Introduction to ∞-topoi and exodromy for schemes

Oberseminar of the Prof. Schmidt's AG "Arithmetische Homotopietheorie"

Pavel Sechin

Time and place: SS19, Tuesday, 11:00-13:00, SR 3, Mathematikon

The goal of this seminar is two-fold. The first part of the semester will be dedicated to introduction of ∞-categories using the language of quasi-categories. The second part will cover the proof of the main theorem of [BGH-Exo] which claims that the functor from reduced normal schemes over a finitely generated field of characteristic zero which assigns to a scheme X a category $Gal(X)$ of points of the étale topos of X is fully faithful to the category of topological categories over the étale fundamental groupoid of k [1]. The idea of the proof is the possibility to reglue the étale ∞-topos of a scheme from the étale ∞-topoi of points (which can be reconstructed from the classifying *spaces* of the absolute Galois groups) using specialization maps between these points. This gluing process is performed in the setting of ∞-categories and, in fact, is a corollary of a purely ∞-categorical statement generalizing Hochster duality.

Quasi-categories are certain simplicial sets which generalize both Kan complexes (i.e. simplicial sets which allow to model the homotopy theory of CW-complexes) and nerves of small categories (i.e. effectively the study of categories, since the nerve functor is fully faithful). The need for infinity categories stems from drawbacks of other approaches to incorporate homotopy theory in different areas of mathematics which we briefly recall.

Arguably, the first such attempt is the notion of a triangulated category due to Verdier. One of the problems which one encounters here is that the category of functors between two triangulated categories in general does not possess a triangulated structure (as a consequence of the non-functoriality of the cone). This comes up when one seeks to glue triangulated categories (in general, to compute a limit of a diagram of triangulated categories), e.g. trying to obtain the derived category of sheaves on a 'space' X from derived categories of sheaves on a closed 'subspace' and its open complement. The problem of gluing can be solved provided that one works with DG-enhancements of triangulated categories. In fact, (pre-trinagulated) DG-categories can be identified in a precise manner with certain 'stable' ∞-categories, thus, showing that this approach is at least not worse than ∞-categorical under appropriate circumstances. However, neither DG-categories nor stable ∞-categories will be discussed in the course of the seminar.

Another approach to abstract homotopy theory are model categories introduced by Quillen. They allow to study localizations of categories with respect to a set of morphisms in efficient manner. The category of functors between model categories has a model structure under mild restrictions on the target model category, thus, allowing to deal with gluing problem. However, a serious drawback comes from the fact that the study of the localization in this approach depends not only on the set of morphisms to localize, but on additional and non-canonical data. Thus, for example, to compare two localizations in this approach one needs also to compare this additional data which may or may be not possible depending on the choices made. Though having disadvantages, model categories remain a tool for the study of quasi-categories as well as a source of examples of ∞-categories via certain nerve construction.

The particular emphasis of the study of ∞-categories in the seminar will be on properties of so-called ∞-topoi, an analogue theory of Grothendieck topoi in the higher category theory. Given a Grothendieck topology on a category C, one can produce an ∞-topos of sheaves of ∞-groupoids[2] on it which is closely related to the study of the homotopy category of simplicial sheaves on C due to Jardine. Broadly speaking, one can view an ∞-topos as a non-abelian generalization of the derived categories of sheaves of abelian groups on a topos.

As an example of a previous construction, there is a way to associate to every space (more precisely, to every Kan complex) an ∞-topos. It turns out that when restricted to the ∞-category of pro-π-finite spaces[3] this functor is fully faithful. Thus, one characterize the topoi in the image of this functor (called Stone or profinite) via certain topological spaces. A particular example of a Stone ∞-topos is the ∞-topos of étale sheaves on a spectrum of a field which corresponds to the classifying space of the profinite absolute Galois group of the field via this construction.

