisting of arcs property embedded in $R$ connecting both vertical edges of $R$;

then we say that $T$ is a traintrack neighborhood of $\lambda$.

In addition suppose that the surface $F$ is a hyperbolic surface and that the branches $R_i$ are smooth rectangles. Then the boundary of $|T|$ is the union of piecewise smooth curves, its non-smooth points are contained in the vertical edges of rectangles. In particular the branch points are non-smooth points.

For $\epsilon > 0$, a (smooth) traintrack $T = \{R_i\}$ is $\epsilon$-nearly straight, if each rectangle $R_i$ is smoothly $(1 + \epsilon)$-bilipschitz to a Euclidean rectangle and at each branch point, the angle of the boundary curve of $|T|$ is $\epsilon$-close to 0. For $K > 0$, a traintrack $T = \{R_i\}$ is $(\epsilon, K)$-nearly straight, if in addition, a horizontal edge of each branch has length at least $K$. Note that for fixed $K > 0$, if $\epsilon > 0$ is sufficiently small, every $(\epsilon, K)$-nearly straight traintrack is hausdorff close to a geodesic lamination.

A round cylinder in the Riemann sphere $\hat{\mathbb{C}}$ is a cylinder bounded by disjoint round circles. The axis of a round cylinder $A$ in $\hat{\mathbb{C}}$ is the geodesic in $\mathbb{H}^3$ orthogonal to both hyperbolic planes in bounded by the boundary circles of $A$. Then $A$ admits a canonical circular foliation by one parameter family of round circles bounding disjoint hyperbolic planes orthogonal to the axis of $A$. A smooth arc $\alpha$ property embedded in $A$ is supported on $A$ if $\alpha$ intersects each leaf of the circular foliation of $A$ transversally (in particular $\alpha$ connects the boundary components of $A$).
Let $C$ be a projective structure on a surface $F$. Suppose that $T = \{R_i\}$ is a smooth traintrack on $C$. Then $C$ induces a projective structure on each rectangle $R_i$. Then $\text{dev}(R_i)$ is an immersion of $R_i$ to $\hat{C}$, defined up to postcomposition with an element of $\text{PSL}(2, \mathbb{C})$.

Then a branch $R_i$ of $T$ is a supported on a round cylinder $A$ on $\hat{C}$ if $\text{dev}(R_i)$ maps into $A$, different vertical edges of $R_i$ immerse into different boundary circles of $A$, and the horizontal edges of $R_i$ are supposed on $A$. Then the circular foliation of $A$ induces a foliation on $R_i$ by circular arcs. Suppose that each branch $R_i$ of $T$ is supposed on a round cylinder. Then the circular foliations on the branches $R_i$ yield a foliation of $|T|$ by circular arcs. Moreover

**Lemma 2.3.** Let $T$ be a traintrack on a projective surface $C$ such that the branches on $T$ is supported on round cylinders on $\hat{C}$. Then, if a loop $\ell$ is carried by $T$, then $\ell$ is admissible. (Lemma 7.2 in [Bab].)

2.6. **Pants Graph.** ([HT80, Bro03]) Recall that $S$ is a closed oriented surface of genus $g \geq 2$. Then a maximal multiloop $M$ on $S$ is a multiloop such that $S \setminus M$ is a union of disjoint pairs of pants. Then $M$ consists of exactly $3(g-1)$ non-parallel loops. In this section multiloops represent their isotopy classes.

An elementary move transforms a maximal multiloop $M$ to a different maximal multiloop by removing a loop $\ell$ of $M$ and adding another loop $m$ disjoint from $M \setminus \ell$ and intersecting $\ell$ minimally, so that $m$ intersects $\ell$ in either one or two points. Then there is a unique connected component $F$ of $S$ minus the multiloop $M \setminus \ell$ such that $F$ contains $\ell$. Then either

- $F$ is a one-holed torus, and $m$ intersects $\ell$ in a single point; or
- $F$ is a four-holed sphere, and $m$ intersects $\ell$ in two points.

Consider the pants graph $\mathcal{P}G$ whose vertices bijectively correspond to (the isotopy classes of) the maximal multiloops on $S$ and the edges bijectively to the elementary moves between the vertices. Then $\mathcal{P}G$ is connected ([HT80]).

2.7. **Purely loxodromic representations.**

**Lemma 2.4.** Let $\rho : \pi_1(S) \to \text{PSL}(2, \mathbb{C})$ be a purely loxodromic representation. Then, if $\gamma, \eta \in \pi_1(S)$ are noncommuting elements, then $\rho(\gamma), \rho(\eta)$ are loxodromic elements and their translation axes share no endpoint.

**Proof.** Since $\gamma$ and $\eta$ do not commute, $\gamma, \eta \neq id$ and $[\gamma, \eta] \neq id$. Thus $\rho(\gamma)$ and $\rho(\eta)$ are loxodromics. Then their axes must be distinct: Otherwise, their commutator $[\rho(\gamma), \rho(\eta)] = id$, and thus $[\gamma, \eta]$ must map to
id under $\rho$. Suppose that their translation axes share exactly one endpoint. Then $\rho$ takes the commutator $[\gamma, \eta]$ to a parabolic element fixing the endpoint. This is a contradiction since $\rho$ is purely loxodromic. \[\square\]

Recall that $S$ is a closed oriented surface of genus at least two. Fix a generating set $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ of $\pi_1(S)$. Then, by the adjoint representation, $\text{PSL}(2, \mathbb{C})$ embeds into $\text{GL}(3, \mathbb{C})$ as a complex affine group (see [Kap01]). Since $\text{GL}(3, \mathbb{C}) \subset \mathbb{C}^9$, with respect to the generating set, a representation $\rho: \pi_1(S) \to \text{GL}$ corresponds to a point of $\text{PSL}(2, \mathbb{C})^m \subset \mathbb{C}^{9m}$. Thus we can regard the space $\mathcal{R}$ of representations $\rho: \pi_1(S) \to \text{GL}$ as an affine algebraic variety. This variety is called the $\text{PSL}(2, \mathbb{C})$-representation variety of $S$.

Then $\text{PSL}(2, \mathbb{C})$ acts on $\mathcal{R}$ by conjugation. This induces the algebra-geometric quotient, called the character variety $\chi$ ([BZ98]). Then $\chi$ has exactly two connected components ([Gol88]). Let $\chi_0$ be the component containing representations that lift to $\pi_1(S) \to \text{SL}(2, \mathbb{C})$. Then $\chi_0$ contains the quasifuchsian space.

For every $\gamma \in \pi_1(S)$, let $\text{Tr}_\gamma^2: \chi \to \mathbb{C}$ denote the square trace function of $\gamma$ given by $\rho \mapsto \text{Tr}_\gamma^2(\rho(\gamma))$. Then $\text{Tr}_\gamma^2$ is a regular function (see ([BZ98])). The singular part of the $\chi$ has complex codimension at least one, and the image of the holonomy map $\text{Hol}: \mathcal{P} \to \chi$ is contained in the smooth part of $\chi_0$.

\textbf{Lemma 2.5.} Almost all elements of $\chi_0$ is purely loxodromic.

\textit{Proof.} Since $\chi_0$ contains the quasifuchsian space, if $\gamma \in \pi_1(S) \setminus \{\text{id}\}$, then $\text{Tr}_\gamma^2$ is nonconstant on $\chi_0$. If $\text{Tr}_\gamma^2(\rho) = 4$ if $\rho(\gamma)$ is parabolic and $\text{Tr}_\gamma^2(\rho) \in [0, 4)$ if $\rho(\gamma)$ is elliptic. Since $[0, 4] \subset \mathbb{R}$ has measure zero in $\mathbb{C}$ and $\text{Tr}_\gamma^2$ is regular, almost every element of $\chi_0$ takes $\gamma$ to a loxodromic element. Since $\pi_1(S)$ contains only countably elements, we have the lemma. \[\square\]

\section*{2.8. Local characterization of projective structure in $\mathcal{G}L(S)$}

Let $\tau$ be a a hyperbolic surface homeomorphic. If two geodesics $\ell$ and $m$ on $\tau$ intersects at a point $p$, then let $\angle_p(\ell, m)$ denote the angle, taking a value in $[0, \pi/2]$, between $\ell$ and $m$ at $p$. Let $\lambda$ and $\nu$ are geodesic laminations on $\tau$. Then the \textit{angle} between $\lambda$ and $\nu$ is

$$\sup \angle_p(\ell, m),$$

where the supremum runs over all points $p \in |\lambda| \cap |\nu|$ and $\ell$ and $m$ are the leaves of $\lambda$ and $\nu$, respectively, intersecting at $p$.

Let $(\lambda_i)$ be a sequence of geodesic laminations on $\tau$. Suppose that $\angle_\tau(\lambda_i, \nu) \to 0$ as $i \to \infty$. Then this convergence does not depend on the choice of $\tau \in \mathcal{T}$. Therefore we may denote $\angle_\tau(\lambda_i, \nu)$ simply by
\(\angle(\lambda, \nu)\) dropping \(\tau\) — In this paper, typically we only need such an angle to be sufficiently small. If \(\sigma\) is a subsurface of \(\tau\), then let \(\angle_\sigma(\lambda, \nu)\) be \(\sup \angle_p(\ell, m)\) over all leaves \(\ell \in \lambda\) and \(m \in \nu\) intersecting at a point \(p\) contained in \(\sigma\).

Given measured geodesic laminations \(M\) and \(L\) on \((S, \tau)\), their angle \(\angle(\tau, M, L)\) is the angle of the underlying geodesic laminations \(|M|\) and \(|L|\).

Note that in order to measure such an angle, we always take geodesic representatives of laminations on a given hyperbolic surface.

**Theorem 2.6** ([Bab], Theorem B). Let \(C \cong (\tau, L)\) be a projective structure on \(S\). Then there exists \(\delta > 0\) such that, if there is another projective structure \(C' = (\tau', L')\) satisfies that \(\angle(\tau, L, L') < \delta\), then we can graft \(C\) and \(C'\) along multiloops to a common projective structure.

That is, there are admissible multiloops \(M\) on \(C\) and \(M'\) on \(C'\) such that

\[
\text{Gr}_M(C) \cong \text{Gr}_{M'}(C').
\]

### 3. Thurston coordinates on \(\mathcal{P}\)

The space \(\mathcal{P}\) of projective structures on \(S\) is naturally homeomorphic to the product of the Teichmüler space of \(S\), and the space of measured laminations on \(S\):

\[
\mathcal{P} \cong \mathcal{T} \times \mathcal{ML}
\]

(see [KT92, KP94]).

Let \(C = (f, \rho) \in \mathcal{P}\) and let \((\tau, L) \in \mathcal{T} \times \mathcal{ML}\) be the corresponding pair by (3). Then there is a measured lamination \(L = (\nu, \omega)\) on \(C\), called the **canonical lamination**, and a continuous map

\[
\kappa: C \to \tau,
\]

called **collapsing map**, such that \(\kappa\) descends \(\mathcal{L}\) to \(L\) and preserved the marking. The leaves of \(\mathcal{L}\) are circular and they have no atomic measure. The collapsing map \(\kappa\) takes each stratum of \((C, \mathcal{L})\) diffeomorphically onto a stratum of \(\mathcal{L}\). If \(\ell\) is a closed leaves of \(L\), which carries positive atomic measure, then \(\kappa^{-1}(\ell)\) is a cylinder \(A_\ell\) foliated by closed leaves of \(\mathcal{L}\). Conversely \(\kappa\) takes those leaves onto \(\ell\) diffeomorphically.

Considering all closed leaves \(\ell\) of \(L\), then we have disjoint cylinders \(A_\ell\) on \(C\); let \(A\) be their union. The **Thurston metric** on \(C\) is the metric given by a canonical Euclidean metric on \(A\) and hyperbolic metric on \(C \setminus A\): With the Euclidean metric on \(A_\ell\), each leaf of \(\mathcal{L}\) in \(A_\ell\) is a closed geodesic whose length is the length of \(\ell\) and the height of the cylinder is the weight of \(\ell\) (given by \(\mu\)). The hyperbolic metric on \(C \setminus A\) is
given as the pullback metric given via $\kappa$. Indeed, the collapsing map $\kappa$ is, on $C \setminus \mathcal{A}$, a $C^1$-diffeomorphism onto its image in $\tau$. In particular, all strata of $(C, \mathcal{L})$ not in $\mathcal{A}$ bijectively correspond to strata of $(\tau, L)$ minus closed leaves of $L$ via $\kappa$.

Naturally a cover of $(\tau, L)$ gives the Thurston coordinates of the corresponding cover of $C$ — in particular this includes the case of their universal covers, which are homeomorphic to open disks. More generally, projective structures on an open disk admits Thurston coordinates unless it is isomorphic to $\mathbb{C}$ ($\S$3.1). We shall define Thurston coordinates for open disks.

Let $X$ be a convex subset of $\mathbb{H}^2$ bounded by disjoint geodesics (possibly $X = \mathbb{H}^2$ or a geodesic segment). We suppose that each boundary geodesic of $X$ is either contained in $X$ or its complement $\mathbb{H}^2 \setminus X$. Let $L$ be a measured lamination on $X$. Then $L$ induces a pleated surface $\beta: X \to \mathbb{H}^3$ by bending $X$ inside $\mathbb{H}^3$ along the geodesic lamination $|L|$ so that the amount of this bending is given by the transversal measure of $L$. Then $\beta$ is unique up to a post composition with an element of $\text{PSL}(2, \mathbb{C})$. In addition we suppose that, given any boundary geodesic $\ell$ of $X$,

- if $\ell$ is contained in $X$, then $\ell$ has weight infinity, and
- otherwise, the transversal measure given by $L$ is infinite “near $\ell$", i.e. if $\alpha$ is an arc in $X$ transversal to $L$ with an (open) end point on $\ell$, then the transversal measure of $\alpha$ is infinite.

Then there is a projective structure $C$ on an open disk corresponding to $(\mathbb{H}^2, L)$. Then $(X, L)$ is called the Thurston coordinates of $C$, and we write $C \cong (X, L)$. Similarly there is a measured lamination $\mathcal{L}$ on $C$ (canonical lamination) and a collapsing map $\kappa: C \to \mathbb{H}^2$ that descends $\mathcal{L}$ to $L$.

Similarly $C$ enjoys Thurston metric. If a leaf $\ell$ of $L$ has atomic measure, then $\kappa^{-1}(\ell)$ is isometric to an infinite strip $[0, w(\ell)] \times \mathbb{R}$ in the Euclidean plane $\mathbb{R}^2$. Then $\kappa$ collapses each horizontal segment $[0, w(\ell)] \times \{r\}$ to a point. Each vertical line $\{h\} \times \mathbb{R}$ is a leaf of $\mathcal{L}$. In the complement of the union of such Euclidean regions $\kappa^{-1}(\ell)$, Thurston metric is hyperbolic so that $\kappa$ is an isometric embedding on to $X$ minus the leaves with positive weights, and this embedding preserves the measured lamination.

Suppose in addition that there is a group $\Gamma$ acting on $X$ freely and property discontinuously preserving $L$. Then accordingly $\Gamma$ acts on $C$ preserving $\mathcal{L}$, and the action commutes with $\kappa$. Then the quotient $(X/\Gamma, L/\Gamma)$ is Thurston coordinates of the quotient projective surface $C/\Gamma$. 
Let $C$ be a projective structure on an open disk. Let $f : C \to \hat{C}$ be $\text{dev}(C)$. Fix a natural spherical metric on $\hat{C}$ unique up to an element of $\text{PSL}(2, \mathbb{C})$. Pullback this metric to $C$ by $f$ and obtain an incomplete spherical metric on $C$. The metric completion of $C$ minus $C$ is called the ideal boundary of $C$ and denoted by $\partial_\infty C$. Note that this completion topologically does not depend on the choice of the spherical metric on $\hat{C}$.

A maximal ball in $C$ is a topological open ball $B$ such that

- $B$ is round, i.e. $f$ embeds $B$ onto a round open ball in $\hat{C}$, and
- $B$ is maximal, i.e. there is no round open ball in $C$ strictly containing $B$.

The idea boundary of a maximal ball $B$ in $C$ is the intersection of $\partial_\infty C$ and $\partial B$ in the metric completion $C \cup \partial_\infty C$. The core $\text{Core}(B)$ of the maximal ball $B$ is, by conformally identifying $B$ with $\mathbb{H}^2$, the convex hull of the ideal boundary of $B$. Then, consider all maximal balls in $C$; then the corresponding cores yield the stratification of $(C, \mathcal{L})$. In other words, for every point $p \in C$, there is a unique maximal ball $B$ in $C$ such that $p \in \text{Core}(B)$. Then we say that $B$ is the maximal ball centered at $p$.

### 3.1. Existence of Thurston coordinates on disks

[KP94, Theorem 11.6] implies

**Theorem 3.1.** Let $C$ be a projective structure on a simply connected surface with $C \neq \mathbb{C}, \hat{C}$. Then $C$ admits unique Thurston coordinates $(X, L)$, where $X$ is a closed convex subset of $\mathbb{H}^2$ bounded by geodesics, and $L$ is a measured lamination on $X$. Each boundary geodesic of $X$ is either entirely contained in or disjoint from $X$ (closed or open) and the closed boundary geodesics of $X$ are the only leaves of $L$ with weight infinity.

**Corollary 3.2.** Let $F$ be a connected surface without boundary. Let $C$ be a projective structure on $F$ with holonomy $\rho : \pi_1(F) \to \text{PSL}(2, \mathbb{C})$. Suppose that $\text{Im} \rho$ is non-elementary. Then $C$ enjoys Thurston coordinates

$$C \cong (\tau, L),$$

where $\tau$ is a convex hyperbolic surface possibly with geodesic boundary such that the interior of $\tau$ is homeomorphic to $F$ and $L$ is a measured geodesic lamination on $\tau$. In addition each boundary component of $\tau$ is either open or closed, and thus the closed boundary components of $\tau$ are the only leaves of $L$ with weight infinity.
Remark 3.3. If $\text{Im}\rho$ is elementary but the limit set of $\text{Im}\rho$ has cardinality two, then $C$ still enjoys Thurston coordinates $(\tau, L)$. However $\tau$ may be a single close geodesic with weight infinity, and the interior of $\tau$ is not homeomorphic to $F$. Nonetheless the regular neighborhood of $\tau$ is still homeomorphic to $F$.

Proof. Since the limit set of $\text{Im}(\rho)$ has cardinality more than one, the universal cover of $C$ can not be isomorphic to $\mathbb{C}$ or $\hat{\mathbb{C}}$. Thus applying Theorem 3.1, we obtain the Thurston coordinates $(\tilde{\tau}, \tilde{L})$ of the universal cover of $C$ so that, if exists, the boundary geodesics of $\tilde{\tau}$ are the only leaves of $L$ with weight infinity. Then $\pi_1(F)$ acts on $\tilde{\tau}$. Then $\tilde{\tau}$ is the convex hull of the limit set of $\pi_1(F)$. Thus, since $\text{Im}(\rho)$ is non-elementary, $\tilde{\tau}$ has interior whose closure is $\tilde{\tau}$. Since $\pi_1(F)$ preserves $(\tilde{\tau}, \tilde{L})$, it descends to the Thurston coordinates $(\tau, L)$ of $C$. Then the interior of $\tau$ is homeomorphic to $F$. □

3.2. Canonical neighborhoods. Suppose that $C$ is a projective structure on an open disk with $C \neq \mathbb{C}$. Then let $(X, L)$ denote its Thurston coordinates, where $X$ is a convex subset of $\mathbb{H}^2$ bounded by geodesics and $L$ is a (possibly heavy) measured lamination on $X$ (Proposition 3.1). Let $\beta : X \to \mathbb{H}^3$ be the corresponding pleated surface and $\kappa : C \to X$ the collapsing map. Let $\mathcal{L}$ be the measured lamination on $C$ that descends to $L$ by $\kappa$.

