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Motivic aspects of anabelian geometry

Alexander Schmidt

Abstract.

We discuss the role which might be played in anabelian geometry by the étale homotopy type of Artin and Mazur. We quote some results around the question how local the construction is. Furthermore, we show that, after completion away from the residue characteristics, the functor étale homotopy type factors through the motivic A^1-homotopy category of Morel–Voevodsky.

§1. Introduction

Grothendieck's anabelian philosophy predicts the existence of a class of "anabelian" schemes X, which have the property that they are reconstructible from their étale fundamental group $\pi_1^{\mathrm{et}}(X)$ (or some sufficiently large characteristic quotient, e.g. the maximal pro-solvable factor group). From the viewpoint of topology it looks somewhat unnatural to reconstruct X solely from its fundamental group, not using the higher homotopy groups. This should only be possible if X is a $K(\pi,1)$-space, i.e. if $\pi_i(X)$ vanishes for all $i \geq 2$.

Higher étale homotopy groups of schemes have been defined by Artin and Mazur in the 1960's [AM]. They constructed a functor which associates to each locally noetherian scheme X its *étale homotopy type* X_{ht}, an object of *pro*-$\mathcal{H}$, the pro-category of the homotopy category $\mathcal{H}$ of simplicial sets. For any geometric point x of X, the (pro)groups $\pi_i((X,x)_{ht})$ give a natural definition of higher homotopy groups in algebraic geometry. So, in order to reconstruct X from $\pi_1^{\mathrm{et}}(X)$ one should expect that X is an étale $K(\pi,1)$-space or at least a $K(\pi,1)$-space at some sufficiently large full class $\mathfrak{c}$ of finite groups in the sense described below.

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In this note, we will quote some results around the question how local the construction of the étale homotopy type is. Furthermore, we will show that, after completion away from the residue characteristics, the functor étale homotopy type factors through the motivic A^1-homotopy category of Morel-Voevodsky. So the best we can hope for a scheme X is to deduce its A^1-homotopy type from its étale homotopy type. Only for schemes which are A^1-local (like hyperbolic curves) we can expect to detect the isomorphism type.

§2. Hypercoverings

Let $\mathcal{H}$ be the homotopy category of simplicial sets and let *pro*-$\mathcal{H}$ be its pro-category. Following [AM], we have a natural functor which assigns to every locally noetherian scheme its 'étale homotopy type' $X_{ht} \in$ *pro*-$\mathcal{H}$. We briefly recall the construction. Let C be any site.

Definition 2.1. *A hypercovering $X.$ of C is a simplicial object with values in C such that the following conditions hold:*

(i) *The natural morphism $X_0 \to e$ to the final object of C is a covering.*

(ii) *For all n the natural morphism*

$$X_{n+1} \longrightarrow (\mathrm{cosk}_n X.)_{n+1}$$

is a covering.

Recall that the functor cosk_n is the right adjoint to the functor 'truncation at level n' and that $(\mathrm{cosk}_n X.)_k = X_k$ for $k \leq n$.

Example. Let C be the site (*Sets*) where coverings are surjective families of maps. A simplicial set is a hypercovering if and only if it satisfies the Kan-condition and is contractible (cf. [AM], 8.5.a).

Let $X.$ be a simplicial object of C and let $T.$ be a simplicial set. We form the simplicial object of C

$$(X \otimes T).$$

which is given in degree n as the coproduct of copies of X_n indexed by T_n. For a non-decreasing map $\alpha : \{0, 1, \dots, m\} \to \{0, 1, \dots, n\}$ the operator α^* acts on the copy of X_n indexed by $t \in T_n$ via $\alpha^* : (X_n)_t \to (X_m)_{\alpha^*(t)}$. Of course, this construction requires the existence of infinite coproducts in C unless T_n is finite for all n.

Two maps $f_0, f_1 : X. \to Y.$ of simplicial objects in C are homotopic if there exists a map

$$H : X. \otimes \Delta(1) \longrightarrow Y.$$

with $H \circ i_\nu = f_\nu$ for $\nu = 0, 1$. Here $i_\nu : X. \to X. \otimes \Delta(1)$, $\nu = 0, 1$, are the obvious inclusions. We have the

Lemma 2.2 ([AM], 8.13). *For a site C, let $HR(C)$ denote the category whose objects are hypercoverings and whose maps are homotopy classes of morphisms. Then $HR(C)$ is left filtering.*

We follow [AM] in defining locally connected categories.

Definition 2.3. *Let $\mathcal{C}$ be a category admitting finite fibre products. $\mathcal{C}$ is called distributive if it has an initial object $\varnothing$, and if the following condition holds: For every family of objects Y_i, $i \in I$, such that the coproduct $\coprod_{i\in I} Y_i$ exists in $\mathcal{C}$, any family of morphisms $Y_i \to S$ and for any morphism $X \to S$, the canonical morphism $\coprod_{i\in I} X \times_S Y_i \to X \times_S (\coprod_{i\in I} Y_i)$ is an isomorphism.*

Every site has an underlying distributive category.