The setting of ∞-categories provides a convenient environment to glue pro-π-finite spaces. As an example, which is used in [BGH-Exo], one can glue classifying spaces of absolute Galois groups of schematic points into what is called a stratified profinite étale homotopy type. It is an object of pro-π-finite spaces together with a stratification over a spectral topological space underlying the scheme.

Gluing ∞-topoi (and topoi), on the other hand, is not so straight-forward. Deligne has defined an oriented pullback topos $X \overleftarrow{\times}_S Y$ for a pair of geometric morphism of topoi $f : X \to S$, $g : Y \to S$ which satisfies a certain universal property in 2-category of topoi given by the diagram

[1] the functors in this category have to be continuous and preserve minimal objects

[2] which are often called sheaves of spaces, or just sheaves

[3] i.e. a pro-completion in ∞-categorical sense of those spaces which have finite homotopy groups and are n-connective for some n

$$\begin{tikzcd} X \overleftarrow{\times}_S Y \arrow[r, "q_*"] \arrow[d, "p_*"'] & Y \arrow[d, "g_*"] \arrow[dl, phantom, "\swArrow \tau"] \\ X \arrow[r, "f_*"'] & S , \end{tikzcd}$$

where $\tau : g_* q_* \to f_* p_*$ is the natural transformation. Moreover, Gabber showed that if f, g are coherent morphisms of coherent topoi, Λ is a ring, then for $F \in \mathrm{D}^+(Y, \Lambda)$ the canonical morphism $f^* R g_* F \to R p_* q^* F$ is an isomorphism in $\mathrm{D}^+(X, \Lambda)$.

The paper [BGH-Exo] performs this construction and proves this base change result for ∞-topoi. This allows to glue the ∞-topos of étale sheaves on points of X to a stratified ∞-topos of étale ∞-sheaves on X. The fully faithfullness result about pro-π-finite spaces above then extends to show that this stratified ∞-topos can be reconstructed from the stratified profinite étale homotopy type. Finally, the main theorem follows from the result of Voevodsky which claims that morphisms between normal reduced schemes over a finitely generated field of characteristic zero can be reconstructed from the morphisms between corresponding étale topoi.

Prerequisites. We will assume familiarity of the participants with simplicial sets, model categories, and, in particular, the Quillen model structure on simplicial sets. In order to have a feeling of the objectives of the first part of the seminar everyone is invited to read through [LurHTT, Chapter 1] which is a discussion of goals, difficulties and some results (mostly, without proofs) about ∞-categories. We will not spend much time introducing and motivating main definitions of the theory of quasi-categories from there.

We will follow [LurHTT], but it seems virtually impossible to cover the whole material even of Chapters 2-6 in one semester. Thus, there bound to be some omissions. The first such omission is the proof of the Joyal extension theorem and related facts from [LurHTT, Chapter 1]. Note, however, that the proof of all these facts is obtained in [LurHTT], though, there exists another approach due to Joyal himself. Many other results will also be used without proof, however, the ambitious goal of the seminar is to cover a 'dense' subset of relevant statements so that the participants of the seminar would be able to read and understand the proofs on their own.

Schedule[4]

Part I. ∞-categories towards ∞-topoi.

Our ultimate goal of the first part is to understand what an ∞-topos is, and, in particular, how can one attach to a Grothendieck topology on 1-category (or ∞-category, provided, we understand what a Grothendieck topology means there) an ∞-topos. With this knowledge we could obtain, for example, the étale ∞-topos $X_{\acute{e}t}$, which plays an important role in [BGH-Exo].

By definition, an ∞-topos is a left exact localization of ∞-category of presheaves of spaces on a ∞-category. Thus, we need to define ∞-categories, spaces and presheaves, as well as obtain methods of working with them first.