Let $p$ be a point on $C$, and let $B(p)$ be the maximal ball in $C$ centered at $p$. Let $U(p)$ denote the union of all maximal balls in $C$ containing $p$.

Let $U(p)$ denote the union of all maximal balls in $C$ contain $p$. Then $U(p)$ is called the canonical neighborhood of $p$ in $C$. It turns out that $U(p)$ is homeomorphic to an open disk and $dev(C)$ embeds $U(p)$ into $\hat{\mathbb{C}}$ (see [KP94] [KT92]). Note since $C \neq \mathbb{C}$, then $U(p) \neq \mathbb{C}$. The idea point of $U(p)$ is the intersection of $\partial_\infty C$ and the closure of $U(p)$ in the completion $C \cup \partial_\infty C$.

Since $C$ is homeomorphic to an open disc, by collapsing each stratum of $(C, \mathcal{L})$ to a point, we obtain a natural quotient map $\Psi : C \to T$ onto a metric $\mathbb{R}$-tree $T$ (see [Kap01]). The metric of $T$ is induced by the transversal measure of $\mathcal{L}$.

Noting that $B(p) \subset U(p)$ and $U(p)$ is embedded in $\hat{\mathbb{C}}$, we have

Lemma 3.4. For $p \in C$, if a neighborhood $V_p$ of $\Psi(p)$ in $T$ is contained in $\Psi(U(p))$, then the ideal boundary $\partial_\infty B(p)$ is the boundary circle $\partial B(p)$ minus the union of maximal balls of $C$ whose cores map into $V_p$ by $\Psi$. 
Proof. For \( x \in C \), if \( \Psi(x) \in V_p \), then \( B(x) \) is contained in \( U(p) \). Then \( B(x) \) contains \( p \). The ideal boundary \( \partial_\infty B(p) \) is naturally embedded in the boundary of \( U(p) \) in \( \hat{C} \), and thus it is disjoint from \( B(x) \) in \( \hat{C} \). Therefore \( \partial_\infty B(p) \) is contained in \( \partial B(p) \setminus \bigcup_x B(x) \) over all \( x \in C \) with \( \Psi(x) \in V_p \).

To show the opposite inclusion, let \( s \) be the connected component of \( \partial B(p) \setminus \partial_\infty B(p) \) (Figure 1). Then \( s \) is a circular arc on \( \hat{C} \) with open ends. Then there is a unique leaf \( \ell \) of \( L \) connecting the endpoints of \( s \) — namely \( \ell \) is a boundary leaf of \( \text{Core } B(p) \).

Consider the connected component of \( C \setminus \text{Core } B(x) \) bounded by \( \ell \). Pick a sequence of points \( x_i \) in the component limiting to an interior point of \( \ell \). Then \( B(x_i) \) converges to \( B(p) \) as \( i \to \infty \), and \( B(x_i) \cap s \) converges to \( s \). Since \( \Psi \) takes \( \text{Core } B(x_i) \) to a point in \( V_p \) for sufficiently large \( i \), we have \( s \subset \bigcup_x B(x) \) over \( x \in C \) with \( \Psi(x) \in V_p \). \( \square \)

![Figure 1.](image-url)

Then let \((\mathbb{H}^2, L_p)\) be the Thurston coordinates of \( U(p) \). Let \( \beta_p : \mathbb{H}^2 \to \mathbb{H}^3 \) be the pleated surface corresponding to \((\mathbb{H}^2, L_p)\). Let \( L_p \) be the circular measured lamination on \( U(p) \) that descends to \( L_p \) by the collapsing map \( \kappa_p : U(p) \to \mathbb{H}^3 \).

Let \( W(p) \) be the union of \( \text{Core}(B(x)) \) for all \( x \in C \) with \( p \in B(x) \). Then \( W(p) \) is the open neighborhood of \( p \) bounded by the leaves \( \ell \) of \( L \) such that their corresponding maximal balls \( B_\ell \) (i.e. \( \text{Core}(B_\ell) = \ell \)) satisfy \( \partial B_\ell \ni p \). If \( B_1 \) and \( B_2 \) are different maximal balls in \( C \), then \( B_1 \) intersects exactly one connected component of \( C \setminus B_2 \); this implies \( W(p) \) is connected.

By \( W(p) \subset U(p) \subset C \), we have
Proposition 3.5.  
• In $W(p)$, $\mathcal{L}_p$ is isomorphic to $\mathcal{L}$ and the Thurston metric on $U(p)$ is isometric to that on $C$.
• There exists a natural isometry $\psi: \kappa_p(W) \to \kappa(W)$ such that $\psi \circ \kappa_p = \kappa$ on $W$ and $\beta \circ \psi = \beta_p$ on $\kappa_p(W)$.

Proof. Recall that $U(p) = \cup_x B(x)$ where $x$ runs over all points $C$ with $p \in B(x)$. Thus such $B(x)$ is a maximal ball not only in $C$ but also in $U(p)$. Since $W(p)$ is connected and it contains no boundary leaves (Figure 2), $\Psi(W(p))$ is an open connected subset of $T$. Therefore, by Lemma 3.4, if $q \in W(p)$, the ideal boundary of the maximal ball $B(q)$ in $C$ is equal to that in $U(p)$. Since the maximal balls and their ideal boundary determine the canonical laminations, $\mathcal{L}$ is isomorphic to $\mathcal{L}_p$ on $W(p)$ by the inclusion $U_p \subset C$. The second assertion similarly holds.

Figure 2.
Part 1. Grafting Conjecture for purely loxodromic holonomy

4. Sequence of pleated surfaces

Fix a purely loxodromic representation $\rho: \pi_1(S) \to \text{PSL}(2, \mathbb{C})$. Let $C_\sharp \cong (\tau_\sharp, L_\sharp)$ and $C_\flat \cong (\tau_\flat, L_\flat)$ be projective structures with holonomy $\rho$. In this section, we construct an infinite family of pleated surfaces, in a coarse sense, connecting the pleated surfaces corresponding to $C_\sharp$ and $C_\flat$. Although $\rho$ is not necessarily discrete or faithful, our ideas here are closely related to families of pleated surfaces sweeping out (some regions of) hyperbolic three-manifolds. In particular, there is a corresponding family of geodesic laminations on $S$ realized by the pleated surfaces. Then this family is similar to the hierarchy construction, due to Maru and Minsky [MM00], of coarse geodesics in curve complexes.

Pick maximal multiloops $M_\sharp$ and $M_\flat$ on $C_\sharp$ and $C_\flat$, respectively, so that

1. $\angle_{\tau_\sharp}(M_\sharp, L_\sharp)$ and $\angle_{\tau_\flat}(M_\flat, L_\flat)$ are sufficiently small, and
2. sufficiently small neighborhoods of $M_\sharp$ and $M_\flat$ contain $L_\sharp$ and $L_\flat$, respectively (on $\tau_\sharp$ and $\tau_\flat$).

Since the pants graph of $S$ is connected ($\S\text{2.6}$), there is a simplicial path in the graph connecting $M_\sharp$ and $M_\flat$. Let $(M_i)_{i=0}^n$ be the corresponding sequence of maximal multiloops on $S$ with $M_0 = M_\sharp$ and $M_n = M_\flat$, so that $M_i$ and $M_{i+1}$ are adjacent vertices of the pants graph for all $i = 0, \ldots, n-1$.

Each connected component $P$ of $S \setminus M_i$ is a pair of pants. Pick a maximal geodesic lamination on $P$; then it consists of three isolated geodesics and each geodesic ray in the lamination (half-leaf) is asymptomatic to a boundary component of $P$, spiraling towards it. We can pick the maximal lamination so that such half-leaves spiral towards left when they approaches towards boundary components (with respect to the orientation of $S$) and, each leaf of the lamination is asymptotic to different boundary components of $P$ depending on the direction of the half-leaf in it. For each $i$, let $\nu_i$ be the maximal lamination of $S$ that is the union of the maximal multiloop $M_i$ and the above maximal laminations on all connected components $P$ of $S \setminus M_i$.

The lamination $\nu_i$ is obtained as the Hausdorff limit of the iteration of the left Dehn twist along $M_i$ of some multiloop $N_i$ on $S$. Indeed we can take the multiloop $N_i$ so that the restriction of $N_i$ to each connected component $P$ of $S \setminus M_i$ is a union of three non-parallel arcs connecting all pairs of boundary components of $P$. Furthermore, for every $k \in \mathbb{Z}_{>0}$, by taking $k$ parallel copies of the arcs on all $P$, we
can also take $N_i$ such that the number of the arcs of $N_i|P$ is $3k$ for all connected components $P$ of $S \setminus M_i$.

The following lemma guarantees that $\nu_i$ is realized by a unique $\rho$-equivariant pleated surface.

**Lemma 4.1.** Suppose that $\rho: \pi_1(S) \rightarrow \text{PSL}(2, \mathbb{C})$ is purely loxodromic. Let $\nu$ be a geodesic lamination on $S$ such that

- $\nu$ is maximal, and
- every half-leaf of $\nu$ accumulates to a closed leaf of $\nu$.

Then there is a unique $\rho$-equivariant pleated surface realizing $\nu$.

**Proof.** Let $\Delta$ be a connected component of $S \setminus |\nu|$, which is an ideal triangle. Let $\tilde{S}$ be the universal cover of $S$, and let $\tilde{\nu}$ be the total lift of $\nu$ to $\tilde{S}$. Let $\tilde{\Delta}$ be a lift of $\Delta$ to $\tilde{S}$. Then $\tilde{\Delta}$ is an ideal triangle property embedded in $\tilde{S}$, and its vertices are at distinct points on the circle at infinity $\partial_{\infty} S$.

By the second assumption, each “vertex” of $\Delta$ corresponds to a closed leaf of $\nu$. This loop lifts to a unique leaf of $\tilde{\nu}$ whose endpoint is the corresponding vertex of $\tilde{\Delta}$ in the boundary circle of $\tilde{S}$ at infinity. Different vertices of $\tilde{\Delta}$ correspond to different leaves of $\tilde{\nu}$. Then the vertices are naturally fixed points of different elements of $\pi_1(S) \setminus \{\text{id}\}$. Since $\rho$ is purely loxodromic, by Lemma 2.4, for different elements of $\pi_1(S) \setminus \{\text{id}\}$, their corresponding loxodromics share no fixed point. Then the vertices of $\tilde{\Delta}$ corresponds to different points on $\hat{\mathbb{C}}$ fixed by different loxodromics, and they spans a unique ideal triangle in $\mathbb{H}^3$.

This correspondence defines a $\rho$-equivariant map $\beta$ from $\tilde{S} \setminus |\tilde{\nu}|$ to $\mathbb{H}^3$. If a leaf of $\tilde{\nu}$ is an isolated leaf, then it separates adjacent complementary ideal triangles, and otherwise, it covers a closed leaf of $\nu$ by the second assumption. Each leaf $\ell$ of $\tilde{\nu}$ either separates adjacent ideal triangles or descends to a closed leaf of $\nu$ on $S$. Clearly the $\rho$-equivariant map continuously extends to the leaves of $\tilde{\nu}$ of the first type. If a leaf $\ell$ of $\tilde{\nu}$ is not isolated, there is a sequence $\{\Delta_i\}$ of ideal triangles of $\tilde{S} \setminus |\tilde{\nu}|$ that converges to $\ell$ uniformly on compacts (in the Hausdorff topology). Then, by the second assumption, if $i \in \mathbb{N}$ is sufficiently large, a vertex of $\Delta_i$ must coincide with an endpoint of $\ell$. Noting that $\tilde{S} \setminus |\tilde{\nu}|$ has only finitely many components, by the equivalency, we see that $\beta(\Delta_i)$ converges to the geodesic axis of the loxodromic corresponding to $\ell$. Therefore we can continuously extend $\beta$ to the leaves of $\tilde{\nu}$ covering closed leaves of $\nu$ and obtain a desired a $\rho$-equivariant pleated surface $\mathbb{H}^2 \rightarrow \mathbb{H}^3$ realizing $\lambda$. □

**4.1. Bi-infinite Sequence of geodesic laminations connecting $\nu_i$ to $\nu_{i+1}$.** Recall that for $i \in \{0, \cdots, n-1\}$, $M_i$ and $M_{i+1}$ are maximal
multiloops on $S$ that are adjacent vertices on the pants graph of $S$. Then let $m_i$ and $m_{i+1}$ be the loops of $M_i$ and $M_{i+1}$, respectively, such that $M_i \setminus m_i = M_{i+1} \setminus m_{i+1}$. Let $F_i$ denote the minimal subsurface of $S$ containing both $m_i$ and $m_{i+1}$, which is either a one-holed torus or a four-holed sphere ($\S 2.6$).

Case One. First suppose that $F_i$ is a one-holed torus. Let $\hat{F}_i$ be the once-punctured torus obtained by pinching the boundary component of $F_i$ to a point. Then every geodesic lamination on $F_i$ descends a unique geodesic lamination on $\hat{F}_i$. In particular the geodesic laminations $\nu_i$ and $\nu_{i+1}$ on $S$ descend to geodesic laminations on $F$, then further to unique laminations $\hat{\nu}_i$ and $\hat{\nu}_{i+1}$, respectively, on $\hat{F}_i$. Then both $\hat{\nu}_i$ and $\hat{\nu}_{i+1}$ are maximal (on $\hat{F}_i$).

Let $T$ be the trivalent tree dual to the Farey tessellation (see for example [Bon09]). Then the vertices of $T$ bijectively correspond to the ideal triangulations of $\hat{F}_i$ and the edges to diagonal exchanges of the ideal triangulations.

Pick a maximal lamination $\hat{\nu}_{i,0}$ of $\hat{F}_i$ that least intersects $m_i$ and $m_{i+1}$ — each leaf of $\hat{\nu}_{i,0}$ intersects $m_i$ and $m_{i+1}$ at most in a single point (Figure 3). Then $\hat{\nu}_i$ is obtained by the infinite iteration of the Dehn twist of $\hat{\nu}_{i,0}$ along $m_i$ and similarly $\hat{\nu}_{i+1}$ by the infinite iteration of the Dehn twist along $m_{i+1}$. The Dehn twists along $m_i$ and $m_{i+1}$ each correspond to a composition of two diagonal exchanges of $\hat{\nu}_{i,0}$. Then we obtain a bi-infinite path in $T$ connecting $\hat{\nu}_i$ to $\hat{\nu}_{i+1}$. Then there is a corresponding bi-infinite sequence $(\hat{\nu}_{i,j})_j$ of maximal laminations on $\hat{F}_i$ that converges to $\hat{\nu}_i$ as $j \to -\infty$ and to $\hat{\nu}_{i+1}$ as $j \to \infty$. Let $\nu_{i,j}$ ($j \in \mathbb{Z}$)

![Figure 3. An ideal triangulation of a once-punctured torus whose edges least intersect $m_i$ and $m_{i+1}$.](image)

be the corresponding maximal laminations on $S$ such that

(i) the restriction of $\nu_{i,j}$ to $F_i$ descends to $\hat{\nu}_{i,j}$ on $\hat{F}_i$, and
(ii) $\nu_{i,j}$ is isomorphic to $\nu_i$ (and $\nu_{i+1}$) in some neighborhood, in $S$, of the closure of $S \setminus F_i$.

In (ii), we need take a neighborhood so that $\nu_{i,j}$ spirals to the left towards $\partial F_i$. Then $\nu_{i,j}$ is a bi-infinite sequence of maximal laminations of $S$ that converges to $\nu_i$ as $i \to -\infty$ and to $\nu_{i+1}$ as $i \to \infty$.

**Case Two.** Suppose that $F_i$ is a four-holed sphere. Similarly let $\hat{F}_i$ be the four-punctured sphere obtained by pinching the boundary components of $F_i$. Then consider the graph $T$ associated with $\hat{F}_i$ such that

- the vertices of $T$ bijectively correspond to the ideal triangulations of $\hat{F}_i$ isomorphic to the triangulation of the boundary of a tetrahedron (see Figure 4) and
- there is a (unique) edge between two vertices of $T$ if and only if the triangulation corresponding to one vertex is obtained form the other by simultaneous (two) diagonal exchanges of opposite edges.

Similarly to Case One, we have

**Lemma 4.2.** This graph $T$ is dual to the Farey tessellation.

**Proof.** Each vertex of $T$ corresponds to a unique ideal triangulation of $\hat{F}_i$. Then for each pair of opposite edges of the triangulation, there is an unique essential loop on $\hat{F}_i$ disjoint from the edges. There are exactly three pairs of opposite edges and their corresponding loops are maximal mutually adjacent vertices of the curve graph of $\hat{F}_i$ (Figure
The curve graph of the four-punctures sphere is the Farey graph ([Sau, §5]). Thus each ideal triangulation of $\hat{F}$ corresponds to a unique triangle of the Farey tessellation, and therefore $T$ is isomorphic to the graph dual to the Farey tessellation. □

The maximal laminations $\hat{\nu}_i$ and $\hat{\nu}_{i+1}$ are distinct endpoints of $T$ at infinity. Then similarly there is a unique bi-infinite sequence $(\hat{\nu}_{i,j})_{j \in \mathbb{Z}}$ of adjacent vertices of $T$ connecting $\hat{\nu}_i$ to $\hat{\nu}_{i+1}$. Indeed there is a vertex of $T$ such that its corresponding triangulation of $\hat{F}_i$ contains two edges disjoint from $m_i$ and two edges disjoint from $m_{i+1}$; then $\hat{\nu}_i$ is obtained by the infinite iteration of the left Dehn twist along $m_i$ and $\hat{\nu}_{i+1}$ by the left Dehn twists along $m_{i+1}$. Then similarly let $\nu_{i,j}$ be the lamination on $S$ satisfying (i) and (ii) in Case One.

In either Case One or Two, we have constructed the sequence $\{\nu_{i,j}\}$ of maximal geodesic laminations on $S$ connecting $\nu_i$ to $\nu_{i+1}$, that is, $\{\nu_{i,j}\}$ converges to $\nu_i$ as $j \to -\infty$ and $\nu_{i+1}$ as $j \to \infty$. Then, by Lemma 4.1, for every $i \in \{0, \ldots, n-1\}$ and $j \in \mathbb{Z}$, there is a $\rho$-equivariant pleated surface $\beta_{i,j}: \hat{S} \to \mathbb{H}^3$ realizing $\nu_{i,j}$. Then, by Theorem 2.2, the convergence of $\nu_{i,j}$ immediately implies

**Proposition 4.3.** For each $i = 0, 1, \ldots, n-1$, the pleated surface $\beta_{i,j}$ converges to $\beta_i$ as $j \to \infty$ and to $\beta_i$ as $j \to -\infty$ (in term of the closeness defined in Theorem 2.2).

Recall that $\nu_i$ is obtained by the infinite iteration of the left Dehn twist of a multiloop $N_i$ along $M_i$. Noting that $M_i \cap M_{i+1}$ is the set of common loops of $M_i$ and $M_{i+1}$, we similarly have

**Lemma 4.4.** For all $i \in \{1, \ldots, n-1\}$ and $j \in \mathbb{Z}$, the lamination $\nu_{i,j}$ is obtained by the infinite iteration of the left Dehn twist of some multiloop on $S$ along $M_i \cap M_{i+1}$.

**Proof.** Recall that, for every $i \in \{1, \ldots, n-1\}$ and every positive integer $k$, we can take the multiloop $N_i$ inducing $\nu_i$ such that, if $P$ is a complementary pants of $M_i$ in $S$, then, for each pair of different boundary components of $P$, the multicurc $N_i \cap P$ contains exactly $k$ parallel arcs of connecting the components. (Thus we can assume that $N_i$ is given by $k = 3$ in Case One and $k = 2$ for Case Two.)

