A Survey on Class Field Theory for Varieties

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Abstract. We give a survey on class field theory for varieties over finite and over algebraically closed fields and explain some recent developments.

The aim of class field theory is the description of the abelian étale coverings of a scheme in terms of its arithmetic/geometric invariants. In this note we will focus on the case of varieties. We use the notation Sch/k for the category of separated schemes of finite type over a field k.

We start with a look at algebraic topology. Let T be a (sufficiently good) topological space and let $x \in T$ be a point. As is well known, there are two descriptions of the fundamental group of (T, x):

1) (The "outer" description): $\pi_1(T, x) = \mathrm{Aut}(F)$, where F is the fibre functor

$$\begin{array}{cccc} F: & \mathrm{Cov}(T) & \longrightarrow & \mathrm{Sets} \\ & (T' \xrightarrow{\pi} T) & \longmapsto & \pi^{-1}(x). \end{array}$$

If $\tilde{T} \to T$ is a universal covering space of T, then $\pi_1(T, x) \cong \mathrm{Aut}(\tilde{T}/T)$, the isomorphism being canonical up to inner automorphisms.

2) (The "inner" description):

$$\begin{array}{rcl} \pi_1(T, x) & = & [(S^1, *), (T, x)] \\ & = & \text{loops modulo homotopy.} \end{array}$$

For a locally noetherian scheme X with geometric base point $\bar{x} \to X$, we have the *étale fundamental group* $\pi_1^{\mathrm{et}}(X, \bar{x})$, a profinite group which is defined by the natural analogue of the outer description 1). It classifies finite, étale coverings of X. For normal schemes, the étale fundamental group can be understood in the language of classical Galois theory as follows (we omit base points from the notation):

- If $X = Spec(K)$ is the spectrum of a field, then $\pi_1^{\mathrm{et}}(X) \cong G_K$ (the absolute Galois group of K).
- Let X be normal connected with function field $k(X)$. Then there is a natural surjection $G_{k(X)} \cong \pi_1^{\mathrm{et}}(k(X)) \twoheadrightarrow \pi_1^{\mathrm{et}}(X)$, which identifies $\pi_1^{\mathrm{et}}(X)$ with the Galois group $\mathrm{Gal}(L/k(X))$ where L is the maximal extension of $k(X)$ inside $k(X)^{\mathrm{sep}}$ with the property that the normalization X_L of X in L is unramified over X.

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The question for an inner description of $\pi_1^{\mathrm{et}}(X)$ occurs naturally but has no satisfying answer so far. The considerably weaker task of describing the maximal abelian quotient $\pi_1^{ab}(X)$ of $\pi_1^{\mathrm{et}}(X)$ runs under the slogan "class field theory". There is no formal definition of what a class field theory for a scheme X should be but in all existing examples it always comes along with the following data:

- for every finite étale covering $Y \to X$ an abelian group $\mathcal{C}_Y$, which is defined in an explicit way out of Y.
- for all $Y' \to Y \to X$ finite étale, compatible norm maps $N_{Y'/Y} : \mathcal{C}_{Y'} \to \mathcal{C}_Y$.
- homomorphisms $rec_Y : \mathcal{C}_Y \to \pi_1^{ab}(Y)$ for all $Y \to X$ finite étale, such that the diagrams

$$\begin{tikzcd} \mathcal{C}_{Y'} \arrow[r, "rec_{Y'}"] \arrow[d, "N_{Y'/Y}"'] & \pi_1^{ab}(Y') \arrow[d, "can"] \\ \mathcal{C}_Y \arrow[r, "rec_Y"] & \pi_1^{ab}(Y) \end{tikzcd}$$

  commute and induce isomorphisms

$$C_Y/N_{Y'/Y}\mathcal{C}_{Y'} \xrightarrow{\sim} \mathrm{Gal}(\widetilde{Y'}/Y),$$

  where $\widetilde{Y}'/Y$ is the maximal abelian subcovering of Y'/Y.

The easiest example of a class field theory is given for the spectrum of a finite field: For $X = \mathit{Spec}(\mathbb{F})$, $\mathbb{F}$ a finite field, put $\mathcal{C}_X = \mathbb{Z}$ and

$$rec : \mathbb{Z} \longrightarrow G_{\mathbb{F}} = G_{\mathbb{F}}^{ab} \cong \hat{\mathbb{Z}},\ 1 \longmapsto \mathrm{Frob}_{\mathbb{F}}.$$

The norm map $N_{\mathbb{F}'/\mathbb{F}} : \mathbb{Z} \to \mathbb{Z}$ is multiplication by $[\mathbb{F}' : \mathbb{F}]$.