The role of spaces in ∞-categories is played by so-called ∞-groupoids, which are just Kan complexes in the approach of quasi-categories. In comparison to the classical theory, we see that the role of a set is replaced by a generalization of a groupoid, which yields a new point of view on presheaves of spaces. In classical category theory functors from a category C to the category of groupoids can be identified via the so-called Grothendieck construction with a category cofibred in groupoids. Left fibrations are generalizations of this phenomenon to ∞-categories, however, the equivalence between them and functors to Kan complexes is not so easy to obtain since it has to be true only up to homotopy (in a suitable sense). This equivalence is obtained using the language of model categories and, in particular, the covariant model structure constructed in Talk 2. Moreover, this result allows to compare the theory of quasi-categories and the theory of simplicial categories (using relevant model structures) in Talk 3.

A more general version of the previous phenomenon comes up when one wants to study functors cofibred in ∞-catgories and not just ∞-groupoids (aka spaces). This will come up naturally, for example, when we will study stratified ∞-topoi, or in general study of relations between different ∞-categories. These fibrations are called Cartesian fibrations and an introduction to them is provided in Talks 4, 5.

After some preliminary work about (co)limits and limits, we will switch to the study of presheaves in Talk 7. However, before studying ∞-topoi in Talk 10 we will need to understand adjoint functors in Talk 8 and some useful smallness assumptions on ∞-categories in Talk 9 which allow to ignore many set-theoretic issues.

All references below in Part I refer to [LurHTT] if not stated otherwise.

Session 1. Overview and left/right fibrations between simplicial sets. (Pavel, 16.04)

In this talk I will sketch the idea of the proof of the main theorem of [BGH-Exo] using ∞-categories and will also explain the scheduled route towards ∞-topoi which we plan to undertake.

[4]Changes can be made in the course of the seminar.

In the second part of this talk I will try to cover the major part of [2.1] introducing left/right fibrations between simplicial sets and proving their properties. Several proofs will be skipped, most notably, of [2.1.2.1-2.1.2.10]. These proofs can also be found in [Rez17].

Finally, I will introduce the covariant model structure using simplicial categories. However, we will see later that its fibrant objects are just left fibrations and cofibrations are monomorphisms (which determines the model structure) and one could give a definition without using simplicial categories.

Session 2. The covariant model structure and (un-)straightening. Simplicial categories. (Pavel, 23.04)

The goal of this talk is to prove a Quillen equivalence [2.2.1.2] between the covariant model structure on the over-category Set_{Δ}/S and the projective model structure on $\mathrm{Fun}(\mathfrak{C}[S], Set_{\Delta})$, where $\mathfrak{C}[S]$ is a simplicial category associated to a simplicial set S. Using this equivalence we will also determine explicitly fibrant objects of the covariant model structure and covariant fibrations between them.

Session 3. Joyal model structure on simplicial sets, Quillen equivalence with simplicial categories. (30.04)

In this talk the covariant model structure will be used to relate the mapping spaces in an ∞-category defined using the simplicial categories and using internal constructions (namely, slices) [2.2.4.1]. The main goal is to prove that there exist the Joyal model structure on simplicial sets which is Quillen equivalent to the model structure on simplicial categories [Th. 2.2.5.1] and to describe explicitly categorical equivalences between quasi-categories [2.2.5.8].

Session 4. Cartesian fibrations I: definitions and properties. (07.05)

The goal of this talk is to explain main definitions and results of [2.4] which studies Cartesian fibrations. A Cartesian fibration between simplicial sets is a generalization of right fibrations (the dual notion is called coCartesian fibration) which can be viewed as a moduli of ∞-categories over a simplicial set. Important results that should be covered, apart from definitions, include [2.4.1.3-2.4.1.8, 2.4.4.1-2.4.4.9] as well as the proof of [1.2.9.3] in [2.4.5], and the description of fibrant objects in the Joyal model structure [Th. 2.4.6.1].