Thus, for each $j \in \mathbb{Z}$, with an appropriate $k$, we can construct a multiloop $N_{i,j}$ such that $N_{i,j} = N_i$ in $S \setminus F_i$ and the restriction of $N_{i,j}$ to $F_i$ induces $\hat{\nu}_{i,j}$ on $\hat{F}_i$ (see Figure 5 for an example of $N_{i,j}$ with $k = 2$ in Case Two). □
5. Existence of admissible loops

**Theorem 5.1** (c.f. §7 in [Bab]). Let $\rho: \pi_1(S) \to \text{PSL}(2, \mathbb{C})$ be a homomorphism. Let $\beta: \mathbb{H}^2 \to \mathbb{H}^3$ be a $\rho$-equivariant pleated surface realizing $(\sigma, \nu) \in \mathcal{T} \times \mathfrak{G}. \mathcal{L}$. Then there exists $\delta > 0$ such that, if a projective structure $C = (\tau, L)$ with holonomy $\rho$ satisfies $\angle(L, \nu) < \delta$ and a loop $\ell$ on $S$ satisfies $\angle(\ell, |L|) < \delta$, then there is an admissible loop on $C$ isotopic to $\ell$.

**Proof.** Suppose that Theorem 5.1 fails. Then there exist a sequence of projective structures $C_i \cong (\tau_i, L_i)$ with holonomy $\rho$ such that, letting $\lambda_i = |L_i|$, we have $\angle(\lambda_i, \nu) \to 0$ as $i \to \infty$ and a sequence of geodesic loops $\ell_i$ on $\tau_i$ with $\angle(\ell_i, \lambda_i) \to 0$ as $i \to \infty$ such that there is no admissible loop on $C_i$ homotopic to $\ell_i$. Then by Theorem 2.2, there are marking-preserving bilipschitz maps $\psi_i: \sigma \to \tau_i$ converging to an isometry as $i \to \infty$ and $\beta$ is the limit of the pleated surfaces $\beta_i$ corresponding to $C_i$ via $\psi_i$.

Since $\mathfrak{G}. \mathcal{L}$ is compact in the Chabauty topology, by taking a subsequence if necessarily, we can in addition assume that $\lambda_i$ converges to a geodesic lamination $\lambda_\infty$ on $S$. Since $\beta_i \to \beta$, thus $\nu$ is a sublamination of $\lambda_\infty$. We can also assume that the sequence of geodesic loops $\ell_i$ converges to a geodesic lamination $\ell_\infty$, taking a subsequence if necessarily. Then $\angle(\ell_\infty, \lambda_\infty) = 0$. Thus $\ell_\infty \cup \lambda_\infty$ is a geodesic lamination on $\sigma$.

There is a constant $K > 0$ depending on $(\sigma, \ell_\infty \cup \lambda_\infty)$, such that for every $\epsilon > 0$, there is an $(\epsilon, K)$-nearly straight traintrack neighborhood $T_\epsilon$ of $\ell_\infty \cup \lambda_\infty$ ([Bab, Lemma 7.8]). Then, if $i \in N$ is sufficiently large, by the above convergences, there is a sufficiently small isotopy of $\psi_i(T_\epsilon)$
into an \((\epsilon, K)\)-nearly straight traintrack \(T_i\) on \(\tau_i\) that carries both \(\lambda_i\) and \(\ell_i\).

For each \(i \in \mathbb{N}\), let \(\kappa_i: C_i \to \tau_i\) denote the collapsing map, and let \(L_i\) be the circular measured lamination on \(C\) that descends to \(L\) via \(\kappa_i\). By [Bab, Proposition 6.13], for sufficiently large \(i\), there is a corresponding traintrack \(T_i\) on \(C_i\) diffeomorphic to \(T_i\) such that

- each branch of \(T_i\) is supported on a round cylinder on \(\hat{C}\),
- \(\kappa_i(\mid T_i \mid) = T_i\), and
- \(\kappa_i(\mid T_i \mid)\) is \(\epsilon\)-close to \(T_i\), i.e. the \(\kappa_i\)-image of each branch \(T_i\) is \(\epsilon\)-close to a corresponding branch of \(T_i\) in the Hausdorff metric on \(\tau_i\).

Then \(T_i\) carries the measured lamination \(L_i\). Since \(T_i\) carries \(\ell_i\), thus \(T_i\) carries a corresponding loop \(m_i\) so that a small homotopy changes \(\kappa_i\mid m_i \mid\) into \(\ell\) in \(\mid T_i \mid\). Since its branches are supported on cylinders, \(T_i\) is foliated by vertical circular arcs (§2.5). Then since \(m_i\) is carried by \(T_i\), we can (further) isotope \(m_i\), through loops carried by \(T_i\), so that \(m_i\) is transversal to this foliation of \(T_i\). Then by Lemma 2.3, \(m_i\) is admissible which is a contradiction.

Theorem 5.1 immediately implies

**Corollary 5.2.** Let \(C \cong (\tau, L)\) be a projective structure on \(S\). Then there exists \(\epsilon > 0\) such that, if a geodesic loop \(\ell\) on \(\tau\) satisfies \(\angle(\lambda, \ell) < \epsilon\), then there is an admissible loop on \(C\) isotopic to \(\ell\).

In addition

**Corollary 5.3.** For every a projective structure \(C \cong (\tau, L)\) on \(S\), there exists sufficiently small \(\epsilon > 0\) such that, if a geodesic multiloop \(M\) on \(\tau\) satisfies

1. \(\angle(\lambda, M, L) < \epsilon\) and
2. \(|L|\) is contained in the \(\epsilon\)-neighborhood of \(M\) in \(\tau\),

then every loop \(\ell\) on \(C\) satisfying \(\angle(\lambda, M, \ell) < \epsilon\) is isotopic to an admissible loop.

**Proof.** Let \(L = (\lambda, \mu)\) denote the measured lamination, where \(\lambda \in \mathfrak{S}\mathcal{L}\) and \(\mu \in \mathcal{T}\mathcal{M}(\lambda)\). Then, for every \(\delta > 0\), such that, there exists \(\epsilon > 0\), such that, if a multiloop \(M\) and a loop \(\ell\) on \(C\) satisfy (1) and (2) for this \(\epsilon\), then \(\angle(\lambda, \lambda, \ell) < \delta\). Thus if \(\delta > 0\) is sufficiently small, then by Corollary 5.2, \(\ell\) is isotopic to an admissible loop on \(C\). \(\Box\)

**5.1. Admissible loops close to \(v_{i,j+1}\) on projective surfaces close to \(v_{i,j}\) in Thurston coordinates.** We carry over the notations from §4. Then
Proposition 5.4. For all \( i \in \{0, 1, \ldots, n-1\} \) and \( j \in \mathbb{Z} \), there exists \( \delta > 0 \), such that, if a projective structure \( C \cong (\tau, L) \) on \( S \) satisfies \( \angle(L, \nu_{i,j}) < \delta \), for every loop \( \ell \) on \( C \) satisfying \( \angle(\ell, \nu_{i,j+1}) < \delta \), there is an admissible loop on \( C \) isotopic to \( \ell \); such a loop \( \ell \) indeed exists.

**Proof.** Let \( \tau_{i,j} \) be the hyperbolic structure on \( S \) such that \( (\tau_{i,j}, \nu_{i,j}) \) is realized by the pleated surface \( \beta_{i,j} \). Given a projective structure \( C \cong (\tau, L) \) on \( S \), let \( L = (\lambda, \mu) \) denote the measured lamination, where \( \lambda \in \mathcal{H} \mathcal{L} \) and \( \mu \in \mathcal{J} \mathcal{M}(\lambda) \). For every \( \epsilon > 0 \), if \( \angle(\nu_{i,j}, \lambda) \) is sufficiently small, then there is a marking preserving \( \epsilon \)-rough isometry \( \psi_{i,j} : \tau_{i,j} \to \tau \) given by Theorem 2.2.

Let \( K \) be any positive number less than one third of the shortest closed leaf of (the geodesic representative of) \( \lambda_{i,j+1} \) on \( \tau_{i,j} \). Then, for every \( \epsilon > 0 \), there is an \( (\epsilon, K) \)-nearly straight traintrack \( T_{i,j+1} \) on \( \tau_{i,j} \) that carries \( \lambda_{i,j+1} \) on \( \tau_{i,j} \); see Lemma 6.11 in [Bab]. In addition, it is easy to show that, for every \( H > 0 \), we can in addition assume, if a branch \( T_{i,j+1} \) intersects no closed leaf of \( \lambda_{i,j} \), then its length is at least \( H \) (since \( K \) is determined by the lengths of closed leaves of \( \lambda_{i,j+1} \)).

We can naturally assume that, for every leaf \( d \in \nu_{i,j+1} \) with \( d \notin \nu_{i,j} \) (removed by the diagonal exchange) there is a unique branch \( R \) of \( T_{i,j+1} \) such that \( d \) passes through \( R \) exactly once and no other leaf of \( \nu_{i,j} \) intersects \( R \).

We show that if \( H > 0 \) is sufficiently large and \( \epsilon > 0 \) are sufficiently small, then the traintrack \( T_{i,j+1} \) (on \( \tau_{i,j} \)) is admissible on \( C \). That is, for every loop \( \ell \) carried by \( T_{i,j+1} \), there is an admissible loop on \( C \) isotopic to it. The proof is similar to that of Theorem 5.1, except more careful arguments for branches corresponding to leaves \( d \) of \( \nu_{i,j+1} \) with \( d \notin \nu_{i,j} \), where \( d \) and \( L \) are not close to being parallel.

**Case One.** First suppose that \( F_i \) is a one-holed torus. Let \( d_j \) be the leaf of \( \nu_{i,j} \) removed by the diagonal exchange of \( \nu_{i,j} \) yielding \( \nu_{i,j+1} \) and \( d_{j+1} \) the leaf of \( \nu_{i,j+1} \) added. Then there is a unique branch \( R_0 \) of \( T_{i,j+1} \) such that \( d_{j+1} \) is the only leaf of \( \nu_{i,j} \) intersecting \( R_0 \). Since \( d_{j+1} \) is not a closed leaf, we can assume that \( R_0 \) has length at least \( H > 0 \). If \( \delta > 0 \) is sufficiently small, then \( \tau \) and \( \tau_{i,j} \) are sufficiently close. Thus, since \( T_{i,j+1} \) is \( (\epsilon, K) \)-nearly straight, there is an \( (\epsilon, K) \)-nearly straight traintrack \( \mathcal{T} \) also on \( \tau \) obtained by a small perturbation of \( \psi_{i,j}(T_{i,j+1}) \). Let \( \mathcal{T} = \bigcup_{k=0}^{m} \mathcal{R}_k \) be the traintrack on \( C \) that descends to \( \mathcal{T} \) via the collapsing map \( \kappa : C \to \tau \). Let \( \mathcal{R}_0 \) be the branch of \( \mathcal{T} \) corresponding to \( R_0 \). Then, again as in [Bab, Proposition 7.10], by a small isotopy of \( \mathcal{T} \) on \( C \) without changing \( |\mathcal{T}| \), we may assume that \( \mathcal{R}_k \) is a rectangle supported a round cylinder for each \( k = 1, \ldots, m \).
It suffices to show that similarly there is an isotopy of $T$ on $C$ that moves only $R_0$ so that $R_0$ is also supported on a round cylinder. Then, the traintrack $T$ is admissible on $C$ by Lemma 2.3; moreover, by Lemma 4.4, there are many loops carried by $T$.

Let $\sigma_{i,j}$ be the subsurface of $\tau_{i,j}$ with geodesic boundary isotopic to the subsurface $F_i$ of $S$. Then the boundary component of $\sigma_{i,j}$ is a closed leaf of $\nu_{i,j}$, and $\nu_{i,j}$ decomposes $\sigma_{i,j}$ into two ideal triangles. Let $\tilde{\sigma}_{i,j}$ be the universal cover of $\sigma_{i,j}$. Then $\tilde{\sigma}_{i,j}$ is a convex subset of $\mathbb{H}^2$ by geodesics that covers the closed leaf of $\nu_{i,j}$. Then $\tilde{\nu}_{i,j}$ yields an ideal triangulation of $\tilde{\sigma}_{i,j}$. Let $\tilde{d}_j$ be a lift of $d_j$ to $\tilde{\sigma}_{i,j}$. Then $\tilde{d}_j$ separates adjacent complementary ideal triangles $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ of $\tilde{\nu}_{i,j}$. Then $\Delta_1 \cup \Delta_2 (=: Q_{i,j})$ is a fundamental domain of $\tilde{\sigma}_{i,j}$, and it is an ideal quadrangle. Different vertices of $Q_{i,j}$ are endpoints of different boundary geodesics of $\tilde{\sigma}_{i,j}$. Then different boundary geodesics $\ell_1, \ell_2$ of $\tilde{\sigma}_{i,j}$ are preserved by different elements $\gamma_1, \gamma_2 \in \pi_1(S) \setminus \{id\}$, respectively. Then, since $\rho$ is purely loxodromic, by Lemma 2.4, the axes of the loxodromic elements $\rho(\gamma_1)$ and $\rho(\gamma_2)$ share endpoint. Then, since the pleated surface $\beta_{i,j}: \mathbb{H}^2 \rightarrow \mathbb{H}^3$ is $\rho$-equivariant, $\beta_{i,j}$ takes different boundary geodesics of $\tilde{\sigma}_{i,j}$ to geodesics in $\mathbb{H}^3$ without common endpoints. In particular $\beta_{i,j}$ takes the vertices of $Q_{i,j}$ to distinct points on $\hat{\mathbb{C}}$.

Let $v_1,v_2$ be the opposite vertices of the quadrangle $Q_{i,j}$ that are not endpoints of the diagonal $\tilde{d}_j$. Let $\tilde{d}_{j+1}$ be the other diagonal of $Q_{i,j}$, which connects $v_1$ and $v_2$. Since the branches $R_1, \ldots, R_m$ are supported on round cylinders, the vertical edges of $R_0$ are circular.

Since $d_{j+1}$ is the only leaf of $\nu_{i,j+1}$ intersecting $R_0 \subset \tau_{i,j}$, there is a unique lift $\tilde{R}_0$ of $R_0$ contained in $Q_{i,j}$. Then, if $H > 0$ and $1/\epsilon > 0$ are sufficiently large, then the vertical edges are sufficiently far in $Q_{i,j}$. Let $\tilde{R}_0$ be the branch of $\tilde{T}$ corresponding to $\tilde{R}_0$. Then the $\tilde{f}$-image of the vertical edges of $\tilde{R}_0$ are contained in round circles $c_1$ and $c_2$ on $\hat{\mathbb{C}}$, and $c_1$ and $c_2$ bound totally geodesic hyperplanes in $\mathbb{H}^3$ that are sufficiently far. In addition the $\beta_{i,j}$-image of edges of $Q_{i,j}$ are geodesics in $\mathbb{H}^3$ almost orthogonal to the hyperplanes. (See Figure 6.) The hyperplanes cut off the $\beta_{i,j}$-images of cusp neighborhoods, in $Q_{i,j}$, of the vertices $v_1$ and $v_2$, and the area of the neighborhoods are sufficiently small.

We will subdivide the branch $R_0$ into three (sub)branches that are supported on round cylinders after an isotopy of $T$ on $C$ moving only the middle branch of $R_0$. Let $A_0$ be the round cylinder in $\hat{\mathbb{C}}$ bounded by $c_1$ and $c_2$. In the quadrangle $Q_{i,j}$, the diagonals $d_j$ and $d_{j+1}$ intersect in a single point. Thus there is a unique round circle $r$ on $\hat{\mathbb{C}}$ such that the hyperbolic plane bounded by $r$ intersects the geodesic $\beta_{i,j}(\tilde{d}_j)$.
orthogonally at the image of the intersection point $\tilde{d}_j \cap \tilde{d}_{j+1}$. Then $\beta_{i,j}(v_1)$ and $\beta_{i,j}(v_2)$ are in different components of $\hat{\mathcal{C}} \setminus r$. Since the hyperplanes bounded by $c_1$ and $c_2$ and the geodesic $\beta_{i,j}(\tilde{d}_j)$ are mutually sufficiently far, $\mathcal{A}_0$ is a regular neighborhood of $r$ in $\hat{\mathcal{C}}$. In addition the image of $d_j \cap d_{j+1}$ is contained in the convex hull of $\mathcal{A}_0$ in $\mathbb{H}^3$, and it is far from the hyperplanes bounded by $c_1$ and $c_2$. Take a small regular neighborhood of $r$ in $\mathcal{A}_0$ bounding by round circles $r_1$ and $r_2$ that bounds hyperbolic planes orthogonal to $\beta_{i,j}(\tilde{d}_j)$, such that, for each $h = 1, 2$, the circle $r_h$ is in the connected component of $\mathcal{A}_0 \setminus r$ bounded by $c_h$ and $r$.

Recall that the quadrangle $Q_{i,j}$ is the union of the ideal triangles $\Delta_1$ and $\Delta_2$ adjacent along $\tilde{d}_j$ (see Figure 7, Right). Since $r_1$ and $r_2$ are orthogonal to the geodesic $\beta_{i,j}(\tilde{d}_j)$, thus the ideal triangles $\beta_{i,j}(\Delta_1)$ and $\beta_{i,j}(\Delta_2)$ orthogonally intersect both hyperbolic planes bounded by $r_1$ and $r_2$. Note that those hyperplane are the convex hulls $\text{Conv}(r_1)$ and $\text{Conv}(r_2)$ in $\mathbb{H}^3$. Let $\tilde{\mathcal{R}}_0$ be a branch of $\tilde{T}$ corresponding $\mathcal{R}_0$, so that the $\tilde{\psi}_{i,j}$-inverse image of $\tilde{\mathcal{R}}_0$ is contained in $Q_{i,j}$. Since $\tilde{d}_j \cap \tilde{d}_{j+1}$ is a single point, $\beta_{i,j}(\tilde{d}_{j+1})$ is a piecewise geodesic with a single singular point at $\tilde{d}_j \cap \tilde{d}_{j+1}$ and it is transversal to $\text{Conv}(r_1)$ and $\text{Conv}(r_2)$. Thus, since $\tilde{T}$ is $(\epsilon, K)$-nearly straight, the $\beta_{i,j}^{-1}$-image of the hyperbolic planes $\text{Conv}(r_1)$ and $\text{Conv}(r_2)$ intersects the branch $\tilde{\mathcal{R}}_0$ in two disjoint short geodesic segments in different components of $Q_{i,j} \setminus \tilde{d}_j$ so that it decomposes $\tilde{\mathcal{R}}_0$ into three consecutive branches.

We shall see that accordingly the branch $\tilde{\mathcal{R}}_0$ is decomposed into three subbranches, cut along circular arcs mapping into $r_1$ and $r_2$. Let $\beta : \mathbb{H}^2 \to \mathbb{H}^3$ be the pleated surface corresponding to $C$. Let $\tilde{\lambda}$ be the total lift of $\lambda$ to $\mathbb{H}^2$. For every $\epsilon > 0$, if $\delta > 0$ is sufficiently small, then $\lambda$ and $\lambda_{i,j}$ are $\epsilon$-close with the Hausdorff metric (on $\tau_{i,j}$). Since $\lambda_{i,j}$ is maximal, if follow from Theorem 2.2 that, for every $\epsilon > 0$ and every compact subset $X$ of $\mathbb{H}^3$, if $\delta > 0$ is sufficiently small, then, for each $h = 1, 2$, there is a connected component $W_h$ of $\mathbb{H}^2 \setminus \tilde{\lambda}$ such that $\beta(W_h)$ is, in $X$, smoothly $\epsilon$-close to $\beta_{i,j}(\Delta_h)$. In particular, if the interior of $X$ contains the intersection of $\beta_{i,j}(Q_{i,j})$ and $\text{Conv}(\mathcal{A}_0)$, then $\beta(W_h)$ is almost orthogonal to the hyperplanes bounded by $c_h$ and $r_h$. Let $\tilde{\kappa} : \hat{\mathcal{C}} \to \mathbb{H}^2$ be the lift of the collapsing map $\kappa : C \to \tau$, and let $W_h = \tilde{\kappa}^{-1}(W_h)$ for each $h = 1, 2$. Then $\text{dev}(C) = f$ embeds $W_h$ into $\hat{\mathcal{C}}$ so that $W_h$ transversally interests $r_h$ in a single arc. If $\epsilon > 0$ is sufficiently small, then those circular arcs embedded in $W_1$ and $W_2$ decompose the branch $\tilde{\mathcal{R}}_0$ into three subbranches so that the outer subbranches are contained in $W_1$ and $W_2$ and supported on round
cylinders bounded by $c_1$ and $r_1$ and by $c_2$ and $r_2$. Let $\mathcal{R}$ be the middle subbranch of $\mathcal{R}_0$. Then, in order to complete the proof, it suffices make $\mathcal{R}$ supported on a round cylinder. The vertical edges of $\mathcal{R}$ embeds into $r_1$ and $r_2$ by $f$. Then

**Claim 5.5.** There is a rectangle $\mathcal{R}$ in the universal cover $\tilde{C}$ containing $\mathcal{R}$ such that

- $\mathcal{R}$ is supported on the round cylinder $\mathcal{A}$ bounded by the round circles $r_1$ and $r_2$ and
- $\mathcal{R} \cap \mathcal{T} = \mathcal{R}$ and different vertical edges of $\mathcal{R}$ contain different vertical edges of $\mathcal{R}$.

