On the $K(\pi, 1)$-property for rings of integers in the mixed case

By

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Abstract

We investigate the Galois group $G_S(p)$ of the maximal $p$-extension unramified outside a finite set $S$ of primes of a number field in the (mixed) case, when there are primes dividing $p$ inside and outside $S$. We show that the cohomology of $G_S(p)$ is ‘often’ isomorphic to the étale cohomology of the scheme $\text{Spec}(\mathcal{O}_k \setminus S)$, in particular, $G_S(p)$ is of cohomological dimension 2 then. We deduce this from the results in our previous paper [Sch2], which mainly dealt with the tame case.

§ 1. Introduction

Let $Y$ be a connected locally noetherian scheme and let $p$ be a prime number. We denote the étale fundamental group of $Y$ by $\pi_1(Y)$ and its maximal pro-$p$ factor group by $\pi_1(Y)(p)$. The Hochschild-Serre spectral sequence induces natural homomorphisms

$$\phi_i : H^i(\pi_1(Y)(p), \mathbb{Z}/p\mathbb{Z}) \longrightarrow H^i_{\text{et}}(Y, \mathbb{Z}/p\mathbb{Z}), \ i \geq 0,$$

and we call $Y$ a ‘$K(\pi, 1)$ for $p$’ if all $\phi_i$ are isomorphisms; see [Sch2] Proposition 2.1 for equivalent conditions. See [Wi2] for a purely Galois cohomological approach to the $K(\pi, 1)$-property. Our main result is the following

Theorem 1.1. Let $k$ be a number field and let $p$ be a prime number. Assume that $k$ does not contain a primitive $p$-th root of unity and that the class number of $k$ is prime to $p$. Then the following holds:

Let $S$ be a finite set of primes of $k$ and let $T$ be a set of primes of $k$ of Dirichlet density $\delta(T) = 1$. Then there exists a finite subset $T_1 \subset T$ such that $\text{Spec}(\mathcal{O}_k \setminus (S \cup T_1))$ is a $K(\pi, 1)$ for $p$.  


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Remarks. 1. If $S$ contains the set $S_p$ of primes dividing $p$, then Theorem 1.1 holds with $T_1 = \emptyset$ and even without the condition $\zeta_p \notin k$ and $\text{Cl}(k)(p) = 0$, see [Sch2], Proposition 2.3. In the tame case $S \cap S_p = \emptyset$, the statement of Theorem 1.1 is the main result of [Sch2]. Here we provide the extension to the ‘mixed’ case $\emptyset \subset S \cap S_p \subset S_p$.

2. For a given number field $k$, all but finitely many prime numbers $p$ satisfy the condition of Theorem 1.1. We conjecture that Theorem 1.1 holds without the restricting assumption on $p$.

Let $S$ be a finite set of places of a number field $k$. Let $k_S(p)$ be the maximal $p$-extension of $k$ unramified outside $S$ and put $G_S(p) = \text{Gal}(k_S(p)|k)$. If $S_R$ denotes the set of real places of $k$, then $G_{S \cup S_R}(p) \cong \pi_1(S\text{pec}(O_k) \setminus S)(p)$ (we have $G_S(p) = G_{S \cup S_R}(p)$ if $p$ is odd or $k$ is totally imaginary). The following Theorem 1.2 sharpens Theorem 1.1.

**Theorem 1.2.** The set $T_1 \subset T$ in Theorem 1.1 may be chosen such that

(i) $T_1$ consists of primes $p$ of degree 1 with $N(p) \equiv 1 \mod p$,

(ii) $(k_{S \cup T_1}(p))_p = k_p(p)$ for all primes $p \in S \cup T_1$.

