Lecture notes: Simplicial sets

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Introduction, motivation

Algebraic topology is the study of topological spaces by using algebraic invariants, such as (co-)homology, the fundamental group or more generally homotopy groups. This lead to a powerful machinery, which also became extremely useful to other fields of mathematics. For example group cohomology (in this setting more often called Galois cohomology) became an important tool in algebraic number theory. Modern algebraic geometry is unimaginable without the study of the various forms of cohomology (Zarisiki, étale, synthomic, fppf, fpqc, ...).

Originally the cohomology of a group G was introduced as the (singular) cohomology of its classifying space $|BG|$, which is a connected space with fundamental group $\pi_1|BG| = G$ and vanishing higher homotopy groups. In algebraic number theory this group G is most likely the Galois group of a field extension. So we are assigning to an algebraic object G a topological space $|BG|$, which we are taking an algebraic invariant of.

$$\begin{tikzcd}
& \text{topological space} \arrow[dr] & \\
\text{algebraic object} \arrow[ur] \arrow[r, dashed] & \text{“construction plan”} \arrow[u] \arrow[r, dashed] & \text{algebraic object}
\end{tikzcd}$$

This looks like a long way round and one might ask, if it is possible to define group cohomology without taking the detour over topological spaces. The idea is to use an abstract “construction plan” BG for the topological space $|BG|$.

The aim of this lecture is the study of these “construction plans”, which in our case are given by simplicial sets. So mainly we will do algebraic topology without topology. Apart from the given example above, simplicial sets have many applications in different fields of mathematics. The first few lectures will be about (abstract) simplicial complexes. Compared to simplicial sets, their geometry is much easier to understand. We point out the difficulties that arise when trying to do homotopy theory in the context of simplicial complexes. Their disadvantages motivate the introduction of simplicial sets, which the main part of these lecture is about.

The theory of simplicial sets is a theory of functors, so there is no way around some techniques provided by category theory. As category theory still enjoys the reputation of being “abstract nonsense”, which may discourage some readers, we present the necessary tools along the way of studying simplicial sets. We intersperse short sections of the abstract theory, which will be applied subsequently in the context of simplicial sets. Although using the abstract language of category theory, we try to motivate every construction by its topological analog. Guided by the geometric realization, we develope the basic constructions like homotopies and mapping cones in the context of simplicial sets.

After this topologically motivated introduction we begin to point out the main advantages. Instead of working with simplicial sets, one can similarly define simplicial objects in any other category. This is a very powerful feature that allows us to use simplicial techniques in any field of mathematics. Probably the most important example is the category of simplicial modules, which turns out to be equivalent to the category of chain complexes via the Dold-Kan correspondence. Coming back to our example of above, when learning group homology, one will most likely end up with studying homological algebra first. Compared to simplicial modules, chain complexes seem much easier to handle. However every tool in homological algebra has its analog in the simplicial context. There are even simplicial constructions, that are not possible in the context of homological algebra (e.g. deriving non-additive functors).

If time allows we will turn to the deeper homotopy theoretic relationship of simplicial sets and topological spaces given by the Quillen equivalence. This involves abstract homotopy theory using model categories. Like in homological algebra, one can define derived functors, a technique which becomes more and more popular also in other fields of mathematics.

1 (Abstract) Simplicial complexes

Recall that an **n-simplex** is the convex hull of a set of $n+1$ points in the $\mathbb{R}^m$, for some $m \geq 0$. A **face** of the simplex is the convex hull of a subset of these points. Moreover a **(geometric) simplicial complex** is a set of simplices in some $\mathbb{R}^m$, such that the intersection of two simplices is a face of each simplex or empty. We can stretch a simplex without changing its homeomorphic type. So topologically the exact coordinates in the ambient space $\mathbb{R}^m$ are redundant, as long as we keep in mind how the simplices intersect. As a preparation for simplicial sets we now introduce (abstract) simplicial complexes, which can be thought of as "construction plans" for (geometric) simplicial complexes. Compared to simplicial sets, they are more easily to understand geometrically. However they have their disadvantages, which motivate the introduction of the more flexible notion of simplicial sets.

Definition 1.1 (i) *An* ***(abstract) simplicial complex***[1] *is a set X, together with a set $S(X)$ of subsets of X with*

a) $0 < \sharp s < \infty, \quad$ *for all* $s \in S(X)$.[2]

b) $\{x\} \in S(X), \quad$ *for all* $x \in X$.

c) $\emptyset \neq t \subset s \in S(X) \quad \Rightarrow \quad t \in S(X)$.

The elements of X are called ***vertices****, the elements of $S(X)$* ***simplices****.*

(ii) *A* ***simplicial map*** *between simplicial complexes is a map $X \xrightarrow{f} Y$ with*

$$f(s) \in S(Y), \quad \text{for all } s \in S(X).$$

Remark 1.2
Let X be a set.

(i) *We define a simplicial complex $D(X)$ by setting*

$$D(X) := X, \qquad SD(X) := \{\{x\}; x \in X\}.$$

It is the smallest simplicial complex with vertices X.

We call it the ***discrete simplicial complex*** *with vertices X.*

[1] As we will only deal with abstract simplicial complexes, we will just write "simplicial complex" for short.

[2] Most commonly the cardinality of a set S is denoted by $|S|$. However to avoid confusion with a later definition, throughout this text the cardinality will be denoted by $\sharp S$.

(ii) *We define another simplicial complex* $I(X)$ *by setting*

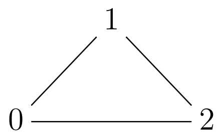
$$I(X) := X, \qquad SI(X) := \{s \subset X; 0 < \sharp s < \infty\}.$$

It is the biggest simplicial complex with vertices X.

We call it the ***indiscrete simplicial complex*** *with vertices* X.

Example 1.3
Let $n \geq 0$.

(i) *The* ***standard combinatorial*** n***-simplex*** *is defined as the simplicial complex* $I(\underline{n})$*, where* $\underline{n} := \{0, ..., n\}$*. For* $n = 2$ *the space we have in mind looks like*



where the triangle in the middle is a 2*-simplex.*

(ii) *The boundary of* $I(\underline{n})$ *is given by*

$$\partial I(\underline{n}) := \underline{n}, \qquad S\partial I(\underline{n}) := \{s \subset \underline{n};\ 0 < \sharp s < n\}.$$

For $n = 2$*, we can draw the same picture as in (i) without the connecting* 2*-simplex.*

Remark 1.4
Every simplicial complex is the union[3] *of its finite simplicial subcomplexes.*
Indeed every simplex is finite and so lies in some finite simplicial subcomplex.

1.1 Geometric realization

In this section we describe the topological space, that an (abstract) simplicial complex stands for. We do this by assigning a topological space to an abstract simplicial complex in a natural way. Explicitly, given an (abstract) simplicial complex X, we define

$$|X| := \left\{ a = \sum_x a_x \cdot x \in \bigoplus_{x \in X} \mathbb{R} \cdot x; \quad a_x \geq 0, \quad \sum_{x \in X} a_x = 1, \quad \{x \in X;\ a_x \neq 0\} \in S(X) \right\},$$

where $\bigoplus_{x \in X} \mathbb{R} \cdot x$ denotes the real vector space of formal linear combinations over the elements $x \in X$.

Example 1.5
Let $n \geq 0$.

[3] meaning that a subset of the union is a simplex, if it is a simplex in some subcomplex.

(i) *The set $|I(\underline{n})|$ coincides with the* ***standard (geometric)*** n***-simplex***

$$|\Delta^n| := \{(a_0, ..., a_n) \in [0,1]^{n+1};\ a_0 + ... + a_n = 1\}.$$

(ii) *The set $|\partial I(\underline{n})|$ is its boundary.*

(iii) *More generally, for every subcomplex $X \leq I(\underline{n})$, we see that $|X| \leq |I(\underline{n})|$ is the union of all geometric simplices in $|I(\underline{n})|$ corresponding to simplices in X.*

Up to now $|X|$ is only a set, that we still need to put a topology on. Every simplicial complex X with $n = \sharp X < \infty$ can be considered as a simplicial subcomplex $X \leq I(\underline{n-1})$, so it seems natural to consider $|X| \subset |\Delta^{n-1}| \subset \mathbb{R}^n = \mathbb{R}^X$ as a subspace of the product space. However for infinite X, we will need a finer topology than the subspace topology $|X| \subset \mathbb{R}^X$.

Definition 1.6
Given a family of maps $F_i \xrightarrow{f_i} X$, whose domains F_i carry a topology.
The ***final topology*** *on X is the topology, for which a subset $U \subset X$ is open, whenever $f_i^{-1}(U) \subset F_i$ is open, for all i.*

Remark 1.7
Suppose X carries the final topology with respect to a family of maps $F_i \xrightarrow{f_i} X$.
Then the following holds.

(i) *f_i is continuous, for all i.*

(ii) *A map $X \xrightarrow{f} Y$ into a topological space Y is continuous, if and only if $f \circ f_i$ is continuous, for all i.*

Remark 1.8
Every simplicial map $X \xrightarrow{f} Y$ induces a map

$$|f| : |X| \longrightarrow |Y|, \quad \sum_x a_x \cdot x \longmapsto \sum_x a_x \cdot f(x).$$

Definition 1.9
The ***geometric realization*** *of a simplicial complex X is defined as the set $|X|$ together with the following topology.*

- *For $\sharp X < \infty$ we give $|X| \subset \bigoplus_{x \in X} \mathbb{R} \cdot x = \prod_{x \in X} \mathbb{R} = \mathbb{R}^X$ the subspace topology of the product topology.*

- *For $\sharp X = \infty$ we give $|X|$ the final topology with respect to all maps $|F| \xhookrightarrow{|j|} |X|$, where $F \xhookrightarrow{j} X$ is the inclusion of a finite subcomplex. That is $U \subset |X|$ is open, iff $U \cap |F|$ is open, for all finite $F \subset X$.*

Proposition 1.10
For every simplicial map $X \xrightarrow{f} Y$, the map $|f|$ is continuous.

Proof.

1) For $\sharp X < \infty$ the map $|f|$ is the restriction of a linear map $\mathbb{R}^X \longrightarrow \mathbb{R}^Y$, which is continuous.

2) For $\sharp X = \infty$, let $F \subset X$ be a finite subcomplex. Then in the commutative square

$$\begin{tikzcd} |F| \arrow[r, hook] \arrow[d] & |X| \arrow[d, "|f|"] \\ |f(F)| \arrow[r, hook] & |Y| \end{tikzcd}$$

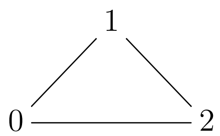
the lower horizontal map is continuous by Definition of the final topology on $|Y|$ and Remark 1.7 (i), because the image $f(F) \subset Y$ is a finite subcomplex. Moreover the left vertical map is the realization of the restriction $F \xrightarrow{f|} f(F)$, which is continuous by 1). Hence their composition is continuous, which coincides with the other composition. By definition of the final topology on $|X|$ and again by Remark 1.7 (ii) also the map $|f|$ must be continuous.

□

1.2 Simplicial approximation

Remark 1.11
Let X, Y be simplicial complexes.
Then there are far more continuous maps $|X| \longrightarrow |Y|$ than those coming from a simplicial map $X \longrightarrow Y$.

The circle can be modeled by the simplicial complex $\partial I(\underline{2})$.



We can construct an explicit homeomorphism

$$h : |\partial I(\underline{2})| \xrightarrow{\sim} S^1 := \{x \in \mathbb{C};\ |x| = 1\}, \quad a = (a_0, a_1, a_2) \longmapsto \frac{(a_1 - a_0) + (a_2 - a_0)i}{|(a_1 - a_0) + (a_2 - a_0)i|}.$$

There is an isomorphism $\mathbb{Z} \xrightarrow{\sim} \pi_1(S^1, 1)$, sending $n \in \mathbb{Z}$ to the homotopy class of the selfmap

$$p_n : S^1 \longrightarrow S^1, \quad x \longmapsto x^n.$$

However we can only find simplicial maps $\partial I(\underline{2}) \xrightarrow{f_n} \partial I(\underline{2})$, whose realizations $|f_n|$ preserve the base point $h^{-1}(1) = (0,1,0) \in |\partial I(\underline{2})|$, which are homotopic to p_n, for $n = -1, 0, 1$. These are given by

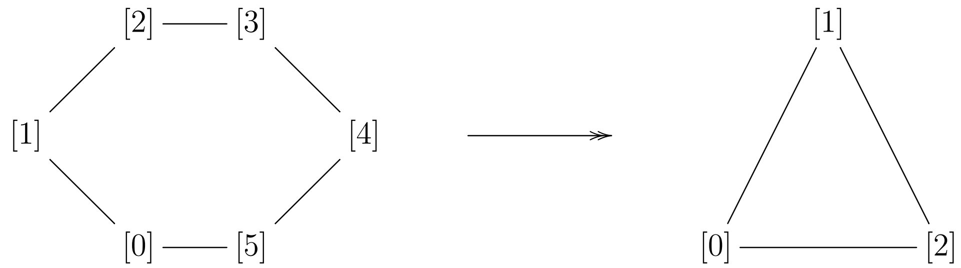
$$f_{-1}(k) = 2 - k, \quad f_0(k) = 1, \quad f_1(k) = k, \qquad 0 \leq k \leq 2.$$

However, given $n \geq 0$ we can choose another simplicial complex X_n modeling S^1, given by

$$X_n = \mathbb{Z}/3n, \quad S(X_n) = \{\{x\};\ x \in \mathbb{Z}/3n\} \cup \{\{x, x+1\};\ x \in \mathbb{Z}/3n\},$$

and chosing the base point $e_1 = (0,1,0,...,0) \in |X_n| \subset \mathbb{R}^{3n}$ we see that the map of degree $n > 0$ can be modeled by the simplicial map given by the canonical quotient map

$$X_n = \mathbb{Z}/3n \longrightarrow\!\!\!\!\!\rightarrow \mathbb{Z}/3 = X_1 \cong \partial I(\underline{2}), \quad [k] \longmapsto [k].$$



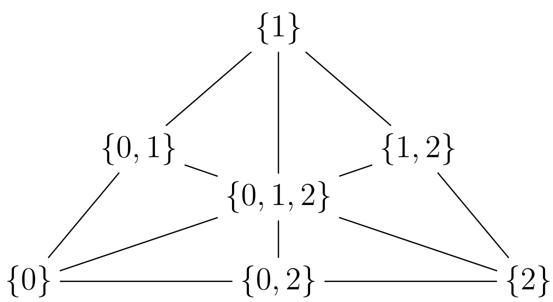
Question 1.12
Given two simplicial complexes X, Y and a continuous map $|X| \xrightarrow{f} |Y|$. Can we always find a simplicial complex X' and a homeomorphism $|X'| \xrightarrow{\sim} |X|$, whose composition with f is homotopic to the realization of a simplicial map?

In case the source X is a finite complex, the answer is yes. The solution lies in subdividing the complex X, as we implicitly did for the circle in the example above. There are many ways of subdividing a simplicial complex. A canonical way of doing this is the barycentric subdivsion.

Definition 1.13
*The **barycentric subdivision** of a simplicial complex is defined as the simplicial complex $\mathrm{sd}_B X$, given by*

$$\mathrm{sd}_B X := S(X), \qquad S(\mathrm{sd}_B X) := \{\{s_0 \subsetneq ... \subsetneq s_m\};\ m \geq 0\}.$$

Example 1.14
The barycentric subdivision of $I(\underline{2})$ is given by



with each appearing triangle being a 2-simplex.

Lemma 1.15
For every finite simplicial complex $|X|$ is a compact Hausdorff space.

Proof. For every simplex $s \in S(X)$ we let A_s be the preimage of 1 under the map

$$[0,1]^X \longrightarrow \mathbb{R}, \quad a \longmapsto \sum_{x\in s} a_x.$$

As this map is continuous and $\{1\} \subset \mathbb{R}$ is a closed subset, A_s is a closed subset of $[0,1]^X$. It is precisely the set of elements $a \in |X|$ with $a_x = 0$, for all $x \in X \backslash s$. Hence by construction $|X| = \bigcup_{s\in S(X)} A_s$ is a finite union of closed subsets. So also $|X| \subset [0,1]^X$ is closed and therefore compact, as $[0,1]^X$ is compact. With $[0,1]^X$ also $|X|$ is Hausdorff.

□

Lemma 1.16
Let $X \xrightarrow{f} Y$ be a continuous bijection from a compact space X to a Hausdorff space Y. Then f is a homeomorphism.

Proof. Let $U \subset X$ be an open subset and $x \in U$. For every $y \in Y \backslash f(U)$ there are open subsets $y \in U_y \subset Y$ and $f(x) \in V_y \subset Y$ with $U_y \cap V_y = \emptyset$, because Y is Hausdorff. As X is compact, the open covering $X = U \cup \bigcup_{y\in Y} f^{-1}U_y$ has a finite subcovering correspondig to elements $y_1, ..., y_n \in Y$. So $X\backslash U \subset f^{-1}U_{y_1} \cup ... \cup f^{-1}U_{y_n}$ or equivalently $Y\backslash f(U) \subset U_{y_1} \cup ... \cup U_{y_n}$. Setting $V := V_{y_1} \cap ... \cap V_{y_n}$, we get

$$V \cap (Y\backslash f(U)) \subset V \cap (U_{y_1} \cup ... \cup U_{y_n}) = (V \cap U_{y_1}) \cup ... \cup (V \cap U_{y_n}) = \emptyset.$$

In other words $x \in V \subset f(U)$ is an open environment of x in $f(U)$. So $f(U) \subset Y$ is open, which proves that f^{-1} is continuous and hence f is a homeomorphism.

□

Theorem 1.17
There is a natural homeomomorphism

$$h_X : |\mathrm{sd}_B X| \xrightarrow{\sim} |X|, \quad \sum_{s\in S(X)} a_s \cdot s \longmapsto \sum_{\substack{s\in S(X),\\ x\in s}} \frac{a_s}{\sharp s} \cdot x.$$

Proof. For $a \in |\mathrm{sd}_B X|$ we have $a_s \geq 0$, for all $s \in S(X)$. Hence

$$\sum_{x\in s\in S(X)} \frac{a_s}{\sharp s} \geq 0, \qquad x \in X.$$

It follows that $h_X(a) \in |X|$, because we also have

$$\sum_{\substack{s\in S(X),\\ x\in s}} \frac{a_s}{\sharp s} = \sum_{s\in S(X)} \sharp s \cdot \frac{a_s}{\sharp s} = \sum_{s\in S(X)} a_s = 1.$$

By construction h is a natural map, i.e. for every simplicial map $X \xrightarrow{f} Y$ we get a commutative square

$$\begin{tikzcd} |\mathrm{sd}_B X| \arrow[r, "h_X"] \arrow[d, "|\mathrm{sd}_B f|"'] & |X| \arrow[d, "|f|"] \\ |\mathrm{sd}_B Y| \arrow[r, "h_Y"] & |Y|. \end{tikzcd}$$

We prove that h_X is continuous by reduction on finite subcomplexes.