Session 5. Cartesian fibrations II: the Cartesian model structure. (14.05)

In this talk the results about Cartesian fibrations from [Chapter 3] which will be often used later should be announced. Please choose which of the following statements you would like to present with a proof and which without (or, perhaps, with a sketch of a proof):
The class of marked anodyne maps: [3.1.1.5-3.1.1.7, 3.1.2.3].
The Cartesian model structure: [3.1.3.5, 3.1.3.7].
Straightening and unstraightening: [3.2.2.10, 3.2.2.11, Th. 3.2.0.1].

Please also include the discussion of universal fibrations in [§3.3.2].

Sessions 6-7. Miscelannous about limits and colimits in ∞-categories. (21.05)

Presheaves; definition, Yoneda embedding, the universal property. (28.05)

The first of these sessions should cover material on computations of (co)limits in ∞-categories which will be needed for the study of presheaves. One should explain [Section 4.2.4] which allows to reduce the calculation of limits and colimits in ∞-categories to the calculation of homotopy colimits in model categories. This will be useful in many cases, e.g. to prove ∞-categorical Yoneda lemma. Perhaps, examples of (co)limits in the ∞-category of spaces could be given.

The results of propositions [4.2.2.4, 4.2.2.7] will be used to prove the existence of small limits and colimits in ∞-categories of presheaves. Proposition [4.2.3.14] is used to prove that Yoneda lemma preserves limits.

The second talk of these sessions should cover main properties of presheaves of spaces as developed in [5.1]. By definition, if $\mathcal{C}$ is an ∞-category, then the ∞-category $\mathcal{P}(C)$ is defined as $Fun(\mathcal{C}^{op}, N(\mathcal{K}an))$ where $\mathcal{K}an$ is the simplicial category of Kan complexes and N is the simplicial nerve functor. Using the techniques we have studied so far, one can produce an ∞-category equivalent to $\mathcal{P}(C)$ using left fibrations [5.1.1.1] and prove that limits and colimits in presheaves can be computed objectwise [§5.1.2]. Moreover, we can prove Yoneda lemma [5.1.3.1] and show that the Yoneda embedding preserves small limits [5.1.3.2].

Please state the universal property of presheaves [5.1.5.6], and sketch the proof if possible.

Session 8. Adjoint functors: definitions, properties, the adjoint functor theorem. (04.06)

The notion of adjoint functors turns out to be a little bit more delicate for ∞-categories than for 1-categories. For example, though one can check the existence of an adjoint functor on the level of enriched homotopy categories of ∞-categories, one can not check adjointness of two functors in this way.

The goal of this talk is to prove fundamental properties of adjoint functors: uniqueness of an adjoint up to equivalence [5.2.1.3-5.2.1.4], possibility to check the adjointness using the unit transformation [5.2.2.8], the fact that left adjoint preserves colimits [5.2.3.5].

It is important to introduce examples of adjoint functors coming from simplicial model categories and Quillen adjunctions between them [5.2.4.6].

Please state the adjoint functor theorem [5.5.2.9] postponing the notion of accesible functor to the next talk.

Session 9. Ind-completions, localizations, accessible and presentable. (11.06)

The goal of this talk is to overview definition and results about accessible and presentable ∞-categories. (Details TBA)

Session 10. ∞-topoi: definitions, constructions from Grothendieck topologies, morphisms. (18.06)

The goal of this talk is to explain the notion of ∞-topos on a particular example arising from the Grothendieck topology on ∞-category. Thus, one needs to prove that the category of sheaves (of spaces) satisfies the general definition of ∞-topos [6.1.0.4] by constructing 'the sheafification' functor and checking its properties [Sections 6.2.1-6.2.2, in particular, 6.2.2.7].

For the purpose of studying ∞-topoi in algebraic geometry, one should explain how geometric morphisms between ∞-topoi [6.3.3.1] arise from morphisms of Grothendieck topologies. Moreover, we will be interested in a particular example of an ∞-topos, where every object is coherent [LurSAG, A.2.1.6]. For this please introduce the notion of finitary Grothendieck topology [LurSAG, A.3.1.1] and prove that it yields a coherent ∞-topos [LurSAG, A.3.1.3] and, moreover, morphisms of finitary Grothendieck topologies yield coherent morphisms of ∞-topoi, e.g. [Hai19, Cor. 2.8]. As examples please mention [Hai19, Prop. 2.19, Example 2.21].

