

DUALITY VIA CYCLE COMPLEXES
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The standard n -simplex over a scheme X is given by $\Delta_X^n = X \times_{\text{Spec } \mathbb{Z}} \Delta_{\mathbb{Z}}^n$, with $\Delta_{\mathbb{Z}}^n = \text{Spec}(\mathbb{Z}[T_0, \dots, T_n]/(\sum T_i - 1))$. The Bloch complex of relative zero cycles of X in (cohomological numbering) is the bounded above complex \mathbb{Z}_X^c of sheaves on $X_{\text{ét}}$ defined as follows: for $W \rightarrow X$ étale and $i \in \mathbb{Z}$

$$(\mathbb{Z}_X^c)^i(W)$$

is given as the free abelian group on the set of integral subschemes of dimension $-i$ in Δ_W^{-i} which intersect all faces properly. The differential is given in the usual way as the alternating sum of face maps. Geißer [4] showed the following remarkable theorem.

Theorem 1. *Let $f : X \rightarrow S$ be a scheme, separated and of finite type over the spectrum of a Dedekind ring. Let A be the ring obtained from \mathbb{Z} by inverting all prime numbers p such that there is a point $s \in S$ with imperfect residue field of characteristic p . Then, for every torsion sheaf F on $X_{\text{ét}}$, there is a canonical isomorphism in $\mathcal{D}(Ab)$*

$$R\text{Hom}_X^\bullet(F, \mathbb{Z}_X^c) \otimes_{\mathbb{Z}} A \simeq R\text{Hom}_S^\bullet(Rf_!F, \mathbb{Z}_S^c) \otimes_{\mathbb{Z}} A.$$

This is an astonishingly general duality theorem. There is no assumption on the smoothness of f , even no regularity assumption on X . Moreover, the torsion of F need not be invertible on X , only the ‘imperfectness’ of S must be inverted.

If duality results for S are known (e.g. $S = \text{Spec } \mathcal{O}_K$, K a number field or a local field, or $S = \text{Spec } k$, k a finite, local or separably closed field) then Theorem 1 implies duality statements for the étale cohomology of X itself. Essentially all known duality theorems in higher dimension are special cases of Theorem 1. This follows from the following calculations of the cohomology of \mathbb{Z}^c :

- Theorem 2.**
- (i) *If $S = \text{Spec } k$ is the spectrum of a field, then $\mathbb{Z}_S^c \simeq \mathbb{Z}$ (constant sheaf placed in degree zero)*
 - (ii) *If S is regular and one-dimensional, then $\mathbb{Z}_S^c \simeq \mathbb{G}_m[1]$.*
 - (iii) *If X is regular of dimension d and $m \in \mathbb{N}$ is invertible on X , then $\mathbb{Z}_X^c/m \simeq \mu_m^{\otimes d}[2d]$.*
 - (iv) *If X is a smooth, d -dimensional and separated scheme of finite type over a perfect field of characteristic $p > 0$, then $\mathbb{Z}_X^c/p^r \simeq \nu_r^d[d]$ (logarithmic deRham-Witt sheaf).*

Here \simeq means canonical isomorphism in the derived category of étale sheaves on X .

The result of Geißer is the culminating point of a development which started with papers of Deninger [3], Nart [12], Milne [10], Spieß [13] and Moser [11]. It uses in an essential way results on motivic cohomology and higher Chow groups obtained by Bloch, Suslin, Voevodky, Rost, Levine and himself.

TALKS

Talk 1: Introduction (Alexander Schmidt)

Talk 2: Higher Chow groups

Explain the definition of higher Chow groups and Chow sheaves for varieties and present the basic properties ([1] 1.3, 2.1, 3.1, 3.2, 3.4, 4.1, 6.1, 10.1). Where possible within reasonable time, present proofs or sketches of proofs. There are some mistakes in [1] which were corrected in [2], [7]. Present what is known for schemes of finite type over a 1-dimensional base following [9], [8].

Talk 2: Higher Chow sheaves mod n on regular schemes

First, for invertible n , construct the cycle map following [6] and [9] and show Theorem 2 (iii). Then introduce the logarithmic de Rham-Witt sheaves, e.g. following [5] and show Theorem 2 (iv). Extend the last result to singular varieties ([4], Prop. 2.3)

Talk 3: Duality in the p - p case

Give a proof of the p - p duality for smooth projective varieties over a perfect field following [10] Thm. 1.11. Deduce duality over an algebraically closed field and give a complete proof of [4], Thm. 4.6.

Talk 4: Etale descent for the Bloch complex

Explain the connection between higher Chow groups and motivic cohomology and formulate without proof the Beilinson-Lichtenbaum conjecture (now a theorem). Explain the method of Thomason [14] 2.8 (here we only need sheaves of abelian groups, not of spectra). Then give a complete proof of Thm. 3.1 and Thm 7.1 of [4].

Talk 5: The main theorem

Show the main theorem Thm. 4.1 and Cor. 4.7 of [4]. Then show the corresponding results over one-dimensional bases Thm. 7.5 and Prop. 7.10.

Talk 6: Applications

Duality over finite and local fields, Kato-homology sequences [4], §5, the generalization of Roitman's theorem [4] Thm. 6.1.

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