- For $\sharp X < \infty$, the map h_X is the restriction of a linear map between finite-dimensional real vector spaces, hence continuous.
- For $\sharp X = \infty$, let $F \subset \mathrm{sd}_B X$ be a finite subcomplex. Considering $F' := \bigcup_{s \in F} s$ as a finite subcomplex of X, we see that $F \subset \mathrm{sd}_B F'$, so the canonical inclusion factors as $F \xhookrightarrow{j} \mathrm{sd}_B F' \xhookrightarrow{\mathrm{sd}_B(j')} \mathrm{sd}_B X$ with $F' \xhookrightarrow{j'} X$. Using that h is natural we get $h_X \circ |\mathrm{sd}_B(j')| \circ |j| = |j'| \circ h_{F'} \circ |j|$ and the latter is continuous by what we have just proven. Hence h_X is continuous by definition of the final topology on $|\mathrm{sd}_B X|$.

Next we prove that h_X is bijective by constructing an inverse map. Defining $\Sigma U := \sum_{x \in U} 1 \cdot x$, for $U \subset X$, every $a \in |X|$ can uniquely be written as

$$a = a_0 \cdot \Sigma s_0 + ... + a_m \cdot \Sigma s_m, \quad \emptyset \neq s_0, ..., s_m \subset X, \quad s_i \cap s_j = \emptyset, \quad 0 < a_0 < ... < a_m. \tag{1.1}$$

We define a map $|X| \xrightarrow{k_X} |\mathrm{sd}_B X|$ by setting

$$k_X(a) := (\sharp s_0 + ... + \sharp s_m) \cdot a_0 \cdot (s_0 \cup ... \cup s_m) + \sum_{1 \leq i \leq m} (\sharp s_i + ... + \sharp s_m) \cdot (a_i - a_{i-1}) \cdot (s_i \cup ... \cup s_m).$$

Every coefficient is greater than zero by assumption on a_i. Moreover $a \in |X|$ implies $s_0 \cup ... \cup s_m \in S(X)$ and hence $\{s_m \subsetneq (s_{m-1} \cup s_m) \subsetneq ... \subsetneq (s_0 \cup ... \cup s_m)\} \in S(\mathrm{sd}_B X)$ by definition of $\mathrm{sd}_B X$. Again using $a \in |X|$ we see that

$$(\sharp s_0 + ... + \sharp s_m) \cdot a_0 + \sum_{1 \leq i \leq m} (\sharp s_i + ... + \sharp s_m) \cdot (a_i - a_{i-1}) = a_0 \cdot \sharp s_0 + ... + a_m \cdot \sharp s_m = 1.$$

This shows that $k_X(a) \in |\mathrm{sd}_B X|$, so k_X is well-defined. We compute

$$\begin{aligned} h_X k_X(a) &= a_0 \cdot \Sigma(s_0 \cup ... \cup s_m) + \sum_{1 \leq i \leq m} (a_i - a_{i-1}) \cdot \Sigma(s_i \cup ... \cup s_m) \\ &= a_0 \cdot \Sigma s_0 + ... + a_m \cdot \Sigma s_m. \end{aligned}$$

Similarly for $b = b_0 \cdot t_0 + ... + b_m \cdot t_m \in |\mathrm{sd}_B X|$ with $t_{m+1} := \emptyset \subsetneq t_m \subsetneq ... \subsetneq t_0 \in S(X)$ and $b_i > 0$, we have

$$h_X(b) = k_X\left(\frac{b_0}{\sharp t_0} \cdot \Sigma t_0 + ... + \frac{b_m}{\sharp t_m} \cdot \Sigma t_m\right) = \sum_{0 \leq i \leq m} \left(\frac{b_0}{\sharp t_0} + ... + \frac{b_i}{\sharp t_i}\right) \cdot \Sigma(t_i \backslash t_{i+1}),$$

which is an element in $|X|$ of the form (1.1). So applying k_X, we get

$$k_X h_X(b) = \sharp t_0 \cdot \frac{b_0}{\sharp t_0} \cdot t_0 + \sum_{1\leq i\leq m} \sharp t_i \cdot \frac{b_i}{\sharp t_i} \cdot t_i = b.$$

We have shown that h_X is a bijection with $h_X^{-1} = k_X$.

It remains to check that also $k_X = h_X^{-1}$ is continuous, which again can be reduced to the finite case.

- For $\sharp X < \infty$, we also have $\sharp \mathrm{sd}_B X < \infty$. Hence $|\mathrm{sd}_B X|$ and $|X|$ are compact and Hausdorff by Lemma 1.15, which using Lemma 1.16 implies that h_X is a homeomorphism.

- For $\sharp X = \infty$ and every finite simplicial subcomplex $F \xhookrightarrow{j} X$ we have $|j| \circ h_F = h_X \circ |\mathrm{sd}_B j|$, hence $h_X^{-1} \circ |j| = |\mathrm{sd}_B j| \circ h_F^{-1}$ is continuous, as we have just proven. Since X is the union of its finite simplicial subcomplexes and carries the final topology for all these inclusions $|F| \xrightarrow{j} |X|$, Remark 1.7 (ii) implies that also $h_X^{-1} = k_X$ is continuous.

□

Lemma 1.18 (Lebesgue)
Let X be a compact metric space and $X = \bigcup_{U\in\mathcal{U}} U$ be an open cover.

*Then there is a number, called the **Lebesgue number** $\delta > 0$, such that:*

$$\forall x \in X \quad \exists U_x \in \mathcal{U}: \qquad B_{<\delta}(x) \subset U_x.$$

Proof. For every $x \in X$ there is an $U_x \in \mathcal{U}$, such that $x \in U_x$. As U_x is open, we find a $\delta_x > 0$, such that $B_{<\delta_x/2}(x) \subset U_x$. Hence $X = \bigcup_{x\in X} B_{<\delta_x}(x)$ is an open cover, which by compactness of X has a finite subcover corresponding to elements $x_1, ..., x_n \in X$. Then $\delta := \min\{\delta_{x_1}, ..., \delta_{x_n}\}$ is the desired number. For if $x \in X$, there is a $1 \leq j \leq n$ with $x \in B_{<\delta_{x_j}}(x_j)$. Moreover, for every $y \in B_{<\delta}(x)$ we have

$$d(y, x_j) \leq d(y, x) + d(x, x_j) < \delta + \delta_{x_j} \leq 2\delta_{x_j},$$

which implies that $y \in B_{<2\delta_{x_j}}(x_j)$, which by the choice of δ_{x_j} is contained in U_{x_j}.

□

Before we can prove the simplicial approximation Theorem we will need the following key lemma. Roughly speaking it says that by iterated subdivision we can make simplices arbitrarily small.

Lemma 1.19
Let X be a finite simplicial complex, $N > 0$ and $a, b \in |\mathrm{sd}_B^N X|$, such that

$$\{x \in \mathrm{sd}_B^N X;\ a_x > 0 \text{ or } b_x > 0\} \in S(\mathrm{sd}_B^N X).$$

Then $\|h_X^N(a) - h_X^N(b)\|_1 < 2 \cdot (1 - \frac{1}{\sharp X})^N$, where $\|\text{-}\|_1$ is the 1-norm on $|X| \subset \mathbb{R}^X$, given by

$$\|v_1 \cdot x_1 + ... + v_n \cdot x_n\|_1 = |v_1| + ... + |v_n|, \qquad v_1, ..., v_n \in \mathbb{R}, \quad x_1, ..., x_n \in X.$$

Proof. As X is finite, we can define $n_X := \max\{\sharp s;\ s \in S(X)\}$. On the one hand, for every chain of X-simplices $s_1 \subsetneq ... \subsetneq s_n$ we have $n_X \geq \sharp s_n \geq n$, which proves $n_X \geq n_{\mathrm{sd}_B X}$. On the other hand, for every $s = \{x_1, ..., x_n\} \in S(X)$ with $n = n_X$ we have

$$n_X = n = \sharp\{\{x_1\} \subsetneq \{x_1, x_2\} \subsetneq ... \subsetneq \{x_1, ..., x_n\}\} \leq n_{\mathrm{sd}_B X}.$$

So all in all $n_X = n_{\mathrm{sd}_B X}$.

By definition the map h_X is the restriction of a linear map $\bigoplus_{s \in \mathrm{sd}_B X} \mathbb{R} \cdot s \xrightarrow{\bar{h}_X} \bigoplus_{x \in X} \mathbb{R} \cdot x$. Let $v \in \bigoplus_{s \in \mathrm{sd}_B X} \mathbb{R} \cdot s$ with $\{s \in \mathrm{sd}_B X;\ v_s > 0\} \in S(\mathrm{sd}_B X)$. Then v is of the form

$$v = v_1 \cdot \{x_1\} + v_2 \cdot \{x_1, x_2\} + ... + v_n \cdot \{x_1, ..., x_n\}, \qquad v_1, ..., v_n \in \mathbb{R}, \quad \{x_1, ..., x_n\} \in S(X).$$

Assuming $v_1 + ... + v_n = 0$, we get

$$\begin{aligned}
\|\bar{h}_X(v)\|_1 &= \left\| v_1 \cdot x_1 + \frac{v_2}{2} \cdot (x_1 + x_2) + ... + \frac{v_n}{n} \cdot (x_1 + ... + x_n) \right\|_1 \\
&= \left\| \left(v_1 + \frac{v_2}{2} + ... + \frac{v_n}{n} \right) \cdot x_1 + \left(\frac{v_2}{2} + ... + \frac{v_n}{n} \right) \cdot x_2 + ... + \frac{v_n}{n} \cdot x_n \right\|_1 \\
&= \left| v_1 + \frac{v_2}{2} + ... + \frac{v_n}{n} \right| + \left| \frac{v_2}{2} + ... + \frac{v_n}{n} \right| + ... + \left| \frac{v_n}{n} \right| \\
&= \left| \left(1 - \frac{1}{n}\right) \cdot v_1 + \left(\frac{1}{2} - \frac{1}{n}\right) \cdot v_2 + ... + \left(\frac{1}{n-1} - \frac{1}{n}\right) \cdot v_{n-1} \right| \\
&\qquad\qquad + \left| \frac{v_2}{2} + ... + \frac{v_n}{n} \right| + ... + \left| \frac{v_n}{n} \right| \\
&\leq \left| 1 - \frac{1}{n} \right| \cdot |v_1| + \left| \frac{1}{2} - \frac{1}{n} \right| \cdot |v_2| + ... + \left| \frac{1}{n-1} - \frac{1}{n} \right| \cdot |v_{n-1}| \\
&\qquad\qquad + \left(\frac{1}{2} \cdot |v_2| + ... + \frac{1}{n} \cdot |v_n| \right) + ... + \frac{1}{n} \cdot |v_n| \\
&= \left(1 - \frac{1}{n}\right) \cdot |v_1| + ... + \left(1 - \frac{1}{n}\right) \cdot |v_n| = \left(1 - \frac{1}{n}\right) \cdot \|v\|_1 \leq \left(1 - \frac{1}{n_X}\right) \cdot \|v\|_1,
\end{aligned}$$

where we used $v_1 + ... + v_n = 0$ for the fourth equality and $n \leq n_X$ for the last inequality.

Now by assumption $v := \bar{h}_X^{N-1}(a - b) = h_X^{N-1}(a) - h_X^{N-1}(b) \in \bigoplus_{s \in \mathrm{sd}_B(X)} \mathbb{R} \cdot s$ is of the stated form, because

$$\sum_{s \in \mathrm{sd}_B^N X} (a_s - b_s) = 1 - 1 = 0.$$

Hence using $n_X = n_{\mathrm{sd}_B X} = ... = n_{\mathrm{sd}_B^N X}$, we can prove by induction that

$$\begin{aligned}
\left\| h_X^N(a) - h_X^N(b) \right\|_1 = \left\| \bar{h}_X^N(a - b) \right\|_1 &\leq \left(1 - \frac{1}{n_X}\right) \cdot \left\| \bar{h}_X^{N-1}(a - b) \right\|_1 \\
&\leq ... \leq \left(1 - \frac{1}{n_X}\right)^N \cdot \|a - b\|_1.
\end{aligned}$$

Since $a, b \in |\mathrm{sd}_B^N X| \subset [0, 1]^{\mathrm{sd}_B^N X}$, we have $\|a - b\|_1 \leq 2$. Moreover $n_X \leq \sharp X$, which concludes the proof.

□

Theorem 1.20 (Simplicial Approximation)
Let X and Y be simplicial complexes and suppose X is finite. Given a continuous map between their geometric realizations $|X| \xrightarrow{f} |Y|$.

Then there is an $N \geq 0$ and a simplicial map $\mathrm{sd}_B^N X \xrightarrow{g} Y$, such that $f \circ h_X^N$ and $|g|$ are homotopic.

Proof. The idea is to subdivide X sufficiently many times, such that the value of f alters inside the realization of a simplex in Y, when moving between adjacent vertices in X. Every $y \in Y$ considered as an element $1 \cdot y \in |Y|$ has a canonically defined environment $U_y := \{a \in |Y|;\ a_y > 0\}$. It is an open subset of $|Y|$, because $U_y \cap |F| \subset \mathbb{R}^F$ is open, for all finite $F \subset Y$. Geometrically every element of U_y lies in the realization of a simplex containing y. By Lemma 1.15 the space $|X|$ is compact, as X is finite. Its topology comes from the metric induced by the 1-norm on $\mathbb{R}^X$. So by Lemma 1.18 there is a Lebesgue number $\delta > 0$ for the open cover $|X| = \bigcup_{y \in Y} f^{-1}U_y$. Let $N > 0$, such that $2 \cdot (1 - \frac{1}{\sharp X})^N < \delta$. For every $x \in \mathrm{sd}_B^N X$ Lebesgue's lemma provides an element $g(x) \in Y$ such that

$$B_{<\delta}(h_X^N(x)) \subset f^{-1}U_{g(x)}.$$

We claim that this defines a simplicial map $\mathrm{sd}_B^N X \xrightarrow{g} Y$. Let $s \in S(\mathrm{sd}_B X)$ and choose $x \in s$. For every $x' \in s$ we have $\|h_X^N(x) - h_X^N(x')\|_1 < 2 \cdot (1 - \frac{1}{\sharp X})^N < \delta$ by Lemma 1.19. Hence $h_X^N(x) \in B_{<\delta}(h_X^N(x')) \subset f^{-1}U_{g(x')}$. In other words $(fh_X^N(x))_{g(x')} > 0$ or

$$g(x') \in \{y \in Y;\ fh_X^N(x)_y > 0\} =: s_x,$$

which is a simplex in Y, because $fh_X^N(x) \in |Y|$. As $x' \in s$ was arbitrary, we have proven that $g(s) \subset s_x \in S(Y)$, so $g(s) \in S(Y)$.

Similarly, for every $a \in |\mathrm{sd}_B^N X|$ we let $s := \{x \in \mathrm{sd}_B^N X;\ a_x > 0\} \in S(\mathrm{sd}_B^N X)$. Then the same argument for a instead of x shows that $g(s) \subset \{y \in Y;\ fh_X^N(a)_y > 0\} =: s_a \in S(Y)$, which proves that $|g|(a)$ and $fh_X^N(a)$ lie in the geometric realization of the same simplex s_a. It follows that

$$H : [0,1] \times |\mathrm{sd}_B^N X| \longrightarrow |Y|, \quad (1-t) \cdot fh_X^N(a) + t \cdot |g|(a)$$

constitutes a well-defined homotopy from fh_X^N to $|g|$.

□

1.3 Products

In our abstraction process of avoiding topology, we would like to have the notion of a homotopy between two simplicial maps.

1) We need a model for the interval. This may be given by the simplicial complex $I(\underline{1})$, whose geometric realization is $|I(\underline{1})| = |\Delta^1|$. The Simplicial Approximation Theorem 1.20 suggests that we may also need to consider its iterated subdivisions.

2) We need the notion of a product of two simplicial complexes, which should be compatible with geometric realization, i.e. the canonical map $|X \times Y| \overset{(\pi_X,\pi_Y)}{\longrightarrow} |X| \times |Y|$ should be a homeomorphism.

There is a natural product construction.

Remark 1.21
The (categorial) product $X \times Y$ of two simplicial complexes X and Y is the simplicial complex, whose set of vertices is the cartesian product $X \times Y$ and whose simplices are

$$S(X \times Y) = \{\emptyset \neq p \subset s \times t;\ s \in S(X),\ t \in S(Y)\}.$$

*The projections $X \overset{\pi_X}{\longleftarrow} X \times Y \overset{\pi_Y}{\longrightarrow} Y$ are simplicial and are **universal** with this property. This means that given two simplicial maps f and g there is a unique simplicial map (f,g) fitting in the commutative diagram*

$$\begin{tikzcd}
 & Z \arrow[dl, "f"'] \arrow[d, dashed, "\exists!(f,g)"] \arrow[dr, "g"] & \\
X & X \times Y \arrow[l, "\pi_X"] \arrow[r, "\pi_Y"'] & Y.
\end{tikzcd}$$

Remark 1.22
For two simplicial complexes X and Y the two projections induce a natural map

$$|X \times Y| \overset{(|\pi_X|,|\pi_Y|)}{\longrightarrow} |X| \times |Y|,$$

which is continuous by definition of the product topology.

Unfortunately this product does not have the desired properties as the following example demonstrates.

Example 1.23
We have $I(\underline{1}) \times I(\underline{1}) = I(\underline{1} \times \underline{1})$, because $\underline{1} \times \underline{1} \in SI(\underline{1} \times \underline{1})$.

(i) *This means we have six 1-simplices, four 2-simplices*

$$\begin{aligned} s_{1,1} &:= \{(0,0),(0,1),(1,0)\}, & s_{1,0} &:= \{(0,0),(0,1),(1,1)\}, \\ s_{0,1} &:= \{(0,0),(1,0),(1,1)\}, & s_{0,0} &:= \{(0,1),(1,0),(1,1)\}, \end{aligned}$$

and moreover one 3-simplex.

$$\begin{tikzcd}
(0,1) \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & (1,1) \arrow[d, dash] \arrow[dl, dash] \\
(0,0) \arrow[r, dash] & (1,0)
\end{tikzcd}$$

The geometric realization $|I(\underline{1}) \times I(\underline{1})|$ is homeomorphic to the standard 3-simplex $|\Delta^3|$, which is not homeomorphic to the product $|I(\underline{1})| \times |I(\underline{1})| = |\Delta^1| \times |\Delta^1|$.

(ii) *However there is a simplicial subcomplex $P \subset I(\underline{1}) \times I(\underline{1})$ with same vertices, such that the composition $|P| \hookrightarrow |I(\underline{1}) \times I(\underline{1})| \longrightarrow |I(\underline{1})| \times |I(\underline{1})|$ is a homeomorphism. Its top dimensional simplices are $s_{1,0}$ and $s_{0,1}$, which are precisely those simplices of $I(\underline{1}) \times I(\underline{1})$ being totally ordered by the product order.*

$$\begin{tikzcd} (0,1) \arrow[r] & (1,1) \\ (0,0) \arrow[u] \arrow[r] \arrow[ur] & (1,0) \arrow[u] \end{tikzcd}$$

We can generalize this product construction by giving simplicial complexes an orientation in form of a partial order.

Definition 1.24
*An **ordered simplicial complex** X is a simplicial complex X together with a partial order on its set of vertices, restricting to a total order on each of its simplices.*

*An **ordered simplicial map** between two ordered simplicial complexes X and Y is a simplicial map $X \longrightarrow Y$ preserving the order.*

Proposition 1.25
The (categorial) product $X \times Y$ of two ordered simplicial complexes X and Y is the subcomplex of their product of simplicial complexes, whose simplices are those being totally ordered by the product order.