Definition 2.4. *Let $\mathcal{C}$ be a distributive category. An object $X \in \mathcal{C}$ is connected if it is not the initial object $\varnothing$ and has no non-trivial coproduct decomposition (i.e. $X = X_1 \coprod X_2 \Rightarrow X_i = \varnothing$ for exactly one i). $\mathcal{C}$ is called locally connected if every object has a coproduct decomposition into connected objects.*

If $\mathcal{C}$ is locally connected, then the expression of an object X as a coproduct of connected objects is essentially unique. Moreover, the rule associating to an object its set of connected components is a functor. We denote this functor by

$$\Pi : \mathcal{C} \longrightarrow (Sets)$$

and call it the connected component functor. Applying Π to each X_n separately, we obtain a simplicial set $\Pi(X.)$ associated to each simplicial object $X.$ of $\mathcal{C}$.

Now assume that C is a locally connected site. Then the above construction yields a simplicial set $\Pi(X.)$ associated to each hypercovering $X.$ of C. This assignment induces a functor $\Pi : HR(C) \to \mathcal{H}$ and therefore defines a pro-object

$$\Pi C \in pro\text{-}\mathcal{H}.$$

Now, if X is a locally noetherian scheme, we obtain its étale homotopy type $X_{ht} \in pro\text{-}\mathcal{H}$ by applying the above construction to the small étale site over X, i.e. $X_{ht} = \Pi X_{\mathrm{et}}$. If, furthermore, $\bar{x}$ is a geometric point of X, we obtain the pointed étale homotopy type $(X,\bar{x})_{ht}$ and the étale homotopy groups

$$\pi_i^{\mathrm{et}}(X,\bar{x}) = \pi_i((X,\bar{x})_{ht}).$$

Note that $\pi_0^{\mathrm{et}}(X,\bar{x})$ is just a pointed set, while, for $i \geq 1$, $\pi_i^{\mathrm{et}}(X,\bar{x})$ is a pro-group, which is commutative for $i \geq 2$. By [AM], 11.1, the groups $\pi_i^{\mathrm{et}}(X,\bar{x})$, $i \geq 1$, are profinite if X is geometrically unibranch. In this case the group $\pi_1^{\mathrm{et}}(X,\bar{x})$ coincides with the étale fundamental group $\pi_1^{\mathrm{et}}(X,\bar{x})$ introduced in [SGA1], while in the general situation, the latter is the profinite completion of the first one.

Let $\mathfrak{c}$ be a full class of finite groups, i.e. a class which is closed under the operations of taking subgroups, factor groups and group extensions. We say that a connected noetherian scheme X is a $\boldsymbol{K(\pi,1)}$ **at** $\mathfrak{c}$, if

$$\pi_i((X,\bar{x})_{ht}^{\mathfrak{c}}) = 0 \text{ for all } i \geq 2.$$

Here $\bar{x}$ is any geometric point of X and $(-)^{\mathfrak{c}}$ is the $\mathfrak{c}$-completion functor defined by Artin–Mazur. This notion is independent of the chosen base point $\bar{x}$.

§3. The local $\boldsymbol{K(\pi,1)}$-property

A (topological) manifold is locally trivial in the classical homotopy theory in the sense that every point has a fundamental system of neighbourhoods which are contractible. This is not true for schemes: already a point $x = Spec(k)$ does not have any contractible étale open subscheme (unless $Gal(\bar{k}|k)$ is finite). But points are $K(\pi,1)$s for the class of all finite groups. So the question arises whether any point on a reasonable (e.g. regular) scheme has a fundamental system of neighbourhoods which are of $K(\pi,1)$-type. This is easily verified for smooth curves. By the existence of 'Artin neighbourhoods' and the behaviour in fibrations, this extends to arbitrary smooth varieties over fields of characteristic zero.

The first result in positive characteristics was achieved by E. Friedlander. For a prime ℓ we denote the class of finite ℓ-groups by (ℓ).

Theorem 3.1 ([Fr])**.** *Let X be a smooth variety over an algebraically closed field of characteristic $p \geq 0$. Let $\ell \neq p$ be a prime number. Then every closed point on X has an étale fundamental system of neighbourhoods which are $K(\pi,1)$ at (ℓ).*

In mixed characteristic, it is well-known that an open subscheme U of the spectrum of a number field is a $K(\pi,1)$ for any class $\mathfrak{c}$ which has the property that ℓ is invertible on U for every prime number ℓ with $\mathbb{Z}/\ell\mathbb{Z} \in \mathfrak{c}$ (combine [NSW], (10.4.8) and [Mi], II, 2.9). For primes occurring as residue characteristic, we have the following result.

Theorem 3.2 ([Sc]). *Let k be a number field and let $x \in Spec(\mathcal{O}_k)$ be a closed point. Let ℓ be the residue characteristic of x. Then x has a fundamental system of Zariski-neighbourhoods which are $K(\pi, 1)$ at (ℓ).*

As a corollary, we see that every geometric point $\bar{x} \in Spec(\mathcal{O}_k)$ of residue characteristic ℓ has a fundamental system of étale neighbourhoods which are $K(\pi, 1)$ at (ℓ). The reason why only the class of ℓ-groups is dealt with in 3.1 and 3.2 lies in the methods of proof which in both cases make use of nilpotency.

