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On the $K(\pi, 1)$-property for rings of integers in the mixed case

By

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Abstract

We investigate the Galois group $G_S(p)$ of the maximal p-extension unramified outside a finite set S of primes of a number field in the (mixed) case, when there are primes dividing p inside and outside S. We show that the cohomology of $G_S(p)$ is 'often' isomorphic to the étale cohomology of the scheme $Spec(\mathcal{O}_k \smallsetminus S)$, in particular, $G_S(p)$ is of cohomological dimension 2 then. We deduce this from the results in our previous paper [Sch2], which mainly dealt with the tame case.

§ 1. Introduction

Let Y be a connected locally noetherian scheme and let p be a prime number. We denote the étale fundamental group of Y by $\pi_1(Y)$ and its maximal pro-p factor group by $\pi_1(Y)(p)$. The Hochschild-Serre spectral sequence induces natural homomorphisms

$$\phi_i : H^i(\pi_1^{et}(Y)(p), \mathbb{Z}/p\mathbb{Z}) \longrightarrow H^i_{et}(Y, \mathbb{Z}/p\mathbb{Z}),\ i \geq 0,$$

and we call Y a '$K(\pi, 1)$ for p' if all ϕ_i are isomorphisms; see [Sch2] Proposition 2.1 for equivalent conditions. See [Wi2] for a purely Galois cohomological approach to the $K(\pi, 1)$-property. Our main result is the following

Theorem 1.1. *Let k be a number field and let p be a prime number. Assume that k does not contain a primitive p-th root of unity and that the class number of k is prime to p. Then the following holds:*

Let S be a finite set of primes of k and let T be a set of primes of k of Dirichlet density $\delta(T) = 1$. Then there exists a finite subset $T_1 \subset T$ such that $Spec(\mathcal{O}_k) \smallsetminus (S \cup T_1)$ is a $K(\pi, 1)$ for p.

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Remarks. 1. If S contains the set S_p of primes dividing p, then Theorem 1.1 holds with $T_1 = \varnothing$ and even without the condition $\zeta_p \notin k$ and $Cl(k)(p) = 0$, see [Sch2], Proposition 2.3. In the tame case $S \cap S_p = \varnothing$, the statement of Theorem 1.1 is the main result of [Sch2]. Here we provide the extension to the 'mixed' case $\varnothing \subsetneq S \cap S_p \subsetneq S_p$.

2. For a given number field k, all but finitely many prime numbers p satisfy the condition of Theorem 1.1. We conjecture that Theorem 1.1 holds without the restricting assumption on p.

Let S be a finite set of places of a number field k. Let $k_S(p)$ be the maximal p-extension of k unramified outside S and put $G_S(p) = Gal(k_S(p)|k)$. If $S_{\mathbb{R}}$ denotes the set of real places of k, then $G_{S \cup S_{\mathbb{R}}}(p) \cong \pi_1(Spec(\mathcal{O}_k) \smallsetminus S)(p)$ (we have $G_S(p) = G_{S \cup S_{\mathbb{R}}}(p)$ if p is odd or k is totally imaginary). The following Theorem 1.2 sharpens Theorem 1.1.

Theorem 1.2. *The set $T_1 \subset T$ in Theorem 1.1 may be chosen such that*

(i) *T_1 consists of primes $\mathfrak{p}$ of degree 1 with $N(\mathfrak{p}) \equiv 1 \bmod p$,*

(ii) *$(k_{S \cup T_1}(p))_{\mathfrak{p}} = k_{\mathfrak{p}}(p)$ for all primes $\mathfrak{p} \in S \cup T_1$.*

Note that Theorem 1.2 provides nontrivial information even in the case $S \supset S_p$, where assertion (ii) was only known when k contains a primitive p-th root of unity (Kuz'min's theorem, see [Kuz] or [NSW], 10.6.4 or [NSW²], 10.8.4, respectively) and for certain CM fields (by a result of Mukhamedov, see [Muk] or [NSW], X §6 exercise or [NSW²], X §8 exercise, respectively).

By Theorem 3.3 below, Theorem 1.2 provides many examples of $G_S(p)$ being a duality group. If $\zeta_p \notin k$, this is interesting even in the case that $S \supset S_p$, where examples of $G_S(p)$ being a duality group were previously known only for real abelian fields and for certain CM-fields (see [NSW], 10.7.15 and [NSW²], 10.9.15, respectively, and the remark following there).