Then $|X \times Y| \overset{(|\pi_X|,|\pi_Y|)}{\longrightarrow} |X| \times |Y|$ is a continuous bijection.

In particular it is a homeomorphism, if X and Y are finite.

Proof. By construction the decscribed categorial product is universal for ordered simplicial maps. To see that the map $(|\pi_X|, |\pi_Y|)$ is bijective, we need another description for the set $|X|$.

Given two ordered simplicial complexes A, B we let $\mathcal{S}imp_c(A, B)$ denote the set of ordered simplicial maps $A \overset{f}{\longrightarrow} b$, which are **cocontinuous**. This means that f preserves suprema, i.e. $\sup f(U) = f(\sup U)$, for all subsets $U \subset A$. Now suppose B is finite and let $A := I(]0,1])$, that is the ordered simplicial complex, whose vertices are elements of the half-open unit interval $]0,1]$ with the canonical partial order. We define a natural bijection

$$\phi_B : |B| \overset{\sim}{\longrightarrow} \mathcal{S}imp_c(I(]0,1]), B)$$

as follows. The image of an element $a = a_1 \cdot x_1 + ... + a_n \cdot x_n \in |B|$ with $x_1 < ... < x_n$ and $a_1, ..., a_n > 0$ is the map

$$\phi_B(a) :]0,1] \longrightarrow B, \quad]a_1 + ... + a_{i-1}, a_1 + ... + a_i] \ni v \longmapsto x_i.$$

Vice versa, for $f \in \mathcal{S}imp_c(I(]0,1]), B)$ there are elements $a_1, ..., a_n \in]0,1]$, such that $f(]0,1]) = \{f(a_1) < ... < f(a_1 + ... + a_n)\} \in S(B)$. As f preserves suprema, we can find maximal $a_1, ..., a_n \in]0,1]$ with this property. Since $f(1) = f(a_1 + ... + a_n)$, this implies $a_1 + ... + a_n = 1$, and hence $a_1 \cdot f(a_1) + ... + a_n \cdot f(a_1 + ... + a_n) \in |B|$ is a unique preimage for f under ϕ_B.

Now for finite X and Y we obtain a commutative square

$$\begin{tikzcd} {|X \times Y|} \arrow[r, "(|\pi_X|, |\pi_Y|)"] \arrow[d, "\phi_{X \times Y}"', "\wr"] & {|X| \times |Y|} \arrow[d, "\phi_X \times \phi_Y"', "\wr"] \\ \mathcal{S}imp_c(I(]0,1]), X \times Y) \arrow[r, "((\pi_X \circ -), (\pi_Y \circ -))"] & \mathcal{S}imp_c(I(]0,1]), X) \times \mathcal{S}imp_c(I(]0,1]), Y). \end{tikzcd}$$

As the product of ordered simplicial complexes is universal for cocontinuous simplicial maps, the lower horizontal map and hence also the upper horizontal map is a bijection. Again finiteness of X, Y and hence $X \times Y$ implies that $|X \times Y|$ and $|X| \times |Y|$ are compact Hausdorff spaces by Lemma 1.15, so $(|\pi_X|, |\pi_Y|)$ is a homeomorphism by Lemma 1.16.

As before infinite X and Y may be written as the union of its finite subcomplexes and the commutative square

$$\begin{tikzcd} {|X \times Y|} \arrow[r, "(|\pi_X|, |\pi_Y|)"] \arrow[d, equal] & {|X| \times |Y|} \arrow[d, equal] \\ \bigcup\limits_{\substack{F \subset X, \\ G \subset Y, \\ \text{finite}}} |F \times G| \arrow[r, "\sim"] & \bigcup\limits_{\substack{F \subset X, \\ G \subset Y, \\ \text{finite}}} |F| \times |G| \end{tikzcd}$$

shows that $(|\pi_X|, |\pi_Y|)$ is a continuous bijection. In general the right equality only holds as sets, because the topology of the union is finer than the product topology.

□

Remark 1.26
Every simplicial complex X can be ordered by choosing an arbitrary total order on its underlying set of vertices.

(i) *This does not lead to a functorial construction.*

(ii) *But given a simplicial map $X \xrightarrow{f} Y$, by the* ***order-extension principle*** *we can choose compatible total orders on X and Y, such that f becomes an oriented simplicial map.*

Remark 1.27
For every simplicial complex X, its barycentric subdivision $\mathrm{sd}_B X$ is an ordered simplicial complex via the inclusion order.

Every simplicial map induces an ordered simplicial map between the barycentric subdivisions.

1.4 Collapsing subspaces

Apart from the bad bevavior of products, simplicial complexes have another flaw. Namely geometric realization does not commute with collapsing subspaces. We will demonstrate this with an example.

Example 1.28
Consider the bounday subcomplex $\partial I(\underline{1}) \subset I(\underline{1})$, whose only simplices are the subsets of cardinality 1.

(i) *We have a homeomorphism $|I(\underline{1})|/|\partial I(\underline{1})| \cong S^1$. So if $|\text{-}|$ would commute with quotients $I(\underline{1})/\partial I(\underline{1})$ would be a very simple model of the circle.*

(ii) *But the quotient simplicial complex $I(\underline{1})/\partial I(\underline{1})$ consists of a single point, so it does not model the circle.*

(iii) *Obviously the problem cannot be fixed by introducing orientations.*

As a consequence, also the constructions of mapping cylinders and cones are getting more complicated.

2 Simplicial sets

2.1 Semi-simplicial sets

We introduced the notion of an abstract simplicial complex as a "construction plan" for a geometric simplicial complex. It was easy to imagine the geometric realization, that a simplicial complex is standing for. But we also saw that the notion behaves badly when it comes to forming products and collapsing subspaces.

Instead of modeling simplicial complexes, we now try to model CW-complexes. We will do this in a "simplicial way", meaning that we will build up the CW-complex by glueing simplices.

Remark 2.1
Let $n > 0$.

(i) *Then there is a homeomorphism*

$$|\partial I(\underline{n})| = \partial|\Delta^n| \xrightarrow{\sim} H \cap S^n \cong S^{n-1}, \quad b + x \longmapsto \frac{x}{\|x\|},$$

where $b = \frac{1}{n+1} \cdot (1, ..., 1) \in |I(\underline{n})| = |\Delta^n|$ is the barycenter of the standard n-simplex and $H \subset \mathbb{R}^{n+1}$ is the hyperplane of elements $a \in \mathbb{R}^{n+1}$ with $a_0 + ... + a_n = 0$.

(ii) *It extends to a homeomorphism*

$$|I(\underline{n})| = |\Delta^n| \xrightarrow{\sim} H \cap D^{n+1} \cong D^n, \quad b + tx \longmapsto \frac{tx}{\|x\|}, \quad 0 \leq t \leq 1.$$

In particular the inclusion $|\partial I(\underline{n})| \hookrightarrow |I(\underline{n})|$ is homeomorphic to $S^{n-1} \hookrightarrow D^n$.

Using this identification a CW-complex is the union of spaces $C = \bigcup_{n \geq 0} C_n$, where C_0 is a discrete set of points and C_n is obtained by glueing a set of standard n-simplices along their boundary to C_{n-1}, for each $n > 0$. More precisely suppose X_n is an index-set of n-simplices we want to glue, for each $n \geq 0$. We consider X_n as a discrete space. We have a homeomorphism $j_0 : X_0 \times |I(\underline{0})| \xrightarrow{\sim} C_0$. Moreover we get C_n by glueing $X_n \times |I(\underline{n})|$ to C_{n-1} and identifying the boundaries along a certain map $X_n \times |\partial I(\underline{n})| \xrightarrow{g_n} C_{n-1}$. This results in a commutative square

$$\begin{tikzcd} X_n \times |\partial I(\underline{n})| \arrow[r, "g_n"] \arrow[d, hook] & C_{n-1} \arrow[d, hook] \\ X_n \times |I(\underline{n})| \arrow[r, "j_n"] & C_n. \end{tikzcd} \tag{2.1}$$

Now instead of working with arbitrary continuous maps g_n, we want this map to be "simplicial" in a certain sense. Let us take a closer look at the boundary $\partial I(\underline{n})$.

Definition 2.2 (i) *For $n > 0$ and $0 \leq i \leq n$ the i-**th coface map** is the unique monotone injection $\underline{n-1} \overset{d^i}{\hookrightarrow} \underline{n}$ with $i \notin d^i(\underline{n-1})$, i.e.*

$$d^i : \underline{n-1} \hookrightarrow \underline{n}, \quad k \longmapsto \begin{cases} k, & 0 \leq k < i, \\ k+1, & i \leq k \leq n-1. \end{cases}$$

(ii) *For $0 \leq i \leq n$ the i-**th codegeneracy map** is the unique monotone surjection $\underline{n+1} \overset{s^i}{\twoheadrightarrow} \underline{n}$ with $s^i(i) = s^i(i+1)$, i.e.*

$$s^i : \underline{n+1} \hookrightarrow \underline{n}, \quad k \longmapsto \begin{cases} k, & 0 \leq k \leq i, \\ k-1, & i < k \leq n+1. \end{cases}$$

These unorthodox names will become clear later.

Remark 2.3
Under the homeomorphism $|I(\underline{n})| \cong |\Delta^n|$ we have the following geometric description of d^i and s^i, where $0 \leq i \leq n$.

(i) *$|I(d^i)| : |I(\underline{n-1})| \longrightarrow |I(\underline{n})|$ corresponds to the inclusion of the i-th face of the standard n-simplex.*

(ii) *$|I(s^i)| : |I(\underline{n+1})| \longrightarrow |I(\underline{n})|$ corresponds to the surjection identifiying the dimensions i and $i+1$ of the n-simplex.*

In particular we also get the description

$$|\partial I(\underline{n})| = \left| \bigcup_{0 \leq i \leq n} d^i I(\underline{n-1}) \right| = \bigcup_{0 \leq i \leq n} |d^i|(|I(\underline{n-1})|).$$

Using this description of $|\partial I(\underline{n})|$ we see that every map $X_n \times |\partial I(\underline{n})| \xrightarrow{g_n} C_n$ is the union of the maps $g_n \circ (\mathrm{id} \times |d^i|)$, where $0 \leq i \leq n$. To make g_n "simplicial", we may use the maps $X_{n-1} \times |I(\underline{n-1})| \xrightarrow{j_{n-1}} C_{n-1}$ and suppose the following diagram commutes

$$\begin{tikzcd} X_n \times |I(\underline{n-1})| \arrow[r, "d_i \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times |d^i|"'] & X_{n-1} \times |I(\underline{n-1})| \arrow[d, "j_{n-1}"] \\ X_n \times |\partial I(\underline{n})| \arrow[r, "g_n"] & C_{n-1}, \end{tikzcd}$$

with certain maps $X_n \xrightarrow{d_i} X_{n-1}$, for $0 \le i \le n$. In other words we assume that the map g_n is the union of the maps $j_{n-1} \circ (d_i \times \mathrm{id})$. Under this assumption an induction on $n \ge 0$ shows

$$C_n = \coprod_{0 \le k \le n} X_k \times |I(\underline{k})| \Big/ (d_i(x), a) \sim (x, |d^i|(a)), \quad x \in X_k, \quad a \in |I(\underline{k-1})|, \quad 0 \le i \le k.$$

So far we did not care, wheather the maps $j_{n-1} \circ (d_i \times \mathrm{id})$ are compatible to define the map g_n. For every $0 \le i < j \le n$ we have a commutative diagram

$$\begin{tikzcd} \underline{n-2} \arrow[r, "d^i"] \arrow[d, "d^{j-1}"'] & \underline{n-1} \arrow[d, "d^j"] \\ \underline{n-1} \arrow[r, "d^i"] & \underline{n}, \end{tikzcd}$$

and $d^j d^i(\underline{n-2}) = d^i d^{j-1}(\underline{n-2}) = d^i(\underline{n-1}) \cap d^j(\underline{n-1})$. In the explicit description of C_n the map j_n is the inclusion followed by the quotient map and we see that we need the corresponding condition

$$d_i d_j = d_{j-1} d_i, \qquad 0 \le i < j \le n. \tag{2.2}$$

Definition 2.4
*A **semi-simplicial set** or **Δ-set** is a collection of sets $X = (X_n)_{n \ge 0}$, together with so-called **face maps***

$$X_n \xrightarrow{d_i} X_{n-1}, \qquad 0 \le i \le n,$$

satisfying (2.2).

A homomorphism of semi-simplicial sets $X \xrightarrow{f} Y$ is a collection of maps

$$X_n \xrightarrow{f_n} Y_n, \qquad n \ge 0,$$

such that $d_i \circ f_n = f_{n-1} \circ d_i$, for all $0 \le i \le n$.

Semi-simplicial sets were first introduced 1950 by Eilenberg-Zilber [**?**], who immediately realized that they still have a flaw. Namely, we are only able to glue an n-simplex by identifying its boundary with a union of $(n-1)$-simplices in C_{n-1}. But often we also want to identify it with lower dimensional simplices (e.g. the simplest CW-model for the n-sphere is given by glueing an n-simplex to a single point). We can come around this problem by adding **degenerate simplices** to X_n, which are formal n-simplices that in fact stand for lower dimensional simplices. This means that some dimensions in their geometric realizations coincide. Such degenerate simplices can be realized by also factoring out along maps corresponding to the already introduced codegeneracy maps (cf. our explicit description of C_n). So similarly to the face maps we are postulating the existence of **degeneracy maps**

$$X_n \xrightarrow{s_i} X_{n+1}, \quad 0 \le i \le n.$$

Geometrically a simplex $x \in X_{n+1}$ lies in the image of the map s_i, if and only if the dimensions i and $i+1$ in its realization coincide. Of course the presence of degeneracy maps involves a lot more relations similar to (2.2) and finally leads to the notion of simplicial sets. Working with all these identities is a mess, but luckily there is a much more compact definition in the language of category theory.

The careful reader may have noticed that, for each map d^i from dimension $n-1$ to dimension n, we required a map d_i in the opposite direction from dimension n to $n-1$. The same correspondence holds for s^i and s_i. The necessary relations for d_i were deduced from the relations that hold for d^i and s^i. The reader familiar to category theory will conclude that a simplicial set must be a contravariant $\mathcal{S}et$-valued functor on a certain category, whose objects are the standard n-simplices.

2.2 Categories

Definition 2.5
*A **category** $\mathcal{C}$ consists of the following data.*

- *A class[1] of **objects** $\mathrm{Obj}(\mathcal{C})$ (also denoted by $\mathcal{C}$) and for two objects $A, B \in \mathcal{C}$ a set of **(homo-)morphisms** $\mathcal{C}(A,B)$ (sometimes also denoted by $\mathrm{Hom}_{\mathcal{C}}(A,B)$), whose elements will also be denoted by arrows $A \xrightarrow{f} B$.*
- *For each object $A \in \mathcal{C}$, there is a homomorphism $\mathrm{id}_A \in \mathcal{C}(A) := \mathcal{C}(A,A)$, called the **identity** on A.*
- *Moreover there are composition maps*

$$\mathcal{C}(B,C) \times \mathcal{C}(A,B) \longrightarrow \mathcal{C}(A,C), \quad (g,f) \longmapsto g \circ f,$$

satisfying the relations

$$h \circ (g \circ f) = (h \circ g) \circ f, \qquad f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f, \qquad A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D.$$

*A **functor** $\mathcal{C} \xrightarrow{F} \mathcal{D}$ between two categories $\mathcal{C}$ and $\mathcal{D}$ is given by the following data.*

- *A "map"[2] between the classes of objects $\mathrm{Obj}(\mathcal{C}) \xrightarrow{F} \mathrm{Obj}(\mathcal{D})$.*
- *For each pair of objects $A, B \in \mathcal{C}$, there is a map*

$$\mathcal{C}(A,B) \longrightarrow \mathcal{D}(F(A), F(B)), \quad f \longmapsto F(f).$$

- *It preserves the structure, i.e.*

$$F(\mathrm{id}_A) = \mathrm{id}_{F(A)}, \qquad F(g \circ f) = F(f) \circ F(g), \quad A \xrightarrow{f} B \xrightarrow{g} C.$$

[1] Classes extend the notion of sets. We need this term to be able to talk about the class of all sets, which cannot be a set itself. Further details about this problem can be found in any book about category theory.

[2] A map between classes is defined in the same way as maps between sets.

A ***natural transformation*** *$F \xrightarrow{t} G$ between two functors $\mathcal{C} \xrightarrow{F,G} \mathcal{D}$ is a family of morphisms $(F(A) \xrightarrow{t_A} G(A))_{A\in\mathcal{C}}$ inducing a commutative square*

$$\begin{tikzcd} F(A) \arrow[r, "t_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "t_B"] & G(B), \end{tikzcd}$$

for all $X, Y \in \mathcal{C}$ and $f \in \mathcal{C}(A,B)$.

Example 2.6
The following are categories.

(i) *The category $\mathcal{S}et$ has the class of sets as objects and for two sets $X, Y \in \mathcal{S}et$, the homomorphisms $\mathcal{S}et(X,Y)$ are the maps from X to Y. Composition and identities are defined in the usual sense.*

(ii) *Similarly we have $\mathcal{T}op$ the category of topological spaces with continuous maps as homomorphisms. There is a functor $\mathcal{T}op \xrightarrow{U} \mathcal{S}et$, which assigns to a space its underlying set. It is called a* ***forgetful functor****, because it "forgets" the topology.*

(iii) *Similarly we have the category $\mathcal{G}rp$ of groups, $\mathcal{R}ing$ of rings, ... All these "algebraic" categories have an obvious forgetful functor U to the category $\mathcal{S}et$.*

(iv) *The category $\mathcal{S}imp$ of simplicial complexes, whose morphisms are simplicial maps.*

Next to the forgetful functor $\mathcal{S}imp \xrightarrow{U} \mathcal{S}et$ we already constructed the geometric realization functor $\mathcal{S}imp \xrightarrow{|\text{-}|} \mathcal{T}op$.

(v) *We will also write $\mathcal{C}\mathrm{AT}$ for the collection of categories and $\mathcal{C}\mathrm{AT}(\mathcal{C},\mathcal{D})$ for the class of functors from $\mathcal{C}$ to $\mathcal{D}$, which are composed in the canonical manner.*

However some attention is required here: Similarly as the collection of sets is not a set, the collection of categories is not a class. In particular $\mathcal{C}\mathrm{AT}$ is not a category in the sense we just defined it.

Definition 2.7
A category is called ***small****, if the class of objects is actually a set.*
We call $\mathcal{C}at$ the category of small categories with functors as morphisms.

Remark 2.8
Let I and $\mathcal{C}$ be categories and suppose I is small.
Then the class of functors $\mathcal{C}\mathrm{AT}(I,\mathcal{C})$ becomes a category, whose morphisms are natural transformations.

Definition 2.9
For a category $\mathcal{C}$ its ***opposite category*** *$\mathcal{C}^{\mathrm{op}}$ is the category with the same objects and*

morphisms
$$\mathcal{C}^{\mathrm{op}}(A,B) := \mathcal{C}(B,A), \qquad A,B \in \mathcal{C}^{\mathrm{op}}.$$
The composition is induced by the composition in $\mathcal{C}$:
$$\mathcal{C}^{\mathrm{op}}(B,C) \times \mathcal{C}^{\mathrm{op}}(A,B) \longrightarrow \mathcal{C}^{\mathrm{op}}(A,C), \quad (g,f) \longmapsto g \circ f := f \circ_{\mathcal{C}} g.$$

A functor $\mathcal{C}^{\mathrm{op}} \xrightarrow{F} \mathcal{D}$ is also called a ***contravariant*** *functor from $\mathcal{C}$ to $\mathcal{D}$.*

The most important example of contravariant functors are the Hom-functors. An example appears in linear algebra, when assigning to a vector space its dual.