For a one dimensional local field $K = \mathbb{F}((t))$ and $X = \mathit{Spec}(K)$ we have

$$\mathcal{C}_X = K^\times,\ rec : K^\times \to G_K^{ab} \text{ (reciprocity map of local class field theory).}$$

As is well known, one obtains a short exact sequence

$$0 \longrightarrow K^\times \longrightarrow G_K^{ab} \longrightarrow \hat{\mathbb{Z}}/\mathbb{Z} \longrightarrow 0.$$

K. Kato [**Ka**] gave a generalization to higher dimensional local fields:

$K = \mathbb{F}((t_1, \dots, t_n))$, $X = \mathit{Spec}(K)$:

$rec : K_n^M(K) \longrightarrow G_K^{ab}$ ($K_n^M = n$-th Milnor K-group).

This can be globalized to

Theorem 1 (Global class field theory of Kato/Saito [**KS2**]). *Let X be a smooth, connected variety of dimension d over a finite field. Then we have a short exact sequence*

$$0 \longrightarrow \varprojlim_{\substack{\mathcal{I} \subset \mathcal{O}_{\bar{X}} \\ \mathcal{I}_{|X} = \mathcal{O}_X}} H^d_{Nis}(\bar{X}, \mathcal{K}_d^M(\mathcal{O}_{\bar{X}}, \mathcal{I})) \xrightarrow{rec} \pi_1^{ab}(X) \longrightarrow \hat{\mathbb{Z}}/\mathbb{Z} \to 0.$$

*Here $\bar{X}$ is a normal compactification of X, $\mathcal{K}_d^M$ is the d-th relative Milnor K-sheaf and H^d_{Nis} is cohomology in dimension d for the Nisnevich topology (cf. [**Ni**]).*

This solves the problem of describing the abelianized fundamental group $\pi_1(X)^{ab}$ in terms of geometric data attached to X. Unfortunately, the class module is difficult to understand and, in particular, contains a cohomology group. It is therefore desirable to find a more direct description.

For a connected topological space X, we have the Hurewicz isomorphism

$$\pi_1(X)^{ab} \cong H_1^{sing}(X, \mathbb{Z}).$$

Recall that the singular homology of X with values in an abelian group A is defined as follows: put

$$\Delta^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0,\ \Sigma\ x_i = 1\},$$

$$C_n(X) = \text{free abelian group on } \mathrm{Map}_{cts}(\Delta^n, X),$$

and set $d = \sum_{i=0}^{n}(-1)^i \delta_i^*$, where $\delta_i : \Delta^{n-1} \to \Delta^n$, $i = 0, \ldots, n$ are the face maps. Then $(C_\bullet(X), d)$ is a complex and

$$H_i^{sing}(X, A) = H_i(C_\bullet(X) \otimes A).$$

Returning to the case of varieties over some field k we can define

$$\Delta_k^n = \mathit{Spec}(k[T_0, \ldots, T_n]/\ \Sigma T_i - 1),$$

but we are confronted with the problem that $\mathrm{Mor}_k(\Delta_k^n, X)$ is too small for a general k-variety X. It was the idea of A. Suslin to use multi-valued morphisms (=finite correspondences) instead.

Definition 2 ([**SV**]).

$$C_n(X) \quad = \quad \begin{array}{l}\text{free abelian group on integral subschemes } Z \subset X \times \Delta^n \\ \text{such that } Z \to \Delta^n \text{ is finite and surjective.}\end{array}$$

The face maps are algebraic and $(C_\bullet(X), d)$ with $d = \sum_{i=0}^{n}(-1)^i \delta_i^*$ is a complex. The group

$$H_i^S(X, A) = H_i(C_\bullet(X) \otimes A)$$

is called the *Suslin homology* of X in dimension i with values in the abelian group A.

Example 3. $C_0(X) = \mathbb{Z}^{(|X|)}$ is the free abelian group on the set of closed points of X, i.e., the group of 0-cycles. Hence

$$H_0^S(X, \mathbb{Z}) = \mathbb{Z}^{(|X|)}/\sim,$$

where $\sim$ is some equivalence relation on the group of 0-cycles.

Fact 4 (cf. [**Sc**], Cor. 5.2). If X is proper, then $\sim$ is rational equivalence.

Fact 5 (Homotopy equivalence, cf. [**Sc**], Thm. 4.1).

$$H_i^S(X, A) \cong H_i^S(X \times \mathbb{A}_k^1, A).$$

However, if $\mathrm{char}(k) = p > 0$, then $\pi_1^{ab}(X) \neq \pi_1^{ab}(X \times \mathbb{A}_k^1)$, hence a Hurewicz isomorphism in perfect analogy to the situation in topology does not exist. This problem is resolved by passing to the *tame fundamental group*.