Previous results in the mixed case had been achieved by K. Wingberg [Wi1], Ch. Maire [Mai] and D. Vogel [Vog]. Though not explicitly visible in this paper, the present progress in the subject was only possible due to the results on mild pro-p groups obtained by J. Labute in [Lab].

I would like to thank K. Wingberg for pointing out that the proof of Proposition 8.1 in my paper [Sch2] did not use the assumption that the sets S and S' are disjoint from S_p. This was the key observation for the present paper. The main part of this text was written while I was a guest at the Department of Mathematical Sciences of Tokyo University and of the Research Institute for Mathematical Sciences in Kyoto. I want to thank these institutions for their kind hospitality.

§ 2. Proof of Theorems 1.1 and 1.2

We start with the observation that the proofs of Proposition 8.1 and Corollary 8.2 in [Sch2] did not use the assumption that the sets S and S' are disjoint from S_p. Therefore, with the same proof (which we repeat for the convenience of the reader) as in loc. cit., we obtain

Proposition 2.1. *Let k be a number field and let p be a prime number. Assume k to be totally imaginary if $p=2$. Put $X = Spec(\mathcal{O}_k)$ and let $S \subset S'$ be finite sets of primes of k. Assume that $X \smallsetminus S$ is a $K(\pi,1)$ for p and that $G_S(p) \neq 1$. Further assume that each $\mathfrak{p} \in S' \smallsetminus S$ does not split completely in $k_S(p)$. Then the following hold.*

(i) *$X \smallsetminus S'$ is a $K(\pi,1)$ for p.*

(ii) *$k_{S'}(p)_\mathfrak{p} = k_\mathfrak{p}(p)$ for all $\mathfrak{p} \in S' \smallsetminus S$.*

Furthermore, the arithmetic form of Riemann's existence theorem holds, i.e., setting $K = k_S(p)$, the natural homomorphism

$$\mathop{\ast}_{\mathfrak{p} \in S' \setminus S(K)} T(K_\mathfrak{p}(p)|K_\mathfrak{p}) \longrightarrow Gal(k_{S'}(p)|K)$$

is an isomorphism. Here $T(K_\mathfrak{p}(p)|K_\mathfrak{p})$ is the inertia group and $\ast$ denotes the free pro-p-product of a bundle of pro-p-groups, cf. [NSW], *Ch. IV, §3. In particular, the group $Gal(k_{S'}(p)|k_S(p))$ is a free pro-p-group.*

Proof. The $K(\pi,1)$-property implies

$$H^i(G_S(p), \mathbb{Z}/p\mathbb{Z}) \cong H^i_{et}(X \smallsetminus S, \mathbb{Z}/p\mathbb{Z}) = 0 \text{ for } i \geq 4,$$

hence $cd\ \ G_S(p) \leq 3$. Let $\mathfrak{p} \in S' \smallsetminus S$. Since $\mathfrak{p}$ does not split completely in $k_S(p)$ and since $cd\ \ G_S(p) < \infty$, the decomposition group of $\mathfrak{p}$ in $k_S(p)|k$ is a non-trivial and torsion-free quotient of $\mathbb{Z}_p \cong Gal(k_\mathfrak{p}^{nr}(p)|k_\mathfrak{p})$. Therefore $k_S(p)_\mathfrak{p}$ is the maximal unramified p-extension of $k_\mathfrak{p}$. We denote the normalization of an integral normal scheme Y in an algebraic extension L of its function field by Y_L. Then $(X \smallsetminus S)_{k_S(p)}$ is the universal pro-p covering of $X \smallsetminus S$. We consider the étale excision sequence for the pair $((X \smallsetminus S)_{k_S(p)}, (X \smallsetminus S')_{k_S(p)})$. By assumption, $X \smallsetminus S$ is a $K(\pi,1)$ for p, hence $H^i_{et}((X \smallsetminus S)_{k_S(p)}, \mathbb{Z}/p\mathbb{Z}) = 0$ for $i \geq 1$ by [Sch2], Proposition 2.1. Omitting the coefficients $\mathbb{Z}/p\mathbb{Z}$ from the notation, this implies isomorphisms