Proposition 2.10
Every $C \in \mathcal{C} \in \mathcal{C}\mathrm{AT}$ induces a functor
$$\mathcal{C}(\text{-},C) : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{S}et, \quad B \longmapsto \mathcal{C}(B,C), \quad \mathcal{C}(B,B') \ni f \longmapsto \mathcal{C}(f,C) := (\text{-} \circ f).$$
The construction is natural in C, meaning that (for small $\mathcal{C}$) it induces a functor
$$\mathcal{C} \longrightarrow \mathcal{C}\mathrm{AT}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}et), \quad C \longmapsto \mathcal{C}(\text{-},C), \quad \mathcal{C}(C,C') \ni f \longmapsto \mathcal{C}(\text{-},f) := (f \circ \text{-}).$$
Proof. Let $C \in \mathcal{C}$.

- For every $B \in \mathcal{C}$ we have $\mathcal{C}(\text{-},C)(\mathrm{id}_B) = \mathrm{id}_{\mathcal{C}(B,C)}$, because
$$\mathcal{C}(\text{-},C)(\mathrm{id}_B)(f) = f \circ \mathrm{id}_B = f = \mathrm{id}_{\mathcal{C}(B,C)}(f), \qquad f \in \mathcal{C}(B,C).$$
- Moreover for morphisms $B \xrightarrow{f} B' \xrightarrow{g} B''$ we have by associativity
$$\mathcal{C}(\text{-},C)(g \circ f) = (\text{-} \circ (g \circ f)) = ((\text{-} \circ g) \circ f) = \mathcal{C}(\text{-},C)(g) \circ \mathcal{C}(\text{-},C)(f).$$

This proves that $\mathcal{C}(\text{-},C)$ is a functor.
For the second statement let $f \in \mathcal{C}(C,C')$. We need to check that f defines a natural transformation
$$\mathcal{C}(\text{-},f) := (\text{-} \circ f) : \mathcal{C}(\text{-},C) \longrightarrow \mathcal{C}(\text{-},C').$$
This holds, because for every $g \in \mathcal{C}(B,B')$ the diagram below commutes
$$\begin{tikzcd}
\mathcal{C}(B',C) \arrow[r, "\mathcal{C}(B',f)=(f\circ\text{-})"] \arrow[d, "\mathcal{C}(g,C)=(\text{-}\circ g)"'] & \mathcal{C}(B',C') \arrow[d, "(\text{-}\circ g)=\mathcal{C}(g,C')"] \\
\mathcal{C}(B,C) \arrow[r, "\mathcal{C}(B,f)=(f\circ\text{-})"] & \mathcal{C}(B,C'),
\end{tikzcd}$$
because again composition is associative.

□

Probably the most important Lemma in category theory is the following one by Yoneda. It also appears in every (modern) algebraic geometry lecture.

Lemma 2.11 (Yoneda)
For every $C \in \mathcal{C} \in \mathcal{C}\mathrm{AT}$ and $F \in \mathcal{C}\mathrm{AT}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}et)$, there is a bijection

$$d : \mathcal{C}\mathrm{AT}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}et)(\mathcal{C}(\text{-}, C), F) \xrightarrow{\sim} F(C), \quad t \longmapsto t_C(\mathrm{id}_C).$$

It is natural in F and C.

Proof. Every $x \in F(C)$ induces a natural transformation $e(x)$ by setting

$$e(x)_B : \mathcal{C}(B, C) \longrightarrow F(B), \quad f \longmapsto F(f)(x).$$

It remains to check that $e(x)$ is a natural transformation, that e and d are inverse to each other and that d is natural in F and C. Although we will provide a full proof here, we advise the reader to do this on his own as an exercise.

Using that F is a functor, for $g \in \mathcal{C}(B, B')$ and $f \in \mathcal{C}(B', C)$ we get

$$e(x)_B \circ \mathcal{C}(g, C)(f) = e(x)_B(f \circ g) = F(f \circ g)(x) = F(g) \circ F(f)(x) = F(g) \circ e(x)_{B'}(f),$$

which proves that $e(x)_B \circ \mathcal{C}(g, C) = F(g) \circ e(x)_{B'}$, so $e(x)$ is in fact natural. Both constructions are inverse to each other:

- $(d \circ e)(x) = e(x)_C(\mathrm{id}_C) = F(\mathrm{id}_C)(x) = x$, for all $x \in F(C)$,
- $(e \circ d)(t) = t$, for all $t \in \mathcal{C}\mathrm{AT}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}et)(\mathcal{C}(\text{-}, C), F)$, because for all $f \in \mathcal{C}(B, C)$ naturality of t implies

$$\begin{aligned}(e \circ d)(t)_B(f) &= e(d(t))_B(f) = e(t_C(\mathrm{id}_C))_B(f) = F(f)(t_C(\mathrm{id}_C)) = (F(f) \circ t_C)(\mathrm{id}_C)\\ &= (t_B \circ \mathcal{C}(f, C))(\mathrm{id}_C) = t_B(\mathcal{C}(f, C)(\mathrm{id}_C)) = t_B(\mathrm{id}_C \circ f) = t_B(f).\end{aligned}$$

Next we prove that d is natural in F and C.

- The right term is a functor in F

$$\mathcal{C}\mathrm{AT}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}et) \longrightarrow \mathcal{S}et, \quad F \longmapsto F(C),$$

  sending a natural transformation $F \xrightarrow{h} F'$ to $F(C) \xrightarrow{h_C} F'(C)$. Moreover the left term is a functor in F by Proposition 2.10. To check that d is natural in F, we need to prove, that

$$d_F \circ \mathcal{C}\mathrm{AT}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}et)(\mathcal{C}(\text{-}, C), h) = h_C \circ d_C.$$

  But this holds, because for every $t \in \mathcal{C}\mathrm{AT}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}et)(\mathcal{C}(\text{-}, C), F)$, we have

$$\begin{aligned}d_F \circ \mathcal{C}\mathrm{AT}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}et)(\mathcal{C}(\text{-}, C), h)(t) &= d_F(h \circ t) = (h \circ t)_C(\mathrm{id}_C) = (h_C \circ t_C)(\mathrm{id}_C)\\ &= h_C(t_C(\mathrm{id}_C)) = h_C(d_C(t)) = h_C \circ d_C(t).\end{aligned}$$

- The right term is a functor in C, because F is a functor. Moreover the left term is a functor in C by Proposition 2.10 applied to $C \in \mathcal{C}$ and then to $\mathcal{C}(\text{-}, \mathcal{C}) \in \mathcal{C}\mathrm{AT}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}et)^{\mathrm{op}}$. To check that d is natural in C, we need to prove, that

$$d_C \circ \mathcal{C}\mathrm{AT}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}et)(\mathcal{C}(\text{-}, g), F) = F(g) \circ t_C, \qquad g \in \mathcal{C}(C, C').$$

But for $t \in \mathcal{C}\mathrm{AT}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}et)(\mathcal{C}(\text{-}, C), F)$, we have

$$\begin{aligned} d_C \circ \mathcal{C}\mathrm{AT}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}et)(\mathcal{C}(\text{-}, g), F)(t) &= d_C(\mathcal{C}\mathrm{AT}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}et)(\mathcal{C}(\text{-}, g), F)(t)) = d_C(t \circ \mathcal{C}(\text{-}, g)) \\ &= (t \circ \mathcal{C}(\text{-}, g))_C(\mathrm{id}_C) = (t_{C'} \circ \mathcal{C}(\text{-}, g)_C)(\mathrm{id}_C) \\ &= t_{C'}(\mathcal{C}(C, g)(\mathrm{id}_C)) = t_{C'}(g \circ \mathrm{id}_C) = t_{C'}(g) \\ &= t_{C'}(\mathcal{C}(g, C)(\mathrm{id}_{C'})) = (t_{C'} \circ \mathcal{C}(g, C))(\mathrm{id}_{C'}) \\ &= (F(g) \circ t_C)(\mathrm{id}_{C'}) = (F(g) \circ d_C)(t). \end{aligned}$$

□

2.3 Simplicial sets

Definition 2.12
The ***simplex category*** Δ *is defined as the category of the finite totally ordered sets* $\underline{n} = \{0 < 1 < ... < n\}$, *for* $n \geq 0$, *whose morphisms are the monotone maps.*

Remark 2.13
Every $f \in \Delta(\underline{m}, \underline{n})$ *can be factored as*

$$f = d^{i_p} \circ ... \circ d^{i_1} \circ s^{j_1} \circ ... \circ s^{j_q},$$

where $\{i_1 < ... < i_p\} := \underline{n} \backslash f(\underline{m})$ *and* $\{j_1 < ... < j_q\} := \{j \in \underline{m};\ f(j) = f(j+1)\}$.

In particular Δ *is generated by the maps* d^i *and* s^j. *It can be shown that* Δ *is completely determined by* d^i, s^j *and the following* ***cosimplicial identities***

(i) $d^j d^i = d^i d^{j-1}, \qquad i < j,$

(ii) $s^j d^i = \begin{cases} d^i s^{j-1}, & i < j, \\ \mathrm{id}, & i = j, j+1, \\ d^{i-1} s^j, & i > j+1, \end{cases}$

(iii) $s^j s^i = s^i s^{j+1}, \qquad i \leq j.$

Definition 2.14
The category of ***simplicial sets*** *is defined as the functor category* $s\mathcal{S}et := \mathcal{C}\mathrm{AT}(\Delta^{\mathrm{op}}, \mathcal{S}et)$.

Remark 2.15
For a simplicial set X *we will write*

$$X_n := X(\underline{n}), \qquad d_i := X(d^i), \qquad s_i := X(s^i), \qquad 0 \leq i \leq n.$$

For a map $f \in \Delta(\underline{m}, \underline{n})$ *we will also write* $f^* := X(f)$.

(i) *Remark 2.13 shows that a simplicial set is completely determined by this data satisfying the* ***simplicial identities***, *dual to the cosimplicial identities of Remark 2.13.*

(ii) *Similarly the category of semi-simplicial sets is the functor category* $\mathcal{C}\mathrm{AT}(\Delta_{inj}^{\mathrm{op}}, \mathcal{S}et)$, *where* $\Delta_{inj} \leq \Delta$ *is the subcategory with same objects but only the injective maps in* Δ *as morphisms.*

2.4 Geometric realization

Definition 2.16
*The **geometric realization** of a simplicial set X is defined as the topological space*

$$|X| := \coprod_{n\geq 0} X_n \times |I(\underline{n})| \Big/ (X(f)(x), a) \sim (x, |I(f)|(a)), \ \ x \in X_n, \ \ a \in |I(\underline{m})|, \ \ f \in \Delta(\underline{m}, \underline{n}),$$

carrying the final topology with respect to the maps $(X_n \times |I(\underline{n})| \longrightarrow |X|)_{n\geq 0}$.

Remark 2.17
Every $f \in s\mathcal{S}et(X, Y)$ induces a continuous map $|f| \in \mathcal{T}op(|X|, |Y|)$.

In particular we get a functor $s\mathcal{S}et \xrightarrow{|-|} \mathcal{T}op$.

Definition 2.18
Let $X \in s\mathcal{S}et$ and $n \geq 0$.

(i) *A simplex $x \in X_n$ is called **nondegenerate**, if $x \notin \sigma X_n := \bigcup_{i=0}^{n-1} s_i(X_{n-1})$.*

(ii) *We denote by $\tilde{X}_n = X_n \backslash \sigma X_n$ the set of nondegenerate n-simplices.*

Lemma 2.19 (Eilenberg-Zilber)
Let $X \in s\mathcal{S}et$.

Then for every $x \in X_n$ there is a unique nondegenerate $y \in X_m$ and a unique monotone surjection $\underline{n} \overset{s}{\twoheadrightarrow} \underline{m}$, such that $x = s^*(y)$.

Proof. For every $x \in X_n$ we consider the set of pairs $y \in X_m$ and $\underline{n} \overset{s}{\twoheadrightarrow} \underline{m}$ with $s^*(y) = x$. This set is non-empty, as it contains $(y, s) := (x, \mathrm{id}_{\underline{n}})$. The set of appearing dimensions m is bounded below by 0, so we can find a pair (y, s) with minimal m. Then y must be nondegenerate, because otherwise we would find a pair of lower dimension.

Now suppose (y', s') is another pair of minimal dimension m. We take a section d for s (i.e. a Δ-morphism d with $sd = \mathrm{id}$) and compute

$$y = (sd)^*(y) = d^*s^*(y) = d^*(x) = d^*(s')^*(y') = (s'd)^*(y').$$

Because y is nondegenerate $s'd \in \Delta(\underline{m}, \underline{m})$ is injective, hence $s'd = \mathrm{id}$. For every $k \in \underline{n}$ we find a section d such that $ds(k) = k$. As observed above d is also a section for s' and so $s'(k) = s'ds(k) = s(k)$ implying $s = s'$.

□

Proposition 2.20
For every $X \in s\mathcal{S}et$ the canonical inclusion induces a continuous bijection (not a homeomorphism!)

$$\beta_X : \coprod_{n\geq 0} \tilde{X}_n \times (|I(\underline{n})| \backslash |\partial I(\underline{n})|) \xrightarrow{\sim} |X|.$$

Proof. For $(x, a) \in X_n \times |I(\underline{n})|$ we let $\underline{m} \overset{d}{\hookrightarrow} \underline{n}$ be the unique monotone injection with image

$$d(\underline{m}) = \{i \in \underline{n};\ a_i \neq 0\}.$$

Then the tuple of nonzero a-coordinates defines an element $b \in |I(\underline{m})|$ with $|I(d)|(b) = a$. Moreover by Eilenberg-Zilber's Lemma 2.19 there is a unique nondegenerate $y \in X_\ell$ and a unique $\underline{m} \overset{s}{\twoheadrightarrow} \underline{\ell}$, such that $d^*(x) = s^*(y)$. In the geometric realization $|X|$ we have

$$(x, a) = (x, |I(d)|(b)) \sim (d^*(x), b) = (s^*(y), b) \sim (y, |I(s)|(b)).$$

By definition of $|I(s)|$, the coordinates of $|I(s)|(b)$ are sums of coordinates of b. So $b_i > 0$, for all $i \in \underline{m}$, implies that also $|I(s)|(b)_i > 0$, for all $i \in \underline{\ell}$. Equivalently $|I(s)|(b) \notin |\partial I(\underline{\ell})|$. This shows that β_X is surjective and uniqueness of d, y and s implies that $(x, a) \longmapsto (y, |I(s)|(b))$ defines an inverse map for β_X.

□

This shows that the n-cells in the geometric realization $|X|$ bijectively correspond to the nondegenerate n-simplices $\tilde{X}_n$.

2.5 Adjunctions

Adjunctions play an important role, whenever one wants to compare two different categories. In this section we will give the definition and some basic facts.

Definition 2.21
An ***adjunction*** *consists of two functors* $G : \mathcal{C} \rightleftarrows \mathcal{D} : F$ *and a bijection*

$$\mathcal{C}(F(X), Y) = \mathcal{D}(X, G(Y)),$$

which is natural in X and Y.

(i) *The functor F is called the* ***left adjoint*** *to G.*

(ii) *The functor G is called the* ***right adjoint*** *to F.*

Proposition 2.22
For an adjunction $\mathcal{C}(F(X), Y) = \mathcal{D}(X, G(Y))$, we define

- $\eta_X \in \mathcal{D}(X, GF(X))$ *as the element corresponding to* $\mathrm{id}_{F(X)} \in \mathcal{C}(F(X))$.
- $\varepsilon_Y \in \mathcal{C}(FG(Y), Y)$ *as the element corresponding to* $\mathrm{id}_{G(Y)} \in \mathcal{D}(G(Y))$.

Then η and ε are natural transformations and the adjunction bijection is given by

$$\begin{aligned} \mathcal{C}(F(X), Y) &\overset{\sim}{\rightleftarrows} \mathcal{D}(X, G(Y)), \\ f &\longmapsto G(f) \circ \eta_X, \\ \varepsilon_Y \circ F(g) &\longleftarrow g. \end{aligned}$$

Proof. We denote the natural bijection of the adjunction by ϕ, i.e.

$$\phi_{X,Y} : \mathcal{C}(F(X), Y) \xrightarrow{\sim} \mathcal{D}(X, G(Y)), \quad f \longmapsto \phi(f).$$

Naturality of ϕ can be expressed as follows:

(i) For all $f \in \mathcal{C}(Y, Y')$ and $g \in \mathcal{C}(F(X), Y)$, we have

$$\phi_{X,Y'}(f \circ g) = (\phi_{X,Y} \circ \mathcal{C}(X, f))(g) = (\mathcal{D}(X, G(f)) \circ \phi_{X,Y'})(g) = G(f) \circ \phi_{X,Y'}(g).$$

(ii) For all $f \in \mathcal{C}(F(X'), Y)$ and $g \in \mathcal{D}(X, X')$, we have

$$\phi_{X,Y}(f \circ F(g)) = (\phi_{X,Y} \circ \mathcal{C}(F(g), Y))(f) = (\mathcal{D}(g, G(Y)) \circ \phi_{X',Y})(f) = \phi_{X',Y}(f) \circ g.$$

For all $g \in \mathcal{D}(X, X')$ this implies

$$\begin{aligned} GF(g) \circ \eta_X &= GF(g) \circ \phi_{X,F(X)}(\mathrm{id}_{F(X)}) = \phi_{X',F(X)}(F(g) \circ \mathrm{id}_{F(X)}) \\ &= \phi_{X',F(X)}(\mathrm{id}_{F(X')} \circ F(g)) = \phi_{X',F(X')}(\mathrm{id}_{F(X')}) \circ g = \eta_{X'} \circ g. \end{aligned}$$

So η is natural. For all $f \in \mathcal{C}(F(X), Y)$ we have by (i)

$$\phi_{X,Y}(f) = G(f) \circ \phi_{X,F(X)}(\mathrm{id}_{F(X)}) = G(f) \circ \eta_X.$$

In the same way one proves that ε is natural and that $\phi^{-1}_{X,Y} = \varepsilon_Y \circ F(\text{-})$. $\square$

Proposition 2.23
Given two functors $G : \mathcal{C} \rightleftarrows \mathcal{D} : F$ and natural transformations $\mathrm{id}_\mathcal{D} \xrightarrow{\eta} GF$ and $FG \xrightarrow{\varepsilon} \mathrm{id}_\mathcal{C}$, such that

(i) $\varepsilon_{F(X)} \circ F(\eta_X) = \mathrm{id}_{F(X)}$, *for all* $X \in \mathcal{D}$,

(ii) $G(\varepsilon_Y) \circ \eta_{G(Y)} = \mathrm{id}_{G(Y)}$, *for all* $Y \in \mathcal{C}$.