By this claim, there is an isotopy of $\mathcal{T}$ on $C$ supported on $\mathcal{R}$ that isotopes $\mathcal{R}$ to $\mathcal{R}$.

**Proof of Claim.** The proof is similar to the proof of [Bab, Proposition 7.10]. Let $(\tilde{\tau}, \tilde{L})$ be the universal cover of $(\tau, L)$. Let $\mathcal{L}$ be the canonical lamination on $C$ descending to $L$, and let $\tilde{\mathcal{L}}$ be the total lift of $\mathcal{L}$ to $\tilde{C}$. Then if a stratum $X$ of $(\tilde{\tau}, \tilde{L})$ intersects $\tilde{\mathcal{R}}_0$, then its image under $\beta$ almost orthogonally intersects the hyperbolic planes $\text{Conv}(r_1)$ and $\text{Conv}(r_2)$ in $\mathbb{H}^3$. Let $\mathcal{X}$ be a strata of $(\tilde{C}, \tilde{\mathcal{L}})$ descending to $X$. Then $f^{-1}(\mathcal{A}) \cap \mathcal{X}$ is either an arc or an rectangle supported on $\mathcal{A}$. Then, since $f$ is a locally homeomorphism, the union of $f^{-1}(\mathcal{A}) \cap \mathcal{X}$ is a rectangle $\mathcal{R}$ supported on $\mathcal{A}$, where the union runs over all strata of $(\tilde{C}, \tilde{\mathcal{L}})$ that descend to strata of $(\mathbb{H}^2, \tilde{\mathcal{L}})$ intersecting $\tilde{R}_0$. Then $\mathcal{R}$ contains $\mathcal{R}$. Yet $\mathcal{R} \cap \mathcal{T}$ is the disjoint union of $\mathcal{R}$ and small neighborhoods of the horizontal edges of $\mathcal{R}$. Then, by removing the small neighborhoods, we can make $\mathcal{R}$ slightly smaller so that it in addition satisfies $\mathcal{R} \cap \mathcal{T} = \mathcal{R}$.  

**Case Two.** Suppose that $F_i$ is a four-holed sphere. Then the proof is similar to Case One, and we leave the proof to the readers. The only difference is that $\nu_{i,j}$ and $\nu_{i,j+1}$ differ by two diagonal exchanges (intend of one). We accordingly deal with two branches of the traintracks more carefully than the other branches.

6. **Sequence of grafts traveling in $\mathcal{G} \mathcal{L}$**.

In this section, we prove our main theorem (Theorem 1.2) modulo Corollary 6.6— this corollary will be independently proved in Part 2. In §4, starting with two projective structures on $S$ sharing purely loxodromic holonomy $\rho: \pi_1(S) \to \text{PSL}(2, \mathbb{C})$, we have constructed finitely many geodesic laminations $\nu_0, \nu_1, \ldots, \nu_n$ on $S$ and then
for each $0 \leq i \leq n$, a family of geodesic laminations $\nu_{i,j}$ ($j \in \mathbb{Z}$) on $S$ “connecting” $\nu_i$ to $\nu_{i+1}$. Then, by Proposition 5.4 and Corollary 6.6, grafting transforms a projective structures close to $\nu_{i,j}$ to a projective structure close to $\nu_{j+1}$ in Thurston coordinates:

**Proposition 6.1.** Given $0 \leq i < n$ and $j \in \mathbb{Z}$, there exists $\delta_{i,j} > 0$ such that, if a projective structure $C \cong (\tau, L)$ with purely loxodromic holonomy $\rho: \pi_1(S) \to \text{PSL}(2, \mathbb{C})$ satisfies $\angle(\lambda, \nu_{i,j}) < \delta_{i,j}$, then, for
every $\epsilon_{i,j} > 0$, there is an admissible loop $\ell$ on $C$ such that, letting $Gr^\ell_\mathcal{G}(C) \cong (\tau_k, L_k)$ in Thurston coordinates, we have $\angle(L_k, \nu_{i,j+1}) < \epsilon_{i,j}$ for sufficiently large $k$.

Using grafting obtained by Proposition 6.1, we can transform a projective structure close to $\nu_i$ to one close to $\nu_{i+1}$:

**Proposition 6.2.** For $i = 0, 1, \ldots, n - 1$, there exists $\delta_i > 0$, such that if $C \cong (\tau, L)$ satisfies $\angle(\lambda, \nu_i) < \delta_i$, for every $\epsilon > 0$, there is a finite composition of grafts starting from $C$,

$$C = C_0 \xrightarrow{Gr_{\ell_1}} C_1 \xrightarrow{Gr_{\ell_2}} C_2 \rightarrow \cdots \xrightarrow{Gr_{\ell_k}} C_k$$

such that the last projective structure $C_k \cong (\tau_k, L_k)$ satisfies $\angle(L_k, \nu_{i+1}) < \epsilon$.

**Proof.** Let $C \cong (\tau, L) \in \mathcal{P}_\rho$ such that $\angle(L, \nu_i) > 0$ is sufficiently small. Recall that $\nu_{i,j} \to \nu_i$ as $j \to -\infty$. Thus, if $j \in \mathbb{Z}$ is sufficiently small and $\ell$ is a loop on $C$ (whose geodesic representative on $\tau_{i,j}$) is sufficiently close to $\nu_{i,j}$ in the Chabauty topology, then $\angle(\ell, \nu_i) > 0$ is also small. Then, by Theorem 5.1, a loop $\ell$ on $C$ is admissible up to an isotopy. Set $Gr^\ell_\mathcal{G}(C) \cong (\tau_h, L_h)$ for each $h \in \mathbb{Z}_{>0}$. Therefore, for every $\delta > 0$, if $j \in \mathbb{Z}$ is sufficiently small and $\ell$ is sufficiently close to $\nu_{i,j}$, then by Corollary 6.6, we have $\angle(L_h, \nu_{i,j}) < \delta$ for sufficiently large $h \in \mathbb{N}$.

Since $\nu_{i,j} \to \nu_{i+1}$ as $j \to \infty$, for every $\epsilon > 0$, we can pick sufficiently large $j_\ell \in \mathbb{N}$ such that $\angle(\nu_{i+1}, \nu_{i,j_\ell}) < \epsilon$. Then let $\delta_{i,j_\ell} > 0$ be such that, if a loop $\ell$ satisfies $\angle(\ell, \nu_{i,j_\ell}) < \delta_{i,j_\ell}$, then $\angle(\ell, \nu_{i+1}) < \epsilon$. For every $j \in \mathbb{Z}$ with $j < j_\ell$, inductively define $\delta_{i,j} > 0$ inductively so that $\delta_{i,j}$ is the constant obtained by applying Proposition 6.1 to $\delta_{i,j+1}$. Thus, for $j < j_\ell$, if there is a projective structure $C' \cong (\tau', L')$ satisfies that $\angle(\nu_{i,j}, L') < \delta_{i,j}$, then there is a composition of grafts starting from $C'$,

$$C' = C_0 \xrightarrow{Gr_{\ell_1}} C_1 \xrightarrow{Gr_{\ell_2}} C_2 \rightarrow \cdots \xrightarrow{Gr_{\ell_k}} C_k = (\tau_k, L_k),$$

such that $\angle(L_k, \nu_{i+1}) < \epsilon$. It follows from the first paragraph that, if $j$ is sufficiently small, then such $C'$ is obtained by iteration of grafts of $C$ along a fixed admissible loop. \qed

Recall that we started with arbitrary projective structures $C_\sharp \cong (\tau_\sharp, L_\sharp)$ and $C_\flat \cong (\tau_\flat, L_\flat)$ on $S$ sharing purely loxodromic holonomy $\rho: \pi_1(S) \to \text{PSL}(2, \mathbb{C})$ and that $L_\sharp = (\lambda_\sharp, \mu_\sharp)$ and $L_\flat = (\lambda_\flat, \mu_\flat)$ are measured laminations. Then
Proposition 6.3. For every \( \epsilon > 0 \), there exists a finite composition of grafts starting from \( C_2 \)
\[
C_2 = C_0 \xrightarrow{Gr_1} C_1 \xrightarrow{Gr_2} C_2 \to \ldots \to C_n
\]
such that the last projective structure \( C_n \cong (\tau_n, L_n) \) satisfies \( \angle(\lambda_n, L_n) < \epsilon \).

Proof of Proposition 6.3. From the construction of the multiloops \( M_2, M_\flat \), given \( \zeta > 0 \), we can assume that \( \angle(M_\flat, \lambda_\flat) < \zeta \), \( \angle(M_\sharp, \lambda_\sharp) < \zeta \) and that \( \lambda_\flat \) and \( \lambda_\sharp \) are contained in the \( \zeta \)-neighborhoods of \( M_\flat \) and \( M_\sharp \), respectively. Since \( \nu_n \) contains \( M_n = M_\flat \), for every \( \epsilon > 0 \), if \( \zeta > 0 \) is sufficiently small, there exists \( \delta_n > 0 \) such that if a geodesic lamination \( \lambda \) on \( S \) satisfies \( \angle(\lambda, \nu_n) < \delta_n \), then \( \angle(\lambda, \lambda_\flat) < \epsilon \). For \( i = 0, 1, \ldots, n-1 \), let \( \delta_i > 0 \) be the constant given by Proposition 6.2.

Claim 6.4. If \( \zeta > 0 \) is sufficiently small, for loops \( \ell \) sufficiently close to \( \nu_0 \) with the Hausdorff metric on \( \tau_\sharp \),
- \( \ell \) is admissible on \( C_2 \) (up to an isotopy), and
- for \( k \in \mathbb{N} \) letting \( \text{Gr}_k^\ell(C_2) \cong (\tau_k, L_k) \) in Thurston coordinates, we have \( \angle(L_k, \nu_0) < \delta_0 \) for sufficiently large \( k \).

Proof. Since \( \ell \) is sufficiently close to \( \nu_0 \), then \( \angle(\nu_0, \ell) \) is sufficiently small. Since \( \angle(\lambda_\sharp, \nu_0) < \zeta \) is sufficiently small and \( \nu_0 \) is maximal, \( \angle(L, \ell) \) is also sufficiently small. Thus, by Corollary 5.2, \( \ell \) is admissible.

By Corollary 6.6, \( \angle(L_k, \ell) \to 0 \) as \( k \to \infty \). Thus, since \( \angle(L, \nu_0) \) is sufficiently small, \( \angle(L_k, \nu_0) < \delta_0 \) for sufficiently large \( k \). \( \square \)

Let \( C_{k_0} \) be \( \text{Gr}_{k_0}^\ell(C) \cong (\tau_{k_0}, L_{k_0}) \), given by Claim 6.4, with a sufficiently large \( k_0 \in \mathbb{N} \). Then, since \( \angle(L_{k_0}, \nu_0) < \delta_0 \), by Proposition 6.2, there is a composition of grafts from \( C_{k_0} \)
\[
C_{k_0} \to C_1 \to C_2 \to \ldots \to C_{k_1} \cong (\tau_{k_1}, L_{k_1})
\]
such that, \( \angle(\nu_1, L_{k_1}) < \delta_1 \). It follows from Proposition 6.2 that, for each \( i = 0, 1, \ldots, n \), by induction in \( i \) we can extend this composition of grafts to
\[
C_{k_0} \to \ldots \to C_{k_1} \to \ldots \to C_{k_2} \to \ldots \to C_{k_i}
\]
so that the last projective structure \( C_{k_i} \cong (\tau_{k_i}, L_{k_i}) \) satisfies \( \angle(\nu_i, L_{k_i}) < \delta_i \). In particular, when \( i = n \), we have \( \angle(\nu_n, L_{k_n}) < \zeta \). Hence \( \angle(\lambda_\flat, L_{k_n}) < \epsilon \). 6.3
Proof of Theorem 1.2. Let \( \delta > 0 \) be the constant obtained by applying Theorem 2.6 to \( C_0 \cong (\tau, L_0) \). Then, by Proposition 6.3, there is a composition of grafts along loops,
\[
C_n = C_0 \xrightarrow{Gr_{\ell_1}} C_1 \xrightarrow{Gr_{\ell_2}} C_2 \to \ldots \to C_n,
\]
such that, letting \( C_n \cong (\tau_n, L_n) \), we have \( \angle(L_n, L_0) < \delta \). Then we can graft \( C_n \) and \( C^\flat \) along multiloops to a common projective structure. Hence there are admissible loops \( M_n \) on \( C_n \) and \( M^\flat \) on \( C^\flat \) such that \( \text{Gr}_{M_n}(C_n) = \text{Gr}_{M^\flat}(C^\flat) \). Since the grafting \( \text{Gr}_{M_n} \) of \( C_n \) is naturally a composition of grafts along loops of \( M_n \), this completes the proof.

Part 2. Limit, in Thurston coordinates, of grafting along a loop

In the second part of this paper, we prove

**Theorem 6.5.** Let \( \ell \) be an admissible loop on a projective surface \( C \). Let \( C \cong (\tau, L) \in T \times M_L \), and let \( L = (\lambda, \mu) \) with \( \lambda \in GL \) and \( \mu \in TM(\lambda) \). For each \( i \in \mathbb{N} \), let \( C_i = (Gr_i)(C) \). Similarly let \( C_i \cong (\tau_i, L_i) \) and \( L_i = (\lambda_i, \mu_i) \). Let \( \beta_i: \mathbb{H}^2 \to \mathbb{H}^3 \) be the pleated surface corresponding to \( C_i \). Then

(i) \( \tau_i \) converges to a hyperbolic surface \( \tau_\infty \in T \) as \( i \to \infty \).

(ii) \( \beta_i \) converges to a \( \rho \)-equivariant pleated surface realizing \( (\tau_\infty, \lambda_\infty) \) for some \( \lambda_\infty \in GL \) containing \( \ell \).

(iii) \( L_i \) converges to an (heavy) measured lamination \( L_\infty \) supported on \( \lambda_\infty \) such that \( \ell \) is the only leaf with weight infinity.

Theorem 6.5 (i) (ii) immediately imply

**Corollary 6.6.** \( \angle(L_i, \ell) > 0 \) converges to 0 as \( i \to \infty \).

7. Limit in the complement of the admissible loop

Let \( C \) be a projective structure on \( S \) with holonomy \( \rho: \pi_1(S) \to PSL(2, \mathbb{C}) \). Let \( \ell \) be an admissible loop on \( C \).

**Definition 7.1.** Let \( \tilde{\ell} \) be a lift of \( \ell \) to the universal cover \( \tilde{C} \), and \( \gamma_\ell \) be the element of \( \pi_1(S) \) that represents \( \ell \) and preserves \( \tilde{\ell} \). Normalize the developing map \( f \) of \( C \) by an element of \( PSL(2, \mathbb{C}) \) so that the loxodromic \( \rho(\gamma_\ell) \in PSL(2, \mathbb{C}) \) fixes 0 and \( \infty \) in \( \hat{C} \). Pick a parametrization \( \tilde{\ell}: \mathbb{R} \to \tilde{C} \) of \( \tilde{\ell} \). Then its \( f \)-image can be written in polar coordinates \((e^r, \theta)\) so that

\[
f \circ \tilde{\ell}(t) = \exp[r(t) + i\theta(t)]
\]
where $r: \mathbb{R} \to \mathbb{R}$ and $\theta: \mathbb{R} \to \mathbb{R}$ are continuous functions. Then we say that $\ell$ spirals if $\theta$ is an unbounded function. Otherwise it is called roughly circular.

**Remark 7.2.** An admissible loop $\ell$ is roughly circular if and only if there is a homotopy (or an isotopy) between $f \circ \ell$ and a circular arc on $\hat{\mathbb{C}}$ connecting the fixed points of $\rho(\gamma_{\ell})$ such that the homotopy is equivariant under the restriction of $\rho: \pi_1(S) \to \text{PSL}(2,\mathbb{C})$ to the infinite cyclic group generated by $\gamma_{\ell}$.

In the setting of Theorem 6.5, since $C_i = \text{Gr}_i^\ell(C)$, then $C \setminus \ell$ isomorphically embeds into $C_i$ so that the complement of $C \setminus \ell$ in $C_i$ is the cylinder inserted by the grafting $\text{Gr}_i^\ell$; this cylinder is naturally cut into $i$ isomorphic copies of a grafted cylinder along parallel copies of $\ell$. Let $M_i$ be the union of $i + 1$ parallel copies of $\ell$ on $C_i$ that decomposes $C$ into the $i$ cylinders and $C \setminus \ell$. Let $\ell_i$ be, if $i$ is even, the middle loop of $M_i$ and, if $i$ is odd, a boundary component of the middle cylinder. Let $C_i = C_i \setminus \ell_i$. Then there is a natural isomorphic embedding of $C_i$ into $C_{i+1}$. Therefore we let

$$C_\infty = \lim_{i \to \infty} (C_i \setminus \ell_i).$$

Then $C_\infty$ is a projective structure on $S \setminus \ell$, and its holonomy is the restriction of $\rho$ to $\pi_1(S \setminus \ell)$. (Equivalently $C_\infty$ is obtained by attaching a half-infinite grafting cylinder along each boundary component of $C \setminus \ell$.)

Recall that the holonomy $\rho$ of $C$ is non-elementary [GKM00]. Noting that $C \setminus \ell$ has either one or two connected components, we have

**Lemma 7.3.** Let $P$ be a connected component of $C \setminus \ell$. Then the restriction of $\rho$ to $\pi_1(P)$ is non-elementary.

**Proof.** Suppose to the contrary that $\rho(\pi_1(P)) =: G$ is elementary. Then, since $\rho(\ell)$ is loxodromic, the limit set $\Lambda$ of $G$ contains only the two fixed points of the loxodromic $\rho(\ell)$. Then the domain of discontinuity of $G$ is identified with $\mathbb{C} \setminus \{0\}$, and it admits a complete Euclidean metric given by the exponential map $\exp: \mathbb{C} \cong \mathbb{R}^2 \to \mathbb{C} \setminus \{0\}$, so that the $G$-action on the domain is isometric.

Let $\mathcal{C}_P$ denote the connected component of $C_\infty$ corresponding to $P$. Consider the inverse image of $\Lambda$ under $\text{dev}(\mathcal{C}_P)$. Since $\Lambda$ is in particular a discrete subset of $\hat{\mathbb{C}}$, the $\text{dev}(\mathcal{C}_P)$-inverse image of $\Lambda$ is a discrete subset preserved by $\pi_1(P)$, and it descends to a set of finitely many points on $\mathcal{C}_P$. By pulling back the Euclidean metric via the developing map, $\mathcal{C}_P$ minus the finitely many points enjoys a complete Euclidean metric. This is a contradiction since the Euler characteristic of $P$ is negative. \qed
Let $\tilde{\mathcal{C}}_i$ and $\tilde{\mathcal{C}}_\infty$ denote the universal covers of $\mathcal{C}_i$ and $\mathcal{C}_\infty$, respectively. Then, by Lemma 7.3, the holonomy of every connected component of $\mathcal{C}_\infty$ is non-elementary. Thus, by Corollary 3.2, let $\mathcal{C}_\infty \cong (\sigma_\infty, N_\infty)$ be the Thurston coordinates, where $\sigma_\infty$ is a convex hyperbolic surface with geodesic boundary whose interior is homeomorphic to $S \setminus \ell$ and $N_\infty$ is a (possibly heavy) measured geodesic lamination on $\sigma_\infty$.