$$H^i_{et}\big((X \smallsetminus S')_{k_S(p)}\big) \xrightarrow{\sim} \bigoplus_{\mathfrak{p} \in S' \smallsetminus S(k_S(p))}{}^{\!\!\prime}\ H^{i+1}_\mathfrak{p}\big(((X \smallsetminus S)_{k_S(p)})_\mathfrak{p}\big)$$

for $i \geq 1$. Here (and in variants also below) we use the notational convention

$$\bigoplus_{\mathfrak{p} \in S' \smallsetminus S(k_S(p))}{}^{\!\!\prime}\ H^{i+1}_\mathfrak{p}\big(((X \smallsetminus S)_{k_S(p)})_\mathfrak{p}\big) := \varinjlim_{K \subset k_S(p)} \bigoplus_{\mathfrak{p} \in S' \smallsetminus S(K)} H^{i+1}_\mathfrak{p}\big(((X \smallsetminus S)_K)_\mathfrak{p}\big),$$

where K runs through the finite extensions of k inside $k_S(p)$. As $k_S(p)$ realizes the maximal unramified p-extension of $k_\mathfrak{p}$ for all $\mathfrak{p} \in S' \smallsetminus S$, the schemes $((X \smallsetminus S)_{k_S(p)})_\mathfrak{p}$, $\mathfrak{p} \in S' \smallsetminus S(k_S(p))$, have trivial cohomology with values in $\mathbb{Z}/p\mathbb{Z}$ and we obtain isomorphisms

$$H^i((k_S(p))_\mathfrak{p}) \xrightarrow{\sim} H^{i+1}_\mathfrak{p}\big(((X \smallsetminus S)_{k_S(p)})_\mathfrak{p}\big)$$

for $i \geq 1$. These groups vanish for $i \geq 2$. This implies

$$H^i_{et}((X \smallsetminus S')_{k_S(p)}) = 0$$

for $i \geq 2$. Since the scheme $(X \smallsetminus S')_{k_{S'}(p)}$ is the universal pro-p covering of $(X \smallsetminus S')_{k_S(p)}$, the Hochschild-Serre spectral sequence

$$E_2^{ij} = H^i\big(\mathit{Gal}(k_{S'}(p)|k_S(p)), H^j_{et}((X \smallsetminus S')_{k_{S'}(p)})\big) \Rightarrow H^{i+j}_{et}((X \smallsetminus S')_{k_S(p)})$$

yields an inclusion

$$H^2(\mathit{Gal}(k_{S'}(p)|k_S(p))) \hookrightarrow H^2_{et}((X \smallsetminus S')_{k_S(p)}) = 0.$$

Hence $\mathit{Gal}(k_{S'}(p)|k_S(p))$ is a free pro-p-group and

$$H^1(\mathit{Gal}(k_{S'}(p)|k_S(p))) \xrightarrow{\sim} H^1_{et}((X \smallsetminus S')_{k_S(p)}) \cong \bigoplus_{\mathfrak{p} \in S' \smallsetminus S(k_S(p))}{}' H^1(k_S(p)_\mathfrak{p}).$$

We set $K = k_S(p)$ and consider the natural homomorphism

$$\phi : \mathop{\ast}_{\mathfrak{p} \in S' \setminus S(K)} T(K_\mathfrak{p}(p)|K_\mathfrak{p}) \longrightarrow \mathit{Gal}(k_{S'}(p)|K).$$

By the calculation of the cohomology of a free product ([NSW], 4.3.10 and 4.1.4), ϕ is a homomorphism between free pro-p-groups which induces an isomorphism on mod p cohomology. Therefore ϕ is an isomorphism. In particular, $k_{S'}(p)_\mathfrak{p} = k_\mathfrak{p}(p)$ for all $\mathfrak{p} \in S' \smallsetminus S$. Using that $\mathit{Gal}(k_{S'}(p)|k_S(p))$ is free, the Hochschild-Serre spectral sequence induces an isomorphism

$$0 = H^2_{et}((X \smallsetminus S')_{k_S(p)}) \xrightarrow{\sim} H^2_{et}((X \smallsetminus S')_{k_{S'}(p)})^{\mathit{Gal}(k_{S'}(p)|k_S(p))}.$$

Hence $H^2_{et}((X \smallsetminus S')_{k_{S'}(p)}) = 0$, since $\mathit{Gal}(k_{S'}(p)|k_S(p))$ is a pro-p-group. Now [Sch2], Proposition 2.1 implies that $X \smallsetminus S'$ is a $K(\pi, 1)$ for p. □

In order to prove Theorem 1.1, we first provide the following lemma. For an extension field $K|k$ and a set of primes T of k, we write $T(K)$ for the set of prolongations of primes in T to K and $\delta_K(T)$ for the Dirichlet density of the set of primes $T(K)$ of K.