Then η and ε form an adjunction

$$\begin{aligned} \mathcal{C}(F(X), Y) &\overset{\sim}{\rightleftarrows} \mathcal{D}(X, G(Y)), \\ f &\longmapsto G(f) \circ \eta_X, \\ \varepsilon_Y \circ F(g) &\longleftarrow g. \end{aligned}$$

Proof. For every $f \in \mathcal{C}(F(X), Y)$ naturality of ε and (i) implies

$$\varepsilon_Y \circ F(G(f) \circ \eta_X) = \varepsilon_Y \circ FG(f) \circ F(\eta_X) = f \circ \varepsilon_{F(X)} \circ F(\eta_X) = f.$$

Similarly for every $g \in \mathcal{D}(X, G(Y))$ naturality of η and (ii) implies

$$G(\varepsilon_Y \circ F(g)) \circ \eta_{G(X)} = G(\varepsilon_Y) \circ GF(g) \circ \eta_{G(X)} = G(\varepsilon_Y) \circ \eta_Y \circ g = g.$$

For all $f \in \mathcal{C}(Y, Y')$ and $g \in \mathcal{C}(F(X), Y)$, we have

$$G(f \circ g) \circ \eta_X = G(f) \circ G(g) \circ \eta_X,$$

which by (i) of the proof of Proposition 2.22 (i) is the same as saying that the map $G(\text{-}) \circ \eta_X$ is natural. Similarly one checks that $\varepsilon_Y \circ F(\text{-})$ is natural. As the two maps are inverse to each other, both are natural in X and Y. $\square$

Definition 2.24
Given an adjunction $\mathcal{C}(F(X), Y) = \mathcal{D}(X, G(Y))$.

(i) *The natural transformation η is called the **unit** of the adjunction.*

(ii) *The natural transformation ε is called the **counit** of the adjunction.*

(iii) *If η and ε are both natural isomorphisms, then the adjunction is called an **equivalence (adjunction)** between the categories $\mathcal{C}$ and $\mathcal{D}$.*

(iv) *Two categories are called **equivalent**, if there is an equivalence adjunction between them.*

Example 2.25
Forgetful functors usually have a left adjoint:

(i) $\mathbb{R}\text{-Mod}(\bigoplus_{x \in X} \mathbb{R} \cdot x, Y) = \mathcal{S}et(X, U(Y))$.

(ii) $\mathcal{T}op(D(X), Y) = \mathcal{S}et(X, U(Y))$,

*where $D(X)$ is the set X with the **discrete topology** (every subset is open).*

In this setting U also has a right adjoint, i.e.

$$\mathcal{S}et(U(X), Y) = \mathcal{T}op(X, I(Y)),$$

*where $I(X)$ is the set X with the **indiscrete topology** ($\emptyset$ and X are the only open subsets).*

(iii) *Similarly we have ajunctions (cf. Remark 1.2)*

$$\mathcal{S}imp(D(X), Y) = \mathcal{S}et(X, U(Y)), \qquad \mathcal{S}et(U(X), Y) = \mathcal{S}imp(X, I(Y)).$$

(iv) *Similarly we have an adjunction*

$$\mathcal{C}at(D(X), Y) = \mathcal{S}et(X, U(Y)),$$

*where U assigns to each small category its set of objects and $D(X)$ is the **discrete category** with objects $\mathrm{Obj}(D(X)) = X$ and only the identities as morphisms.*

2.6 The singular nerve

It turns out that the geometric realization functor $s\mathcal{S}et \xrightarrow{|-|} \mathcal{T}op$ has a right adjoint, which appears in every lecture on algebraic topology.

Definition 2.26
Let X be a topological space.

*The **singular nerve** of X is defined as the composite functor*

$$S(X) : \Delta^{\mathrm{op}} \xrightarrow{U^{\mathrm{op}}} \mathcal{S}et^{\mathrm{op}} \xrightarrow{I^{\mathrm{op}}} \mathcal{S}imp^{\mathrm{op}} \xrightarrow{|\text{-}|} \mathcal{T}op^{\mathrm{op}} \xrightarrow{\mathcal{T}op(\text{-},X)} \mathcal{S}et.$$

*By construction it is a simplicial set, whose n-simplices are the **singular n-simplices***

$$S_n(X) = \mathcal{T}op(|I(\underline{n})|, X) = \mathcal{T}op(|\Delta^n|, X), \qquad n \geq 0.$$

Remark 2.27
The singular nerve is used to define singular homology.

(i) *Recall that the **singular n-chains** are defined as the abelian group*

$$C_n(X, \mathbb{Z}) := \bigoplus_{\sigma \in S_n(X)} \mathbb{Z} \cdot \sigma.$$

It becomes a chain complex by defining a differential $C_n(X, \mathbb{Z}) \xrightarrow{d} C_{n-1}(X, \mathbb{Z})$ via

$$d\Big(\sum_{\sigma \in S_n(X)} a_\sigma \cdot \sigma \Big) = \sum_{0 \leq i \leq n} (-1)^i \cdot \sum_{\sigma} a_\sigma \cdot \underbrace{(\sigma \circ |I(d^i)|)}_{d_i(\sigma)=}.$$

(ii) *The **n-th singular homology group** of X is defined as the n-th homology group of the singular chain complex $C_*(X, A)$, i.e.*

$$H_n(X, \mathbb{Z}) = \ker(C_n(X, \mathbb{Z}) \xrightarrow{d} C_{n-1}(X, \mathbb{Z}))/d(C_{n+1}(X, \mathbb{Z})), \qquad n \geq 0.$$

Proposition 2.28
The singular nerve induces a functor $\mathcal{T}op \xrightarrow{S} s\mathcal{S}et$. Together with the geometric realization an adjunction $\mathcal{T}op(|X|, Y) = s\mathcal{S}et(X, S(Y))$.

Proof. Recall that the geometric realization was defined as

$$|X| := \coprod_{n \geq 0} X_n \times |I(\underline{n})| \Big/ (X(f)(x), a) \sim (x, |I(f)|(a)), \;\; x \in X_n, \;\; a \in |I(\underline{m})|, \;\; f \in \Delta(\underline{m}, \underline{n}).$$

For every $g \in \mathcal{T}op(|X|, Y)$ and every $n \geq 0$, we get a map

$$\phi_{X,Y}(g)_n : X_n \longrightarrow \mathcal{T}op(|I(\underline{n})|, Y), \quad x \longmapsto g(x, \text{-}).$$

By construction of the geometric realization this defines a homomorphism of simplicial sets, because for every $f \in \Delta(\underline{m}, \underline{n})$ and $x \in X_n$ we have

$$\phi_{X,Y} \circ f^*(x) = g(f^*(x), \text{-}) = g(x, |I(f)|(\text{-})) = g(x, \text{-}) \circ |I(f)| = f^* \circ \phi_{X,Y}(x).$$

Vice versa every $t \in s\mathcal{S}et(X, S(Y))$ glues to a map

$$\psi_{X,Y}(f) : |X| \longrightarrow Y, \quad X_n \times |I(\underline{n})| \ni (x, a) \longmapsto t_n(x)(a),$$

because for every $f \in \Delta(\underline{m}, \underline{n})$ we have

$$t_n(f^*(x))(a) = (t_n \circ f^*)(x)(a) = (f^* \circ t_m)(x)(a) = (t_m(x) \circ |I(f)|)(a) = t_m(x)(|I(f)|(a)).$$

By construction $\phi_{X,Y}$ and $\psi_{X,Y}$ are inverse to each other and natural in X and Y. □

Remark 2.29
The definition of H_* can be extended to arbitrary simplicial sets $X \in s\mathcal{S}et$.

- *The relation between simplicial sets and chain complexes will be studied in greater detail later on.*
- *One might further study the map $H_*(X, \mathbb{Z}) \longrightarrow H_*(|X|, \mathbb{Z})$ induced by the adjunction unit $X \xrightarrow{\eta_X} S(|X|)$. It can be proven, that it is a natural isomorphism. We will deduce this from the Quillen equivalence between the model categories $\mathcal{T}op$ and $s\mathcal{S}et$.*

2.7 Isomorphisms, monomorphisms and epimorphisms

Definition 2.30
Let $\mathcal{C}$ be a category and $f \in \mathcal{C}(B, C)$ a morphism.

(i) *f is called a **retraction**, if there is a morphism $g \in \mathcal{C}(C, B)$ with $fg = \mathrm{id}_C$.*

(ii) *f is called a **section**, if there is a morphism $g \in \mathcal{C}(C, B)$ with $gf = \mathrm{id}_B$.*

(iii) *f is called an **isomorphism**, if there is a morphism $g \in \mathcal{C}(C, B)$ with $gf = \mathrm{id}_B$ and $fg = \mathrm{id}_A$.*

(iv) *f is called an **epi(-morphism)**, if $gf = hf$ implies $g = h$, for all $g, h \in \mathcal{C}(C, D)$.*

Epimorphisms are denoted by arrows $B \overset{f}{\twoheadrightarrow} C$.

(v) *f is called a **mono(-morphism)**, if $fg = fh$ implies $g = h$, for all $g, h \in \mathcal{C}(A, B)$.*

Monomorphisms are denoted by arrows $B \overset{f}{\hookrightarrow} C$.

Example 2.31 (i) *In the categories $\mathcal{S}et$, $\mathcal{T}op$, $\mathcal{G}rp$, $\mathcal{S}imp$ the monomorphisms are precisely the injective homomorphisms.*

(ii) *In the category $\mathcal{S}et$ and $\mathcal{G}rp$ the epimorphisms are precisely the surjective homomorphisms.*

(iii) *Every surjective continuous map is a an epimorphism in the category of Hausdorff spaces.*

The converse is false, e.g. the inclusion of the subspace $\mathbb{Q} \hookrightarrow \mathbb{R}$ is a non-surjective epimorphism.

(iv) *Every surjective ring homomorphism is an epimorphism in the category $\mathcal{R}ing$ of rings.*

Again the converse is false, e.g. the homomorphism $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is a non-surjective epimorphism.

Remark 2.32
Let $\mathcal{C}$ be a category.

(i) *Every retraction is an epimorphism.*

(ii) *Every section is a monomorphism.*

(iii) *Every monomorphic retraction is an isomorphism.*

(iv) *Every epimorphic section is an isomorphism.*

Remark 2.33
Let $\mathcal{C}$ be a category and $f \in \mathcal{C}(B, C)$ a morphism.

(i) *If $\mathcal{C}(\text{-}, B) \xrightarrow{\mathcal{C}(\text{-},f)} \mathcal{C}(\text{-}, C)$ is a natural bijection, then f is a monomorphic retraction.*

(ii) *If $\mathcal{C}(C, \text{-}) \xrightarrow{\mathcal{C}(f,\text{-})} \mathcal{C}(B, \text{-})$ is a natural bijection, then f is an epimorphic section.*

In particular in both cases f is an isomorphism by Remark 2.32.

2.8 Simplicial standard simplices

Definition 2.34
*For $m \geq 0$ the **simplicial standard m-simplex** is defined as the simplicial set*

$$\Delta^m := \Delta(\text{-}, \underline{m}) \in s\mathcal{S}et.$$

Proposition 2.35
There is a natural homeomorphism

$$e_{\underline{m}} : |\Delta^m| \xrightarrow{\sim} |I(\underline{m})|, \quad \Delta^m_n \times |I(\underline{n})| \ni (f, a) \longmapsto |I(f)|(a), \qquad \underline{m} \in \Delta.$$

Proof. Instead of working directly with the geometric realization we will prove this by using category theory. Let $Y \in \mathcal{T}op$ and consider the compostion c

$$\mathcal{T}op(|I(\underline{m})|, Y) \xrightarrow{\mathcal{T}op(e_{\underline{m}},Y)} \mathcal{T}op(|\Delta^m|, Y) \xrightarrow{\phi_{\Delta^m,Y}} s\mathcal{S}et(\Delta^m, S(Y)) \xrightarrow{d} S_m(Y) = \mathcal{T}op(|I(\underline{m})|, Y),$$

where the map in the middle is the adjunction bijection of Proposition 2.28 and the map on the right is Yoneda's isomorphism of Lemma 2.11. For every $g \in \mathcal{T}op(|I(\underline{m})|, Y)$ we have by definition of the three maps

$$\begin{aligned}(d \circ \phi_{\Delta^m,Y} \circ \mathcal{T}op(e_{\underline{m}}, Y))(g) &= (d \circ \phi_{\Delta^m,Y})(g \circ e_{\underline{m}}) = \phi_{\Delta^m,Y}(g \circ e_{\underline{m}})_m(\mathrm{id}_{\underline{m}}) \\ &= (g \circ e_{\underline{m}})(\mathrm{id}_{\underline{m}}, \text{-}) = g \circ |I(\mathrm{id}_{\underline{m}})| = g.\end{aligned}$$

This proves that c equals the identity. Since $\phi_{\Delta^m,Y}$ and d are bijections, also $\mathcal{T}op(e_{\underline{m}}, Y)$ is a bijection. Remark 2.33 implies that $e_{\underline{m}}$ is an isomorphism in $\mathcal{T}op$, which is a homeomorphism. By the usual arguments e is natural in $\underline{m} \in \Delta$.

□

2.9 Limits and colimits

Definition 2.36
Let $I \xrightarrow{X} \mathcal{C}$ *be a functor.*

(i) *A* ***cone*** *on* X *is a family of morphisms* $(C \xrightarrow{c_i} X(i))_{i \in I}$*, such that*

$$X(f) \circ c_i = c_j, \qquad \text{for all } f \in I(i,j).$$

(ii) *A* ***limit*** *for* X *is a* ***universal*** *cone* $(\lim X \xrightarrow{\pi_i} X(i))_{i \in I}$*. That is, for every other cone* $(C \xrightarrow{c_i} X(i))_{i \in I}$ *there is a unique morphism* $C \xrightarrow{c} \lim X$*, such that*

$$\pi_i \circ c = c_i, \qquad \text{for all } i \in I.$$

Another common notation for the limit object is $\varprojlim_{i \in I} X(i) := \lim_{i \in I} X(i) := \lim X$.

(iii) *Dually a* ***cocone*** *on* X *is a family of morphisms* $(X(i) \xrightarrow{c_i} C)_{i \in I}$*, such that*

$$c_j \circ X(f) = c_i, \qquad \text{for all } f \in I(i,j).$$

(iv) *A* ***colimit*** *for* X *is a* ***universal*** *cocone* $(X(i) \xrightarrow{\iota_i} \operatorname{colim} X)_{i \in I}$*. That is, for every other cocone* $(X(i) \xrightarrow{c_i} C)_{i \in I}$ *there is a unique morphism* $\operatorname{colim} X \xrightarrow{c} C$*, such that*

$$c \circ \iota_i = c_i, \qquad \text{for all } i \in I.$$

Another common notation for the colimit object is $\operatorname*{colim}_{i \in I} X(i) := \operatorname*{colim}_{i \in I} X(i) := \operatorname{colim} X$.

Remark 2.37
Let $I \xrightarrow{X} \mathcal{C}$ *be a functor.*

(i) *In general* X *may not have a limit or colimit.*

(ii) *If a limit/colimit of* X *exists, it is unique up to unique isomorphism.*

(iii) *Considering* X *as a functor* $I^{\mathrm{op}} \xrightarrow{X^{\mathrm{op}}} \mathcal{C}^{\mathrm{op}}$ *we see that every limit/cone for* X^{op} *can be considered as a colimt/cocone for* X.

There is a canonical construction for limits and colimits of $\mathcal{S}et$-valued functors.

Proposition 2.38
For $X \in \mathcal{C}\mathrm{AT}(I, \mathcal{S}et)$ *with small* $I \in \mathcal{C}at$*, the following holds.*

(i) *A limit of* X *is given by*

$$\lim X := \left\{ x \in \prod_{i \in I} X(i);\ X(f)(x_i) = x_j,\ \forall f \in I(i,j) \right\}.$$

The map $\lim X \xrightarrow{\pi_i} X(i)$ *is the projection onto the factor* $X(i)$*, for each* $i \in I$.

(ii) *A colimit of X is given by*

$$\operatorname{colim} X := \coprod_{i \in I} X(i) \Big/ x \sim X(f)(x), \quad x \in X(i), \quad f \in I(i,j).$$

The map $X(i) \xrightarrow{\iota_i} \operatorname{colim} X$ is the map induced by the inclusion of the disjoint summand $X(i)$, for each $i \in I$.

Proof.

(i) By construction $(\lim X \xrightarrow{\pi_i} X(i))_{i \in I}$ is a limit cone for X. Given another cone $(C \xrightarrow{c_i} X(i))_{i \in I}$ we can define

$$c : C \longrightarrow \lim X, \quad x \longmapsto (c_i(x))_{i \in I}.$$

Then $\pi_i \circ c = c_i$, for all $i \in I$, which also implies the uniqueness of c.

(ii) Again by construction $(X(i) \xrightarrow{\iota_i} \operatorname{colim} X)_{i \in I}$ is a colimit cone. Given another cocone $(X(i) \xrightarrow{c_i} C)_{i \in I}$ the map

$$c : \operatorname{colim} X \longrightarrow C, \quad X(i) \ni x \longmapsto c_i(x),$$

is well-defined. Again uniqueness of c follows from the condition $c_i = c \circ \iota_i$, for all $i \in I$.

□

Corollary 2.39
For $X \in \mathcal{C}\mathrm{AT}(I, \mathcal{T}op)$ with small $I \in \mathcal{C}at$, the following holds.

(i) *A limit $\lim X$ is given as the $\mathcal{S}et$-limit together with the initial topology, which in this case is the subspace topology of the product topology.*

(ii) *A colimit $\operatorname{colim} X$ is given as the $\mathcal{S}et$-limit together with the final topology with respect to the inclusions.*

Definition 2.40
Let $\mathcal{C} \in \mathcal{C}\mathrm{AT}$ and $S \in \mathcal{S}et$. Like in Example 2.25 a family of objects $(X_s)_{s \in S} \in \mathcal{C}^S$ corresponds to a functor

$$X \in \mathcal{C}\mathrm{AT}(D(S), \mathcal{C}) = \mathcal{S}et(S, \mathrm{Obj}(\mathcal{C})) = \mathcal{C}^S.$$

(i) $\prod_{s \in S} X_s := \lim X$ *is called a* ***product*** *over the set S.*

(ii) $\coprod_{s \in S} X_s := \operatorname{colim} X$ *is called a* ***coproduct*** *over the set S.*

Consider the empty functor $D(\emptyset) \xrightarrow{X} \mathcal{C}$.

(i) $* := \lim X$ *is called a* ***terminal*** *or* ***final*** *object in $\mathcal{C}$.*

(ii) $\emptyset := \operatorname{colim} X$ *is called an* ***initial*** *or* ***cofinal*** *object in* $\mathcal{C}$.

Remark 2.41
From the explicit construction of limits and colimits in $\mathcal{S}et$*, we see that the notion of product, coproduct,* $*$ *and* $\emptyset$ *has the usual meaning.*

Remark 2.42
Let $I \in \mathcal{C}at$ *and* $t \in \mathcal{C}\mathrm{AT}(I, \mathcal{C})(X, Y)$.

(i) *Given limits* $\lim X$ *and* $\lim Y$*, the universal property of* $\lim Y$ *applied to the cone* $(\lim X \xrightarrow{\pi_i} X(i) \xrightarrow{t_i} Y(i))_{i \in I}$ *yields a unique* $\lim t \in \mathcal{C}(\lim X, \lim Y)$ *with* $\pi_i \circ \lim t = t_i \circ \pi_i$*, for all* $i \in I$.