**Proposition 7.4.** The boundary of $\sigma$ is the union of two geodesic loops corresponding the (open) boundary circles of $S \setminus \ell$. The lengths of boundary components are the translation length of $\rho(\ell)$. Furthermore

(i) Suppose that $\ell$ is roughly circular. Then $\sigma$ contains both boundary geodesic loops (i.e. closed boundary), and they are isolated leaves of $N_\infty$ with weight infinity.

(ii) Suppose that $\ell$ spirals. Then $\sigma_\infty$ contains no boundary geodesic (i.e. open boundary). Leaves of the lamination $N_\infty$ spiral towards each boundary component of $\sigma_\infty$ in the same direction with respect to the orientation on the boundary components of $\tau$ (induced by the orientation of $S$); see Figure 8. In particular $N_\infty$ contains no heavy leaves.

![Figure 8. Geodesics spiraling to the left towards both boundary components (when you stand on the surface facing toward boundary).](image)

**Remark 7.5.**

- In (ii), the metric completion of $\sigma_\infty$ is the union of $\sigma_\infty$ and the boundary loops. Then $N_i$ naturally extends to yet a heavy measured lamination on the completion, so that both boundary loops are leaves of weight infinity.
- Let $\zeta_\infty : \tilde{\sigma}_\infty \to \mathbb{H}^3$ be the pleated surface associate with $\mathcal{C}_\infty$, where $\tilde{\sigma}_\infty$ is the universal cover of $\sigma_\infty$ (note that, if $\ell$ is separating, $\tilde{\sigma}_\infty$ has two connected components). Then, let $m$ be a boundary geodesic of $\tilde{\sigma}_\infty$ and let $\gamma_m$ be a non-trivial deck transformation preserving $m$. Then, in both Case (i) and (ii), $\zeta_\infty$
isometrically takes $m$ onto the axis of the loxodromic element $ho(\gamma_m)$.

Proof of Proposition 7.4. Case (i). Suppose that $\ell$ is roughly circular. Let $\mathcal{A}$ be a component of $\mathcal{C}_\infty \setminus \mathcal{C}_0$, which is a half infinite grafting cylinder attached a boundary component of $\mathcal{C}_0$. Then, since $\ell$ is roughly circular, there is a circular loop $\alpha$ in $\mathcal{A}$ homotopy equivalent to $\mathcal{A}$. Then $\alpha$ bounds a (smaller) half-infinite cylinder $\mathcal{A}'$ isotopic to $\mathcal{A}$, and $\mathcal{A}'$ is uniquely foliated by circular loops. Let $\ell'$ be such a circular loop in $\mathcal{A}'$.

Let $\mathcal{N}_\infty$ be the circular measured lamination on $\tilde{\mathcal{C}}_\infty$, which descends to $N_\infty$. Then, since the cylinder $\mathcal{A}'$ is half-infinite, by taking $\ell'$ sufficiently far from the boundary component of $\mathcal{A}'$ if necessarily, we may assume that $\ell$ is a closed leaf of $\mathcal{N}_\infty$ (there is enough space in $\mathcal{A}'$ to find a maximal ball corresponding to such $\ell'$). In addition we can assume that $\mathcal{A}'$ is a maximal cylinder in $\mathcal{C}_\infty$ that is isotopic to $\mathcal{A}$ such that $\mathcal{A}'$ is foliated by closed leaves of $\mathcal{N}_\infty$. (Then $\mathcal{A}'$ may not be contained in $\mathcal{A}$ anymore.) Then $\mathcal{A}'$ is still half-infinite and the total transversal measure of $\mathcal{N}_\infty$ on $\mathcal{A}'$ is infinite. Let $\iota_\infty: \mathcal{C}_\infty \to \sigma_\infty$ be the collapsing map. The $\iota_\infty$ takes $\mathcal{A}'$ to a boundary geodesic loop of $\sigma_\infty$ of infinite weight. Conversely the inverse-image of the geodesic loop is $\mathcal{A}'$ since $\mathcal{A}'$ is maximal. Since $\mathcal{N}_\infty$ has infinite measure on $\mathcal{A}'$, the boundary component is a leaf of $N_\infty$ with infinite weight. Then, since the transversal measure of $\mathcal{N}_\infty$ is locally finite, no leaf of $\mathcal{N}_\infty$ spirals towards the boundary loop of $\mathcal{A}'$. Thus the boundary geodesic of $\sigma_\infty$ is an isolated leaf of $N_\infty$.

(ii). Suppose that $\ell$ spirals. Since $\ell$ is admissible, the restriction of $\text{dev}(C)$ to a lift $\tilde{\ell}$ to $\hat{S}$ is a simple curve on $\hat{\mathcal{C}}$, and we can assume that it connects 0 and $\infty$. Then, as in Definition 7.1, it lifts to a curve $\mathbf{l}: \mathbb{R} \to \mathbb{R}^2$

$$t \mapsto (\theta(t), r(t))$$

through $\exp: \mathbb{C} \cong \mathbb{R}^2 \to \mathbb{C} \setminus \{0\}$, where $\theta: \mathbb{R} \to \mathbb{R}$ and $r: \mathbb{R} \to \mathbb{R}$ are continuous functions, so that $\exp(r(t) + i\theta(t))$ is the curve $\text{dev}(C)|\tilde{\ell}$. Since $\ell$ is admissible, $\mathbf{l}$ is a simple curve. Since $\text{dev}(C)|\tilde{\ell}$ is preserved by the loxodromic $\rho(\ell)$, accordingly $\mathbf{l}$ is preserved by a nontrivial translation of $\mathbb{R}^2$ (along a geodesic). Since $\rho(\ell)$ is loxodromic and $\ell$ is spiraling, the axis of this translation intersects both $\theta$ and $r$-axes transversally. Thus one connected component of $\mathbb{R}^2 \setminus \mathbf{l}$ lies above $\mathbf{l}$, i.e. it contains $\{0\} \times [R, \infty)$ for sufficiently large $R > 0$, and the other component lies below. Let $c$ be a boundary component of $\mathcal{C}_0$, which is isomorphic to $\ell$. Let $\tilde{c}$ be a lift of $c$ to the universal cover $\tilde{\mathcal{C}}_0$, so that $\tilde{c}$ is isomorphic to $\tilde{\ell}$. Then we can assume that $\text{dev}(\mathcal{C}_0)$ takes the small neighborhood of $\tilde{c}$
into the region below 1 in $\mathbb{R}^2$, if necessary, by exchanging 0 and $\infty \in \hat{C}$ by an element of $\text{PSL}(2,\mathbb{C})$.

Clearly $\tilde{C}_0$ is isomorphically embedded in $\tilde{C}_\infty$. Then in particular $\tilde{c}$ is embedded in $\tilde{C}_\infty$ and we can regard the endpoints of $\tilde{c}$ as distinct ideal points of both $\tilde{C}_0$ and $\tilde{C}_\infty$. Then let $p^+$ and $p^-$ be the ideal points corresponding to $\infty$ and 0, respectively, via $l$.

**Lemma 7.6.** There is a maximal ball $B$ in $\tilde{C}_\infty$ such that $p^+$ is an ideal point of $B$.

**Proof.** Let $A$ be the connected component of $C_\infty \setminus C_0$ bounded by $c$, so that $A$ is a half-infinite grafting cylinder. Let $\tilde{A}$ be the corresponding connected component of $\tilde{C}_\infty \setminus \tilde{C}_0$ bounded by $\tilde{c}$, so that $\tilde{A}$ covers $A$. Then $\text{dev}(A)$ lifts, through $\exp$, to an embedding onto the component of $\mathbb{R}^2 \setminus l$ above $l$. Thus we can find a round ball $B$ contained in $\tilde{A}$ such that $p^+$ is an ideal point of $B$ (Figure 9). Since $\tilde{A} \subset \tilde{C}_\infty$, there is a desired maximal ball in $\tilde{C}_\infty$ that contains $B$. □

![Figure 9](image)

**Lemma 7.7.** There is no maximal ball $B$ in $\tilde{C}_\infty$ such that $p^+$ and $p^-$ are both its ideal points.

**Proof.** Suppose, to the contrary, that there is a maximal ball $B$ in $\tilde{C}_\infty$ such that $\partial_\infty B$ contains both $p^+$ and $p^-$. Then its core $\text{Core}(B)$ contains a circular arc $\alpha$ connecting $p^+$ and $p^-$. The surface $\tilde{C}_\infty$ is of hyperbolic type (topologically). Then since $\alpha$ and $\tilde{c}$ share their ideal end points, they must project to isotopic loops on $C_\infty$. Thus $\alpha$ covers a circular loop on $C$ isotopic to $c$. This contradicts that $c$ spirals. □

Next we show that $\sigma_\infty$ has an open boundary component that is a closed geodesic homotopic to $c$. Let $B$ be a maximal ball of $\tilde{C}_\infty$ given by Lemma 7.6 so that $\partial_\infty B \ni p^+$. Then the collapsing map $i_\infty : \tilde{C}_\infty \rightarrow$
\( \tilde{\sigma}_\infty \) projects Core\((B)\) onto a convex subset \( X \) of \( \tilde{\sigma}_\infty \). Moreover \( \iota_\infty \) continuously extends to a map from the ideal boundary of \( \tilde{\mathcal{C}}_\infty \) to the ideal boundary of \( \tilde{\sigma}_\infty \). Since \( c \) is an essential loop, there are points \( p^+ \) and \( p^- \) on the ideal boundary of \( \tilde{\sigma}_\infty \) corresponding to \( p^+ \) and \( p^- \), respectively. Then, by Lemma 7.7, \( \partial_\infty X (\subset \partial_\infty \mathbb{H}^2) \) contains \( p^+ \) but not \( p^- \).

By regarding \( \tilde{\sigma}_\infty \) as a convex subset of \( \mathbb{H}^2 \), there is a unique geodesic \( g \) connecting \( p^+ \) and \( p^- \) in \( \mathbb{H}^2 \). Let \( \gamma_c \) be a deck transformation corresponding to \( c \) so that \( \gamma_c \) preserves \( g \). Then we see that \( (\gamma_c)^j X \) converges to \( g \) uniformly on compacts as \( j \to \infty \), if necessarily, changing \( \gamma_c \) to its inverse (Figure 10). By Lemma 7.7, the geodesic \( g \) is not contained in \( \tilde{\sigma}_\infty \). Thus \( g \) descends to a desired open boundary component of \( \sigma_\infty \). Let \( \zeta_\infty : \tilde{\sigma}_\infty \to \mathbb{H}^3 \) be the pleated surface for \( C_\infty \). Then since \( \zeta_\infty \) is equivariant and 1-Lipschitz, the continuous extension of \( \zeta_\infty \) takes \( g \) isometrically onto the axis of the loxodromic \( \rho(c) \). Therefore the translation length of \( \rho(c) \) is the length of the boundary component of \( \sigma_\infty \) homotopic to \( c \). In addition the convergence \( (\gamma_c)^j X \to g \) implies that leaves of \( N_\infty \) spiral towards \( c \).

\[ \text{Figure 10.} \]

Recall that a small neighborhood of the boundary component \( c \) of \( C_0 \) in \( \mathcal{A} \) develops above \( l \), which is used to distinguish \( p^+ \) and \( p^- \). Letting \( c' \) be the other boundary component of \( C_0 \), then \( \text{dev}(C_0) \) takes a small neighborhood of \( c \) to the region below \( l \) (with respect to \( c = c' \) on \( C \)). Then it follows that the labels \( p^+ \) and \( p^- \) are opposite for \( c' \) and \( c \). However, since the normal directions of \( c \) and \( c' \) of \( C_0 \) are the opposite on \( C \), leaves of \( N_\infty \) spiral towards both boundary components in the same direction with respect to the normal directions of the boundary.
components.

8. IDENTIFICATION OF BOUNDARY COMPONENTS OF THE LIMIT STRUCTURE

We have obtained the Thurston coordinates \((\sigma_\infty, N_\infty)\) of \(C_\infty\) (Proposition 7.4). In particular the boundary components of \(\sigma_\infty\) are two closed geodesics whose lengths are equal to the translation length of the loxodromic \(\rho(\ell)\) corresponding to the admissible loop \(\ell\) on \(C\). Thus we can identify the boundary components of \(\sigma_\infty\) and obtain a hyperbolic structure \(\tau_\infty\) on \(S\) and a heavy measured lamination \(L_\infty\) with a unique heavy leaf homotopic to \(\ell\). Although this (isometric) identification is a priori unique up to sharing along the heavy leaf, in fact

**Lemma 8.1.** There is a unique identification of the boundary components of \(\sigma_\infty\) so that the resulting pair \((\tau_\infty, L_\infty)\) is realized by a \(\rho\)-equivariant pleated surface that coincides, in the complement of \(\ell\), with the pleated surface \(\iota_\infty\) corresponding to \(C_\infty\).

**Proof.** Let \(\tilde{\ell}\) be a lift of \(\ell\) to \(\tilde{S}\). Let \(\hat{\ell}\) be the total lift of \(\ell\) to \(\tilde{S}\). Let \(\gamma_\ell\) be the element of \(\pi_1(S)\) that corresponds to \(\ell\) and preserves \(\tilde{\ell}\). Let \(P_1\) and \(P_2\) be the connected components of \(\tilde{S} \setminus \tilde{\ell}\) that are adjacent along \(\tilde{\ell}\).

For each \(k = 1, 2\), let \(\iota_{P_k} : X_k \rightarrow \mathbb{H}^3\) denote the pleated surface for the connected component of \(C_\infty\) corresponding to \(P_k\) so that \(\iota_{P_k}\) is equivariant under the restriction of \(\rho\) to the subgroup \(\pi_1(S)\) that preserves \(P_k\). Let \(g_k\) be the boundary geodesic of \(X_k\) corresponding to \(\tilde{\ell}\). Then \(\iota_{P_k}\) isometrically takes \(g_k\) to the axis of \(\rho(\gamma_\ell)\). Thus there is a unique identification of \(g_1\) and \(g_2\) so that \(\iota_{P_1}\) and \(\iota_{P_2}\) continuously extends the union \(X_1 \cup X_2\) given by the identification. Then, by quotienting out \(X_1 \cup X_2\) by the infinite cyclic group generated by \(\gamma_\ell\), the identification of \(g_1\) and \(g_2\) descends to a unique isometry between the boundary components of \(\sigma_\infty\).

Since \(C_\infty\) is obtained by grafting \(C\) and \(dev(C)\) is \(\rho\)-equivariant, the identification of the boundary components of \(\sigma_\infty\) is independent of the choice of the lift \(\tilde{\ell}\). Then, applying the identification of boundary components for all adjacent components of \(\tilde{S} \setminus \tilde{\ell}\), we obtain a \(\rho\)-equivariant pleated surface from \(\mathbb{H}_2\) to \(\mathbb{H}_3\) realizing \((\tau_\infty, L_\infty)\). \(\Box\)

9. CONVERGENCE OF MEASURED LAMINATIONS UNDER GRAFTINGS

By Lemma 8.1, identifying the boundary components of \((\sigma_\infty, N_\infty)\), we obtained a hyperbolic structure \(\tau_\infty\) on \(S\) and the heavy measured
lamination $L_{\infty}$ on $\tau_{\infty}$, so that $\ell$ is the unique heavy leaf of $L_{\infty}$. Recalling $\text{Gr}_A^{\delta}(C) \cong (\tau_i, L_i)$, we show

**Theorem 9.1.** $(\tau_i, L_i)$ converges to $(\tau_{\infty}, L_{\infty})$ as $i \to \infty$.

For each $i \in \mathbb{N}$, let $e_i : C_i \to C_\infty$ denote the canonical isomorphic embedding. Then $e_i$ gives an exhaustion of $C_\infty$,

$C_1 \subset C_2 \subset C_3 \subset \cdots \subset C_\infty$.

Let $p$ be a point on $C_\infty$, and let $\tilde{p}$ be a lift of $p$ to the universal cover $\tilde{C}_\infty$. Let $U_\infty(\tilde{p}) \subset \tilde{C}_\infty$ be the canonical neighborhood (§3.2) of $\tilde{p}$.

For each $i \in \mathbb{N}$, let $\tilde{C}_i$ be the universal cover of $C_i$. If $\ell$ is non-separating, then let $\tilde{C}_i$ be the quotient of $\tilde{C}_i$ by $\pi_1(S \setminus \ell)$. If $\ell$ is separating, for each connected component $F$ of $S \setminus \ell$, quotient $\tilde{C}_i$ by $\pi_1(F)$, and let $\tilde{C}_i$ be the disjoint union of both quotients. Then $C_i$ is isomorphically embedded in $\tilde{C}_i$.

For sufficiently large $i$, we have $p \in C_i \subset \tilde{C}_i$. Accordingly $\tilde{p} \in \tilde{C}_i \subset \tilde{C}_i$. Let $U_i$ and $U_\infty$ be the canonical neighborhoods of the point $\tilde{p}$ in $\tilde{C}_i$ and $\tilde{C}_\infty$, respectively. Fix any metric on $\tilde{C}$ inducing the (standard) topology of $\tilde{C}$ (e.g. a spherical metric). Note that canonical neighborhoods embeds into $\tilde{C}$ by developing maps. We consider the following relative version of the hausdorff metric: two non-empty subsets $X$ and $Y$ of $\tilde{C}$ are $\epsilon$-close if, in the Hausdorff metric, $X$ is $\epsilon$-close to $Y$ and $\partial X$ is $\epsilon$-close to $\partial Y$. Then, with this distance on the subsets on $\tilde{C}$,

**Proposition 9.2.** $U_i$ converges to $U_\infty$ as $i \to \infty$.

**Proof of Proposition 9.2.** Let $U_i$ be the canonical neighborhood of $\tilde{p}$ in $\tilde{C}_i$. Then, since $\tilde{C}_i$ embeds into $\tilde{C}_i$ and $\tilde{C}_\infty$, canonically $U_i \subset U_\infty$ and $U_i \subset U_\infty$.

**Lemma 9.3.** $U_i$ converges to $U_\infty$ as $i \to \infty$.

**Proof.** For every $\epsilon > 0$, there exits finitely many closed round balls $B_1, B_2, \ldots, B_n$ in $\tilde{C}_\infty$ containing $\tilde{p}$ such that $\bigcup_{k=1}^n B_k$ is $\epsilon$-close to $U_\infty$ in $\tilde{C}$. Since $\{\tilde{C}_i\}$ exhausts $\tilde{C}_\infty$, thus each $B_k$ is also contained in $\tilde{C}_i$ for sufficiently large $i$. Thus $\tilde{C}_i$ contains $\bigcup_{k=1}^n B_k$, and therefore $\bigcup_{k=1}^n B_k$ is contained in $U_i$.  \(\square\)
Since canonical neighborhoods are topologically open balls, by $U_i \subset \mathcal{U}_i$ and Lemma 9.3, it suffices to show that for every $\epsilon > 0$, if $i$ is large enough, then $\partial U_i$ is contained in the (honest) $\epsilon$-neighborhood of $\partial \mathcal{U}_\infty$.

**Lemma 9.4.** Given any $x \in \partial_\infty \mathcal{U}_\infty$ and any neighborhood $\mathcal{V}_x$ of $x$ in $\hat{\mathcal{C}}$, then $\mathcal{V}_x$ is not a subset of $U_i$ for sufficiently large $i \in \mathbb{N}$.