Lemma 2.2. *Let k be a number field, p a prime number and S a finite set of nonarchimedean primes of k. Let T be a set of primes of k with $\delta_{k(\mu_p)}(T) = 1$. Then there exists a finite subset $T_0 \subset T$ such that all primes $\mathfrak{p} \in S$ do not split completely in the extension $k_{T_0}(p)|k$.*

Proof. By [NSW], 9.2.2 (ii) or [NSW^2], 9.2.3 (ii), respectively, the restriction map

$$H^1(G_{T \cup S \cup S_p \cup S_\mathbb{R}}(p), \mathbb{Z}/p\mathbb{Z}) \longrightarrow \prod_{\mathfrak{p} \in S \cup S_p \cup S_\mathbb{R}} H^1(k_\mathfrak{p}, \mathbb{Z}/p\mathbb{Z})$$

is surjective. A class in $\alpha \in H^1(G_{T \cup S \cup S_p \cup S_\mathbb{R}}(p), \mathbb{Z}/p\mathbb{Z})$ which restricts to an unramified class $\alpha_\mathfrak{p} \in H^1_{nr}(k_\mathfrak{p}, \mathbb{Z}/p\mathbb{Z})$ for all $\mathfrak{p} \in S \cup S_p \cup S_\mathbb{R}$ is contained in $H^1(G_T(p), \mathbb{Z}/p\mathbb{Z})$. Therefore the image of the composite map

$$H^1(G_T(p), \mathbb{Z}/p\mathbb{Z}) \hookrightarrow H^1(G_{T \cup S \cup S_p \cup S_\mathbb{R}}(p), \mathbb{Z}/p\mathbb{Z}) \to \prod_{\mathfrak{p} \in S} H^1(k_\mathfrak{p}, \mathbb{Z}/p\mathbb{Z})$$

contains the subgroup $\prod_{\mathfrak{p} \in S} H^1_{nr}(k_\mathfrak{p}, \mathbb{Z}/p\mathbb{Z})$. As this group is finite, it is already contained in the image of $H^1(G_{T_0}(p), \mathbb{Z}/p\mathbb{Z})$ for some finite subset $T_0 \subset T$. We conclude that no prime in S splits completely in the maximal elementary abelian p-extension of k unramified outside T_0. □

Proof of Theorems 1.1 and 1.2. As $p \neq 2$, we may ignore archimedean primes. Furthermore, we may remove the primes in $S \cup S_p$ and all primes of degree greater than 1 from T. In addition, we remove all primes $\mathfrak{p}$ with $N(\mathfrak{p}) \not\equiv 1 \bmod p$ from T. After these changes, we still have $\delta_{k(\mu_p)}(T) = 1$.

By Lemma 2.2, we find a finite subset $T_0 \subset T$ such that no prime in S splits completely in $k_{T_0}(p)|k$. Put $X = \mathit{Spec}(\mathcal{O}_k)$. By [Sch2], Theorem 6.2, applied to T_0 and $T \smallsetminus T_0$, we find a finite subset $T_2 \subset T \smallsetminus T_0$ such that $X \smallsetminus (T_0 \cup T_2)$ is a $K(\pi, 1)$ for p. Then Proposition 2.1 applied to $T_0 \cup T_2 \subset S \cup T_0 \cup T_2$, shows that also $X \smallsetminus (S \cup T_0 \cup T_2)$ is a $K(\pi, 1)$ for p. Now put $T_1 = T_0 \cup T_2 \subset T$.

It remains to show Theorem 1.2. Assertion (i) holds by construction of T_1. Again by construction, $X \smallsetminus T_1$ is a $K(\pi, 1)$ for p. By [Sch2], Theorem 3, the field $k_{T_1}(p)$ realizes $k_\mathfrak{p}(p)$ for $\mathfrak{p} \in T_1$, showing (ii) for these primes. Finally, assertion (ii) for $\mathfrak{p} \in S$ follows from Proposition 2.1. □

§ 3. Duality

We start by investigating the relation between the $K(\pi, 1)$-property and the universal norms of global units.