Note that Theorem 1.2 provides nontrivial information even in the case $S \supset S_p$, where assertion (ii) was only known when $k$ contains a primitive $p$-th root of unity (Kuz’min’s theorem, see [Kuz] or [NSW], 10.6.4 or [NSW$^2$], 10.8.4, respectively) and for certain CM fields (by a result of Mukhamedov, see [Muk] or [NSW], X §6 exercise or [NSW$^2$], X §8 exercise, respectively).

By Theorem 3.3 below, Theorem 1.2 provides many examples of $G_S(p)$ being a duality group. If $\zeta_p \notin k$, this is interesting even in the case that $S \supset S_p$, where examples of $G_S(p)$ being a duality group were previously known only for real abelian fields and for certain CM-fields (see [NSW], 10.7.15 and [NSW$^2$], 10.9.15, respectively, and the remark following there).

Previous results in the mixed case had been achieved by K. Wingberg [Wil], Ch. Maire [Mai] and D. Vogel [Vog]. Though not explicitly visible in this paper, the present progress in the subject was only possible due to the results on mild pro-$p$ groups obtained by J. Labute in [Lab].

I would like to thank K. Wingberg for pointing out that the proof of Proposition 8.1 in my paper [Sch2] did not use the assumption that the sets $S$ and $S'$ are disjoint from $S_p$. This was the key observation for the present paper. The main part of this text was written while I was a guest at the Department of Mathematical Sciences of Tokyo University and of the Research Institute for Mathematical Sciences in Kyoto. I want to thank these institutions for their kind hospitality.
§ 2. Proof of Theorems 1.1 and 1.2

We start with the observation that the proofs of Proposition 8.1 and Corollary 8.2 in [Sch2] did not use the assumption that the sets $S$ and $S'$ are disjoint from $S_p$. Therefore, with the same proof (which we repeat for the convenience of the reader) as in loc. cit., we obtain

**Proposition 2.1.** Let $k$ be a number field and let $p$ be a prime number. Assume $k$ to be totally imaginary if $p = 2$. Put $X = \text{Spec}(\mathcal{O}_k)$ and let $S \subset S'$ be finite sets of primes of $k$. Assume that $X \setminus S$ is a $K(\pi, 1)$ for $p$ and that $G_S(p) \neq 1$. Further assume that each $p \in S' \setminus S$ does not split completely in $k_S(p)$. Then the following hold.

(i) $X \setminus S'$ is a $K(\pi, 1)$ for $p$.

(ii) $k_{S'}(p)_p = k_p(p)$ for all $p \in S' \setminus S$.

Furthermore, the arithmetic form of Riemann’s existence theorem holds, i.e., setting $K = k_S(p)$, the natural homomorphism

$$T(K_p(p)|K_p) \rightarrow \text{Gal}(k_{S'}(p)|K)$$

is an isomorphism. Here $T(K_p(p)|K_p)$ is the inertia group and $\ast$ denotes the free pro-$p$-product of a bundle of pro-$p$-groups, cf. [NSW], Ch. IV, §3. In particular, the group $\text{Gal}(k_{S'}(p)|k_S(p))$ is a free pro-$p$-group.

**Proof.** The $K(\pi, 1)$-property implies

$$H^i(G_S(p), \mathbb{Z}/p\mathbb{Z}) \cong H^i_{et}(X \setminus S, \mathbb{Z}/p\mathbb{Z}) = 0 \text{ for } i \geq 4,$$

hence $cd \ G_S(p) \leq 3$. Let $p \in S' \setminus S$. Since $p$ does not split completely in $k_S(p)$ and since $cd \ G_S(p) < \infty$, the decomposition group of $p$ in $k_S(p)|k$ is a non-trivial and torsion-free quotient of $Z_p \cong \text{Gal}(k_p^{nr}|k_p)$. Therefore $k_S(p)_p$ is the maximal unramified $p$-extension of $k_p$. We denote the normalization of an integral normal scheme $Y$ in an algebraic extension $L$ of its function field by $Y_L$. Then $(X \setminus S)_{k_S(p)}$ is the universal pro-$p$ covering of $X \setminus S$. We consider the étale excision sequence for the pair $((X \setminus S)_{k_S(p)}, (X \setminus S')_{k_S(p)})$. By assumption, $X \setminus S$ is a $K(\pi, 1)$ for $p$, hence $H^i_{et}((X \setminus S)_{k_S(p)}, \mathbb{Z}/p\mathbb{Z}) = 0$ for $i \geq 1$ by [Sch2], Proposition 2.1. Omitting the coefficients $\mathbb{Z}/p\mathbb{Z}$ from the notation, this implies isomorphisms