Note that the uniqueness implies:

a) $\lim \mathrm{id}_X = \mathrm{id}_{\lim X}$*, for all* $X \in \mathcal{C}\mathrm{AT}(I, X)$.

b) $\lim(s \circ t) = \lim s \circ \lim t$*, for natural transformations* $X \xrightarrow{t} Y \xrightarrow{s} Z$.

(ii) *Suppose every functor* $I \xrightarrow{X} \mathcal{C}$ *has a* ***constructible limit*** $\lim X$*. That is a canonical construction of the limit for each functor* X *(like e.g. we have for the category* $\mathcal{S}et$*)*[3].

Then the universal property for limits can be expressed as an adjunction

$$\mathcal{C}\mathrm{AT}(I, \mathcal{C})(\operatorname{const} X, Y) = \mathcal{C}(X, \lim Y),$$

where

- const *is the functor, sending an object* X *to the constant functor*

$$\operatorname{const} X : I \longrightarrow \mathcal{C}, \quad i \longmapsto X, \quad I(i, j) \ni f \longmapsto \mathrm{id}_X.$$

- $\lim$ *is the functor (by (i)), sending a functor* $Y \in \mathcal{C}\mathrm{AT}(I, \mathcal{C})$ *to its (canonical) limit.*

The subsequent Proposition shows that limits and colimits of functors targetting at functor categories can be constructed objectwise. Together with Proposition 2.38 this provides a general construction for limits and colimits of $s\mathcal{S}et$-valued functors.

Proposition 2.43
Let $I, J \in \mathcal{C}at$ *be small categories and* $\mathcal{C}$ *be an arbitrary category. Let* $I \xrightarrow{X} \mathcal{C}\mathrm{AT}(J, \mathcal{C})$ *and consider the functors*

$$X_j : I \longrightarrow \mathcal{C}, \quad i \longmapsto X(i)(j), \qquad j \in J.$$

Then the following holds.

[3] Often it is assumed that every limit/colimit is constructible, which is implied by the axiom of choice for classes. If you feel uncomfortable with that, note that in most cases the existence of limits/colimits in a given category is verified by giving an explicit construction. So existence of limits is obtained by proving the existence of constructible limits

(i) *If X_j has a limit* $\lim X_j$*, for each $j \in J$, then X has a limit in* $\mathcal{C}\mathrm{AT}(J,\mathcal{C})$*, given by*

$$L : J \longrightarrow \mathcal{C}, \qquad j \longmapsto \lim X_j, \quad J(j,j') \ni f \longmapsto \lim f.$$

(ii) *If X_j has a colimit* colim X_j*, for each $j \in J$, then X has a colimit in* $\mathcal{C}\mathrm{AT}(J,\mathcal{C})$*, given by*

$$L : J \longrightarrow \mathcal{C}, \qquad j \longmapsto \operatorname{colim} X_j, \quad J(j,j') \ni f \longmapsto \operatorname{colim} f.$$

Proof.

(i) By Remark 2.42 L is a functor and $(L \xrightarrow{\pi_i} X(i))_{i\in I}$ is a cone of functors $J \longrightarrow \mathcal{C}$. Given another cone of functors $(C \xrightarrow{c_i} X(i))_{i\in I}$, we get unique morphisms $(\lim X_j \xrightarrow{d_j} C(j))_{j\in J}$ being compatible with the projections $(L \xrightarrow{\pi_i} X(i))_{i\in I}$. Uniqueness implies that $(d_j)_{j\in J}$ defines a natural transformation $d \in \mathcal{C}\mathrm{AT}(J,\mathcal{C})(L,C)$ and that it is unique with the property $\pi_i \circ d = c_i$, for all $i \in I$. We conclude that $(L \xrightarrow{\pi_i} X(i))_{i\in I}$ is a limit cone for X.

(ii) This is dual to (i), i.e. by (i) we can construct a limit for

$$X^{\mathrm{op}} : I^{\mathrm{op}} \longrightarrow \mathcal{C}\mathrm{AT}(J,\mathcal{C})^{\mathrm{op}} = \mathcal{C}\mathrm{AT}(J^{\mathrm{op}},\mathcal{C}^{\mathrm{op}}),$$

which then will be a colimit for X.

□

2.10 Preservation of (co-)limits

Definition 2.44
Let $F \in \mathcal{C}\mathrm{AT}(\mathcal{C},D)$ be a functor.

(i) *We say F **preserves limits**, if for each functor $I \xrightarrow{X} \mathcal{C}$ having a limit* $\lim X$*, the cone $(F(\lim X) \xrightarrow{F(\pi_i)} F(X(i)))_{i\in I}$ is a limit cone for $F \circ X$.*

(ii) *We say F **preserves colimits**, if for each functor $I \xrightarrow{X} \mathcal{C}$ having a colimit* colim X*, the cocone $(F(X(i)) \xrightarrow{F(\iota_i)} F(\operatorname{colim} X))_{i\in I}$ is a colimit cocone for $F \circ X$.*

Remark 2.45
Given two functors $I \xrightarrow{X} \mathcal{C} \xrightarrow{F} \mathcal{D}$.

(i) *If there are limits for X and $F \circ X$, then the universal property for* $\lim(F \circ X)$ *applied to the cone $(F(\lim X) \xrightarrow{F(\pi_i)} F(X(i)))_{i\in I}$ induces a unique map*

$$F(\lim X) \longrightarrow \lim(F \circ X).$$

It is an isomorphism, if and only if F preserves limits.

(ii) *If there are colimits for X and $F \circ X$, then the universal property for* colim $(F \circ X)$ *applied to the cocone* $(F(X(i)) \xrightarrow{F(\iota_i)} F(\text{colim } X))_{i \in I}$ *induces a unique map*

$$\text{colim}\,(F \circ X) \longrightarrow F(\text{colim } X).$$

It is an isomorphism, if and only if F preserves colimits.

Proposition 2.46
For a functor $X \in \mathcal{C}\mathrm{AT}(I, \mathcal{C})$ the following holds.

(i) *An arbitrary cone $(L \xrightarrow{p_i} X(i))_{i \in I}$ for X is a limit cone for X, if and only if $\Big(\mathcal{C}(C, L) \xrightarrow{\mathcal{C}(C, p_i)} \mathcal{C}(C, X(i))\Big)_{i \in I}$ is a limit cone, for all $C \in \mathcal{C}$.*

(ii) *An arbitrary cocone $(X(i) \xrightarrow{j_i} L)_{i \in I}$ for X is a colimit cocone for X, if and only if $\Big(\mathcal{C}(X(i), C) \xrightarrow{\mathcal{C}(j_i, C)} \mathcal{C}(L, C)\Big)_{i \in I}$ is a colimit cocone, for all $C \in \mathcal{C}$.*

In particular hom-functors preserve (co-)limits.

Proof.

(i) Consider the canonical map

$$\begin{aligned} \phi_C : \mathcal{C}(C, L) &\longrightarrow \lim \mathcal{C}(C, X) = \left\{ c \in \prod_{i \in I} \mathcal{C}(C, X(i));\ X(f) \circ c_i = c_j,\ \forall f \in I(i,j) \right\}, \\ c &\longmapsto (c \circ p_i)_{i \in I}, \end{aligned}$$

where the equality on the right follows from the construction of the $\mathcal{S}et$-limit in Proposition 2.38. By definition of a cone the elements of the right object are cones with source C. Hence by definition of a limit cone the following are equivalent:

- $(L \xrightarrow{p_i} X(i))_{i \in I}$ is a limit cone.
- The map ϕ_C is bijective, for all $C \in \mathcal{C}$.
- $\Big(\mathcal{C}(C, L) \xrightarrow{\mathcal{C}(C, p_i)} \mathcal{C}(C, X(i))\Big)_{i \in I}$ is a limit cone, for all $C \in \mathcal{C}$.

(ii) This follows from (i) applied to $I^{\mathrm{op}} \xrightarrow{X^{\mathrm{op}}} \mathcal{C}^{\mathrm{op}}$ using Remark 2.37.

□

Corollary 2.47
For an adjunction $\mathcal{C}(F(X), Y) = \mathcal{D}(X, G(Y))$ the following holds.

(i) *G preserves limits.*[4]

[4] Preservation of limits is a strong criterion for a functor being a right adjoint. Freyd's adjoint functor Theorem states that almost every functor preserving limits has a left adjoint. As we will not need it here, for a precise formulation the reader again is advised to books about category theory.

(ii) *F preserves colimits.*

Proof.

(i) Given a limit cone $(\lim Y \xrightarrow{\pi_i} Y(i))_{i\in I}$, Proposition 2.46 implies that

$$\Big(\mathcal{C}(F(X), \lim Y) \overset{\mathcal{C}(F(X),\pi_i)}{\longrightarrow} \mathcal{C}(F(X), Y(i))\Big)_{i\in I}$$

is a limit cone, for all $X \in \mathcal{D}$. Using the adjunction bijection, it is isomorphic to the cone

$$\Big(\mathcal{D}(X, G(\lim Y)) \overset{\mathcal{D}(X,G(\pi_i))}{\longrightarrow} \mathcal{D}(X, G(Y(i)))\Big)_{i\in I},$$

which is therefore also a limit cone, for all $X \in \mathcal{D}$. Equivalently $(G(\lim Y) \overset{G(\pi_i)}{\longrightarrow} G(Y(i)))_{i\in I}$ is a limit cone by Proposition 2.46 again.

(ii) This is dual to (i).

□

Given a functor $J \xrightarrow{X} \mathcal{C}\mathrm{AT}(I,\mathcal{C})$, where $I, J \in \mathcal{C}at$ and $\mathcal{C}$ has constructible limits. Then by Remark 2.42 the limit induces a right adjoint functor $\mathcal{C}\mathrm{AT}(I,\mathcal{C}) \xrightarrow{\lim_I} \mathcal{C}$ and using that limits in functor categories are computed degreewise by Proposition 2.43 we get a natural isomorphism

$$\lim_{i\in I}\lim_{j\in J} X(j)(i) = \lim_{I}(\lim_{j\in J} X(j)) \xrightarrow{\sim} \lim_{j\in J}(\lim_{I} X)(j) = \lim_{j\in J}\lim_{i\in I} X(j)(i).$$

Dually also colimits preserve colimits. One might wonder, if also the natural transformation

$$\operatorname*{colim}_{j\in J}\lim_{i\in I} X(j)(i) = \operatorname*{colim}_{j\in J}(\lim_{I} X)(j) \longrightarrow \lim_{I}(\operatorname*{colim}_{j\in J} X(j)) = \lim_{i\in I}\operatorname*{colim}_{j\in J} X(j)(i)$$

is an isomorphism. In general this is false for the most categories. However under certain conditions on I and J this is true in the category $\mathcal{S}et$.

Proposition 2.48
Given two small categories $I, J \in \mathcal{C}at$, such that

(i) *I is finite, i.e. has finitely many objects and morphisms.*

(ii) *J is **filtered**, i.e.*
   a) *For all $i, j \in J$ there are morphisms $i \longrightarrow k \longleftarrow j$, for some object $k \in J$.*
   b) *For all morphisms $f, g \in J(i,j)$ there is a morphism $h \in J(j,k)$ with $hf = hg$.*

Then for every functor $J \xrightarrow{X} \mathcal{C}\mathrm{AT}(I, \mathcal{S}et)$ the natural map below is bijective

$$\operatorname*{colim}_{j\in J}\lim_{i\in I} X(j)(i) \xrightarrow{\sim} \lim_{i\in I}\operatorname*{colim}_{j\in J} X(j)(i).$$

Proof. First we claim that for every functor $J \xrightarrow{S} \mathcal{S}et$ we have

$$\iota_j(s) = \iota_{j'}(s') \quad \Longleftrightarrow \quad \exists j \xrightarrow{f} k \xleftarrow{f'} j' : \quad S(f)(s) = S(f')(s'), \qquad \forall\, s \in S(j),\ s' \in S(j').$$

Using the explicit construction for the colimit given in Proposition 2.38

$$\operatorname{colim} X := \coprod_{i \in J} S(j) \Big/ s \sim S(f)(s), \quad s \in S(i), \quad f \in J(i,j),$$

it follows that the given condition on s, s' implies that $s \sim s'$ and hence $\iota_j(s) = \iota_{j'}(s')$. Moreover the condition holds for $s' = S(f)(s)$ with $j \xrightarrow{f} j'$ by setting $f' = \mathrm{id}_{j'}$. So it remains to prove that the condition defines an equivalence relation. By definition the defined relation is reflexive and symmetric. For transitivity, suppose we are given maps

$$j \xrightarrow{f} k \xleftarrow{f'} j' \xrightarrow{g'} \ell \xleftarrow{g''} j''$$

and $s \in S(j), s' \in S(j'), s'' \in S(j'')$ with $S(f)(s) = S(f')(s')$ and $S(g')(s') = S(g'')(s'')$. Using that J is filtered there are morphisms

$$\begin{tikzcd}
 & & j \arrow[d, "f"'] & \\
 & j' \arrow[r, "f'"] \arrow[d, "g'"'] & k \arrow[d, dashed, "\exists a"'] & \\
j'' \arrow[r, "g''"] & \ell \arrow[r, dashed, "\exists b"] & m \arrow[dr, dashed, "\exists c"] & \\
 & & & n,
\end{tikzcd}$$

such that $c(af') = c(bg')$ and hence

$$\begin{aligned} S(caf)(s) &= S(ca)S(f)(s) = S(ca)S(f')(s') = S(caf)(s') \\ &= S(cbg')(s') = S(cb)S(g')(s') = S(cb)S(g'')(s'') = S(cbg'')(s''), \end{aligned}$$

which proves transivity.

Now we return to our problem of question.

- For surjectivity of the map let $x \in \lim_{i \in I} \operatorname{colim}_{j \in J} X(j)(i)$. Then using the explicit description of limits in colimits in $\mathcal{S}et$, we can consider x as a tuple

$$x = (x_i)_{i \in I} \in \prod_{i \in I} \operatorname*{colim}_{j \in J} X(j)(i),$$

with $x_i = \iota_{j_i}(y_i)$, for some $y_i \in X(j_i)(i)$, for each $i \in I$. Using that J is filtered, by induction on $\sharp I < \infty$ we find an object $j \in J$ and morphisms $j_i \xrightarrow{f_i} j$, for each $i \in I$. So replacing each y_i by its image $(f_i)_*(y_i)$ we may assume that $y_i \in X(j)(i)$, for each $i \in I$. Now for every $g \in I(i, i')$ we have $g_*(x_i) = x_{i'}$ and by our explicit description of the colimits over J, there are J-morphisms $f, f' \in J(j, k)$, such that

$$f_* g_*(y_i) = f'_*(y_{i'}).$$

Using that J is filtered, there is a J-morphism $h \in J(k, \ell)$, such that $hf = hf'$ and hence

$$g_*(hf)_*(y_i) = (hf)_* g_*(y_i) = (hf')_*(y_{i'}).$$

So by replacing every y_i by its image $(hf)_*(y_i)$, we may assume that $g_*(y_i) = y_{i'}$. Doing this for all $g \in \mathrm{Mor}(i, i')$ one after another, we may assume that $(y_i) \in \lim_{i \in I} X(j)(i)$ and since

$$x = (x_i)_i = (\iota_j y_i)_i$$

the tuple $\iota_(y_i)_i \in \operatorname{colim}_{j \in J} \lim_{i \in I} X(j)(i)$ defines a preimage for x.

- For injectivity let $x, x' \in \operatorname{colim}_{j \in J} \lim_{i \in I} X(j)(i)$ being mapped to the same element. Then x and x' are represented by elements $y = (y_i)_i \in \lim_{i \in I} X(j)(i)$ and $y' = (y'_i)_i \in \lim_{i \in I} X(j')(i)$ respectively. As before we may assume that $j = j'$. As x and x' are mapped to the same element $\iota_j(y_i) = \iota_{j'}(y'_i)$, for all $i \in I$. So by our explicit description of J-colimits, there are J-morphisms $f_i, f'_i \in J(j, k_i)$, such that $(f_i)_*(y_i) = (f'_i)_*(y'_i)$, for all $i \in I$. Using that I is finite and J is filtered, there is an object $\ell \in J$ and morphisms $g_i \in J(k_i, \ell)$. Moreover we may assume that all the maps $g_i f_i$ and $g_i f'_i$ coincide. In other words there are maps $h, h' \in J(j, \ell)$, such that $h_*(y_i) = (h')_*(y_i)$, for all $i \in I$. Equivalently $h_*(y) = (h')_*(y')$ and hence $x = \iota_j(y) = \iota_j(y') = x'$ by our description of J-colimits.

$\square$

2.11 Comma categories

Definition 2.49
*Given a two functors $\mathcal{C} \xrightarrow{F} \mathcal{E} \xleftarrow{G} \mathcal{D}$, the **comma category** $F \downarrow G$ is the following category.*

- *Its objects are tuples (C, D, α), where $C \in \mathcal{C}, D \in \mathcal{D}$ and $\alpha \in \mathcal{E}(F(C), G(D))$.*
- *Its morphisms are defined as*

  $$F \downarrow G((C, D, \alpha), (C', D', \alpha')) = \{(f, g) \in \mathcal{C}(C, C') \times \mathcal{D}(D, D');\ G(g) \circ \alpha = \alpha' \circ F(f)\}$$

  and composition is induced by that of $\mathcal{C}$ and $\mathcal{D}$.

(i) *If $* \xrightarrow{F} \mathcal{E}$ is the functor sending the only object $*$ to E, we write $E/G := F \downarrow G$ and X for the object $(*, X, E \xrightarrow{\eta_X} G(X))$.*

*If moreover G is the identity functor on $\mathcal{E} = \mathcal{C}$, we write $\mathcal{E}/E := F/E$ and call it the **slice** or **under-category** of $\mathcal{E}$-objects under E.*

(ii) *Dually if* $* \xrightarrow{G} \mathcal{E}$ *is the functor sending* $*$ *to* E, *we write* $F/E := F \downarrow G$ *and* X *for the object* $(X, *, F(X) \xrightarrow{\varepsilon_X} E)$.

If moreover F *is the identity functor on* $\mathcal{E} = \mathcal{C}$, *we write* $\mathcal{E}/E := F/E$ *and call it the* ***coslice*** *or* ***over-category*** *of* $\mathcal{E}$*-objects over* E.

Remark 2.50
Given $F \in \mathcal{C}\mathrm{AT}(\mathcal{C}, \mathcal{E})$, *every* $f \in \mathcal{E}(E, E')$ *induces a functor*

$$f_* : F/E \longrightarrow F/E', \quad X = (F(X) \xrightarrow{\varepsilon_X} E) \longmapsto f_*X = (F(X) \xrightarrow{\varepsilon_X} E \xrightarrow{f} E').$$

In particular for small $\mathcal{C}$ *we get a functor* $\mathcal{E} \xrightarrow{F/\text{-}} \mathcal{C}at$.
Moreover there is a functor

$$F/E \longrightarrow \mathcal{C}, \quad (F(X) \xrightarrow{\varepsilon_X} E) \longmapsto X.$$

Lemma 2.51 (co-Yoneda Lemma)
For $I \in \mathcal{C}at$, *we will denote the Yoneda functor of Proposition 2.10 by the same letter*

$$I : I \longrightarrow \mathcal{C}\mathrm{AT}(I^{\mathrm{op}}, \mathcal{S}et), \quad i \longmapsto I(\text{-}, i).$$

Then for $X \in \mathcal{C}\mathrm{AT}(I^{\mathrm{op}}, \mathcal{S}et)$ *the the following holds.*

(i) *The category* I/X *can be described as follows:*

$$\mathrm{Obj}(I/X) = \coprod_{i \in I} X(i), \quad I/X(x, y) = \{f \in I(i, j);\ f^*(y) = x\}, \quad x \in X(i),\ y \in X(j).$$

(ii) *The maps* $I(\text{-}, i) \xrightarrow{\varepsilon_i} X$ *induce a natural isomorphism* $\operatorname*{colim}_{i \in I/X} \varepsilon_i : \operatorname*{colim}_{i \in I/X} I(\text{-}, i) \xrightarrow{\sim} X$.