Definition 6 (cf. [**KSc**]). Let $C \in Sch/k$ be a regular curve and C' the unique regular compactification of C. We call a finite étale morphism $D \to C$ *tame* if every $v \in C' \smallsetminus C$ (considered as a discrete valuation of $k(C)$) is tamely ramified in $k(D)/k(C)$.

For an arbitrary $X \in Sch(k)$, a finite étale morphism $Y \to X$, is *tame* if for every morphism $C \to X$ with C a regular curve, the base change $C \times_X Y \to C$ is tame.

The *tame fundamental group* $\pi_1^t(X)$ of $X \in Sch(X)$ classifies tame finite étale coverings of X (cf. [**KSc**]) and is a quotient of the usual étale fundamental group $\pi_1^{\mathrm{et}}(X)$ in a natural way. Dually, for every $m \in \mathbb{N}$, we have the subgroup $H_t^1(X, \mathbb{Z}/m\mathbb{Z}) \subset H_{\mathrm{et}}^1(X, \mathbb{Z}/m\mathbb{Z})$ which classifies isomorphism classes of tame $\mathbb{Z}/m\mathbb{Z}$-torsors over X. The inclusion is equality if X is proper or if $p \nmid m$.

Next we construct a reciprocity map for the abelianized tame fundamental group. Let $k = \mathbb{F}$ be a finite field. Sending a closed point $x \in X$ to its Frobenius automorphism Frob_x defines a homomorphism

$$C_0(X) \longrightarrow \pi_1^{ab}(X)$$

from the group of zero cycles of X to its abelianized fundamental group. By [**Sc**, Thm. 8.1], the composite $C_0(X) \to \pi_1^{ab}(X) \twoheadrightarrow \pi_1^{t,ab}(X)$ factors through $H_0^S(X, \mathbb{Z})$ inducing

$$rec_X : H_0^S(X, \mathbb{Z}) \to \pi_1^{t,ab}(X). \tag{1}$$

We denote the kernel of the degree map $H_0^S(X, \mathbb{Z}) \to H_0^S(\mathbb{F}, \mathbb{Z}) \cong \mathbb{Z}$ by $H_0^S(X, \mathbb{Z})^0$ and the kernel of $\pi_1^{t,ab}(X) \to \pi_1^{t,ab}(\mathbb{F}) \cong \widehat{\mathbb{Z}}$ by $\pi_1^{t,ab}(X)^0$.

Theorem 7 (Artin if $\dim X = 1$, Kato-Saito if X is proper, Schmidt/Spieß for general X). *If X is smooth, then rec_X fits into an exact sequence*

$$0 \longrightarrow H_0^S(X, \mathbb{Z}) \xrightarrow{rec} \pi_1^{t,ab}(X) \longrightarrow \widehat{\mathbb{Z}}/\mathbb{Z} \longrightarrow 0.$$

The induced map on the degree zero subgroups $rec_X^0 : H_0^S(X, \mathbb{Z})^0 \to \pi_1^{t,ab}(X)^0$ is an isomorphism of finite abelian groups.

See [**SS**], [**Sc**] for a proof of Theorem 7. It generalizes the unramified class field theory of Kato and Saito [**KS1**], [**Sa**] to the case of smooth, not necessarily proper schemes. Recently, Kerz and Saito [**KeS**] found a generalization which describes the full fundamental group $\pi_1^{ab}(X)$ by using "Chow groups with modulus" instead of Suslin homology.

Note that the assumption on X being smooth is vital in Theorem 7. The cokernel of rec_X classifies completely split coverings and can be large if X is not geometrically unibranch. Furthermore, even for proper, normal schemes there are examples where rec_X is not injective [**MAS**].

Next we are going to construct a reciprocity map for varieties over algebraically closed fields. For U, $X \in Sch/k$ with U regular, the group of finite correspondences from U to X is defined by

$\mathrm{Cor}(U, X) =$ free abelian group on integral subschemes $Z \subset X \times U$ such that $Z \to U$ is finite and surjective over a connected component of U.

FACT 8. Let $\alpha \in \mathrm{Cor}(U, X)$ be a finite correspondence and let m be a positive integer. There exists a functor

$$\alpha^* : \mathrm{PHS}(X, \mathbb{Z}/m\mathbb{Z}) \longrightarrow \mathrm{PHS}(U, \mathbb{Z}/m\mathbb{Z})$$

from the category of étale $\mathbb{Z}/m\mathbb{Z}$-torsors on X to those on U which gives back the usual pull-back map $\alpha^* : H^1_{\mathrm{et}}(X, \mathbb{Z}/m\mathbb{Z}) \to H^1_{\mathrm{et}}(U, \mathbb{Z}/m\mathbb{Z})$ on isomorphism classes and which sends tame torsors to tame torsors.