Let us first remove redundant primes from S: If $\mathfrak{p} \nmid p$ is a prime with $\zeta_p \notin k_{\mathfrak{p}}$, then every p-extension of the local field $k_{\mathfrak{p}}$ is unramified (see [NSW], 7.5.1 or [NSW^2], 7.5.9, respectively). Therefore primes $\mathfrak{p} \notin S_p$ with $N(\mathfrak{p}) \not\equiv 1 \bmod p$ cannot ramify in a p-extension. Removing all these redundant primes from S, we obtain a subset $S_{\min} \subset S$, which has the property that $G_S(p) = G_{S_{\min}}(p)$. Furthermore, by [Sch2], Lemma 4.1, $X \smallsetminus S$ is a $K(\pi,1)$ for p if and only if $X \smallsetminus S_{\min}$ is a $K(\pi,1)$ for p.

Theorem 3.1. *Let k be a number field and let p be a prime number. Assume that k is totally imaginary if $p = 2$. Let S be a finite set of nonarchimedean primes of k. Then any two of the following conditions* (a) – (c) *imply the third.*

(a) *$Spec(\mathcal{O}_k) \smallsetminus S$ is a $K(\pi,1)$ for p.*

(b) $\varprojlim_{K \subset k_S(p)} \mathcal{O}_K^\times \otimes \mathbb{Z}_p = 0.$

(c) *$(k_S(p))_{\mathfrak{p}} = k_{\mathfrak{p}}(p)$ for all primes $\mathfrak{p} \in S_{\min}$.*

The limit in (b) *runs through all finite extensions K of k inside $k_S(p)$. If* (a)–(c) *hold, then also*

$$\varprojlim_{K \subset k_S(p)} \mathcal{O}_{K,S_{\min}}^\times \otimes \mathbb{Z}_p = 0.$$

Remarks: 1. Assume that $\zeta_p \in k$ and $S \supset S_p$. Then (a) holds and condition (c) holds for $p > 2$ if $\#S > r_2 + 2$ (see [NSW^2], Remark 2 after 10.9.3). In the case $k = \mathbb{Q}(\zeta_p)$, $S = S_p$, condition (c) holds if and only if p is an irregular prime number.
2. Assume that $S \cap S_p = \varnothing$ and $S_{\min} \neq \varnothing$. If condition (a) holds, then either $G_S(p) = 1$ (which only happens in very special situations, see [Sch2], Proposition 7.4) or (c) holds by [Sch2], Theorem 3 (or by Proposition 3.2 below).

Proof of Theorem 3.1. We may assume $S = S_{\min}$ in the proof. Let K run through the finite extensions of k in $k_S(p)$ and put $X_K = Spec(\mathcal{O}_K)$. Applying the topological Nakayama-Lemma ([NSW], 5.2.18) to the compact $\mathbb{Z}_p$-module $\varprojlim \mathcal{O}_K^\times \otimes \mathbb{Z}_p$, we see that condition (b) is equivalent to

(b)' $\quad \varprojlim_{K \subset k_S(p)} \mathcal{O}_K^\times / p = 0.$

Furthermore, by [Sch2], Proposition 2.1, condition (a) is equivalent to

(a)' $\quad \varinjlim_{K \subset k_S(p)} H^i_{et}((X \smallsetminus S)_K, \mathbb{Z}/p\mathbb{Z}) = 0$ for $i \geq 1$.