Finally, there is the following result of J. to Baben for semi-stable relative curves.

Theorem 3.3 ([Ba]). *Let A be a strictly henselian discrete valuation ring with residue characteristic $p \geq 0$ and let $X \to S = Spec(A)$ be a regular, proper and semi-stable relative curve. Let $\mathfrak{c}$ be any full class of finite groups with $\mathbb{Z}/p\mathbb{Z} \notin \mathfrak{c}$. Then every closed point x of X has a fundamental system of Zariski-neighbourhoods which are $K(\pi, 1)$ at $\mathfrak{c}$.*

§4. Local fibrations versus hypercoverings

Next we recall several definitions and facts on simplicial sheaves from [MV], only that we work with the étale site.

Let S be a locally noetherian scheme. Slightly changing the usual convention, we call an S-scheme X smooth if it is the disjoint union of separated smooth schemes of finite type over S and we denote the category of these schemes by $Sm(S)$. We say that a morphism in $Sm(S)$ is étale if it is the disjoint union of étale morphisms in the usual sense. As a sheaf is uniquely given by its values on connected schemes, our change in terminology does not affect the notion of étale sheaves on the category of smooth schemes over S: denoting the category of separated smooth schemes of finite type over S by $sm(S)$, the pull-back defines an isomorphism $Shv_{\mathrm{et}}(sm(S)) \xrightarrow{\sim} Shv_{\mathrm{et}}(Sm(S))$.

We work in the category $\Delta^{op} Shv_{\mathrm{et}}(Sm(S))$ of simplicial étale sheaves (of sets) on $Sm(S)$. By a point we will always mean a geometric point. A map of simplicial sheaves $f : F \to G$ is called a simplicial weak equivalence if for every point x the map $F_x \to G_x$ is a weak equivalence of simplicial sets. f is called a (trivial) cofibration if it is injective (injective and a weak equivalence). Fibrations are maps satisfying the right lifting property with respect to trivial cofibrations. The category of simplicial sheaves together with these three classes of morphisms is a simplicial closed model category and we denote the associated homotopy category by $\mathcal{H}_{s,et}(Sm(S))$.

A map of simplicial sheaves $F \to G$ is called a (trivial) local fibration if for every point x the map $F_x \to G_x$ is a fibration (fibration and weak equivalence). A local fibration has the right lifting property after an étale refinement. Kan-simplicial sets considered as constant simplicial sheaves are locally fibrant.

For simplicial sheaves $\mathcal{X}$, $\mathcal{Y}$ denote by $\pi(\mathcal{X},\mathcal{Y})$ the quotient of $Hom(\mathcal{X},\mathcal{Y}) = S_0(\mathcal{X},\mathcal{Y})$ with respect to the equivalence relation generated by simplicial homotopies, i.e. the set of connected components of the simplicial function object $S(\mathcal{X},\mathcal{Y})$, and call it the set of *simplicial homotopy classes of morphisms* from $\mathcal{X}$ to $\mathcal{Y}$. One easily checks that the simplicial homotopy relation is compatible with composition and thus one gets a category $\pi\Delta^{op} Shv_{\rm et}(Sm(S))$ with objects the simplicial sheaves and morphisms the simplicial homotopy classes of morphisms. For any simplicial sheaf $\mathcal{X}$ denote by $\pi\, Triv/\mathcal{X}$ the category whose objects are the trivial local fibrations to $\mathcal{X}$ and whose morphisms are the obvious commutative triangles in $\pi\Delta^{op} Shv_{\rm et}(Sm(S))$. This category is filtering and essentially small ([MV], 2.1.12). We will need the following

Proposition 4.1 ([MV], 2.1.13)**.** *For any simplicial sheaves $\mathcal{X}$, $\mathcal{Y}$, with $\mathcal{Y}$ locally fibrant, the canonical map*

$$colim_{p:\mathcal{X}'\to\mathcal{X}\in\pi Triv/\mathcal{X}}\ \pi(\mathcal{X}',\mathcal{Y}) \longrightarrow Hom_{\mathcal{H}_{s,et}(Sm(S))}(\mathcal{X},\mathcal{Y})$$

is a bijection.

As usual, we consider a sheaf F as a simplicial sheaf with F in each degree and all face and degeneracy morphisms the identity of F. We will also make no difference in notation between a smooth scheme X and the sheaf that it represents. A hypercovering $U.$ of X is a hypercovering in the small étale site over X and will also be considered as an object in $\Delta^{op} Shv_{\rm et}(Sm(S))$.

Lemma 4.2. *Let X be a smooth scheme and let $U.$ be a hypercovering of X. Then, considered as a map in $\Delta^{op} Shv_{\rm et}(Sm(S))$, the projection*

$$U. \longrightarrow X$$

is a trivial local fibration.

Proof. For every point x, the associated map $U._x \to X_x$ is a hypercovering in $(Sets)$, hence a Kan-fibration and a weak equivalence. Q.E.D.