**Proof.** Suppose that the assertion fails; then there is a neighborhood $\mathcal{V}_x$ of $x$ in $\hat{\mathcal{C}}$, such that, for every $n \in \mathbb{N}$, there is $i > n$ with $\mathcal{V}_x \subset U_i$.

Let $\mathcal{N}_\infty$ be the circular lamination of $\mathcal{C}_\infty$ that descends to $\mathcal{N}_\infty$, and let $\tilde{\mathcal{N}}_\infty$ be the total lift of $\mathcal{N}_\infty$ to the universal cover $\tilde{\mathcal{C}}_\infty$.

Since $\mathcal{N}_\infty$ is nonempty, the endpoints of leaves of $\tilde{\mathcal{N}}_\infty$ is dense in the ideal boundary $\partial_\infty \mathcal{C}_\infty$. Therefore we can in addition assume that $x$ is an endpoint of a leaf of $\tilde{\mathcal{N}}_\infty$. Then the leaf contains a ray $\tilde{r}: [0, \infty) \to \tilde{\mathcal{C}}_\infty$ ending at $x$. Let $r$ be the projection of $\tilde{r}$ to $\mathcal{C}_\infty$.

For every $s > 0$, since $r|[0, s]$ is a compact subset of $\mathcal{C}_\infty$, thus, for sufficiently large $i$, it is also a circular curve in $\mathcal{C}_i$ and thus in $\mathcal{C}_i$. Accordingly $\tilde{r}|[0, s]$ is a circular arc embedded in $\tilde{\mathcal{C}}_i$. Since $\tilde{r}$ ends at $x$, if $s > 0$ is sufficiently large, $\tilde{r}|[s, \infty)$ is contained in $\mathcal{V}_x$. Therefore $\tilde{r}$ is contained in a compact subset of $U_i$ for sufficiently large $i$. Thus we induce a contradiction, showing that $x$ is an ideal point of $\tilde{\mathcal{C}}_i$ for sufficiently large $i$.

First suppose that $r$ stays in the compact subset of $\mathcal{C}_\infty$. Then, since every compact subset of $\mathcal{C}_\infty$ naturally embeds into $\mathcal{C}_i$ for sufficiently large $i$, accordingly $\tilde{r}$ is naturally a circular ray in $\tilde{\mathcal{C}}_i$ limiting to a point of $\partial_\infty \tilde{\mathcal{C}}_i$.

Next suppose that no compact subset of $\mathcal{C}_\infty$ contains $r$. Then the admissible $\ell$ loop must spiral— otherwise $\ell$ is roughly circular, and every leaf of $\mathcal{N}_\infty$ is contained in a compact subset of $\mathcal{C}_\infty$. The projection of $r$ to $\sigma_\infty$ is an (eventually simple) geodesic ray spiraling towards a boundary component $c$ of $\sigma_\infty$ (with respect to the Thurston metric). Then $x$ is a fixed point of the corresponding loxodromic element. For each $j \in \mathbb{N}$, let $b_j$ be the boundary component of $\mathcal{C}_j$ isotopic to $c$, so that $b_j$ isomorphic to $\ell$. Then $b_j$ are parallel in $\mathcal{C}_\infty$ and $r \in \mathcal{C}_\infty$ intersects $b_i$ for sufficiently large $i$. Let $\tilde{b}_j: \mathbb{R} \to \tilde{\mathcal{C}}_j$ be a (parametrized) lift of $b_j$ so that $\tilde{b}_j(t)$ limits to $x$ as $t \to \infty$. Then $\mathcal{V}_x$ contains $\tilde{b}_j(t)$ for all $t > t_j$ with some $t_j$. Therefore $U_i$ must contain $b_j(t)$ ($t > t_j$) as well. Since $x$ is also an ideal point of $\tilde{\mathcal{C}}_i$. 

Then Lemma 9.4 implies that
Corollary 9.5. For every $\epsilon > 0$, if $i \in \mathbb{N}$ is large enough, then the $\epsilon$-neighborhood of $U_\infty$ contains all maximal balls $B_i$ in $\hat{C}$ containing $\tilde{p}$.

Proof. The point $\tilde{p}$ is contained in $\hat{C} \setminus \partial_\infty U_\infty$. Note $\partial_\infty U_\infty$ is a compact subset of $\hat{C}$. Then $\hat{C} \setminus \partial_\infty U_\infty$ enjoys a canonical projective structure, and $U_\infty$ naturally is isomorphic to the canonical neighborhood of $\tilde{p}$ in the complement.

For every $\delta > 0$, take finitely many points $x_1, \ldots, x_n(\delta)$ in $\partial_\infty U_\infty$ so that their $\delta$-neighborhoods $\mathcal{V}_1, \ldots, \mathcal{V}_n(\delta)$ cover $\partial_\infty U_\infty$. Then their union $\mathcal{V}_1 \cup \cdots \cup \mathcal{V}_n(\delta)$ converges to $\partial_\infty U_\infty$ as $\delta \to 0$ in the Hausdorff topology. By Lemma 9.4, if $i$ is sufficiently large, there is a point $y_k$ in $\mathcal{V}_k$ that is not contained in $U_i$ for each $k = 1, \ldots, n(\delta)$. Then $U_i$ is contained in the canonical neighborhood of $\tilde{p}$ in the punctured sphere $\hat{C} \setminus \{y_1, \ldots, y_n(\delta)\}$. Since $\{y_1, \ldots, y_n(\delta)\}$ converges $\partial_\infty U_\infty$ as $\delta \to 0$, for every $\epsilon > 0$, if $i$ is sufficiently large, $U_i$ is contained in the $\epsilon$-neighborhood of $U_\infty$. \qed

Corollary 9.5 immediately implies that $U_i$ is contained in the $\epsilon$-neighborhood of $U_\infty$. Then $\partial U_i$ is contained in the $\epsilon$-neighborhood of $\partial U_i$, since $U_i \subset U_i$ and $U_i \to U_\infty$.  

10. Convergence of domains in $\hat{C}$ and Thurston coordinates

Let $R$ be a subset of $\mathbb{C}$ homeomorphic to an open disk. By Theorem 3.1, the projective structure on $R$ has Thurston coordinates $(\mathbb{H}^2, L)$, where $L$ is a measured lamination on $\mathbb{H}^2$ (note that $L$ contains no heavy leaf since $R$ is embedded in $\hat{C}$). Let $\beta: \mathbb{H}^2 \to \mathbb{H}^3$ denote the corresponding pleated surface. Let $\mathcal{L} = (\nu, \omega)$ be the circular measured lamination on $R$ that descends to $L = (\lambda, \mu)$ via the collapsing map $\kappa: R \to \mathbb{H}^2$, where $\lambda = |L|$ and $\mu \in \mathcal{T} \mathcal{M}(\lambda)$.
Fix a conformal identification of \( \hat{C} \) with \( S^2 \) in order to fix a spherical Riemannian metric on \( \hat{C} \). Then let \( \{ R_i \} \) be a sequence of regions in \( \hat{C} \) homeomorphic to an open disk, such that \( \partial R_i \to \partial R \) and \( \hat{C} \setminus R_i \to \hat{C} \setminus R \) in the Hausdorff topology. For each \( i \), we similarly let \((\mathbb{H}^2, L_i)\) denote Thurston coordinates of the projective structure on \( R_i \); let \( \beta_i: \mathbb{H}^2 \to \mathbb{H}^3 \) be the corresponding pleated surface; let \( L_i = (\nu_i, \omega_i) \) be the circular measured lamination on \( R_i \) that descends to \( L = (\lambda, \mu) \) via the collapsing map \( \kappa_i: R_i \to \mathbb{H}^2 \).

Every compact subset of \( R \) is also a compact subset of \( R_i \) for sufficiently large \( i \). Then for every compact subset \( K \) of the domain \( \mathbb{H}^2 \) of \( \beta \) if \( i \) is large enough, there is a (not necessarily continuous) map \( \psi_i: \mathbb{H}^2 \to \mathbb{H}^3 \) such that, for every \( k \in K \), there exists \( z \in R \) with \( \kappa(z) = k \) and \( \kappa_i(z) = \psi_i \circ \kappa(z) \).

**Theorem 10.1.**

(i) \( L_i \) converges to \( L \), uniformly on compacts, via the convergence of \( R_i \) to \( R \).

(ii) \( L_i \) converges to \( L \) pointwise.

(iii) \( \psi_i \) converges to an isometry uniformly on compacts; \( \beta_i \circ \psi_i: \mathbb{H}^2 \to \mathbb{H}^3 \) converges to \( \beta: \mathbb{H}^2 \to \mathbb{H}^3 \) uniformly on compacts.

**Remark 10.2.** In (i), by the uniform convergence, we mean that, for every \( \epsilon > 0 \) and every compact subset \( K \) of \( R \), if \( i \) is sufficiently large, then given any \( p, q \in K \), \( \omega(p,q) \) is \( \epsilon \)-close to \( \omega_i(p,q) \), where \( \omega_i(p,q) \) and \( \omega(p,q) \) denote the transversal measures of the geodesic segments connecting \( p \) to \( q \) on \( R_i \) and \( R \), respectively, in the Thurston metric. In (ii), by the pointwise convergence, for any \( p, q \in R \) not on leaves with positive weight, \( \mu_i(p,q) \to \mu(p,q) \) as \( i \to \infty \). In (iii), for every compact subset \( K \) of \( \mathbb{H}^2 \), \( \psi_i \) is \( \epsilon_i \)-close isometry with the sequence \( \epsilon_i > 0 \) converging to 0. The convergence, \( \beta_i \circ \psi_i \to \beta \) is with respect to the sup norm.

Note that (i) implies (ii) by the definition of \( \psi_i \). The rest of §10 is the proof of Proposition 10.1. For each point \( x \in R \), let \( B(x) \) be the maximal ball in \( R \) centered at \( x \). For sufficiently large \( i \), we have \( x \in R_i \). Thus let \( B_i(x) \) be the maximal ball in \( R_i \) centered at \( x \).

**Proposition 10.3.**

(i) For every \( \epsilon > 0 \) and every compact subset \( K \) of \( R \), if \( i \in \mathbb{N} \) is sufficiently large, then \( B_1(x) \) is \( \epsilon \)-close to \( B(x) \) for every \( x \in K \). (ii) For every \( \epsilon > 0 \) and \( x \in R \), there is a neighborhood \( U_x \) of \( x \) in \( R \), such that, if \( i \) is sufficiently large, then the ideal boundaries \( \partial_\infty B_1(y) \) and \( \partial_\infty B(y) \) are contained in the \( \epsilon \)-neighborhood of \( \partial_\infty B(x) \) in \( \hat{C} \) for all \( y \in U_x \).

**Proof of 10.3.** (See also the proof of Theorem 4.4 in [KP94].) For every compact subset \( X \) of the Euclidean plane \( \mathbb{R}^2 \), there is a unique closed
round ball \( D = D(X) \) of least radius containing \( X \). Let \( \partial_X(D) \) be the intersection of \( X \) with the boundary circle of \( D \). Then the minimality implies that the convex hull of \( \partial_X(D) \) (with the Euclidean metric) contains the center of \( D \). In addition that the uniqueness of \( D \) implies that \( D \) changes continuously when \( X \) changes continuously in the Hausdorff metric. Therefore, for every \( \epsilon > 0 \), there exists \( \delta > 0 \) such that if \( Y \) is a compact subset of \( \mathbb{R}^2 \) that is \( \delta \)-close to \( X \), then, letting \( D_Y \) be the round ball of least radius containing \( Y \), the \( \epsilon \)-neighborhood of \( \partial_X(D) \) contains \( \partial_Y(D_Y) \).

If \( x \) is a point in \( R \), regarding \( \hat{C} = \mathbb{R}^2 \cup \{ \infty \} \), we can assume that \( x = \{ \infty \} \) by an element of \( \text{PSL}(2, \mathbb{C}) \). Note that the round \( B(x) \) is the complement of \( D(\hat{C} \setminus R) \) in \( \hat{C} \). In addition \( \partial_{\infty}B(x) = \partial_XD(\hat{C} \setminus R) \). If \( U \) is a neighborhood of the identity element in \( \text{PSL}(2, \mathbb{C}) \), then \( Ux \) and \( U^{-1}x \) are neighborhoods of \( x = \{ \infty \} \) in \( \hat{C} \).

Thus, \( Ux \) is the complement of \( D(\hat{C} \setminus R) \) in \( \hat{C} \). In addition \( \partial_{\infty}B(x) = \partial_XD(\hat{C} \setminus R) \). If \( U \) is a neighborhood of the identity element in \( \text{PSL}(2, \mathbb{C}) \), then \( Ux \) and \( U^{-1}x \) are neighborhoods of \( x = \{ \infty \} \) in \( \hat{C} \). Thus for every \( \delta > 0 \), if the neighborhood \( U \) is sufficiently small, for every \( \gamma \in U \), \( R \) and \( \gamma R \) are \( \delta \)-close. Therefore, it follows from the preceding paragraph that, for \( \epsilon > 0 \), if \( U \) is sufficiently small, then, letting \( y = \gamma^{-1}x \), \( B(x) \) and \( B(y) \) are \( \epsilon \)-close and the \( \epsilon \)-neighborhood of \( \partial_{\infty}B(x) \) contains \( \partial_{\infty}B(y) \).

Since \( \hat{C} \setminus R_i \to \hat{C} \setminus R \) and \( \gamma R_i \) changes continuously in \( \gamma \in U \), for every \( \delta > 0 \), if \( i \) is sufficiently large and \( U \) is sufficiently small, then \( \mathbb{R}^2 \setminus \gamma R_i \) is \( \delta \)-close to \( \mathbb{R}^2 \setminus R \). Therefore, for every \( \epsilon > 0 \), we can assume that the maximal balls \( B(y) \) and \( B_i(y) \) are \( \epsilon \)-close to \( B(x) \) and the \( \epsilon \)-neighborhood of \( \partial_{\infty}B(x) \) contains \( \partial_{\infty}B(y) \), which proves (ii).

Thus \( B(y) \) and \( B_i(y) \) are \( 2\epsilon \)-close for all \( y \) in the small neighborhood \( U \) of \( x \). Since \( K \) is compact, this implies (i).

Recalling \( \hat{C} \setminus R_i \) converges to \( \hat{C} \setminus R \) in \( \hat{C} \) as \( i \to \infty \), let \( e: R \cap R_i \to R \) and \( e_i: R \cap R_i \to R_i \) be the obvious embeddings. Let \( \phi = \beta \circ \kappa \circ e: R \cap R_i \to \mathbb{H}^3 \) and \( \phi_i = \beta_i \circ \kappa_i \circ e_i: R_i \cap R \to \mathbb{H}^3 \).

Note that \( \nu \) and \( \nu_i \) are circular and we can measured their intersection angle with respect to the spherical Riemannian metric on \( \hat{C} \). Then

**Corollary 10.4.** \( \phi_i \) converges to \( \phi \) uniformly on compacts in \( R \) as continuous maps; therefore, \( \beta \circ \psi_i \) converges to \( \beta_i \) uniformly on compacts as \( i \to \infty \) in the sup norm.

**Proof.** By Proposition 10.3 (i), for every \( \epsilon > 0 \) and every compact subset \( K \) of \( R \), if \( i \in \mathbb{N} \) is sufficiently large, then for every \( p \in K \),
the maximal balls $B_i(p)$ and $B(p)$ of $R_i$ and $R$, respectively, at $p$ are $\epsilon$-close. Thus, for sufficiently large $i$, the orthogonal projection of $p$ into the totally geodesic hyperplane in $\mathbb{H}^3$ bounded by $\partial B_i$ is $\epsilon$-close to that into the hyper bounded by $\partial B_i$ for all $p \in K(\subset \hat{\mathcal{C}})$. Since these projections of $p$ are $\phi_i(p)$ and $\phi(p)$, the first assertion holds. Then the second assertion immediately follows from the definition of $\psi_i$. □

**Proposition 10.5.** Let $K$ be an arbitrary compact subsurface in $R$. Then $\angle_K(\nu, \nu) \to 0$ as $i \to \infty$.

**Proof.** Since $K$ is compact, it suffices to show that, for every $\epsilon > 0$ and $x \in R$, if an open neighborhood $U_x$ of $x$ in $R$ is sufficiently small, then $\angle_{U_x}(\nu, \nu) < \epsilon$ for sufficiently large $i$.

Suppose that $x$ is contained in a leaf $\ell_x$ of $\nu$. Let $\ell$ and $\ell_i$ be leaves of $\nu$ and $\nu_i$, respectively, that intersect in $U_x$. Let $B(\ell)$ be the maximal ball in $R$ whose core contains $\ell$, and let $B_i(\ell_i)$ be the maximal ball in $R_i$ whose core contains $\ell_i$. Then, it follows from Proposition 10.3 (ii) that, for every $\epsilon > 0$, if $U_x$ is small enough and $i$ is sufficiently large, then the endpoints of $\ell$ and $\ell_i$ are sufficiently close to the endpoints of $\ell_x$ so that $\angle_{U_x}(\ell_i, \ell) < \epsilon$. Hence $\angle_{U_x}(\nu, \nu) < \epsilon$.

Suppose that $x \in R \setminus \{\nu\}$. Then take $U_x$ disjoint from $\nu$. Then $\angle_{U_x}(\nu, \nu) = 0$. □

Let $d: R \times R \to \mathbb{R}_{\geq 0}$ be the continuous map obtained by pulling back of the hyperbolic distance function $\mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{R}_{\geq 0}$ via $\kappa: R \to \mathbb{H}^2$. Then $d$ is a pseudometric on $R$. (In fact $d$ coincides with the Thurston metric $d_{Th}$ in the subsurface where $d_{Th}$ is hyperbolic. In the subsurface where $d_{Th}$ is Euclidean, they also coincide along each leaf of $\nu$; however, circular arcs orthogonal to $\nu$ have, with respect to $d_{Th}$, length zero since they map to single points.) Similarly $d_i: R_i \times R_i \to \mathbb{R}_{\geq 0}$ be the pseudometric on $R_i$ obtained via $\kappa_i: R_i \to \mathbb{H}^2$.

**Proposition 10.6.** Let $K$ be a compact subset of $R$. Then, for every $\epsilon > 0$, if $i \in \mathbb{N}$ is sufficiently large, then $d$ and $d_i$ are $\epsilon$-close on $K \times K$, i.e. $|d(x, y) - d_i(x, y)| < \epsilon$ for all $x, y$ in $K$.

In the sense of Remark 10.2, we have

**Corollary 10.7.** $\psi_i$ converges to an isometry uniformly on compacts; Theorem 10.1 (iii) holds.

**Proof of Proposition 10.6.** First suppose that the Thurston metrics on $R$ and $R_i$ are hyperbolic on $K$—in other words, $|\nu|$ and $|\nu_i|$ have measure zero in $K$. Then $\kappa_i|K$ and $\kappa_i|\nu$ are $C^1$-diffeomorphism onto their images, and $d$ and $d_i$ are both hyperbolic metrics on $K$. Thus,
by Proposition 10.3 (i), for every $\epsilon > 0$, if $i$ is sufficiently large, then for every unit tangent vector $v$ at a point $x$ in $K$, the length of the derivative $\kappa'(v)$ is $\epsilon$-close to that of $\kappa_i'(v)$. Thus $d$ and $d_i$ are $\epsilon$-close in $K$. This special case extends to general Thurston metrics by:

**Proposition 10.8.** Let $P$ be a topological open disk in $\hat{\mathbb{C}}$ with $P \neq \mathbb{C}$, so that $P$ admits Thurston coordinates (by Proposition 3.1). Then, for every $\epsilon > 0$ and every compact subset $K$ of $P$ homeomorphic to a closed disk, there is another topological open disk $Q$ in $\hat{\mathbb{C}}$ containing $K$ such that

1. $P$ and $Q$ are $\epsilon$-close on $\hat{\mathbb{C}}$ in the Hausdorff metric,
2. the Thurston metric is hyperbolic on $Q$, and
3. letting $d_P: P \times P \to \mathbb{R}_{\geq 0}$ be the pseudo metric and $d_Q: Q \times Q \to \mathbb{R}_{\geq 0}$ be the metric defined as above, then $d_P$ and $d_Q$ are $\epsilon$-close in $K \times K$.