If time permits, please explain the notion of hypercomplete ∞-topos, its relation to the Jardine model structure on simplicial sheaves [Section 6.5.2]. Please also define a notion of a point of an ∞-topos and state the Deligne's Completeness theorem for the existence of points of hypercomplete ∞-topoi [LurSAG, Prop. A.4.0.5].

Part II[5]. Exodromy.

References below refer to [BGH-Exo] if not specified otherwise.

Session 11. The comparison between quasi-categories and Segal complete spaces. (25.06)

This talk should explain the theorem of Joyal and Tierney which compares two approaches to ∞-categories: quasi-categories and Segal complete spaces (the latter were introduced by Rezk).

First, introduce the 'classifying diagram' construction of Rezk from $\mathbf{Cat}$ to bisimplicial sets [Rez01, 3.5]. If time permits, show that 'vertical' weak equivalences between such nerves are in 1-to-1 correspondence with equivalences of categories [Rez01, Th. 3.7].

Second, introduce Segal spaces and complete Segal spaces, and briefly explain how one can view them as a model of ∞-categories (e.g. explain the construction of the homotopy category, mapping spaces) [Rez01, 4.1, Section 5]. Introduce the model structure on the category of bisimplicial sets where fibrant objects are complete Segal spaces and cofibrations are monomorphisms [Rez01, Th. 7.2]. Perhaps, also show that classifying diagrams yield complete Segal spaces [Rez01, 6.1].

Finally, prove the main result of [JoyTie], constructing a pair of Quillen equivalences between the Joyal model structure on simplicial sets and the model structure on bisimplicial sets defined above [JoyTie, Th. 4.11, 4.12]. If time does not permit to explain the whole proof, please at least explain why (and if) the functors involved preserve fibrant objects.

Results of this talk will be used in the décollage statements for stratified spaces and stratified ∞-topoi.

Session 12. Décollage. Pro-completions. (02.07)

In this talk the application of complete Segal spaces to the study of stratified spaces should be explained. Namely, main results and definitions of [Chapter 1] should be covered: stratified spaces as conservative functors to a poset [Def 2.1], their materialisations, strata and links, and, most important, equivalence between stratified spaces and décollage of them [Th. 4.7].

However, the main goal is to extend these results to stratified pro-(π-finite) spaces [4.9, 4.11] which seems purely formal but better be explained in all details. For the construction of pro-completions please exploit [LurHTT, §5.3] as well as remarks about existence of pro-adjoints in [0.6].

Finally, please introduce stratified ∞-topoi over a poset [9.6-9.6] and over a proposet [9.26]. If time permits, please explain the natural stratification of a coherent ∞-topos [9.40]. Also, construct a functor from stratified spaces to stratified ∞-topoi [5.72, 9.16] and its pro-version [9.50]. Introduce spectral stratified ∞-topoi [10.3] and state the Hochster duality theorem [10.10] which we will prove in the last talk.

Session 13-14. Evanescent ∞-topos of Deligne and localization of an ∞-topos at a point. (09.07)

Beck-Chevalley conditions and base change for ∞-topoi. Gluing of bounded and coherent ∞-topoi. (16.07)

[5]The schedule of this part is very tentative at the moment.

The goal of these two session is to introduce the ∞-generalization of oriented pullback of topoi by Deligne and generalize the Gabber's base change result in this context. The 1-toposic statements constitute [Gab14, Exp. XI, Sections 1&2], you may wish to consult it or even explain some material from there first.