Lemma 4.3. *Let X be a smooth scheme and let $\mathcal{X} \to X$ be a trivial local fibration of simplicial sheaves. Then there exists a hypercovering*

$U. \to X$ of X and a map of simplicial sheaves $U. \to \mathcal{X}$ that commutes with the respective projections to X.

Proof. For each geometric point $x : Spec(K) \to X$, the stalk $\mathcal{X}_x$ is a contractible Kan-simplicial set. In particular, these stalks are nonempty and, for all n, the restriction of the map

$$\mathcal{X}_{n+1} \longrightarrow (\mathrm{cosk}_n \mathcal{X})_{n+1}$$

to the small étale site over X is an epimorphism of sheaves.

Now we construct the required hypercovering by induction. First of all, there exists an étale covering $U_0 \to X$ such that $\mathcal{X}_0(U_0) \neq \varnothing$. This gives a map in degree zero. Assume we have already constructed the hypercovering $U.$ up to level n together with a map from $U.$ to the level n truncation of $\mathcal{X}$. This gives a map of sheaves $\alpha : (\mathrm{cosk}_n U.)_{n+1} \to (\mathrm{cosk}_n \mathcal{X})_{n+1}$ or, equivalently, a section $\alpha \in (\mathrm{cosk}_n \mathcal{X})_{n+1}\big((\mathrm{cosk}_n U.)_{n+1}\big)$. Since the map of sheaves $\mathcal{X}_{n+1} \longrightarrow (\mathrm{cosk}_n \mathcal{X})_{n+1}$ is an epimorphism in the small étale site over X, we find an étale covering $W \to (\mathrm{cosk}_n U.)_{n+1}$ and a lift of α to $\mathcal{X}_{n+1}$ over W. Then we put $U_{n+1} = W$. Q.E.D.

Summarizing, we have proven the following

Proposition 4.4. *Let X be a smooth scheme. Every hypercovering $U. \to X$ defines an object of $\pi\, Triv/X$. The category $HR(X)$ is a full and cofinal subcategory of $\pi\, Triv/X$.*

In order not to overload notation, given a simplicial set, we will denote several associated objects by the same letter: the associated constant pro-simplicial set, the associated constant object in *pro-*$\mathcal{H}$, the associated constant simplicial sheaf in $Shv_{\mathrm{et}}(Sm(S))$, its image in $\mathcal{H}_{s,et}(Sm(S))$, and so on.

Corollary 4.5. *Let M be a simplicial set and let X be a smooth scheme. Then we have a natural isomorphism*

$$Hom_{\mathcal{H}_{s,et}(Sm(S))}(X, M) = Hom_{pro\text{-}\mathcal{H}}(X_{ht}, M).$$

Proof. First we may replace M by a weakly equivalent Kan-simplicial set. Then M is locally fibrant as an object in $\Delta^{op} Shv_{\mathrm{et}}(Sm(S))$. By 4.1 and 4.4, we obtain

$$\begin{aligned} Hom_{\mathcal{H}_s(Sm(S)_{et})}(X, M) &= colim_{p\colon \mathcal{X}\to X \in \pi Triv/X}\ \pi(\mathcal{X}, M) \\ &= colim_{p\colon U.\to X \text{ hypercovering}}\ \pi(U., M) \\ &= colim_{p\colon U.\to X \text{ hypercovering}}\ \pi(\Pi(U.), M) \\ &= Hom_{pro\text{-}\mathcal{H}}(X_{ht}, M), \end{aligned}$$

where, in the third line, we considered $\Pi(U.)$ as a constant simplicial sheaf. This proves the corollary. Q.E.D.

Proposition 4.4 suggests to define the étale homotopy type of a simplicial sheaf $\mathcal{X}$ as the functor of connected components

$$\Pi : \pi Triv/\mathcal{X} \longrightarrow \mathcal{H}.$$

In order to do this, one has to show that the category of sheaves is locally connected. We will do this in the next section.

§5. Local connectedness of $\mathit{Shv}_{\mathrm{et}}(\mathit{Sm}(S))$

The goal of this section is to show that $\mathit{Shv}_{\mathrm{et}}(\mathit{Sm}(S))$ is a locally connected category. The results of this section are rather formal and extend to sheaves for any subcanonical topology.

The category $\mathit{Shv}_{\mathrm{et}}(\mathit{Sm}(S))$ has fibre products, in contrast to $\mathit{Sm}(S)$. If $X \to Z$ and $Y \to Z$ are morphisms in $\mathit{Sm}(S)$ and if at least one of them is smooth, then the scheme-theoretical fibre product $X \times_Z Y$ is smooth over S and represents the sheaf-theoretical fibre product. The fact that $\mathit{Shv}_{\mathrm{et}}(\mathit{Sm}(S))$ is distributive is obvious. The sheaf pt represented by S is the final object of $\mathit{Shv}_{\mathrm{et}}(\mathit{Sm}(S))$. As long as there is no danger of confusion, we will denote the sheaf represented by a scheme X by the same letter. The sheaf $\varnothing$ represented by the empty scheme is the initial object of $\mathit{Shv}_{\mathrm{et}}(\mathit{Sm}(S))$.