Condition (a)' always holds for $i = 1$, $i \geq 4$, and it holds for $i = 3$ provided that $G_S(p)$ is infinite or S is nonempty or $\zeta_p \notin k$ (see [Sch2], Lemma 3.7). The flat Kummer sequence $0 \to \mu_p \to \mathbb{G}_m \xrightarrow{\cdot p} \mathbb{G}_m \to 0$ induces exact sequences

$$0 \longrightarrow \mathcal{O}_K^\times / p \longrightarrow H^1_{fl}(X_K, \mu_p) \longrightarrow {}_p Pic(X_K) \to 0$$

for all K. As the field $k_S(p)$ does not have nontrivial unramified p-extensions, class field theory implies

$$\varprojlim_{K\subset k_S(p)} {}_pPic(X_K) \subset \varprojlim_{K\subset k_S(p)} Pic(X_K)\otimes \mathbb{Z}_p = 0.$$

As we assumed k to be totally imaginary if $p=2$, the flat duality theorem of Artin-Mazur ([Mil], III Corollary 3.2) induces natural isomorphisms

$$H^2_{et}(X_K,\mathbb{Z}/p\mathbb{Z}) = H^2_{fl}(X_K,\mathbb{Z}/p\mathbb{Z}) \cong H^1_{fl}(X_K,\mu_p)^\vee.$$

We conclude that

$$(*)\qquad \varinjlim_{K\subset k_S(p)} H^2_{et}(X_K,\mathbb{Z}/p\mathbb{Z}) \cong \Big(\varprojlim_{K\subset k_S(p)} \mathcal{O}_K^\times/p\Big)^\vee.$$

We first show the equivalence of (a) and (b) in the case $S=\varnothing$. If (a)' holds, then $(*)$ shows (b)'. If (b) holds, then $\zeta_\mathfrak{p}\notin k$ or $G_S(p)$ is infinite. Hence we obtain (a)' for $i=3$. Furthemore, (b)' implies (a)' for $i=2$ by $(*)$. This finishes the proof of the case $S=\varnothing$.

Now we assume that $S\neq\varnothing$. For $\mathfrak{p}\in S(K)$, a standard calculation of local cohomology shows that

$$H^i_\mathfrak{p}(X_K,\mathbb{Z}/p\mathbb{Z}) \cong \begin{cases} 0 & \text{for } i\le 1,\\ H^1(K_\mathfrak{p},\mathbb{Z}/p\mathbb{Z})/H^1_{nr}(K_\mathfrak{p},\mathbb{Z}/p\mathbb{Z}) & \text{for } i=2,\\ H^2(K_\mathfrak{p},\mathbb{Z}/p\mathbb{Z}) & \text{for } i=3.\\ 0 & \text{for } i\ge 4.\end{cases}$$

For $\mathfrak{p}\in S=S_{\min}$, every proper Galois subextension of $k_\mathfrak{p}(p)|k_\mathfrak{p}$ admits ramified p-extensions. Hence condition (c) is equivalent to

(c)' $\varinjlim_{K\subset k_S(p)} \bigoplus_{\mathfrak{p}\in S(K)} H^i_\mathfrak{p}(X_K,\mathbb{Z}/p\mathbb{Z}) = 0$ for all i,

and to

(c)" $\varinjlim_{K\subset k_S(p)} \bigoplus_{\mathfrak{p}\in S(K)} H^2_\mathfrak{p}(X_K,\mathbb{Z}/p\mathbb{Z}) = 0.$

Consider the direct limit over all K of the excision sequences

$$\cdots\to \bigoplus_{\mathfrak{p}\in S(K)} H^i_\mathfrak{p}(X_K,\mathbb{Z}/p\mathbb{Z}) \to H^i_{et}(X_K,\mathbb{Z}/p\mathbb{Z}) \to H^i_{et}((X\smallsetminus S)_K,\mathbb{Z}/p\mathbb{Z}) \to \cdots.$$

Assume that (a)' holds, i.e. the right hand terms vanish in the limit for $i\ge 1$. Then $(*)$ shows that (b)' is equivalent to (c)".

Now assume that (b) and (c) hold. As above, (b) implies the vanishing of the middle term for $i=2$ in the limit. Condition (c)' then shows (a)'.

We have proven that any two of the conditions (a)–(c) imply the third.