First, explain oriented pushouts of ∞-topoi as oriented pullbacks in $\mathbf{Cat}_\infty$: perhaps, start with recollements of ∞-categories ([LurHA, A.8]). Then give a brief introduction to bounded and coherent (aka bc) ∞-topoi. Explain the bc-version of the pushout of ∞-topoi [BGH-Exo, Constructions 6.12, 6.22]. Please also note that Beck-Chevalley condition is satisfied for bounded coherent oriented pushout.

Second, introduce the oriented fibre product of ∞-topoi [BGH-Exo, Def. 6.29], and explain the Deligne's construction of the computation of it when ∞-topoi come from finitary ∞-sites [Constructions 6.42-6.46]. Please also explain [Lemma 6.48] which says that oriented pullbacks of bc-∞-topoi are bc. For the proof of the Gabber's result you will also need statements [6.51-6.55].

Third, introduce local geometric morphism of ∞-topoi and their properties. The important result here is [Prop. 7.27] which says the localization of an ∞-topos at a point can be calculated as a limit of ∞-topoi of neighbourhoods of the point.

To prove the base change result, one uses the Deligne Completeness Theorem which reduces the situation to the study of stalks at points. Thus, first one proves the base change result in the case when the down map is a localization at a point [Prop. 8.8]. Then one generalizes the statement to a local morphism of ∞-topoi [8.22-8.24]. Finally, one shows how to reduce the general case to the local case [pp. 80-81].

Session 15. The main theorem. (23.07)

First, explain gluing squares [8.29-8.34] using the results of the previous talk.

Second, introduce décollage of stratified ∞-topoi and prove the comparison between stratifications and décollage [Def. 9.19, Th. 9.25]. Construct the functor $\hat{\lambda}$ from stratified pro-π-finite spaces to stratified ∞-topoi [9.50] and show that it is fully faithful [Prop. 9.51] appealing to a theorem of Lurie.

Third, define spectral ∞-topoi and prove Hochster duality for them [Def. 10.3, Th. 10.10].

Finally, explain how to apply this theorem to schemes [Construction 13.5] and combine it with Voevodsky's result [Voe] to obtain the main theorem [Th. 14.16, Th. A].

References

[BGH-Exo] Barwick C., Glasman S. and Haine P., *Exodromy*, arXiv preprint, Link, 2018.

[Gab14] Illusie L., Laszlo Y., Orgogozo F., Eds., *Travaux de Gabber sur l'uniformisation locale et la cohomologie étale des schémas quasi-excellents*. Société Mathéma- tique de France, Paris, 2014, i–xxiv and 1–625, Séminaire à l'École Polytechnique 2006–2008. With the collaboration of Frédéric Déglise, Alban Moreau, Vincent Pilloni, Michel Raynaud, Joël Riou, Benoît Stroh, Michael Temkin and Weizhe Zheng, Astérisque No. 363-364 (2014).

[Hai19] Haine P. J., *On coherent topoi & coherent 1-localic ∞-topoi*, preprint, arXiv:1904.01877 (2019).

[JoyTie] Joyal A., Tierney M., *Quasi-categories vs Segal spaces*, in "Categories in algebra, geometry and mathematical physics", Contemp. Math. Vol. 431, Providence, RI: Amer. Math. Soc., 2007, pp. 277–326.

[LurHTT] Lurie J., *Higher Topos Theory*, (AM-170). Vol. 189. Princeton University Press, 2009.

[LurHA] Lurie J., *Higher algebra*, preprint, Link, Sep. 2017.

[LurSAG] Lurie J., *Spectral algebraic geometry*, preprint, Link, Feb. 2018.

[Rez01] Rezk C., *A model for the homotopy theory of homotopy theory*, Transactions of the American Mathematical Society 353.3 (2001): 973-1007.

[Rez17] Rezk C., *Stuff about quasicategories*, Unpublished notes, Link.

[Voe] Voevodsky V.A., *Etale topologies of schemes over fields of finite type over* $\mathbf{Q}$, Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya 54.6 (1990): 1155-1167.