Lemma 5.1. *Let $X \in \mathit{Sm}(S)$ and let $X = \coprod_{i\in I} \mathcal{F}_i$ be a disjoint union decomposition of X as an object in $\mathit{Shv}_{\mathrm{et}}(\mathit{Sm}(S))$. Then each F_i is represented by an open subscheme $U_i \subset X$ such that $X = \coprod_{i\in I} U_i$. In particular, a connected scheme in $\mathit{Sm}(S)$ represents a connected sheaf.*

Proof. Let $B = \coprod_{i\in I} pt_i$, where $pt_i = pt$ for all $i \in I$. The decomposition $X = \coprod_{i\in I} \mathcal{F}_i$ and the projections $\mathcal{F}_i \to pt_i$ define a morphism (of sheaves, hence of schemes) $\varphi : X \to B$. We find $U_i \subset X$ as the open subscheme $\varphi^{-1}(pt_i)$. Q.E.D.

Lemma 5.2. *Let $F \in \mathit{Shv}_{\mathrm{et}}(\mathit{Sm}(S))$ be connected. Then every map from F to a constant sheaf is constant, i.e. factors through $F \to pt$.*

Proof. The constant sheaf over a set M is the disjoint union over the final sheaf pt indexed by M. If $f : F \to M$ is a map of sheaves, then F is the disjoint union $F = \coprod_{m\in M} f^{-1}(pt_m)$ and therefore only one of these components can be nontrivial if F is connected. Q.E.D.

The following proposition (well known for locally noetherian schemes) will be essential for the construction of the étale homotopy type.

Proposition 5.3. *Each $G \in Shv_{\mathrm{et}}(Sm(S))$, $G \neq \varnothing$, can be written in a unique way as the disjoint union of connected subsheaves.*

Proof. If $F \to G$ is a morphism with F connected then the image sheaf $\mathrm{im}(F) \subset G$ is connected. If $F_1, F_2 \subset G$ are connected subsheaves with $F_1 \cap F_2 \neq \varnothing$, then also $F_1 \cup F_2 \subset G$ is connected.

Now let $G \neq \varnothing$ be arbitrary. We consider the natural surjection

$$\coprod_{(X,\alpha)} X \twoheadrightarrow G,$$

where (X, α) runs through the pairs $X \in Sm(S)$ connected, $\alpha \in G(X)$. Let I be the set of equivalence classes of such pairs with respect to the smallest equivalence relation containing the relations '$(X_1, \alpha_1) \sim (X_2, \alpha_2)$ if $\mathrm{im}(\alpha_1) \cap \mathrm{im}(\alpha_2) \neq \varnothing$'. Then G is the disjoint union of its connected subsheaves

$$G_i := \bigcup_{(X,\alpha)\in i} \mathrm{im}(\alpha), \quad i \in I.$$

The uniqueness of the decomposition is obvious. Q.E.D.

§6. The étale homotopy type

By the results of the last section, the rule associating to a sheaf its set of connected components defines the connected component functor

$$\Pi : Shv_{\mathrm{et}}(Sm(S)) \longrightarrow (Sets).$$

This functor naturally extends to simplicial sheaves (taking values in simplicial sets). Furthermore, simplicial homotopies between simplicial sheaves carry over to homotopies between simplicial sets.

Definition 6.1. *The étale homotopy type $\mathcal{X}_{ht}$ of an $\mathcal{X} \in \Delta^{op} Shv_{\mathrm{et}}(Sm(S))$ is the induced functor*

$$\Pi : \pi Triv/\mathcal{X} \longrightarrow \mathcal{H}.$$

By 4.4, the étale homotopy type of a smooth scheme is naturally isomorphic to the étale homotopy type of the sheaf that it represents. Let $\mathcal{X} \to \mathcal{Y}$ be a morphism in $\Delta^{op} Shv_{\mathrm{et}}(Sm(S))$. As the pull-back of a trivial fibration is a trivial fibration, we obtain a (strict) morphism $\mathcal{X}_{ht} \to \mathcal{Y}_{ht}$ in *pro*-$\mathcal{H}$.

Corollary 6.2. *Let M be a simplicial set and let $\mathcal{X} \in \Delta^{op} Shv_{\mathrm{et}}(Sm(S))$. Then we have a natural isomorphism*

$$Hom_{\mathcal{H}_{s,et}(Sm(S))}(\mathcal{X}, M) = Hom_{pro\text{-}\mathcal{H}}(\mathcal{X}_{ht}, M).$$

Proof. We may replace M by a weakly equivalent Kan-simplicial set. Then the statement follows from 4.1 and 5.2. Q.E.D.

Corollary 6.3. *If a morphism $\mathcal{X} \to \mathcal{Y}$ in $\Delta^{op} Shv_{\text{et}}(Sm(S))$ is a simplicial weak equivalence, then the induced map $\mathcal{X}_{ht} \to \mathcal{Y}_{ht}$ is an isomorphism in pro-$\mathcal{H}$.*

Proof. This follows from 6.2 and from the fact that pro-objects in a category are uniquely determined by their morphisms to constant objects. Q.E.D.