Indeed, by Proposition 10.8, we can take $Q$ and $Q_i$ for each $i$ that are close to $R$ and $R_i$ on $\hat{\mathbb{C}}$, respectively, so that $d_Q$ and $d_{Q_i}$ are sufficiently close to $d_R$ and $d_{R_i}$ on $K \times K$. Then, by (ii), the general case is reduced to the case that $|\nu|$ and $|\nu_i|$ have measure zero.

**Proof of Proposition 10.8.** For every $\delta_1 > 0$, pick another open topological disk $Q$ contained in $P$ such that

1. $\partial Q$ is a smooth loop embedded in $\hat{\mathbb{C}}$ and the sign of its curvature changes at most finitely many times, and
2. the Hausdorff distance of $\hat{\mathbb{C}} \setminus P$ and $\hat{\mathbb{C}} \setminus Q$ is less than $\delta_1$ ((i)).

We can in addition assume that $Q$ also contains $K$ by taking sufficiently small $\delta_1$. Let $(\mathbb{H}^2, L_Q)$ be the Thurston coordinates of the projective surface $Q$. Set $L_Q = (\nu_Q, \omega_Q)$ to be the circular measured lamination on $Q$ descending to $L_Q$. By the smoothness in (1), for each point of $\partial Q$, there is a unique maximal ball in $Q$ tangent, at the point, to $\partial Q$. Thus every leaf of $L_Q$ has no atomic measure. Thus the Thurston metric on $Q$ is hyperbolic ((iii)). Then, by the assumption on the curvature in (1), the two-dimensional strata of $(\mathbb{H}^2, L_Q)$ are isolated. Then, since $\partial Q$ is compact, there are only finitely many two-dimensional strata of $(Q, L_Q)$, and they mutually share no ideal point. Let $\text{pr}_P: \mathbb{H}^3 \to \text{Conv}(\hat{\mathbb{C}} \setminus P)$ denote the nearest point projection onto the convex hull of $\hat{\mathbb{C}} \setminus P$ in $\mathbb{H}^3$. Let $(P, L_P) \to (\mathbb{H}^2, L_P)$ denote the collapsing map of $P$. Then $\partial \text{Conv}(\hat{\mathbb{C}} \setminus P)$ is the pleated surface induced by $(\mathbb{H}^2, L_P)$. 

For \( \delta_2 > 0 \), consider the \( \delta_2 \)-neighborhood of \( \text{Conv}(\mathcal{C} \setminus P) \). Then its boundary surface \( S_{\delta_2} \) is \( C^1 \)-smooth and it enjoys an intrinsic Riemannian metric induced from \( \mathbb{H}^3 \) (see [EM87]). Similarly let \( \text{pr}_{\delta_2} : P \to S_{\delta_2} \) denote the orthogonal projection along geodesics in \( \mathbb{H}^3 \); then \( \text{pr}_{\delta_2} \) is a \( C^1 \)-diffeomorphism. Consider the Riemannian metric on \( P \) obtained by pulling back the Riemannian metric on \( S_{\delta_2} \) via \( \text{pr}_{\delta_2} \), let \( d_{\delta_2} : P \times P \to \mathbb{R}_{\geq 0} \) be the associated distance function of \( P \). Then (iii) follows from:

**Claim 10.9.** For every \( \epsilon > 0 \), if \( \delta_1 > 0 \) and \( \delta_2 > 0 \) are sufficiently small then

1. \( d_{\delta_2} \) and \( d_Q \) are \((1 + \epsilon)\)-bilipschitz on \( K \times K \), i.e.

\[
1 - \epsilon < d_{\delta_2}(x, y)/d_Q(x, y) < 1 + \epsilon
\]

for all distinct \( x, y \) in \( K \).

2. \( d_{\delta_2} \) and \( d_P \) are \( \epsilon \)-close on \( K \times K \).

**Proof.** (1) For each point \( x \in P \), let \( H_{\delta_2}(x) \) be the unique hyperbolic plane in \( \mathbb{H}^3 \) tangent to \( S_{\delta_2} \) at \( \text{pr}_{\delta_2}(x) \). Then the boundary circle \( \partial H_{\delta_2}(x) \) is contained in \( P \). The boundary of the maximal ball \( B_P(x) \) bounds another hyperbolic plane supporting, at \( \text{pr}_P(x) \), the pleated surface bounding \( \text{Conv}(\mathcal{C} \setminus P) \). Those two hyperbolic planes are perpendicular to the geodesic through at \( \text{pr}_{\delta_2}(x) \) and \( \text{pr}_P(x) \). Then the distance between the planes is exactly \( \delta_2 \).

Let \( B_{\delta_2}(x) \) be the round open ball in \( P \) bounded by \( \partial H_{\delta_2}(x) \). Then \( B_{\delta_2}(x) \) contains \( x \). Let \( B_P(x) \) and \( B_Q(x) \) be the maximal balls in \( P \) and \( Q \), respectively, centered at \( x \). Then, for every \( \epsilon > 0 \), if \( \delta_2 > 0 \) is sufficiently small, then \( B_P(x) \) is \( \epsilon \)-close to \( B_{\delta_2}(x) \) on \( \mathcal{C} \) for every \( x \in P \). In addition, by Proposition 10.3 (i), if \( \delta_1 > 0 \) is sufficiently small then, \( B_P(x) \) and \( B_Q(x) \) are \( \epsilon \)-close for all \( x \in K \). Then \( B_Q(x) \) and \( B_{\delta_2}(x) \) are \( 2\epsilon \)-close. Therefore if \( \delta_1 > 0 \) and \( \delta_2 > 0 \) are sufficiently small then, for every unite tangent vector \( v \) at a point in \( K \), the derivatives \( d\text{pr}_{\delta_2}(v) \) and \( d\text{pr}_P(v) \) are tangent vectors in \( \mathbb{H}^3 \), that are \( \epsilon \)-close. Thus \( d_{\delta_2} \) and \( d_Q \) are \( \epsilon \)-bilipschitz on \( K \).

(2) Let \( H \) be the subsurface of \( P \) where the Thurston metric is hyperbolic. Then \( \text{pr}_P \) takes \( H \) isometrically onto its image in \( \partial \text{Conv}(\mathcal{C} \setminus P) \) with the intrinsic metric induced by \( \mathbb{H}^3 \). By denitifying \( L_P \) and its image on \( \partial \text{Conv}(\mathcal{C} \setminus P) \), then, \( x \in H \) if and only if \( \text{pr}_P(x) \) is not on a leaf of \( L_P \) with positive weight. Then \( \text{pr}_P \) is \( C^1 \)-smooth on \( H \). Thus similarly to (1), for every \( \epsilon > 0 \), if \( \delta_2 > 0 \) is sufficiently small, then \( d_{\delta_2} \) and \( d_P \) are \( \epsilon \)-bilipschitz on each connected component of \( H \).

Each connected component \( E_{\ell} \) of the Euclidean subsurface of \( P \) corresponds to a leaf \( \ell \) of \( L_P \) with positive weight, so that \( E_{\ell} = \text{pr}_P^{-1}(\ell) \). we
show that, for every $\epsilon > 0$, if $\delta_2 > 0$ is sufficiently small, then, for every leaf $\ell$ of $L_P$ with weight $w(\ell) > 0$, $d_P$ and $d_{\delta_2}$ are $(1 + \epsilon, w(\ell)\delta_2)$-quasi isometric on $E_\ell$.

Then $E_\ell$ is, in the Thurston metric, an infinite Euclidean strip with width $w(\ell)$. Thus we may regard $E_\ell$ as a subset of $\mathbb{R}^2$ so that it is infinite in the vertical direction. On $\hat{\mathbb{C}}$, the strip $E_\ell$ is regarded as a wedge, i.e. a region bounded by two circular arcs sharing both endpoints. Consider the $\delta_2$-neighborhood $M$, in $\mathbb{H}^3$, of the geodesic $m$ connecting the vertices of $E_\ell$ — it is an infinite solid cylinder invariant under any hyperbolic translation along $m$. Then $d_{\delta_2}$ on $E_\ell$ is given by pulling back the intrinsic metric on $\partial M$ by $pr_{\delta_2}$. The boundary of $M$ is foliated by round loops bounding (geometric) disks of radius $\delta_2$ orthogonal to $m$ in $\mathbb{H}^3$. Then, by the nearest point projection $\mathbb{H}^3 \to m$, each loop map to a single point on $m$. In addition there is another foliation of $\partial M$ by straight lines (with its intrinsic metric) that are orthogonal to the round loops. Then each straight line diffeomorphically projects onto $m$ by the projection $\mathbb{H}^3 \to m$.

For different points $p, q$ in $E_\ell$, a geodesic connecting $p$ to $q$ with $d_P$ can be realized as a union of a vertical geodesic segment and horizontal geodesic segment. On vertical lines in $E_\ell$, for every $\epsilon > 0$, if $\delta_2 > 0$ is sufficiently small, then $d_P$ and $d_{\delta_2}$ are $(1 + \epsilon)$-bilipschitz. On the other hand, on the horizontal lines, $d_P$ and $d_{\delta_2}$ are $w(\ell)\delta_2$ rough isometric. Therefore for even $\epsilon > 0$, if $\delta_2 > 0$ is small, then for every leaf $\ell$ of $L_P$ of positive weight, the projection $d_P$ and $d_{\delta_2}$ are $(1 + \epsilon, w(\ell)\delta_2)$-quasiisometric on $E_\ell$.

The total transversal measure on $K$ given by $\mathcal{L}_P$ is finite. Therefore, for every $\epsilon > 0$, if $\delta_2 > 0$ is sufficiently small, then $d_P$ and $d_{\delta_2}$ are $(1 + \epsilon, \epsilon)$-quasiisometric on $K$. Since $K$ is a compact subset, we can in addition assume that they are $\delta_2$-rough isometric. \hfill $\Box$

In the rest of this section, we show the convergence of the transversal measures for Theorem 10.1 (i).

**Proposition 10.10.** Let $p_1, p_2$ be (distinct) points in a single stratum of $(\mathcal{R}, \mathcal{L})$. The $\omega_i(p_1, p_2) \to \omega(p_1, p_2) = 0$ as $i \to \infty$, where $\omega(p_1, p_2)$ and $\omega_i(p_1, p_2)$ are the transversal measures of, in Thurston metrics, the geodesic segment from $p_1$ to $p_2$ on $R$ and $R_i$, respectively.

**Proof.** Let $P$ be the strata of $(\mathcal{R}, \mathcal{L})$ containing $p_1$ and $p_2$. Then let $\alpha : [1, 2] \to P$ be the geodesic segment from $p_1$ to $p_2$. Let $Q = \kappa(P)$, the corresponding strata of $(\mathbb{H}^2, L)$. We can naturally identify $Q$ and $\beta(Q)$.
For each \( j = 1, 2 \), let \( q_j = \kappa(p_j) \) and \( q_{i,j} = \kappa_i(p_j) \) for all sufficiently large \( i \in \mathbb{N} \). Then, for each \( j \), the point \( \beta(q_{i,j}) \) converges to the point \( \beta(q_j) \) as \( i \to \infty \) (Corollary 10.4). Let \( N \) be the totally geodesic hyperplane in \( \mathbb{H}^3 \) orthogonally intersecting \( \beta(Q) \) in the geodesic segment from \( \beta(q_1) \) to \( \beta(q_2) \).

For each \( t \in [1, 2] \), considering the maximal ball in \( R_t \) centered at \( \alpha(t) \), let \( H_{i,t} \) be the totally hyperbolic plan in \( \mathbb{H}^3 \) bounded by the boundary of the maximal ball. By Proposition 10.3, \( \sup_{i \in [1,2]} \angle (N, H_{i,t}) \to \pi/2 \) as \( i \to \infty \). Thus, for every \( \epsilon > 0 \), if \( i \) is sufficiently large, then \( H_{i,s} \) and \( H_{i,t} \) intersect and \( \angle(H_{i,s}, H_{i,t}) < \epsilon \) for all \( s, t \in [1, 2] \). This implies that \( \mu_i(q_1, q_2) < \epsilon \) (see [EM87]). Therefore \( \omega_i(p_1, p_2) < \epsilon \). \( \square \)

Let \( d_R \) denote Thurston metric on \( R \). Then

**Proposition 10.11.** For every \( \epsilon > 0 \), there exists \( \delta > 0 \) such that, if \( p_1, p_2 \in R \) are points contained in different strata of \( (R, L) \) satisfying \( d_R(p_1, p_2) < \delta \) and \( \angle([p_1, p_2], L) < \delta \), then

\[
1 - \epsilon < \frac{\omega(p_1, p_2)}{\omega_i(p_1, p_2)} < 1 + \epsilon,
\]

for all sufficiently large \( i \in \mathbb{N} \).

**Proof of 10.11.** Let \( p_1, p_1 \) be points in \( R \) satisfying the assumptions. Let \( q_1 = \kappa(p_1) \) and \( q_2 = \kappa(p_2) \). Then for every \( \epsilon > 0 \), \( \delta > 0 \) is sufficiently small, then there is a hyperbolic plane \( N \) in \( \mathbb{H}^3 \) passing \( \beta(q_1), \beta(q_2) \) so that \( N \) is \( \epsilon \)-nearly orthogonal to \( \beta(Q) \) for all strata \( Q \) of \( (\mathbb{H}^2, L) \) intersecting \([q_1, q_2]\).

Let \( \alpha: [1, 2] \to R \) be the geodesic connecting \( p_1 \) to \( p_2 \). Then, for each \( t \in [1, 0] \), the maximal ball in \( R \) centered at \( \alpha(t) \) shares its boundary circle with a unique hyperbolic plane \( H_t \) in \( \mathbb{H}^3 \). Note that, if \( d_R(p_1, p_2) > 0 \) is sufficiently small, then \( H_s \) and \( H_t \) must intersect for all \( s, t \in [1, 2] \). Let \( \theta \) be a subdivision of \([1, 2]\) as \( 1 = t_0 < t_1 < \cdots < t_{n_\theta} = 2 \). Let \(|\theta|\) be the maximal width of the subintervals \([t_k, t_{k+1}]\) \((0 \leq k < n_\theta - 1)\). Then the transversal measure \( \omega([p_1, p_2]) \) is the limit of

\[
\sum_{k=1}^{n_\theta} \angle_{\mathbb{H}^3}(H_{t_k}, H_{t_{k+1}})
\]

as \(|\theta| \to 0 \), where \( \angle_{\mathbb{H}^3}(H_{t_k}, H_{t_{k+1}}) \) be the angle taking its value in \([0, \pi/2]\) between the hyperbolic planes \( H_{t_k} \) and \( H_{t_{k+1}} \) in \( \mathbb{H}^3 \). Note that this summation decreases when the subdivision \( \theta \) is refined ([EM87, II.1.10]).

For \( s, t \in [1, 2] \), the geodesic \( H_s \cap N \) intersect the geodesic \( H_t \cap N \); let \( \angle_N(H_s, H_t) \in [0, \pi/2] \) denote their intersection angle in \( N \). Since \( \angle_{\mathbb{H}^3}(H_t, N) \) is \( \epsilon \)-close to \( \pi/2 \) for all \( t \in [0, 1] \), by taking sufficiently small
\[ \delta > 0, \text{ we can assume that} \]
\[ 1 - \epsilon < \frac{\angle_{\mathbb{H}^3}(H_s, H_t)}{\angle_N(H_s, H_t)} < 1 + \epsilon, \]
for all \( s, t \in [1, 2] \). Thus
\[ 1 - \epsilon < \frac{\sum_{k=1}^{n_\theta} \angle_{\mathbb{H}^3}(H_{t_k}, H_{t_{k+1}})}{\sum_{k=1}^{n_\theta} \angle_N(H_{t_k}, H_{t_{k+1}})} < 1 + \epsilon. \]

Since \( R_i \) contains the geodesic segment \( \alpha \) for sufficiently large \( i \), similarly let \( H_{i,t} \) be a copy of \( \mathbb{H}^2 \) such that \( \partial_\infty H_{i,t} \) bounds the maximal ball in \( R_i \) centered at \( \alpha(t) \). Then the transversal measure \( \omega_i(p_1, p_2) \) is the limit of
\[ \sum_{k=1}^{\infty} \angle_{\mathbb{H}^3}(H_{i,t_k}, H_{i,t_{k+1}}) \]
as \( |\theta| \to 0 \). For every \( \epsilon > 0 \), if \( i \) is sufficiently large, the hyperbolic planes \( H_t \) and \( H_{i,t} \) are \( \epsilon \)-close for all \( t \in [1, 2] \). Thus, if \( \delta > 0 \) is sufficiently small and \( i \) is sufficiently large, then \( H_{i,t} \) intersects \( N \) at an angle \( \epsilon \)-close to \( \pi/2 \). Thus we can in addition assume that
\[ 1 - \epsilon < \frac{\sum_{k=1}^{\infty} \angle_{\mathbb{H}^3}(H_{i,t_k}, H_{i,t_{k+1}})}{\sum_{k=1}^{\infty} \angle_N(H_{i,t_k}, H_{i,t_{k+1}})} < 1 + \epsilon, \]
for any subdivision \( \theta \).

Therefore it remains to show that, if \( |\theta| \) is sufficiently small, then
\[ -\epsilon < \sum_{k=1}^{n_\theta} \angle_N(H_{t_k}, H_{t_{k+1}}) - \sum_{k=1}^{n_\theta} \angle_N(H_{t_k}, H_{t_{k+1}}) < \epsilon. \]

Consider the convex subset \( X_\theta \) of \( \mathbb{H}^3 \) bounded by the hyperbolic planes \( H_{t_1}, \ldots, H_{t_n} \) so that \( X_\theta \) contains \( \text{Conv}(\mathbb{C} \setminus R) \). Then \( N \) intersects \( \partial X_\theta \) nearly orthogonally, and the intersection is a piecewise geodesic that is a convex bi-infinite curve through \( \beta(q_1) \) and \( \beta(q_2) \), and its non-smooth points are between \( \beta(q_1) \) and \( \beta(q_2) \). Pick a segment \( \eta_\theta \) of this curve that is slightly larger than the segment from \( \beta(q_1) \) to \( \beta(q_2) \) so that the interior of \( \eta_\theta \) contains \( \beta(q_1) \) and \( \beta(q_2) \). Then \( \sum_{k=1}^{n_\theta} \angle_N(H_{t_k}, H_{t_{k+1}}) \) is equal to the sum of the exterior angles of \( \eta_\theta \).

Similarly let \( X_{i,\theta} \) be the convex subset of \( \mathbb{H}^3 \) bounded by \( H_{i,t_1}, \ldots, H_{t_n} \), such that \( X_{i,\theta} \) contains \( \text{Conv}(\mathbb{C} \setminus R_i) \). Then \( \partial X_{i,\theta} \cap N \) is a piecewise geodesic convex curve in \( N \), which converges to the convex curve \( \partial X_\theta \cap N \) above as \( i \to \infty \). For sufficiently large \( i \), each endpoint \( \eta_\theta \) has a unique closest point on \( \partial X_{i,\theta} \cap N \). Then those closest points cut off a segment \( \eta_{i,\theta} \) of \( \partial X_{i,\theta} \cap N \) that contains all non-smooth points. Then \( \sum_{k=1}^{n_\theta} \angle_N(H_{i,t_k}, H_{i,t_{k+1}}) \) is the sum of the exterior angles of \( \eta_{i,\theta} \).

Consider the loop \( \ell_i \) that obtained by connecting the corresponding endpoints of \( \eta_\theta \) and \( \eta_{i,\theta} \) by geodesic segments. Then, since \( \eta_{i,\theta} \) converges to \( \eta_\theta \) as \( i \to \infty \), the area in \( N \) bounded by \( \ell_i \) converges to 0 as \( i \to \infty \).
By applying Gauss-Bonnet Theorem to \( \ell_i \) in the hyperbolic plane \( N \), we obtain (4).