$$H^i_{et}((X \setminus S')_{k_S(p)}) \cong \bigoplus_{p \in S' \setminus S(k_S(p))} H^{i+1}_{p}(((X \setminus S)_{k_S(p)})_p)$$

for $i \geq 1$. Here (and in variants also below) we use the notational convention

$$\bigoplus_{p \in S' \setminus S(k_S(p))} H^{i+1}_{p}(((X \setminus S)_{k_S(p)})_p) := \lim_{\rightarrow \ K \subset k_S(p)} \bigoplus_{p \in S' \setminus S(K)} H^{i+1}_{p}(((X \setminus S)_{K})_p),$$
where $K$ runs through the finite extensions of $k$ inside $k_S(p)$. As $k_S(p)$ realizes the maximal unramified $p$-extension of $k_p$ for all $p \in S' \setminus S$, the schemes $((X \times S)_{k_S(p)}), p \in S' \setminus S(k_S(p))$, have trivial cohomology with values in $\mathbb{Z}/p\mathbb{Z}$ and we obtain isomorphisms

$$H^i((k_S(p))_p) \cong H^{i+1}_p(((X \times S)_{k_S(p)})_p)$$

for $i \geq 1$. These groups vanish for $i \geq 2$. This implies

$$H^i_{et}((X \times S')_{k_S(p)}) = 0$$

for $i \geq 2$. Since the scheme $(X \times S')_{k_S(p)}$ is the universal pro-$p$ covering of $(X \times S')_{k_S(p)}$, the Hochschild-Serre spectral sequence

$$E_2^{ij} = H^i(Gal(k_{S'}(p)|k_S(p)), H^j_{et}((X \times S')_{k_S(p)})) \Rightarrow H^{i+j}_{et}((X \times S')_{k_S(p)})$$

yields an inclusion

$$H^2(Gal(k_{S'}(p)|k_S(p))) \hookrightarrow H^2_{et}((X \times S')_{k_S(p)}) = 0.$$ 

Hence $Gal(k_{S'}(p)|k_S(p))$ is a free pro-$p$-group and

$$H^1(Gal(k_{S'}(p)|k_S(p))) \cong H^1_{et}((X \times S')_{k_S(p)}) \cong \bigoplus_{p \in S' \setminus S(k_S(p))} H^1(k_S(p)_p).$$

We set $K = k_S(p)$ and consider the natural homomorphism

$$\phi : \bigoplus_{p \in S' \setminus S(K)} T(K_p(p)|K_p) \longrightarrow Gal(k_{S'}(p)|K).$$

By the calculation of the cohomology of a free product ([NSW], 4.3.10 and 4.1.4), $\phi$ is a homomorphism between free pro-$p$-groups which induces an isomorphism on mod $p$ cohomology. Therefore $\phi$ is an isomorphism. In particular, $k_{S'}(p)_p = k_p(p)$ for all $p \in S' \setminus S$. Using that $Gal(k_{S'}(p)|k_S(p))$ is free, the Hochschild-Serre spectral sequence induces an isomorphism

$$0 = H^2_{et}((X \times S')_{k_S(p)}) \cong H^2_{et}((X \times S')_{k_S(p)})^{Gal(k_{S'}(p)|k_S(p))}.$$ 

Hence $H^2_{et}((X \times S')_{k_S(p)}) = 0$, since $Gal(k_{S'}(p)|k_S(p))$ is a pro-$p$-group. Now [Sch2], Proposition 2.1 implies that $X \times S'$ is a $K(\pi, 1)$ for $p$. 