Summarizing, we have proven the

Theorem 6.4. *The assignment $\mathcal{X} \mapsto \mathcal{X}_{ht}$ defines a functor*

$$ht : \mathcal{H}_{s,et}(Sm(S)) \longrightarrow pro\text{-}\mathcal{H}.$$

The composite $Sm(S) \xrightarrow{can} \mathcal{H}_{s,et}(Sm(S)) \xrightarrow{ht} pro\text{-}\mathcal{H}$ is the functor 'étale homotopy type' of Artin–Mazur.

Remark 6.5. There is another approach of extending the notion of étale homotopy type: one starts with the (already defined) étale homotopy type of representable (pre)sheaves and then extends the notion by passing to limits. This less intuitive approach was used by Isaksen [Is] and Quick [Qu]. It has the great advantage that (working with rigid hypercoverings introduced by Friedlander [Fr]) one obtains an actual pro-simplicial set, and not just an element in *pro-*$\mathcal{H}$, as we do. Furthermore, Quick, who works in the category $\hat{\mathcal{S}}$ of simplicial profinite sets, can stabilize his construction, obtaining a realization functor on the category of $\mathbb{P}^1$-spectra.

The advantage of our approach is twofold. First, we have a natural interpretation of π_0. Secondly, we have the very useful adjunction formula 4.5.

§7. Weak equivalences and $\mathfrak{c}$-completions

We say that $X \in \mathcal{H}$ is of finite homotopical dimension if there is a bound d such that the homotopy groups of all connected components of X are zero in dimension greater than d. For an arbitrary $X \in \mathcal{H}$ all coskeletons $\operatorname{cosk}_n X$ of X are of finite homotopical dimension.

Now let $X = \{X_i\}$ be in *pro-*$\mathcal{H}$. The various coskeletons $\operatorname{cosk}_n X_i$ form a pro-object $X^\natural$ indexed by pairs (i, n). We have a natural map $X \to X^\natural$ and $X^\natural \to X^{\natural\natural}$ is an isomorphism. We call a map $f : X \to Y$ in *pro-*$\mathcal{H}$ a weak equivalence ($\natural$-isomorphism in [AM]) if $f^\natural : X^\natural \to Y^\natural$

is an isomorphism. This is equivalent to the statement that f induces an isomorphism

$$Hom_{pro\text{-}\mathcal{H}}(Y, M) \longrightarrow Hom_{pro\text{-}\mathcal{H}}(X, M)$$

for every $M \in \mathcal{H}$ of finite homotopical dimension. Let

$$(pro\text{-}\mathcal{H})_w$$

be the full subcategory in *pro*-$\mathcal{H}$ consisting of objects isomorphic to $X^\natural$ for some X. Then $(pro\text{-}\mathcal{H})_w$ is the localization of *pro*-$\mathcal{H}$ with respect to the class of weak equivalences and $\natural : pro\text{-}\mathcal{H} \to (pro\text{-}\mathcal{H})_w$ is the localization functor.

Furthermore, we recall the notion of $\mathfrak{c}$-completion. Let $\mathfrak{c}$ be any full class of finite groups and let $\mathfrak{c}\mathcal{H}$ be the subcategory of objects such that the homotopy groups of all connected components are in $\mathfrak{c}$. We call a simplicial set a $\mathfrak{c}$-simplicial set if its class is in $\mathfrak{c}\mathcal{H}$. Following Artin and Mazur, we have a $\mathfrak{c}$-completion functor:

$$\mathfrak{c} : pro\text{-}\mathcal{H} \longrightarrow pro\text{-}\mathfrak{c}\mathcal{H}, \quad X \longmapsto X^{\mathfrak{c}},$$

which is left adjoint to the inclusion of the subcategory $pro\text{-}\mathfrak{c}\mathcal{H} \hookrightarrow pro\text{-}\mathcal{H}$. Combining all occurring definitions, we obtain

Lemma 7.1. *A morphism $f : X \to Y$ in pro-$\mathcal{H}$ induces a weak equivalence*

$$f^{\mathfrak{c}} : X^{\mathfrak{c}} \longrightarrow Y^{\mathfrak{c}}$$

on $\mathfrak{c}$-completions if an only if for every $\mathfrak{c}$-simplicial set M of finite homotopical dimension the induced homomorphism $Hom_{pro\text{-}\mathcal{H}}(Y, M) \to Hom_{pro\text{-}\mathcal{H}}(X, M)$ is an isomorphism.

Let $\mathcal{H}_{*c}$ be the homotopy category of pointed connected simplicial sets. A $\mathfrak{c}$-twisted coefficient group on $X \in pro\text{-}\mathcal{H}_{*c}$ is an abelian group $A \in \mathfrak{c}$ together with a map (of pro-groups) $\pi_1(X) \to Aut(A)$ which factors through some subgroup of $Aut(A)$ contained in $\mathfrak{c}$. In other words, A is a finite, discrete $\mathfrak{c}$-torsion module over the maximal pro-$\mathfrak{c}$ factor group $\pi_1(X)(\mathfrak{c})$ of $\pi_1(X)$.