In this case 'natural' means every $f \in \mathcal{C}\mathrm{AT}(I^{\mathrm{op}}, \mathcal{S}et)(X, Y)$ *induces a commutative diagram*

$$\begin{tikzcd}
\operatorname*{colim}_{i \in I/X} I(\text{-}, i) \arrow[r, "\operatorname{colim}_i \varepsilon_i"] \arrow[d, "\operatorname{colim}_i \iota_{f_*(i)}"'] & X \arrow[d, "f"] \\
\operatorname*{colim}_{i \in I/Y} I(\text{-}, i) \arrow[r, "\operatorname{colim}_i \varepsilon_i"] & Y.
\end{tikzcd}$$

Proof.

(i) Using Yoneda's isomorphism

$$\mathcal{C}\mathrm{AT}(I^{\mathrm{op}}, \mathcal{S}et)(I(\text{-}, i), X) = X(i), \qquad i \in I,$$

an object $i = (I(\text{-}, i) \xrightarrow{\varepsilon_i} X) \in I/\mathcal{C}$ corresponds to an element $x_i \in X(i)$, for some $i \in I$. Moreover for $f \in I(i, j)$ we have

$$f \in I/X(i, j) \quad \Longleftrightarrow \quad \varepsilon_j \circ f = \varepsilon_i \quad \Longleftrightarrow \quad f^*(x_j) = x_i.$$

(ii) Using the description of I/X from (i), for every $Y \in \mathcal{C}\mathrm{AT}(I^{\mathrm{op}}, \mathcal{S}et)$ we get

$$\begin{aligned}
&\mathcal{C}\mathrm{AT}(I^{\mathrm{op}}, \mathcal{S}et)(\operatorname*{colim}_{i \in I/X} I(\text{-}, i), Y) \overset{\sim}{\longrightarrow} \lim_{i \in I/X} \mathcal{C}\mathrm{AT}(I^{\mathrm{op}}, \mathcal{S}et)(I(\text{-}, i), Y) \overset{\sim}{\longrightarrow} \lim_{i \in I/X} Y(i) \\
=&\{y \in \prod_{\substack{x \in X(i), \\ i \in I}} Y(i);\ f^*(y_{x',i'}) = y_{x,i} = y_{f^*(x'),i} \quad \forall f \in I/X((x,i),(x',i'))\} \\
=&\{t \in \prod_{i \in I} \mathcal{S}et(X(i), Y(i));\ f^* \circ t_j = t_i \circ f^*, \quad \forall f \in I(i,j)\} \\
=&\mathcal{C}\mathrm{AT}(I^{\mathrm{op}}, \mathcal{S}et)(X, Y).
\end{aligned}$$

The first isomorphism is the canonical one induced by composition with the inclusion maps (cf. Remark 2.45), using that hom-functors preserve limits by Proposition 2.46. Next we have Yoneda's isomorphism of Lemma 2.11. The explicit description of limits in $\mathcal{S}et$ given in Proposition 2.38 gives the next equality. Rewriting the product gives the next description, which by definition is the set of natural transformations.

We claim that composition of the upper isomorphism with the map given by composition with $\operatorname{colim}_i \varepsilon_i : \operatorname{colim}_{i \in I/X} I(\text{-}, i) \longrightarrow X$ is the identity. Given $f \in \mathcal{C}\mathrm{AT}(I^{\mathrm{op}}, \mathcal{S}et)(X, Y)$ the map $f \circ \varepsilon_i$ corresponding to $x \in X(i)$ sends $g \in I(j,i)$ to $f_j \circ g^*(x)$. As Yoneda's isomorphism is given by evaluation at $g = \mathrm{id}_i$, we get $f_i \circ \mathrm{id}^*(x) = f_i(x)$. This proves the claim, which implies that also composition by $\operatorname{colim}_i \varepsilon_i$ is bijective, for all $Y \in \mathcal{C}\mathrm{AT}(I^{\mathrm{op}}, \mathcal{S}et)$. So $\operatorname{colim}_i \varepsilon_i$ is an isomorphism by Remark 2.33.

□

2.12 Internal homomorphisms

Remark 2.52
There is a bijection

$$\mathcal{S}et(X \times Y, Z) = \mathcal{S}et(X, \mathcal{S}et(Y, Z)), \quad f \longmapsto [x \longmapsto f(x, \text{-})],$$

which is natural in $X, Y, Z \in \mathcal{S}et$.

In particular the functor $(\text{-} \times Y)$ is left adjoint to the functor $\mathcal{S}et(Y, \text{-})$.

We want to construct a similar natural bijection for the category of simplicial sets.

Definition 2.53
The simplicial set of ***internal homomorphisms*** *between $X, Y \in s\mathcal{S}et$ is defined as*

$$\underline{s\mathcal{S}et}(X, Y) := s\mathcal{S}et(\Delta^\bullet \times X, Y) : \Delta^{\mathrm{op}} \longrightarrow \mathcal{S}et, \quad \underline{n} \longmapsto s\mathcal{S}et(\Delta^n \times X, Y).$$

Proposition 2.54
There is a bijection

$$s\mathcal{S}et(X \times Y, Z) = s\mathcal{S}et(X, \underline{s\mathcal{S}et}(Y, Z)),$$

which is natural in $X, Y, Z \in s\mathcal{S}et$.

Proof. By Lemma 2.51 there is a natural isomorphism $\varepsilon : \operatorname{colim}_{\underline{m}\in\Delta/X} \Delta^m \xrightarrow{\sim} X$. The natural map

$$\delta : \operatorname*{colim}_{\underline{m}\in\Delta/X} (\Delta^m \times Y) \xrightarrow{\sim} (\operatorname*{colim}_{\underline{m}\in\Delta/X} \Delta^m) \times Y, \qquad Y \in s\mathcal{S}et$$

is an isomorphism. Indeed by Proposition 2.43 colimits in functor categories are computed dimensionwise and evaluated at $\underline{n} \in \Delta$ the corresponding map δ_n is a bijection. Indeed by Remark 2.52 (- $\times\, Y_n$) is a left adjoint and thus preserves colimits by Corollary 2.47.

Now the desired isomorphism is given as the composition of isomorphisms

$$\begin{tikzcd}
s\mathcal{S}et(X \times Y, Z) \arrow[r, equal] \arrow[d, "\varepsilon^*"', "\wr"] & s\mathcal{S}et(X, \underline{s\mathcal{S}et}(Y, Z)) \arrow[d, equal] \\
s\mathcal{S}et((\operatorname*{colim}_{\underline{m}\in\Delta/X} \Delta^m) \times Y, Z) \arrow[d, "\delta^*"', "\wr"] & s\mathcal{S}et(X, s\mathcal{S}et(\Delta^\bullet \times Y, Z)) \arrow[d, "\varepsilon^*"', "\wr"] \\
s\mathcal{S}et(\operatorname*{colim}_{\underline{m}\in\Delta/X} (\Delta^m \times Y), Z) \arrow[d, "\wr"] & s\mathcal{S}et(\operatorname*{colim}_{\underline{m}\in\Delta/X} \Delta^m, s\mathcal{S}et(\Delta^\bullet \times Y, Z)) \arrow[d, "\wr"] \\
\lim_{\underline{m}\in\Delta/X} s\mathcal{S}et(\Delta^m \times Y, Z) & \lim_{\underline{m}\in\Delta/X} s\mathcal{S}et(\Delta^m, s\mathcal{S}et(\Delta^\bullet \times Y, Z)). \arrow[l, "\sim"']
\end{tikzcd}$$

Here the lower vertical maps are the natural maps, which are isomorphisms as hom-functors preserve limits. The lower horizontal isomorphism is the Yoneda isomorphism of Lemma 2.11. By construction all isomorphisms are natural in X, Y and Z.

□

2.13 Ordered simplicial complexes as simplicial sets

Definition 2.55
Let $X \in \mathcal{S}imp_o$ be an ordered simplicial complex.
*The **nerve** of X is defined as the composite functor*

$$BX : \Delta^{\mathrm{op}} \xrightarrow{I} \mathcal{S}imp_o \xrightarrow{\mathcal{S}imp_o(-,X)} \mathcal{S}et, \qquad \underline{n} \longmapsto B_n X.$$

Remark 2.56 (i) *The nerve induces a functor $\mathcal{S}imp_o \xrightarrow{B} s\mathcal{S}et$.*

As hom-functors preserve limits by Proposition 2.46, so does B by Proposition 2.43.

(ii) *The Yoneda embedding $\Delta \longrightarrow s\mathcal{S}et$ coincides with the composition*

$$\Delta \xrightarrow{I} \mathcal{S}imp_o \xrightarrow{B} s\mathcal{S}et.$$

Definition 2.57
*A simplicial set $X \in s\mathcal{S}et$ is called **finite**, if the set of nondgenerate simplices is finite.*

Remark 2.58 (i) *Every simplicial set is the union of its finite simplicial subsets.*

(ii) *The geometric realization $|X|$ of a finite simplicial set X is compact, since Proposition 2.20 yields a surjection*

$$\coprod_{n\geq 0} \tilde{X}_n \times |I(\underline{n})| \twoheadrightarrow |X|,$$

and the left space is compact being a finite disjoint union of compact spaces.

Proposition 2.59
For every ordered simplicial complex $X \in \mathcal{S}imp_o$ there is a natural homeomorphism

$$e_X : |BX| = |\mathcal{S}imp_o(I(\text{-}), X)| \xrightarrow{\sim} |X|, \quad \mathcal{S}imp_o(I(\underline{n}), X) \times |I(\underline{n})| \ni (f, a) \longmapsto |f|(a).$$

Proof. Unfortunately this is not as easy as the proof for the simplicial standard simplex given in Proposition 2.35. There are bijections

$$\begin{aligned} im_X : \widetilde{B_n X} = \{f \in \mathcal{S}imp_o(I(\underline{n}), X); f \text{ injective}\} &\xrightarrow{\sim} \{s \in S(X);\ \sharp s = n+1\}, \quad n \geq 0, \\ f &\longmapsto f(\underline{n}). \end{aligned}$$

inducing the upper bijection in the commutative diagram of sets below[5].

$$\begin{tikzcd} \coprod_{n\geq 0} \widetilde{B_n X} \times (|I(\underline{n})| \backslash |\partial I(\underline{n})|) \arrow[r, "\coprod_n im_X \times \mathrm{id}", "\sim"'] \arrow[d, "\beta_{BX}"', "\wr"] & \coprod_{n\geq 0} \{s \in S(X);\ \sharp s = n+1\} \times (|I(\underline{n})| \backslash |\partial I(\underline{n})|) \arrow[d, equal] \\ |BX| \arrow[r, "e_X"] & |X|. \end{tikzcd}$$

Moreover the left vertical map is the continuous bijection of Proposition 2.20 and it follows that the lower horizontal map must be a bijection, too.

Now for every finite $F \in \mathcal{S}imp_o$, the set $\coprod_{n\geq 0} \widetilde{B_n F} \cong S(F)$ is finite and hence BF is a finite simplicial set, so $|BF|$ is compact and e_F is a homeomorphism by Lemma 1.16. For general $X \in \mathcal{S}imp_o$ we consider the usual commutative square

$$\begin{tikzcd} \operatorname*{colim}_{\substack{F\subset X,\\ F \text{ finite}}} |BF| \arrow[r, "\operatorname{colim}_F e_F", "\sim"'] \arrow[d, "\wr"] & \operatorname*{colim}_{\substack{F\subset X,\\ F \text{ finite}}} |F| \arrow[d, "\wr"] \\ |BX| \arrow[r, "e_X"] & |X|. \end{tikzcd}$$

As the colimit here is a union of sets containing each other[6], it is preserved by $\mathcal{S}imp_o \xrightarrow{B} s\mathcal{S}et$ (this is false for arbitrary colimits!). Moreover $s\mathcal{S}et \xrightarrow{|\text{-}|} \mathcal{T}op$ preserves arbitrary colimits, as it is a left adjoint. This proves that the left vertical map is a homeomorphism. By definition of the final topology on $|X|$ and the description of colimits in $\mathcal{T}op$ of Corollary 2.39, also the right vertical map is a homeomorphism. The colimit of homeomorphism is a homeomorphism (its inverse is the colimit of the inverse maps), so the upper horizontal map is a homeomorphism. Finally it follows that also the lower horizontal map is one. □

[5] To simplify the notation we will skip the forgetful functor $\mathcal{T}op \xrightarrow{U} \mathcal{S}et$.
[6] In particular it is a filtered colimit.

Proposition 2.60
There is a natural continuous bijection

$$(|\pi_X|, |\pi_Y|) : |X \times Y| \longrightarrow |X| \times |Y|, \qquad X, Y \in s\mathcal{S}et,$$

which is a homeomorphism, if X and Y are finite.

Proof. For every $m, n \geq 0$ we have a commutative diagram

$$\begin{tikzcd}
|\Delta^m \times \Delta^n| \arrow[r, equal] \arrow[d, "(|\pi_1|,|\pi_2|)"'] & |BI(\underline{m}) \times BI(\underline{n})| & |B(I(\underline{m}) \times I(\underline{n}))| \arrow[l, "|(B\pi_1, B\pi_2)|"', "\sim"] \arrow[r, "e_{I(\underline{m})\times I(\underline{n})}", "\sim"'] \arrow[d, "(|B\pi_1|,|B\pi_2|)"'] & |I(\underline{m}) \times I(\underline{n})| \arrow[d, "(|\pi_1|,|\pi_2|)"', "\wr"] \\
|\Delta^m| \times |\Delta^n| \arrow[rr, equal] & & |BI(\underline{m})| \times |BI(\underline{n})| \arrow[r, "e_{I(\underline{m})} \times e_{I(\underline{n})}", "\sim"'] & |I(\underline{m})| \times |I(\underline{n})|,
\end{tikzcd}$$

where the upper horizontal map in the middle is a homeomorphism, because B preserves products by Remark 2.84 (i). The right two horizontal maps are homeomorphisms by Proposition 2.59 and the right vertical map is a homeomorphism by Proposition 1.25. As the diagram commutes also the vertical map in the middle and the left vertical map are homeomorphisms.

The general statement follows from the commutative diagram of sets (not spaces!)

$$\begin{tikzcd}
\operatorname*{colim}_{\underline{m}\in\Delta/X} \operatorname*{colim}_{\underline{n}\in\Delta/Y} |\Delta^m \times \Delta^n| \arrow[r, "\sim"] \arrow[d, "\wr"'] & |(\operatorname*{colim}_{\underline{m}\in\Delta/X} \Delta^m) \times (\operatorname*{colim}_{\underline{n}\in\Delta/Y} \Delta^n)| \arrow[r, "\sim"] \arrow[d] & |X \times Y| \arrow[d, "(|\pi_X|,|\pi_Y|)"] \\
\operatorname*{colim}_{\underline{m}\in\Delta/X} \operatorname*{colim}_{\underline{n}\in\Delta/Y} |\Delta^m| \times |\Delta^n| \arrow[r, "\sim"] & |(\operatorname*{colim}_{\underline{m}\in\Delta/X} \Delta^m)| \times |(\operatorname*{colim}_{\underline{n}\in\Delta/Y} \Delta^n)| \arrow[r, "\sim"] & |X| \times |Y|,
\end{tikzcd}$$

where the right two isomorphisms are induced by the co-Yoneda isomorphism of Lemma 2.51 and the left two horizontal maps are isomorphisms, because all appearing functors are left adjoints (targetting to $\mathcal{S}et$) and thus commute with colimits.

As in the case of simplicial complexes treated in Proposition 1.25 the lower left horizontal map does not need to be a homeomorphism, for infinite X and Y.

□

Remark 2.61
For a (locally) compact space $Y \in \mathcal{T}op$, there is an adjunction

$$\mathcal{T}op(X \times Y, Z) = \mathcal{T}op(X, \underline{\mathcal{T}op}(Y, Z)),$$

where $\underline{\mathcal{T}op}(Y, Z)$ is the set $\mathcal{T}op(Y, Z)$ together with the ***compact open topology****. A subset $U \subset \underline{\mathcal{T}op}(Y, Z)$ is open, if it is the union of finite intersections of sets*

$$N(C, U) := \{f \in \mathcal{T}op(Y, Z);\ f(C) \subset U\}, \qquad C \subset Y \text{ compact}, \quad U \subset Z \text{ open}.$$

In particular $(\text{-} \times Y)$ commutes with arbitrary colimits.

Corollary 2.62
We can use this to extend the proof of Proposition 2.60 to get a homeomorphism

$$(|\pi_X|, |\pi_Y|) : |X \times Y| \xrightarrow{\sim} |X| \times |Y|, \qquad X, Y \in s\mathcal{S}et,$$

if at least one of the simplicial sets X and Y is finite.

Remark 2.63 (i) *There is no space of internal homomorphisms in the category $\mathcal{T}op$ of topological spaces generalizing the construction of Remark 2.61.*

(ii) *However there is a subcategory $\mathcal{CGH}aus \leq \mathcal{T}op$, which has this property. It can be shown that the geometric realization of every simplicial set is infact an object in $\mathcal{CGH}aus$.*

(iii) *One can show that $s\mathcal{S}et \xrightarrow{|\text{-}|} \mathcal{CGH}aus$ preserves finite products. Infact it even preserves finite limits (i.e. limits over finite categories), which can be proven in the same way as Propositions 1.25 and Proposition 2.60.*

2.14 Homotopies

Definition 2.64 (i) *A **simplicial homotopy** from $f \in s\mathcal{S}et(X, Y)$ to $g \in s\mathcal{S}et(X, Y)$, written $f \underset{h}{\simeq} g$, is a morphism $h \in s\mathcal{S}et(\Delta^1 \times X, Y)$ such that*

$$\begin{tikzcd}
\Delta^0 \times X = X \arrow[d, "d^1 \times \mathrm{id}"'] \arrow[dr, bend left, "f"] & \\
\Delta^1 \times X \arrow[r, "h"] & Y. \\
\Delta^0 \times X = X \arrow[u, "d^0 \times \mathrm{id}"] \arrow[ur, bend right, "g"'] &
\end{tikzcd}$$

Note that d^i corresponds to the inclusion of the subset $\{1 - i\} \subset \underline{1}$. This is why d^1 corresponds to f and d^0 to g.