In order to prove Theorem 1.1, we first provide the following lemma. For an extension field $K/k$ and a set of primes $T$ of $k$, we write $T(K)$ for the set of prolongations of primes in $T$ to $K$ and $\delta_K(T)$ for the Dirichlet density of the set of primes $T(K)$ of $K$. 

...
Lemma 2.2. Let $k$ be a number field, $p$ a prime number and $S$ a finite set of nonarchimedean primes of $k$. Let $T$ be a set of primes of $k$ with $\delta_{k(\mu_p)}(T) = 1$. Then there exists a finite subset $T_0 \subset T$ such that all primes $p \in S$ do not split completely in the extension $k_{T_0}(p)|k$.

Proof. By [NSW], 9.2.2 (ii) or [NSW2], 9.2.3 (ii), respectively, the restriction map

$$H^1(G_{T\cup S_p \cup S_\infty}(p), \mathbb{Z}/p\mathbb{Z}) \to \prod_{p \in S \cup S_p \cup S_\infty} H^1(k_p, \mathbb{Z}/p\mathbb{Z})$$

is surjective. A class in $\alpha \in H^1(G_{T\cup S_p \cup S_\infty}(p), \mathbb{Z}/p\mathbb{Z})$ which restricts to an unramified class $\alpha_p \in H^1_{nr}(k_p, \mathbb{Z}/p\mathbb{Z})$ for all $p \in S \cup S_p \cup S_\infty$ is contained in $H^1(G_T(p), \mathbb{Z}/p\mathbb{Z})$. Therefore the image of the composite map

$$H^1(G_T(p), \mathbb{Z}/p\mathbb{Z}) \to H^1(G_{T\cup S_p \cup S_\infty}(p), \mathbb{Z}/p\mathbb{Z}) \to \prod_{p \in S} H^1(k_p, \mathbb{Z}/p\mathbb{Z})$$

contains the subgroup $\prod_{p \in S} H^1_{nr}(k_p, \mathbb{Z}/p\mathbb{Z})$. As this group is finite, it is already contained in the image of $H^1(G_{T_0}(p), \mathbb{Z}/p\mathbb{Z})$ for some finite subset $T_0 \subset T$. We conclude that no prime in $S$ splits completely in the maximal elementary abelian $p$-extension of $k$ unramified outside $T_0$.

Proof of Theorems 1.1 and 1.2. As $p \neq 2$, we may ignore archimedean primes. Furthermore, we may remove the primes in $S \cup S_p$ and all primes of degree greater than 1 from $T$. In addition, we remove all primes $p$ with $N(p) \not\equiv 1 \mod p$ from $T$. After these changes, we still have $\delta_{k(\mu_p)}(T) = 1$.

By Lemma 2.2, we find a finite subset $T_0 \subset T$ such that no prime in $S$ splits completely in $k_{T_0}(p)|k$. Put $X = \text{Spec}(\mathcal{O}_k)$. By [Sch2], Theorem 6.2, applied to $T_0$ and $T \setminus T_0$, we find a finite subset $T_2 \subset T \setminus T_0$ such that $X \setminus (T_0 \cup T_2)$ is a $K(\pi, 1)$ for $p$. Then Proposition 2.1 applied to $T_0 \cup T_2 \subset S \cup T_0 \cup T_2$, shows that also $X \setminus (S \cup T_0 \cup T_2)$ is a $K(\pi, 1)$ for $p$. Now put $T_1 = T_0 \cup T_2 \subset T$.