Theorem 7.2 ([AM], 4.3)**.** *Let $\mathfrak{c}$ be a complete class of finite groups and let $f : X \to Y$ be a morphism in pro-$\mathcal{H}_{*c}$. Then the following are equivalent:*

(i) *$f^{\mathfrak{c}} : X^{\mathfrak{c}} \to Y^{\mathfrak{c}}$ is a weak equivalence.*

(ii) *f induces an isomorphism $\pi_1(X)(\mathfrak{c}) \xrightarrow{\sim} \pi_1(Y)(\mathfrak{c})$, and for every $\mathfrak{c}$-twisted coefficient group A on Y the induced homomorphism*

$$H^i(Y, A) \to H^i(X, A)$$

is an isomorphism for all i.

Finally, under some finiteness condition, weak equivalences may be identified as actual isomorphisms. We say that an object X of a site C has $\mathfrak{c}$-dimension $\leq d$ if for every locally constant sheaf F of abelian groups on C, locally isomorphic to some constant sheaf $\underline{A}$ for some $A \in \mathfrak{c}$, we have $H^q(X, F) = 0$ for $q > d$. The site C is said to have local $\mathfrak{c}$-dimension $\leq d$ if for every $X \in C$, there is a covering $X' \to X$ such that X' has $\mathfrak{c}$-dimension $\leq d$.

Theorem 7.3 ([AM], Theorem 12.5)**.** *Let $f : C \to D$ be a morphism of locally connected sites. Suppose that C and D have finite local $\mathfrak{c}$-dimension and that $\Pi f : (\Pi C)^{\mathfrak{c}} \to (\Pi D)^{\mathfrak{c}}$ is a weak equivalence in pro-$\mathcal{H}$. Then Πf is an isomorphism in pro-$\mathcal{H}$.*

§8. $\mathbf{A^1}$-factorization

The A^1-homotopy category $\mathcal{H}_{\mathrm{A}^1,et}(Sm(S))$ is obtained from $\mathcal{H}_{s,et}(Sm(S))$ by a process which essentially inverts the morphism $\mathrm{A}^1_S \to S$, see [MV]. The aim of this section is to consider the question whether the realization functor

$$ht : \mathcal{H}_{s,et}(Sm(S)) \longrightarrow \textit{pro-}\mathcal{H},$$

constructed in Section 6 factors through $\mathcal{H}_{\mathrm{A}^1,et}(Sm(S))$. A first observation is that the projection $\mathrm{A}^1_S \to S$ induces an isomorphism on mod ℓ étale cohomology only for those primes ℓ which are invertible on S. Therefore we can expect factorization only after completion away from the residue characteristics.

Proposition 8.1. *Let $\mathfrak{c}$ be a full class of finite groups such that ℓ is invertible on S for all primes ℓ with $\mathrm{Z}/\ell\mathrm{Z} \in \mathfrak{c}$. Then for any smooth scheme U over S the projection $\mathrm{A}^1_S \times_S U \to U$ induces a weak equivalence*

$$(\mathrm{A}^1_S \times_S U)^{\mathfrak{c}}_{ht} \to U^{\mathfrak{c}}_{ht}$$

in pro-$\mathcal{H}$. If S_{et} has finite local $\mathfrak{c}$-dimension, then this map is an isomorphism.

Proof. First of all, we may assume that U is connected. Choosing a (geometric) point u of U and the point $u' = (0, u)$ of $\mathbb{A}^1_S \times_S U$, we obtain a morphism of smooth connected pointed schemes. By 7.2, it suffices to show that the projection induces an isomorphism on pro-$\mathfrak{c}$ fundamental groups and on étale cohomology with finite $\mathfrak{c}$-twisted coefficients. The statement on the pro-$\mathfrak{c}$ fundamental groups follows from the fact that the geometric fibres of $\mathbb{A}^1_S \times_S U \to U$ do not admit nontrivial connected étale coverings of degree prime to their residue characteristics (use Hurwitz' genera formula). The statement on the étale cohomology follows from smooth base change.

If S_{et} has finite local $\mathfrak{c}$-dimension, then the same is true for $(\mathbb{A}^1_S \times_S U)_{\mathrm{et}}$ and for U_{et}. Therefore the weak equivalence $(\mathbb{A}^1_S \times_S U)^{\mathfrak{c}}_{ht} \to U^{\mathfrak{c}}_{ht}$ is an isomorphism by 7.3. Q.E.D.

The above results enable us to show $\mathbb{A}^1$-factorization. Isaksen [Is] and Quick [Qu] obtained a similar result in their approaches (completing at a single prime ℓ invertible on S).

Theorem 8.2. *Let $\mathfrak{c}$ be a full class of finite groups such that ℓ is invertible on S for all primes ℓ with $\mathbb{Z}/\ell\mathbb{Z} \in \mathfrak{c}$. Then the functor*

$$\natural \circ (-^{\mathfrak{c}}) \circ ht : \mathcal{H}_{s,et}(Sm(S)) \longrightarrow (pro\text{-}\mathcal{H})_w$$

factors through the étale $\mathbb{A}^1$-homotopy category $\mathcal{H}_{\mathbb{A}^1,et}(Sm(S))$. If S_{et} has finite local $\mathfrak{c}$-dimension, then already

$$(-^{\mathfrak{c}}) \circ ht : \mathcal{H}_{s,et}(Sm(S)) \longrightarrow pro\text{-}\mathcal{H}$$

factors through $\mathcal{H}_{\mathbb{A}^1,et}(Sm(S))$.