Now let k be algebraically closed. For a tame $\mathbb{Z}/m\mathbb{Z}$-torsor $\mathcal{T}$ on X and a finite correspondence $\alpha : \Delta^1 \to X$ we obtain the tame, hence trivial torsor $\alpha^*(\mathcal{T})$ on $\Delta^1 \cong \mathbb{A}^1$. Parallel transport therefore induces an isomorphism

$$\Phi_{par} : 0^*(\alpha^*(\mathcal{T})) \xrightarrow{\sim} 1^*(\alpha^*(\mathcal{T}))$$

of $\mathbb{Z}/m\mathbb{Z}$-torsors over Δ^0. If α represents a 1-cocycle in the $\mathrm{mod}\, m$ Suslin complex, we furthermore obtain the tautological identification

$$\Phi_{taut} : 0^*(\alpha^*(\mathcal{T})) \xrightarrow{\sim} 1^*(\alpha^*(\mathcal{T}))$$

in addition. We conclude that there is a unique $\langle \alpha, \mathcal{T} \rangle \in \mathbb{Z}/m\mathbb{Z}$ such that

$$\Phi_{par} = (\text{translation by } \langle \alpha, \mathcal{T} \rangle) \circ \Phi_{taut}.$$

THEOREM 9 (Geisser/Schmidt). *For any $X \in Sch/k$ the assignment $(\alpha, \mathcal{T}) \mapsto \langle \alpha, \mathcal{T} \rangle$ induces a pairing*

$$H_1^S(X, \mathbb{Z}/m\mathbb{Z}) \times H_t^1(X, \mathbb{Z}/m\mathbb{Z}) \longrightarrow \mathbb{Z}/m\mathbb{Z}.$$

The induced homomorphism

$$rec_X : H_1^S(X, \mathbb{Z}/m\mathbb{Z}) \longrightarrow \pi_1^{t,ab}(X)/m$$

is surjective. It is an isomorphism of finite abelian groups if $(m, char(k)) = 1$, and for general m if resolution of singularities for schemes of dimension $\leq \dim X + 1$ holds over k.

A proof of Theorem 9 can be found in [**GS1**].

Returning to the case that $k = \mathbb{F}$ is finite, we recall the notion of Weil-Suslin homology introduced by Geisser [**Ge**]: Let $\bar{\mathbb{F}}$ be an algebraic closure of $\mathbb{F}$, $X \in Sch/\mathbb{F}$ and $\bar{X} = X \times_{\mathbb{F}} \bar{\mathbb{F}}$. The Frobenius automorphism $\mathrm{Frob} \in \mathrm{Gal}(\bar{\mathbb{F}}/\mathbb{F})$ acts on $C_n(\bar{X}) = \mathrm{Cor}(\bar{\Delta}^n, \bar{X})$ for all n and the Weil-Suslin homology of X with values in an abelian group A is defined by

$$H_n^{WS}(X, A) = H_n\big(\mathrm{cone}(C_\bullet(\bar{X}) \otimes A \xrightarrow{1-\mathrm{Frob}} C_\bullet(\bar{X}) \otimes A)\big).$$

The obvious homomorphism $H_0^S(X, \mathbb{Z}) \to H_1^{WS}(X, \mathbb{Z})$ is conjectured to be an isomorphism if X is smooth.

In a similar spirit as above, one constructs compatible pairings for all m

$$H_1^{WS}(X, \mathbb{Z}/m\mathbb{Z}) \times H_t^1(X, \mathbb{Z}/m\mathbb{Z}) \longrightarrow \mathbb{Z}/m\mathbb{Z}.$$

These pairings and the natural maps $H_1^{WS}(X, \mathbb{Z}) \to H_1^{WS}(X, \mathbb{Z}/m\mathbb{Z})$ induce a homomorphism

$$rec_X^{WS} : H_1^{WS}(X, \mathbb{Z}) \to \pi_1^{t,ab}(X) \tag{2}$$

such that composition with $H_S^0(X, \mathbb{Z}) \to H_1^{WS}(X, \mathbb{Z})$ is the map rec_X defined in (1) above.

THEOREM 10 (Geisser/Schmidt). *Let $X \in Sch/\mathbb{F}$ and assume that resolution of singularities holds for schemes of dimension $\leq \dim X + 1$ over $\mathbb{F}$. Then rec_X^{WS} induces an isomorphism*

$$H_1^{WS}(X, \mathbb{Z})^\wedge \to \pi_1^{t,ab}(X)$$

on profinite completions.

A proof of Theorem 10 can be found in [**GS2**].

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