It remains to show Theorem 1.2. Assertion (i) holds by construction of $T_1$. Again by construction, $X \setminus T_1$ is a $K(\pi, 1)$ for $p$. By [Sch2], Theorem 3, the field $k_{T_1}(p)$ realizes $k_p(p)$ for $p \in T_1$, showing (ii) for these primes. Finally, assertion (ii) for $p \in S$ follows from Proposition 2.1.

§ 3. Duality

We start by investigating the relation between the $K(\pi, 1)$-property and the universal norms of global units.
Let us first remove redundant primes from $S$: If $p \nmid p$ is a prime with $\zeta_p \notin k_p$, then every $p$-extension of the local field $k_p$ is unramified (see [NSW], 7.5.1 or [NSW$^2$], 7.5.9, respectively). Therefore primes $p \notin S_p$ with $N(p) \neq 1 \mod p$ cannot ramify in a $p$-extension. Removing all these redundant primes from $S$, we obtain a subset $S_{\text{min}} \subset S$, which has the property that $G_S(p) = G_{S_{\text{min}}(p)}$. Furthermore, by [Sch2], Lemma 4.1, $X \setminus S$ is a $K(\pi, 1)$ for $p$ if and only if $X \setminus S_{\text{min}}$ is a $K(\pi, 1)$ for $p$.

**Theorem 3.1.** Let $k$ be a number field and let $p$ be a prime number. Assume that $k$ is totally imaginary if $p = 2$. Let $S$ be a finite set of nonarchimedean primes of $k$. Then any two of the following conditions (a) – (c) imply the third.

(a) $\text{Spec}(O_k) \setminus S$ is a $K(\pi, 1)$ for $p$.
(b) $\lim_{K \subset k_S(p)} O_k^\times \otimes \mathbb{Z}_p = 0$.
(c) $(k_S(p))_p = k_p(p)$ for all primes $p \in S_{\text{min}}$.

The limit in (b) runs through all finite extensions $K$ of $k$ inside $k_S(p)$. If (a)–(c) hold, then also

$$\lim_{K \subset k_S(p)} O_{K,S_{\text{min}}}^\times \otimes \mathbb{Z}_p = 0.$$

**Remarks:** 1. Assume that $\zeta_p \in k$ and $S \supset S_p$. Then (a) holds and condition (c) holds for $p > 2$ if $\#S > r_2 + 2$ (see [NSW$^2$], Remark 2 after 10.9.3). In the case $k = \mathbb{Q}(\zeta_p)$, $S = S_p$, condition (c) holds if and only if $p$ is an irregular prime number.
2. Assume that $S \cap S_p = \emptyset$ and $S_{\text{min}} \neq \emptyset$. If condition (a) holds, then either $G_S(p) = 1$ (which only happens in very special situations, see [Sch2], Proposition 7.4) or (c) holds by [Sch2], Theorem 3 (or by Proposition 3.2 below).

**Proof of Theorem 3.1.** We may assume $S = S_{\text{min}}$ in the proof. Let $K$ run through the finite extensions of $k$ in $k_S(p)$ and put $X_K = \text{Spec}(O_K)$. Applying the topological Nakayama-Lemma ([NSW], 5.2.18) to the compact $\mathbb{Z}_p$-module $\lim_{K \subset k_S(p)} O_K^\times \otimes \mathbb{Z}_p$, we see that condition (b) is equivalent to

$$(b)' \lim_{K \subset k_S(p)} O_K^\times /p = 0.$$

Furthermore, by [Sch2], Proposition 2.1, condition (a) is equivalent to

$$(a)' \lim_{K \subset k_S(p)} H^i_{et}(X \setminus S)_K, \mathbb{Z}/p\mathbb{Z}) = 0 \text{ for } i \geq 1.$$  