**Proposition 10.12.** For all \( p, q \in R \), \( \omega_i(p, q) \to \omega(p, q) \) as \( i \to \infty \).

*Proof.* For every \( \delta > 0 \), pick a simple piecewise geodesic path \( \eta = \bigcup_{k=1}^n [p_k, p_{k+1}] \) in \( R \) connecting \( p \) to \( q \), where \( p_k \) are points in \( R \), such that \( d_R(p_k, p_{k+1}) < \delta \) and \( \pi/2 - \delta < \angle(\mathcal{L}, [p_k, p_{k+1}]) < \pi/2 - \delta \) for all \( k = 0, 1, \ldots, n - 1 \). By Proposition 10.11, if \( \delta > 0 \) is sufficiently small, then if \( p_k \) and \( p_{k+1} \) are in different strata of \( (R, \mathcal{L}) \), then

\[
1 - \epsilon < \frac{\omega(p_k, p_{k+1})}{\omega_i(p_k, p_{k+1})} < 1 + \epsilon
\]

For sufficiently large \( i \). If \( p_k \) and \( p_{k+1} \) are in a single stratum of \( (R, \mathcal{L}) \), then by Proposition 10.10, \( \omega_i(p_k, p_{k+1}) \to 0 = \omega(p_k, p_{k+1}) \). Clearly \( \omega_i(p, q) = \sum_{k=1}^n \omega_i(p_k, p_{k+1}) \) and \( \omega(p, q) = \sum_{k=1}^n \omega(p_k, p_{k+1}) \). Thus for every \( \epsilon > 0 \), if \( \delta > 0 \) is sufficiently small, then \( |w(p, q) - w_i(p, q)| < \epsilon \) for sufficiently large \( i \). \( \square \)

**Corollary 10.13.** Let \( K \) be a compact subset of \( R \). Then for every \( \epsilon > 0 \), if \( i \in \mathbb{N} \) is sufficiently large, then \( -\epsilon < w(p, q) - w_i(p, q) < \epsilon \) for all \( p, q \in K \).

*Proof.* For every point \( x \in K \), there is a neighborhood \( U_x \) such that It follows from Proposition 10.3 that, for every \( \epsilon > 0 \), if \( i \) is sufficiently large, then \( -\epsilon < w(y, z) - w_i(y, z) < \epsilon \) for all \( y, z \in U_x \). Since \( K \) is compact, there are finitely many points \( x_1, \ldots, x_n \) such that \( U_{x_1}, \ldots, U_{x_n} \) cover \( K \). Applying Proposition 10.12 to all pairs of points in \( x_1, \ldots, x_n \), we have \( -\epsilon < w(x_j, x_k) - w_i(x_j, x_k) < \epsilon \) for all \( 0 < j, k \leq n \). Then the Triangle Inequality implies the corollary. \( \square \)

11. **Proof of Theorem 6.5**

Let \( \mathcal{K} \) be a compact connected surface \( \pi_1 \)-injectively embedded in \( \mathcal{C} \). Recalling the natural embedding \( e_i : \mathcal{C} \to \mathcal{C}_\infty \), let \( \mathcal{K}_i = e_i^{-1}(\mathcal{K}) \) for each \( i \in \mathbb{N} \). Since \( \mathcal{C}_1 \subset \mathcal{C}_2 \subset \ldots \) exhausts \( \mathcal{C}_\infty \), if \( i \) is sufficiently large, \( \mathcal{K}_i \) is isomorphic to \( \mathcal{K} \) by \( e_i \), and thus \( \mathcal{K}_i \) is a compact subsurface of \( \mathcal{C}_i \). Since \( \mathcal{C}_1 \subset \mathcal{C}_i \), naturally \( \mathcal{K}_1 \subset \mathcal{K}_i \). Recall that \( \tau_i \) and \( \tau_\infty \) are homeomorphic to \( S \) and that \( \tau_\infty \) is obtained by identifying the boundary geodesics of \( \sigma_\infty \). Then, since \( \kappa_i : \mathcal{C}_i \to \tau_i \) and \( \kappa_\infty : \mathcal{C}_\infty \to \tau_\infty \) are collapsing maps, when \( \mathcal{K}_i \) is isomorphic to \( \mathcal{K} \), then \( \kappa_i |_{\mathcal{K}_i} \) and \( \kappa_\infty |_{\mathcal{K}_i} \) are homotopic as maps to \( S \). Let \( \tilde{\kappa}_i(\tilde{\mathcal{K}}_i) \) and \( \tilde{\kappa}_\infty(\tilde{\mathcal{K}}) \) denote the universal covers of \( \kappa_i(\mathcal{K}_i) \) and \( \kappa_\infty(\mathcal{K}) \), respectively. Then, recalling that \( \mathcal{N}_\infty \) is the canonical circular lamination on \( \mathcal{C}_\infty \), we have
Proposition 11.1. There exists a sequence of (not necessarily continuous) maps $\psi_i: \iota_\infty(\mathcal{K}) \to \kappa_i(\mathcal{K}_i)$ in $i \in \mathbb{N}$, such that, letting $\tilde{\psi}_i: \iota_\infty(\tilde{\mathcal{K}}) \to \tilde{\kappa}_i(\tilde{\mathcal{K}}_i)$ be the lift of $\psi_i$, which commutes with deck transformations, we have

(i) $L_i$ on $\mathcal{K}_i$ converges to $\mathcal{N}_\infty$ on $\mathcal{K}$ uniformly,

(ii) $\psi_i$ converges to an isometry uniformly as $i \to \infty$,

(iii) the sup distance between $\kappa_i \circ \iota_i^{-1}$ and $\psi_i \circ \iota_\infty$ converges to zero on $\mathcal{K}$ as $i \to \infty$ (Figure 12),

(iv) the sup distance between $\beta_i \circ \tilde{\psi}_i$ and $\beta_\infty$ converges to 0 on $\iota_\infty(\mathcal{K}_\infty)$ as $i \to \infty$,

and therefore

(v) for $x, y \in \iota_\infty(\mathcal{K})$ not on leaves of positive atomic measure, let $[x, y]$ be a geodesic segment connecting $x$ to $y$ in $\sigma_\infty$ and let $[\tilde{\psi}_i(x), \tilde{\psi}_i(y)]$ be the geodesic segment on $\tau_i$ that is homotopic to $\psi_i([x, y])$ with its endpoints fixed; then the transversal measure on $[\psi_i(x), \psi_i(y)]$ by $L_i$ converges to the transversal measure on the geodesic segment $[x, y]$ by $\mathcal{N}_\infty$.

More precisely, in (i), we mean that for every $\epsilon > 0$, if $i$ is sufficiently large, then for all $x, y \in \mathcal{K}$, then the transversal measure of $[x, y]$ given by $L_i$ is $\epsilon$-close to that given by $\mathcal{N}_\infty$. By (ii), we mean that for every $\epsilon > 0$, if $i$ is sufficiently large, then

$$-\epsilon < \text{dist}_{\mathbb{H}^2}(\tilde{\psi}_i(x), \tilde{\psi}_i(y)) - \text{dist}_{\mathbb{R}^2}(x, y) < \epsilon$$

for every $x, y \in \tilde{\iota}_\infty(\tilde{\mathcal{K}})$.

Proof of 11.1. It suffices to show that, for every $p$ in $\tilde{\mathcal{K}}_\infty$, there is a compact neighborhood of $p$ with the desired properties. Consider all maximal balls in $\tilde{\mathcal{C}}_\infty$ containing $p$; then the union of their cores is a neighborhood of $p$ contained in the canonical neighborhood of $p$. Thus we can assume that $\mathcal{K}$ is a simply connected region contained this union.

For sufficiently large $i \in \mathbb{N}$, let $p_i \in \tilde{\mathcal{C}}_i$ such that $\tilde{e}_i(p_i) = p$. Let $U_i$ be the canonical neighborhood of $p_i$ in $\tilde{\mathcal{C}}_i$. By Proposition 9.2, $U_i$
converges to $\mathcal{U}_\infty$ and $\partial \mathcal{U}_i$ converges to $\partial \mathcal{U}_\infty$ on $\hat{\mathbb{C}}$ as $i \to \infty$ in the Hausdorff metric for $\hat{\mathbb{C}}$ (by fixing a natural metric on $\hat{\mathbb{C}}$). Hence, by Theorem 10.1 (including the definition of $\phi_i$) and Proposition 3.5, we have (i) - (iv).

Let $l_\infty$ and $l_i$ be the geodesic representatives of $\ell$ in $\tau_\infty$ and $\tau_i$, respectively. We first show that $\tau_i \to \tau_\infty$ and $\beta_i \to \beta_\infty$ as $i \to \infty$. Let $\sigma_i$ be $\tau_i \setminus l_i$. Then, by Proposition 11.1 (ii), $\sigma_i$ converges to $\sigma_\infty(=\tau_\infty \setminus l_\infty)$ as $i \to \infty$. In other words, $\tau_i$ converges to $\tau_\infty$ possibly up to a "twist" along $l_\infty$. By Proposition 11.1 (iii), the restriction of $\beta_i$ to a lift of $\sigma_i$ converges to the restriction of $\beta_\infty$ to the corresponding lift of $\sigma_\infty(\subset \tau_\infty)$ to $\mathbb{H}^2$. Since $\beta_i$ and $\beta$ are both $\rho$-equivariant, $\beta_i$ must converge to $\beta_\infty$ (c.f. §8) as $i \to \infty$, which proves (ii). Therefore $\tau_i$ must converge to $\tau_\infty$.

Last we show that the convergence of $L_i$. By Proposition 11.1 (v), the restriction of $L_i$ to $\sigma_i$ converges to the restriction of $L_\infty$ to $\sigma_\infty$ as $i \to \infty$ uniformly on compacts. Thus it is left to show that the transversal measure of $L_i$ near $l_i$ must diverges to $\infty$. Each connected component of $C_\infty \setminus C_0$ is a half-infinite grafting cylinder. Then this cylinder has infinite total transversal measure given by $N_\infty$. Thus, by Proposition 11.1 (iv), for any fixed $j \in \mathbb{N}$, the total transversal measure on $C_i \setminus C_j$ given by $L_i$ diverges to $\infty$ as $i \to \infty$. Let $\alpha_\infty$ be a smooth arc on $\tau_\infty$ transversal to $L_\infty$ such that $\alpha_\infty$ intersects $l_\infty$ in a single point. Then the transversal measure of $\alpha_\infty$ by $L_\infty$ is infinity. By the convergence $\tau_i \to \tau$, we have $|L_i| \to |L_\infty|$ (in $\mathcal{G} L$) as $i \to \infty$. Thus let $(\alpha_i)$ be a sequence of arcs $\alpha_i$ on $\tau_i$ smoothly converges to an arc $\alpha_\infty$, so that $\alpha_i$ is transversal to $L_i$ for sufficiently large $i$. Since the total transversal measure of $C_i \setminus C_j$ diverges as $i \to \infty$ as above, the divergence, accordingly the transversal measure of $\alpha_i$ given by $L_i$ must diverge to $\infty$. Therefore $L_i$ converges to $L_\infty$ as $i \to \infty$.

### Part 3. Appendix: density of holonomy map fibers $\mathcal{P}_\rho$ in $\mathcal{PML}$

Recall Thurston coordinates $\mathcal{P} \cong \mathcal{T} \times \mathcal{ML}$ (§3) on the space $\mathcal{P}$ of all (marked) projective structures on $S$. This gives an obvious projection from $\mathcal{P}$ to $\mathcal{ML}$. Then the obvious projection $\mathcal{ML} \setminus \{\emptyset\} \to \mathcal{PML}$ extends to $\Phi: \mathcal{ML} \to \mathcal{PML} \sqcup \{\emptyset\}$ so that the empty lamination $\emptyset$ maps to $\emptyset$.

Recall from §1 that $\mathcal{P}_\rho$ is the set of all projective structures with fixed holonomy $\rho: \pi_1(S) \to \text{PSL}(2,\mathbb{C})$ and that $\mathcal{P}_\rho$ is a discrete subset of $\mathcal{P}$. If $\rho$ is fuchsian, letting $\tau \in \mathcal{T}$ be the corresponding hyperbolic
structure, we have
\[ \mathcal{P}_\rho \cong \{ (\tau, M) \mid \text{multiloops } M \text{ with } 2\pi \text{- multiple weights} \}, \]
in Thurston coordinates ([Gol87], c.f. [Bab]). Thus \( \Phi(\mathcal{P}_\rho) \) is the union of \( \emptyset \) and a dense subset of \( \mathcal{PML} \). Note that a projective structure \( C \in \mathcal{P} \) maps to \( \emptyset \) via \( \Phi \) if and only if \( C \) is a hyperbolic structure ([Gol87]). Thus, for almost all \( \rho: \pi_1(S) \to \text{PSL}(2, \mathbb{C}) \), we have \( \Phi(\mathcal{P}_\rho) \not\ni \emptyset \).

**Theorem 11.2.** Given arbitrary \( \rho: \pi_1(S) \to \text{PSL}(2, \mathbb{C}) \), if \( \mathcal{P}_\rho \) is nonempty, then \( \Phi(\mathcal{P}_\rho) \setminus \{\emptyset\} \) is a dense subset of \( \mathcal{PML} \).

A Schottky decomposition of a representation \( \rho: \pi_1(S) \to \text{PSL}(2, \mathbb{C}) \) is a decomposition of \( S \) into pairs of pants along a (maximal) multiloop \( M \) such that the restriction of \( \rho \) to \( \pi_1(P_k) \) is an isomorphism onto a Schottky group for each pants \( P_k \). A Schottky decomposition of a projective structure \( C = (f, \rho) \) is a decomposition of \( C \) into pairs of pants along a multiloop \( M \) on consisting of admissible loops such that \( M \) realizes a Schottky decomposition of \( \rho \).

**Proposition 11.3.** Let \( \rho: \pi_1(S) \to \text{PSL}(2, \mathbb{C}) \) be the holonomy representation of some projective structure on \( S \). Then for every uniquely ergodic measured lamination \( L \), there is a sequence of projective structures \( C_i \) with holonomy \( \rho \) such that there is, for each \( i \), a Schottky decomposition of \( C_i \) along some admissible multiloop containing a loop \( \ell_i \) and \( [\ell_i] \to [L] \) in \( \mathcal{PML} \) as \( i \to \infty \).

**Proof.** Given a non-elementary representation \( \rho: \pi_1(S) \to \text{PSL}(2, \mathbb{C}) \) that lifts to \( \pi_1(S) \to \text{SL}(2, \mathbb{C}) \), Gallo Kapovich and Marden gave a Schottky decomposition of \( \rho \) along a multiloop \( M \) and then constructed a projective structure \( C = (f, \rho) \) with holonomy \( \rho \) that admits a Schottky decomposition along \( M \) ([GKM00, §4, 5]). We sketch their construction and explains how it implies the proposition, following the notations in [GKM00].

Let \( a \) be an arbitrary element of \( \pi_1(S) \) representing an essential loop on \( S \); then modify \( a \) in several steps to another loop (namely \( d^a \) in [GKM00, p.650]) that represents to a loop \( d' \). Then \( d' \) extends to a multiloop realizing a Schottky decomposition of \( \rho \). Thus, for a sequence of \( \alpha_i \in \pi_1(S) \) representing simple loops \( \alpha_i \) with \( [\alpha_i] \to [L] \) as \( i \to \infty \), letting \( d'_i \) be the loop given by appropriately applying the above contraction to \( d_i \), we claim that \( d'_i \) also converges to \( [L] \) because each modification step in the construction preserves the convergence property.

We can assume that \( \alpha_i \) are non-separating loop, by replacing \( \alpha_i \) to a non-separating loop disjoint from \( \alpha_i \). Then the convergence \( [\alpha_i] \to [L] \)
still holds. By abusing notation, we let elements of $\pi_1(S)$ also denote their corresponding loops on $S$. Then given $\rho : \pi_1(S) \rightarrow \text{PSL}(2, \mathbb{C})$, a handle is a pair of elements $a, b \in \pi_1(S)$ such that

- $a$ and $b$ are simple loops on $S$ intersecting in a single point, and
- $\rho(a), \rho(b)$ are loxodromic, and they generate a non-elementary subgroup of $\text{PSL}(2, \mathbb{C})$.

By modifying $a_i \in \pi_1(S)$, possibly in a few steps, we obtain a handle $H_i$ by Proposition 3.1.1 in [GKM00] for each $i$. If necessarily after changing $a_i$, let $H_i = \langle a_i, b_i \rangle$ with some $b_i \in \pi_1(S)$. By the proof of the proposition, we can assume that the projective classes $[a_i]$ and $[b_i]$ also converge to $[L]$ as $i \rightarrow \infty$ in $\mathcal{PML}$: Basically each modification is given by chaining a loop $\ell$ to another loop with a bounded intersection number with $\ell$ or to a loop “twisted” along $\ell$ many times.

For each handle $H_i$, we let $\alpha_i = \rho(a_i)$ and $\beta_i = \rho(b_i) \in \text{PSL}(2, \mathbb{C})$, which are loxodromic elements. Then we may in addition assume that $\beta_i$ does not take a fixed point of $\alpha_i$ to the other ([GKM00], §4.2). Indeed this modification is done by, if necessarily, replacing $\langle a_i, b_i \rangle$ by a new handle of the form either $\langle a_i b_i^k, b_i \rangle$ or $\langle b_i, a_i b_i^k \rangle$. This modification also preserves the convergence to $[L]$ since $a_i$ and $b_i$ intersects in a single point.

Pick another pair of non-separating loops $x_i, y_i$ in $\pi_1(S)$ such that $x_i, y_i$ intersect in a single point and they are disjoint from $a_i$ and $b_i$ ([GKM00], §4.3). Clearly $[x_i]$ and $[y_i]$ converge to $[L]$ as $i \rightarrow \infty$. Then the induced multiloop for the Schottky decomposition of $\rho$ contains a loop of the form $d_i^{n_i} x_i$, where $d_i = y_i b_i a_i^{k_i}$, for some $k_i, n_i \in \mathbb{Z}$. Then $d_i^{n_i} x_i$ is a non-separating loop disjoint from $b_i a_i^{k_i}$ ([GKM00], §4.5). Since $\langle a_i, b_i \rangle$ is a handle, the loop $b_i a_i^{k_i}$ intersects the loop $a_i$ in a single point, and thus the projective class $[b_i a_i^{k_i}]$ also converges to $[L]$ as $i \rightarrow \infty$. Hence $[d_i]$ and thus $[d_i^{n_i} x_i]$ converge to $[L]$ as $i \rightarrow \infty$.

\[\square\]

Proof of Theorem 11.2. Let $L$ be a uniquely ergodic measured lamination on $S$. By Proposition 11.3, there are sequences of projective structures $C_i$ with holonomy $\rho$ and admissible loops $\ell_i$ on $C_i$ converging to $[L]$ in $\mathcal{PML}$.

For $n_i \in \mathbb{N}$, consider the projective structure $\text{Gr}^{n_i}_i(C_i)$ obtained by $n_i$ times grafting $C_i$ along $\ell_i$. Then its measured lamination $L_{i,n_i}$ in Thurston coordinates, converges to $[\ell_i]$ as $n_i \rightarrow \infty$ in $\mathcal{PML}$ by Theorem 6.5. We can pick sufficiently large $n_i$ for each $i$ so that $[L_{i,n_i}]$ converges to $[L]$ as $i \rightarrow \infty$. Therefore $[L]$ is an accumulation point of $\Phi(\mathcal{P}_\rho)$. Almost all measured laminations are uniquely ergodic lamination and
in particular they are dense in $\mathcal{PML}$. Thus $\Phi(\mathcal{P}_P)$ is dense in $\mathcal{PML}$.

11.3

References