The proof of 8.2 will occupy the rest of this section. Let $\mathcal{B}$ be the class of morphisms $f : \mathcal{X} \to \mathcal{Y}$ in $\mathcal{H}_{s,et}(Sm(S))$ such that for every $\mathfrak{c}$-simplicial set M of finite homotopical dimension the induced map

$$Hom_{\mathcal{H}_{s,et}(Sm(S))}(\mathcal{Y}, M) \longrightarrow Hom_{\mathcal{H}_{s,et}(Sm(S))}(\mathcal{X}, M)$$

is an isomorphism. We say that a morphism of simplicial sheaves is in $\mathcal{B}$ if its class in $\mathcal{H}_{s,et}(Sm(S))$ is in $\mathcal{B}$. The proof of the next lemma is strictly parallel to the proof of [MV] 2.2.12 (where the class of $\mathbb{A}^1$-weak equivalences is considered), and therefore we omit it.

Lemma 8.3. *Let I be a small category, $\mathcal{X}, \mathcal{Y}$ functors from I to $\Delta^{op} Shv_{\mathrm{et}}(Sm(S))$ and $f : \mathcal{X} \to \mathcal{Y}$ a natural transformation such that all the morphisms f_i, $i \in I$, are in $\mathcal{B}$. Then the morphism*

$$hocolim_I \mathcal{X} \longrightarrow hocolim_I \mathcal{Y}$$

is in $\mathcal{B}$.

Lemma 8.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of simplicial sheaves such that for each $n \geq 0$ the morphism of sheaves $f_n : \mathcal{X}_n \to \mathcal{Y}_n$ is in $\mathcal{B}$. Then f is in $\mathcal{B}$.*

Proof. (cf. the proof of [MV] 2.2.14) Consider $\mathcal{X}$ and $\mathcal{Y}$ as diagrams of simplicial sheaves of simplicial dimension zero indexed by Δ^{op}. By [BK], XII, 3.4, the obvious morphisms $hocolim_{\Delta^{op}}\mathcal{X} \to \mathcal{X}$ and $hocolim_{\Delta^{op}}\mathcal{Y} \to \mathcal{Y}$ are simplicial weak equivalences. Therefore the statement follows from 8.3. Q.E.D.

Proposition 8.5. (i) *The class $\mathcal{B}$ contains the class of* A^1-*weak equivalences.*

(ii) *A $\mathfrak{c}$-simplicial set M of finite homotopical dimension considered as an element in $\mathcal{H}_{s,et}(Sm(S))$ is* A^1*-local.*

Proof. Let M be a $\mathfrak{c}$-simplicial set of finite homotopical dimension. By definition, M is A^1-local if for every $\mathcal{X} \in \mathcal{H}_{s,et}(Sm(S))$ the projection $\mathrm{A}^1_S \times_S \mathcal{X} \to \mathcal{X}$ induces an isomorphism

$$Hom_{\mathcal{H}_{s,et}(Sm(S))}(\mathcal{X}, M) \longrightarrow Hom_{\mathcal{H}_{s,et}(Sm(S))}(\mathrm{A}^1_S \times_S \mathcal{X}, M).$$

In other words, we have to show that the projections $\mathrm{A}^1_S \times_S \mathcal{X} \to \mathcal{X}$ are in $\mathcal{B}$. If $\mathcal{X}$ is a smooth scheme, this follows from 8.1 and 6.2. The case of a smooth simplicial scheme follows from 8.4. Finally, by [MV], 2.1.16, applied to the class of representable sheaves, each object in $\Delta^{op}Shv_{\mathrm{et}}(Sm(S))$ is simplicially weakly equivalent to a smooth simplicial scheme. This shows (ii), and (i) follows easily. Q.E.D.

Now, 8.2 follows from 8.5, 6.2 and 7.1.

§9. Conclusion

Assume that $k|\mathrm{Q}$ is a finitely generated field extension. Then k_{et} has finite local cohomological dimension. For any smooth scheme X over k the étale homotopy groups are already profinite, i.e. completion at the class of all finite groups is the identity. We have proven that X_{ht} only depends on the isomorphism class of X in the motivic category $\mathcal{H}_{\mathrm{A}^1,\mathrm{et}}(Sm(k))$. So the anabelian question (for higher dimensional varieties) should be formulated as follows:

Which are the smooth varieties over k such that the isomorphism class of X in $\mathcal{H}_{\mathrm{A}^1,\mathrm{et}}(Sm(k))$ is already determined by X_{ht}?

The A^1-local objects among those are anabelian in the strict sense. This coincides with the classical notion of anabelian varieties if, in addition, the $K(\pi, 1)$-property holds.

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Universität Heidelberg
Mathematisches Institut
Im Neuenheimer Feld 288, D-69120 Heidelberg
Deutschland
E-mail address: schmidt@mathi.uni-heidelberg.de