Condition (a)$'$ always holds for $i = 1, i \geq 4$, and it holds for $i = 3$ provided that $G_S(p)$ is infinite or $S$ is nonempty or $\zeta_p \notin k$ (see [Sch2], Lemma 3.7). The flat Kummer sequence $0 \to \mu_p \to G_m \overset{p}{\to} G_m \to 0$ induces exact sequences

$$0 \to O_K^\times /p \to H^1_f(X_K, \mu_p) \to_p \text{Pic}(X_K) \to 0$$
for all $K$. As the field $k_S(p)$ does not have nontrivial unramified $p$-extensions, class field theory implies
\[
\lim_{K \subset k_S(p)} \nu P_{ic}(X_K) \subset \lim_{K \subset k_S(p)} P_{ic}(X_K) \otimes \mathbb{Z}_p = 0.
\]
As we assumed $k$ to be totally imaginary if $p = 2$, the flat duality theorem of Artin-Mazur ([Mil], III Corollary 3.2) induces natural isomorphisms
\[
H_{et}^2(X_K, \mathbb{Z}/p\mathbb{Z}) = H_\text{fl}^2(X_K, \mathbb{Z}/p\mathbb{Z}) \cong H_\text{fl}^1(X_K, \mu_p) \vee.
\]
We conclude that
\[
(*) \quad \lim_{K \subset k_S(p)} H_{et}^2(X_K, \mathbb{Z}/p\mathbb{Z}) \cong \left( \lim_{K \subset k_S(p)} \mathcal{O}_K^\times /p \right) \vee.
\]
We first show the equivalence of (a) and (b) in the case $S = \emptyset$. If (a)' holds, then (*) shows (b)'. If (b) holds, then $\zeta_p \notin k$ or $G_S(p)$ is infinite. Hence we obtain (a)' for $i = 3$. Furthermore, (b)' implies (a)' for $i = 2$ by (*). This finishes the proof of the case $S = \emptyset$.

Now we assume that $S \neq \emptyset$. For $p \in S(K)$, a standard calculation of local cohomology shows that
\[
H_i^p(X_K, \mathbb{Z}/p\mathbb{Z}) \cong \begin{cases} 
0 & \text{for } i \leq 1, \\
H^1(K_p, \mathbb{Z}/p\mathbb{Z})/H^1_{nr}(K_p, \mathbb{Z}/p\mathbb{Z}) & \text{for } i = 2, \\
H^2(K_p, \mathbb{Z}/p\mathbb{Z}) & \text{for } i = 3, \\
0 & \text{for } i \geq 4.
\end{cases}
\]
For $p \in S = S_{\text{min}}$, every proper Galois subextension of $k_p(p)|k_p$ admits ramified $p$-extensions. Hence condition (c) is equivalent to
\[
(c)' \quad \lim_{K \subset k_S(p)} \bigoplus_{p \in S(K)} H_i^p(X_K, \mathbb{Z}/p\mathbb{Z}) = 0 \text{ for all } i,
\]
and to
\[
(c)'' \quad \lim_{K \subset k_S(p)} \bigoplus_{p \in S(K)} H_i^2(X_K, \mathbb{Z}/p\mathbb{Z}) = 0.
\]
Consider the direct limit over all $K$ of the excision sequences
\[
\cdots \to \bigoplus_{p \in S(K)} H_i^p(X_K, \mathbb{Z}/p\mathbb{Z}) \to H_i^i(X_K, \mathbb{Z}/p\mathbb{Z}) \to H_i^i((X \setminus S)_K, \mathbb{Z}/p\mathbb{Z}) \to \cdots.
\]
Assume that (a)' holds, i.e. the right hand terms vanish in the limit for $i \geq 1$. Then (*) shows that (b)' is equivalent to (c)''.

Now assume that (b) and (c) hold. As above, (b) implies the vanishing of the middle term for $i = 2$ in the limit. Condition (c)' then shows (a)'.