Finally, assume that (a)–(c) hold. Tensoring the exact sequences (cf. [NSW], 10.3.11 or [NSW[2]], 10.3.12, respectively)

$$0 \to \mathcal{O}_K^\times \to \mathcal{O}_{K,S}^\times \to \bigoplus_{\mathfrak{p}\in S(K)} (K_\mathfrak{p}^\times/U_\mathfrak{p}) \to Pic(X_K) \to Pic((X\smallsetminus S)_K) \to 0$$

by (the flat $\mathbb{Z}$-algebra) $\mathbb{Z}_p$, we obtain exact sequences of finitely generated, hence compact, $\mathbb{Z}_p$-modules. Passing to the projective limit over the finite extensions K of k inside $k_S(p)$ and using $\varprojlim Pic(X_K)\otimes\mathbb{Z}_p = 0$, we obtain the exact sequence

$$0 \to \varprojlim_{K\subset k_S(p)} \mathcal{O}_K^\times\otimes\mathbb{Z}_p \to \varprojlim_{K\subset k_S(p)} \mathcal{O}_{K,S}^\times\otimes\mathbb{Z}_p \to \varprojlim_{K\subset k_S(p)} \bigoplus_{\mathfrak{p}\in S(K)} (K_\mathfrak{p}^\times/U_\mathfrak{p})\otimes\mathbb{Z}_p \to 0.$$

Condition (c) and local class field theory imply the vanishing of the right hand limit. Therefore (b) implies the vanishing of the projective limit in the middle. □

If $G_S(p)\neq 1$ and condition (a) of Theorem 1.1 holds, then the failure in condition (c) can only come from primes dividing p. This follows from the next

Proposition 3.2. *Let k be a number field and let p be a prime number. Assume that k is totally imaginary if $p=2$. Let S be a finite set of nonarchimedean primes of k. If $Spec(\mathcal{O}_k)\smallsetminus S$ is a $K(\pi,1)$ for p and $G_S(p)\neq 1$, then every prime $\mathfrak{p}\in S$ with $\zeta_p\in k_\mathfrak{p}$ has an infinite inertia group in $G_S(p)$. Moreover, we have*

$$k_S(p)_\mathfrak{p} = k_\mathfrak{p}(p)$$

for all $\mathfrak{p}\in S_{\min}\smallsetminus S_p$.

Proof. We may assume $S=S_{\min}$. Suppose $\mathfrak{p}\in S$ with $\zeta_p\in k_\mathfrak{p}$ does not ramify in $k_S(p)|k$. Setting $S'=S\smallsetminus\{\mathfrak{p}\}$, we have $k_{S'}(p)=k_S(p)$, in particular,

$$H^1_{et}(X\smallsetminus S',\mathbb{Z}/p\mathbb{Z}) \xrightarrow{\sim} H^1_{et}(X\smallsetminus S,\mathbb{Z}/p\mathbb{Z}).$$

In the following, we omit the coefficients $\mathbb{Z}/p\mathbb{Z}$ from the notation. Using the vanishing of $H^3_{et}(X\smallsetminus S)$, the étale excision sequence yields a commutative exact diagram

$$\begin{tikzcd}
 & H^2(G_{S'}(p)) \arrow[r, "\sim"] \arrow[d, hook] & H^2(G_S(p)) \arrow[d, "\wr"] & & \\
H^2_\mathfrak{p}(X) \arrow[r, hook] & H^2_{et}(X\smallsetminus S') \arrow[r, "\alpha"] & H^2_{et}(X\smallsetminus S) \arrow[r] & H^3_\mathfrak{p}(X) \arrow[r, two heads] & H^3_{et}(X\smallsetminus S').
\end{tikzcd}$$

Hence α is split-surjective and $\mathbb{Z}/p\mathbb{Z}\cong H^3_\mathfrak{p}(X)\xrightarrow{\sim} H^3_{et}(X\smallsetminus S')$. This implies $S'=\varnothing$, hence $S=\{\mathfrak{p}\}$, and $\zeta_p\in k$. The same applies to every finite extension of k in $k_S(p)$, hence $\mathfrak{p}$ is inert in $k_S(p)=k_\varnothing(p)$. This implies that the natural homomorphism

$$Gal(k_\mathfrak{p}^{nr}(p)|k_\mathfrak{p}) \longrightarrow G_\varnothing(k)(p)$$

is surjective. Therefore $G_S(p) = G_\varnothing(p)$ is abelian, hence finite by class field theory. Since this group has finite cohomological dimension by the $K(\pi, 1)$-property, it is trivial, in contradiction to our assumptions.