We have proven that any two of the conditions (a)–(c) imply the third.
Finally, assume that (a)–(c) hold. Tensoring the exact sequences (cf. [NSW], 10.3.11 or [NSW²], 10.3.12, respectively)

$$0 \to \mathcal{O}_K^\times \to \mathcal{O}_{K,S}^\times \to \bigoplus_{p \in S(K)} (K_p^\times /U_p) \to \text{Pic}(X_K) \to \text{Pic}((X\setminus S)_K) \to 0$$

by (the flat $\Z$-algebra) $\Z_\mu$, we obtain exact sequences of finitely generated, hence compact, $\Z_\mu$-modules. Passing to the projective limit over the finite extensions $K$ of $k$ inside $k_S(p)$ and using $\lim\limits_{\leftarrow} \text{Pic}(X_K) \otimes \Z_\mu = 0$, we obtain the exact sequence

$$0 \to \lim_{K \subset k_S(p)} \mathcal{O}_K^\times \otimes \Z_\mu \to \lim_{K \subset k_S(p)} \mathcal{O}_{K,S}^\times \otimes \Z_\mu \to \lim_{K \subset k_S(p)} \bigoplus_{p \in S(K)} (K_p^\times /U_p) \otimes \Z_\mu \to 0.$$

Condition (c) and local class field theory imply the vanishing of the right hand limit. Therefore (b) implies the vanishing of the projective limit in the middle. □

If $G_S(p) \neq 1$ and condition (a) of Theorem 1.1 holds, then the failure in condition (c) can only come from primes dividing $p$. This follows from the next

**Proposition 3.2.** Let $k$ be a number field and let $p$ be a prime number. Assume that $k$ is totally imaginary if $p = 2$. Let $S$ be a finite set of nonarchimedean primes of $k$. If Spec($\mathcal{O}_k$)$\setminus S$ is a $K(\pi,1)$ for $p$ and $G_S(p) \neq 1$, then every prime $p \in S$ with $\zeta_p \in k_p$ has an infinite inertia group in $G_S(p)$. Moreover, we have

$$k_S(p)_p = k_p(p)$$

for all $p \in S_{\text{min}} \setminus S_p$.

**Proof.** We may assume $S = S_{\text{min}}$. Suppose $p \in S$ with $\zeta_p \in k_p$ does not ramify in $k_S(p)|k$. Setting $S' = S \setminus \{p\}$, we have $k_{S'}(p) = k_S(p)$, in particular,

$$H^1_{et}(X \setminus S', \Z/p\Z) \cong H^1_{et}(X \setminus S, \Z/p\Z).$$

In the following, we omit the coefficients $\Z/p\Z$ from the notation. Using the vanishing of $H^3_{et}(X \setminus S)$, the étale excision sequence yields a commutative exact diagram

$$
\begin{array}{ccc}
H^2(G_{S'}(p)) & \cong & H^2(G_S(p)) \\
\downarrow & & \downarrow \\
H^3_p(X) & \cong & H^3_{et}(X \setminus S').
\end{array}
$$

Hence $\alpha$ is split-surjective and $\Z/p\Z \cong H^3_p(X) \cong H^3_{et}(X \setminus S')$. This implies $S' = \emptyset$, hence $S = \{p\}$, and $\zeta_p \in k$. The same applies to every finite extension of $k$ in $k_S(p)$, hence $p$ is inert in $k_S(p) = k_\emptyset(p)$. This implies that the natural homomorphism

$$\text{Gal}(k_p^{nr}(p)|k_p) \to G_\emptyset(k)(p)$$
is surjective. Therefore $G_S(p) = G_G(p)$ is abelian, hence finite by class field theory. Since this group has finite cohomological dimension by the $K(\pi, 1)$-property, it is trivial, in contradiction to our assumptions.

This shows that all $p \in S$ with $\zeta_p \in k_p$ ramify in $k_S(p)$. As this applies to every finite extension of $k$ inside $k_S(p)$, the inertia groups must be infinite. For $p \in S_{\min} \setminus S_p$, this implies $k_S(p)_p = k_p(p)$. \hfill \Box

**Theorem 3.3.** Let $k$ be a number field and let $p$ be a prime number. Assume that $k$ is totally imaginary if $p = 2$. Let $S$ be a finite nonempty set of nonarchimedean primes of $k$. Assume that conditions (a)–(c) of Theorem 3.1 hold and that $\zeta_p \in k_p$ for all $p \in S$. Then $G_S(p)$ is a pro-$p$ duality group of dimension 2.

**Proof.** Condition (a) implies $H^3(G_S(p), \mathbb{Z}/p\mathbb{Z}) \cong H^3_{et}(X \setminus S, \mathbb{Z}/p\mathbb{Z}) = 0$. Hence $cd \ G_S(p) \leq 2$. On the other hand, by (c), the group $G_S(p)$ contains $Gal(k_p(p)|k_p)$ as a subgroup for all $p \in S$. As $\zeta_p \in k_p$ for $p \in S$, these local groups have cohomological dimension 2, hence so does $G_S(p)$.

In order to show that $G_S(p)$ is a duality group, we have to show that $$D_i(G_S(p), \mathbb{Z}/p\mathbb{Z}) := \lim_{U \subset G_S(p)} H^i(U, \mathbb{Z}/p\mathbb{Z})^\vee$$ vanish for $i = 0, 1$, where $U$ runs through the open subgroups of $G_S(p)$ and the transition maps are the duals of the corestriction homomorphisms; see [NSW], 3.4.6. The vanishing of $D_0$ is obvious, as $G_S(p)$ is infinite. We therefore have to show that $$\lim_{K \subset k_S(p)} H^1((X \setminus S)_K, \mathbb{Z}/p\mathbb{Z})^\vee = 0.$$ We put $X = Spec(\mathcal{O}_k)$ and denote the embedding by $j : (X \setminus S)_K \rightarrow X_K$. By the flat duality theorem of Artin-Mazur, we have natural isomorphisms $$H^1((X \setminus S)_K, \mathbb{Z}/p\mathbb{Z})^\vee \cong H^2_{fl,c}((X \setminus S)_K, \mu_p) = H^2_{fl}(X_K, j_!\mu_p).$$ The excision sequence together with a straightforward calculation of local cohomology groups shows an exact sequence

\[
\bigoplus_{p \in S(K)} K_p^\times / K_p^{\times p} \rightarrow H^2_{fl}(X_K, j_!\mu_p) \rightarrow H^2_{fl}((X \setminus S)_K, \mu_p).
\]

As $\zeta_p \in k_p$ and $k_S(p)_p = k_p(p)$ for $p \in S$ by assumption, the left hand term of (*) vanishes when passing to the limit over all $K$. We use the Kummer sequence to obtain an exact sequence

\[
Pic((X \setminus S)_K)/p \rightarrow H^2_{fl}((X \setminus S)_K, \mu_p) \rightarrow pBr((X \setminus S)_K).
\]
The left hand term of (**) vanishes in the limit by the principal ideal theorem. The Hasse principle for the Brauer group induces an injection

\[ p \mathrm{Br}((X \setminus S)_K) \hookrightarrow \bigoplus_{p \in S(K)} p \mathrm{Br}(K_p). \]

As \( k_S(p) \) realizes the maximal unramified \( p \)-extension of \( k_p \) for \( p \in S \), the limit of the middle term in (**) vanishes, and hence also the limit of the middle term in (*) vanishes. This shows that \( G_S(p) \) is a duality group of dimension 2.

\[ \square \]

**Remark:** The dualizing module can be calculated to

\[ D \cong \text{tor}_p(C_S(k_S(p))), \]

i.e. \( D \) is isomorphic to the \( p \)-torsion subgroup in the \( S \)-idèle class group of \( k_S(p) \). The proof is the same as in ([Sch1], Proof of Thm. 5.2), where we dealt with the tame case.

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**References**