This shows that all $\mathfrak{p} \in S$ with $\zeta_p \in k_\mathfrak{p}$ ramify in $k_S(p)$. As this applies to every finite extension of k inside $k_S(p)$, the inertia groups must be infinite. For $\mathfrak{p} \in S_{\min} \smallsetminus S_p$ this implies $k_S(p)_\mathfrak{p} = k_\mathfrak{p}(p)$. □

Theorem 3.3. *Let k be a number field and let p be a prime number. Assume that k is totally imaginary if $p = 2$. Let S be a finite nonempty set of nonarchimedean primes of k. Assume that conditions* (a)–(c) *of Theorem 3.1 hold and that $\zeta_p \in k_\mathfrak{p}$ for all $\mathfrak{p} \in S$. Then $G_S(p)$ is a pro-p duality group of dimension* 2.

Proof. Condition (a) implies $H^3(G_S(p), \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\sim} H^3_{et}(X \smallsetminus S, \mathbb{Z}/p\mathbb{Z}) = 0$. Hence $cd\ G_S(p) \leq 2$. On the other hand, by (c), the group $G_S(p)$ contains $Gal(k_\mathfrak{p}(p)|k_\mathfrak{p})$ as a subgroup for all $\mathfrak{p} \in S$. As $\zeta_p \in k_\mathfrak{p}$ for $\mathfrak{p} \in S$, these local groups have cohomological dimension 2, hence so does $G_S(p)$.

In order to show that $G_S(p)$ is a duality group, we have to show that

$$D_i(G_S(p), \mathbb{Z}/p\mathbb{Z}) := \varinjlim_{\substack{U \subset G_S(p) \\ cor^\vee}} H^i(U, \mathbb{Z}/p\mathbb{Z})^\vee$$

vanish for $i = 0, 1$, where U runs through the open subgroups of $G_S(p)$ and the transition maps are the duals of the corestriction homomorphisms; see [NSW], 3.4.6. The vanishing of D_0 is obvious, as $G_S(p)$ is infinite. We therefore have to show that

$$\varinjlim_{K \subset k_S(p)} H^1((X \smallsetminus S)_K, \mathbb{Z}/p\mathbb{Z})^\vee = 0.$$

We put $X = Spec(\mathcal{O}_k)$ and denote the embedding by $j : (X \smallsetminus S)_K \to X_K$. By the flat duality theorem of Artin-Mazur, we have natural isomorphisms

$$H^1((X \smallsetminus S)_K, \mathbb{Z}/p\mathbb{Z})^\vee \cong H^2_{fl,c}((X \smallsetminus S)_K, \mu_p) = H^2_{fl}(X_K, j_! \mu_p).$$

The excision sequence together with a straightforward calculation of local cohomology groups shows an exact sequence

$$(*) \qquad \bigoplus_{\mathfrak{p} \in S(K)} K_\mathfrak{p}^\times / K_\mathfrak{p}^{\times p} \to H^2_{fl}(X_K, j_! \mu_p) \to H^2_{fl}((X \smallsetminus S)_K, \mu_p).$$

As $\zeta_p \in k_\mathfrak{p}$ and $k_S(p)_\mathfrak{p} = k_\mathfrak{p}(p)$ for $\mathfrak{p} \in S$ by assumption, the left hand term of $(*)$ vanishes when passing to the limit over all K. We use the Kummer sequence to obtain an exact sequence

$$(**) \qquad Pic((X \smallsetminus S)_K)/p \longrightarrow H^2_{fl}((X \smallsetminus S)_K, \mu_p) \longrightarrow {}_pBr((X \smallsetminus S)_K).$$

The left hand term of $(**)$ vanishes in the limit by the principal ideal theorem. The Hasse principle for the Brauer group induces an injection

$$ {}_pBr((X \smallsetminus S)_K) \hookrightarrow \bigoplus_{\mathfrak{p}\in S(K)} {}_pBr(K_\mathfrak{p}). $$

As $k_S(p)$ realizes the maximal unramified p-extension of $k_\mathfrak{p}$ for $\mathfrak{p} \in S$, the limit of the middle term in $(**)$, and hence also the limit of the middle term in $(*)$ vanishes. This shows that $G_S(p)$ is a duality group of dimension 2. □

Remark: The dualizing module can be calculated to

$$ D \cong \mathrm{tor}_p\big(C_S(k_S(p))\big), $$

i.e. D is isomorphic to the p-torsion subgroup in the S-idèle class group of $k_S(p)$. The proof is the same as in ([Sch1], Proof of Thm. 5.2), where we dealt with the tame case.

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