(ii) *A **simplicial homotopy equivalence** is a map $f \in s\mathcal{S}et(X, Y)$, for which we have a $g \in s\mathcal{S}et(Y, X)$ with $\mathrm{id}_X \simeq gf$ and $\mathrm{id}_Y \simeq fg$.*

(iii) *A **simplicial deformation section** is a section $s \in s\mathcal{S}et(X, Y)$ having a retraction $r \in s\mathcal{S}et(Y, X)$ and a simplicial homotopy $\mathrm{id}_X \underset{h}{\simeq} sr$. It is called a **strong simplicial deformation section**, if h fits in a commutative diagram*

$$\begin{tikzcd}
\Delta^1 \times Y \arrow[r, "\pi_Y"] \arrow[d, "\mathrm{id} \times s"'] & Y \arrow[d, "s"] \\
\Delta^1 \times X \arrow[r, "h"] & X.
\end{tikzcd}$$

*We also call X a **(strong) simplicial deformation retract** of Y.*

(iv) *A* ***simplicial deformation retraction*** *is a retraction* $r \in s\mathcal{S}et(X,Y)$ *having a section* $s \in s\mathcal{S}et(Y,X)$ *and a simplicial homotopy* $\mathrm{id}_Y \underset{h}{\simeq} sr$*. It is called a* ***strong simplicial deformation retraction****, if* h *fits in a commutative diagram*

$$\begin{tikzcd} \Delta^1 \times X \arrow[r, "h"] \arrow[d, "\mathrm{id}\times r"'] & X \arrow[d, "r"] \\ \Delta^1 \times Y \arrow[r, "\pi_X"] & Y. \end{tikzcd}$$

Remark 2.65
As Δ^1 *is a finite simplicial set, by Corollary 2.62 every simplicial homotopy* $f \underset{h}{\simeq} g$ *induces a homotopy* $|f| \underset{|h|}{\simeq} |g|$ *in the topological sense.*

(i) *In particular every simplicial homotopy equivalence resp. (strong) simplicial deformation section/retraction induces a homotopy equivalence resp. (strong) deformation section/retraction after realization.*

(ii) *In contrast to homotopies between continuous maps, '*$\simeq$*' is* <u>*not*</u> *an equivalence relation on* $s\mathcal{S}et(X,Y)$*, for general* X, Y*. It is reflexive, but need neither be transitive nor symmetric. We will later introduce fibrant simplicial sets* Y*, for which '*$\simeq$*' defines an equivalence relation.*

Definition 2.66
Let $f \in s\mathcal{S}et(X,Y)$*.*

(i) *The* ***mapping cylinder*** *of* f *is defined as*

$$M(f) := (\Delta^1 \times X) +_X Y = \operatorname{colim}(\Delta^1 \times X \xleftarrow{d^0 \times \mathrm{id}_X} \Delta^0 \times X = X \xrightarrow{f} Y).$$

(ii) *The* ***mapping cone*** *of* f *is defined as*

$$C(f) := * +_X M(f) = \operatorname{colim}(* \longleftarrow X = \Delta^0 \times X \xrightarrow{d^1 \times \mathrm{id}_X} \Delta^1 \times X \xrightarrow{\iota_{\Delta^1 \times X}} M(f)).$$

Remark 2.67
We have seen that $s\mathcal{S}et \xrightarrow{|-|} \mathcal{T}op$ *by Corollary 2.62 behaves well with the appearing products. Moreover by Corollary 2.47 it also preserves arbitrary colimits, as it is a left adjoint by Proposition 2.28. So we have natural homeomorphisms*

(i) $|M(f)| = |(\Delta^1 \times X) +_X Y| \xleftarrow{\sim} |\Delta^1 \times X| +_{|X|} |Y| \xrightarrow{\sim} (|\Delta^1| \times |X|) +_{|X|} |Y| = M(|f|)$.

(ii) $|C(f)| = |* +_X M(f)| = |*| +_{|X|} |M(f)| \cong * +_{|X|} M(|f|) = C(|f|)$.

Proposition 2.68
Every $f \in s\mathcal{S}et(X,Y)$ *has a natural factorization*

$$f : X \xrightarrow{d^1} \Delta^1 \times X \xrightarrow{\iota_{\Delta^1 \times X}} M(f) \xrightarrow{(f\pi_X) \cup \mathrm{id}} Y.$$

Denoting $f' := \iota_{\Delta^1 \times X} d^1$ *and* $f' := (f\pi_X) \cup \mathrm{id}$*, the following holds.*

(i) *f' is a monomorphism,*

(ii) *f'' has a section ι_Y and there is a homotopy $h \in s\mathcal{S}et(\Delta^1 \times M(f), M(f))$ with $\mathrm{id}_{M(f)} \underset{h}{\simeq} \iota_Y f''$ inducing a commutative diagram*

$$\begin{tikzcd}
\Delta^1 \times Y \arrow[r, "\mathrm{id}\times\iota_Y"] \arrow[d, "\pi_Y"'] & \Delta^1 \times M(f) \arrow[r, "\mathrm{id}\times f''"] \arrow[d, "h"'] & \Delta^1 \times Y \arrow[d, "\pi_Y"] \\
Y \arrow[r, "\iota_Y"] & M(f) \arrow[r, "f''"] & Y.
\end{tikzcd}$$

In particular f'' is a strong deformation retraction and ι_Y is a strong deformation section.

Proof.

(i) Monomorphisms in $s\mathcal{S}et$ are simply dimensionwise injections.

(ii) There is a homomorphism of simplicial sets

$$V : \Delta^1 \times \Delta^1 \longrightarrow \Delta^1, \quad \Delta^1_n \times \Delta^1_n \ni (s,t) \longmapsto V(s,t) : [k \longmapsto \max\{s(k), t(k)\}].$$

inducing a commutative diagram

$$\begin{tikzcd}
\Delta^1 \times (\Delta^1 \times X) \arrow[d, "V\times\mathrm{id}"'] & \Delta^1 \times X \arrow[l, "d^0"'] \arrow[r, "\mathrm{id}\times f"] \arrow[d, "\pi_X"'] & \Delta^1 \times Y \arrow[d, "\pi_Y"'] \\
\Delta^1 \times X & X \arrow[l, "d^0"'] \arrow[r, "f"] & Y,
\end{tikzcd}$$

because

$$(V \times \mathrm{id})d^0(f,x) = (V \times \mathrm{id})(1,f,x) = (1,x) = d^0\pi_X(f,x), \qquad (f,x) \in \Delta^1_n \times X_n.$$

Taking the colimit induces a homomorphism

$$h : \Delta^1 \times M(f) \overset{\sim}{\longleftarrow} M(\Delta^1 \times f) = (\Delta^1 \times \Delta^1 \times X) +_{\Delta^1 \times X} (\Delta^1 \times Y) \longrightarrow M(f).$$

which by construction has all the desired properties.

□

2.15 Connected components

Definition 2.69
The set of ***connected components*** *of an $X \in s\mathcal{S}et$ is defined as*

$$\pi_0 X := \mathrm{colim}\ X = \mathrm{colim}\left(X_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} X_0 \right).$$

We call X ***connected****, if $\pi_0 X = *$.*

Proposition 2.70 (i) *A simplicial set X is connected, if and only if any two 0-simplices $x, y \in X_0$ can be connected by a chain of 1-simplices $c_1, ..., c_n \in X_1$, i.e.*

$$x \xleftarrow{d_0} c_1 \xmapsto{d_1} d_1(c_1) = d_0(c_2) \xleftarrow{d_0} ... \xleftarrow{d_0} c_n \xmapsto{d_1} d_1(c_n) = y.$$

In this case $|X|$ is path-connected.

(ii) *For general $X \in s\mathcal{S}et$ we get a natural bijection $\pi_0|X| \xrightarrow{\sim} \pi_0 X$.*

Proof.

(i) By definition X is connected, if and only if $[x] = [y]$ in $\pi_0 X$, for all $x, y \in X_0$. By definition of $\pi_0 X$, this is the case precisely if there is a chain as described.

Now given $(x, a), (y, b) \in \coprod_{n \geq 0} X_n \times |I(\underline{n})|$ corresponding to two elements in $|X|$. Supposing $(x, a) \in X_n \times |I(\underline{n})|$ we define

$$p : [0, 1] \longrightarrow |I(\underline{n})|, \quad t \longmapsto t \cdot [0] + (1 - t) \cdot a,$$

where $[0] \in |I(\underline{n})|$ is the element with $[0]_0 = 1$. This defines a path from (x, a) to $(x, [0])$. In the same way we construct a path q from $(y, [0])$ to (y, b). Let $\underline{0} \xrightarrow{\iota} \underline{n}$ denote the canonical inclusion. Then in $|X|$ we have

$$(x, [0]) = (x, \iota_*[0]) \sim (\iota^* x, [0]).$$

By (i) there is a chain $c_1, ..., c_n \in X_1$ linking $\iota^* x$ and $\iota^* y$. It corresponds to a chain of maps $c_1, ..., c_n \in s\mathcal{S}et(\Delta^1, X) \xrightarrow{\sim} X_1$ by using Yoneda's isomorphism. Naturality of Yoneda's isomorphism implies that the realization $|c_i|$ is a path linking the two elements $(d_0(c_i), [0]), (d_1(c_i), [0]) \in X_0 \times |I(\underline{0})|$. So glueing $|c_i| \in \mathcal{T}op(|\Delta^1|, |X|)$ with p and q gives rise to a path from (x, a) to (y, b):

$$[0, n+2] \longrightarrow |X|, \quad t \longmapsto \begin{cases} p(t), & t \in [0, 1], \\ |c_{2i+1}|(2i + 2 - t), & t \in [2i + 1, 2i + 2], \\ |c_{2i}|(t - 2i), & t \in [2i, 2i + 1], \\ q(t - n - 1), & t \in [n + 1, n + 2]. \end{cases}$$

This proves that $|X|$ is path-connected, i.e. $\pi_0|X| = *$.

(ii) Realization of the unit $X \xrightarrow{\eta_X} \operatorname{const} \pi_0 X$ of the colimit adjunction of Remark 2.42 induces the continuous map

$$|X| \longrightarrow |\operatorname{const} \pi_0 X| = \pi_0 X, \qquad X_n \times |I(\underline{n})| \ni (x, a) \longmapsto [\iota^* x].$$

The right object is the discrete space $\pi_0 X$, proving that the map is surjective. Hence applying π_0 induces a natural surjection $\pi_0|X| \longrightarrow \pi_0 X$ and the argument of (i) shows that it is also injective.

□

Remark 2.71
For $X, Y \in s\mathcal{S}et$, we have by definition

$$\underline{s\mathcal{S}et}(X,Y)_0 = s\mathcal{S}et(\Delta^1 \times X, Y).$$

Hence $\pi_0\underline{s\mathcal{S}et}(X,Y)$ is the set of homomorphisms $s\mathcal{S}et(X,Y)$ modulo the equivalence relation spanned by '$\simeq$'.

This defines a category with same objects as $s\mathcal{S}et$, that we will simply denote by $\pi_0\underline{s\mathcal{S}et}$.

2.16 Skeleton and coskeleton

Definition 2.72
Let $X \in \mathcal{S}imp$ and $n \geq 0$.

*The n-**skeleton** of X is the simplicial complex $\mathrm{sk}_n X$ with the same vertices, but*

$$S(\mathrm{sk}_n X) := \{s \in S(X);\ \sharp s \leq n+1\}.$$

Remark 2.73 (i) $\mathrm{sk}_n X \subset X$ *is the subcomplex of all simplices of dimension $\leq n$.*

(ii) $\mathcal{S}imp \xrightarrow{\mathrm{sk}_n} \mathcal{S}imp$ *is a functor.*

(iii) $\partial I(\underline{n}) = \mathrm{sk}_{n-1} I(\underline{n})$.

We will later need an extension of this construction to simplicial sets.

Definition 2.74
Let $n \geq 0$.

(i) $\Delta_{\leq n} \xhookrightarrow{i_n} \Delta$ *is the inclusion of the full subcategory with objects $\underline{0}, ..., \underline{n}$.*

(ii) $s_n\mathcal{S}et := \mathcal{C}\mathrm{AT}(\Delta^{\mathrm{op}}_{\leq n}, \mathcal{S}et)$.

(iii) *The n-**skeleton** of $X \in s\mathcal{S}et$ is defined as*

$$\mathrm{sk}_n X := \operatorname*{colim}_{\underline{m} \in i_n/(i_n)^* X} \Delta^m \in s\mathcal{S}et.$$

(iv) *The n-**coskeleton** of $X \in s\mathcal{S}et$ is defined as*

$$\mathrm{cosk}_n X := s\mathcal{S}et(\mathrm{sk}_n \Delta^\bullet, X) : \Delta^{\mathrm{op}} \longrightarrow \mathcal{S}et, \quad m \longmapsto s\mathcal{S}et(\mathrm{sk}_n \Delta^m, X).$$

Proposition 2.75
For every $n \geq 0$ there are adjunctions

$$s\mathcal{S}et((i_n)_! X, Y) = s_n\mathcal{S}et(X, (i_n)^* Y), \qquad s_n\mathcal{S}et((i_n)^* Y, Z) = s\mathcal{S}et(Y, (i_n)^* Z),$$

where

- $(i_n)_! X := \operatorname*{colim}_{\underline{m} \in \Delta_{\leq n}/X} \Delta^m$ *is called the* ***left Kan extension*** *of* X *along* i_n*,*
- $(i_n)^* Y := Y \circ i_n$*,*
- $(i_n)_* Z := s_n\mathcal{S}et((i_n)^*\Delta^\bullet, Z)$ *is called the* ***right Kan extension*** *of* X *along* i_n*.*

We have $\mathrm{sk}_n = (i_n)_!(i_n)^*$ *and* $\mathrm{cosk}_n = (i_n)_*(i_n)^*$ *and get an adjunction*

$$s\mathcal{S}et(\mathrm{sk}_n X, Y) = s\mathcal{S}et(X, \mathrm{cosk}_n Y).$$

Moreover counit of the first and unit of the second adjunction induce natural transformations

$$\mathrm{sk}_n X = (i_n)_!(i_n)^* X \xrightarrow{\varepsilon_X} X, \qquad Y \xrightarrow{\eta_Y} (i_n)_*(i_n)^* Y = \mathrm{cosk}_n Y, \qquad X, Y \in s\mathcal{S}et.$$

Proof. Using that Hom-functors preserve limits by Proposition 2.46 and twice Yoneda's isomorphism 2.11 we get the adjunction

$$\begin{aligned} s\mathcal{S}et((i_n)_! X, Y) = s\mathcal{S}et(\operatorname*{colim}_{\underline{m}\in\Delta_{\leq n}/X} \Delta^m, Y) &\xrightarrow{\sim} \lim_{\underline{m}\in\Delta_{\leq n}/X} s\mathcal{S}et(\Delta^m, Y) \xrightarrow{\sim} \lim_{\underline{m}\in\Delta_{\leq n}/X} Y_m \\ &= \left\{ y \in \prod_{\substack{x \in X_m,\\ 0\leq m\leq n}} Y_m;\ g^* y_{x'} = y_x = y_{g^* x'},\ g \in \Delta_{\leq n}(x, x') \right\} = s_n\mathcal{S}et(X, Y). \end{aligned}$$

Using the the isomorphism $\varepsilon : \operatorname{colim}_{\underline{m}\in\Delta/X} \Delta^m \xrightarrow{\sim} X$ of the co-Yoneda Lemma 2.51, we get the other adjunction as the composition

$$\begin{tikzcd} s_n\mathcal{S}et((i_n)^*X, Y) \arrow[rr, equal] \arrow[d, "\varepsilon^*", "\wr"'] & & s\mathcal{S}et(X, (i_n)_*Y) \arrow[d, "\varepsilon^*", "\wr"'] \\ s_n\mathcal{S}et(\operatorname*{colim}_{\underline{m}\in\Delta/X} (i_n)^*\Delta^m, Y) \arrow[d, "\wr"] & & s\mathcal{S}et(\operatorname*{colim}_{\underline{m}\in\Delta/X} \Delta^m, s_n\mathcal{S}et((i_n)^*\Delta^\bullet, Y)) \arrow[d, "\wr"] \\ \lim_{\underline{m}\in\Delta/X} s_n\mathcal{S}et((i_n)^*\Delta^m, Y) \arrow[rr, "\sim"] & & \lim_{\underline{m}\in\Delta/X} s\mathcal{S}et(\Delta^m, s_n\mathcal{S}et((i_n)^*\Delta^\bullet, Y)), \end{tikzcd}$$

where the vertical maps are isomorphisms, because hom-functors preserve (co-)limits by Proposition 2.46 and the lower horizontal map is Yoneda's isomorphism of Lemma 2.11. Moreover $(i_n)^*$ preserves (co-)limits, as colimits in functor categories are constructed dimensionwise by Proposition 2.43. By definition we have $\mathrm{sk}_n = (i_n)_!(i_n)^*$ and

$$\mathrm{cosk}_n Y = s\mathcal{S}et(\mathrm{sk}_n \Delta^\bullet, Y) = s_n\mathcal{S}et((i_n)^*\Delta^\bullet, (i_n)^*Y) = (i_n)_*(i_n)^*Y, \qquad Y \in s\mathcal{S}et.$$

$\square$

Proposition 2.76
For $X \in s\mathcal{S}et$ there are natural cocartesian squares (meaning that the lower right object is the colimit of the rest of the diagram)

$$\begin{tikzcd} \tilde{X}_n \times \partial\Delta^n \arrow[r] \arrow[d, hook] & \mathrm{sk}_{n-1}X \arrow[d] \\ \tilde{X}_n \times \Delta^n \arrow[r] & \mathrm{sk}_n X, \end{tikzcd}$$

where $\partial\Delta^n = B\partial I(\underline{n})$ and the horizontal maps send (x, f) to f^*x and the vertical maps are the canonical maps.

Proof. By Proposition 2.46 we can equivalently prove that, for every $Y \in s\mathcal{S}et$, applying $s\mathcal{S}et(\text{-}, Y)$ yields a cartesian square

$$\begin{tikzcd} s\mathcal{S}et(\mathrm{sk}_n X, Y) \arrow[r] \arrow[d] & s\mathcal{S}et(\tilde{X}_n \times \partial\Delta^n, Y) \arrow[d] \\ s\mathcal{S}et(\mathrm{sk}_{n-1}X, Y) \arrow[r] & s\mathcal{S}et(\tilde{X}_n \times \Delta^n, Y), \end{tikzcd}$$

which by using the adjunction bijections of Proposition 2.75 is naturally isomorphic to the diagram

$$\begin{tikzcd} s_n\mathcal{S}et((i_n)^*X, (i_n)^*Y) \arrow[r] \arrow[d] & \mathcal{S}et(\tilde{X}_n, Y_n) \arrow[d] \\ s_{n-1}\mathcal{S}et((i_{n-1})^*X, (i_{n-1})^*Y) \arrow[r] & \mathcal{S}et(\tilde{X}_n, s\mathcal{S}et(\partial\Delta^n, Y)). \end{tikzcd}$$

The left vertical map sends $f = (f_0, ..., f_n)$ to the tuple $(f_0, ..., f_{n-1})$, while the upper horizontal map sends f to the restriction of f_n to $\tilde{X}_n$. Now composition with the natural surjection

$$d := \coprod_{0\leq i\leq n} d^i : \coprod_{0\leq i\leq n} \Delta^{n-1} = \coprod_{0\leq i\leq n} BI(\underline{n-1}) \twoheadrightarrow B\partial I(\underline{n}) = \partial\Delta^n,$$

yields an injection

$$d^* : \mathcal{S}et(\tilde{X}_n, s\mathcal{S}et(\partial\Delta^n, Y)) \hookrightarrow \mathcal{S}et(\tilde{X}_n, \prod_i s\mathcal{S}et(\Delta^{n-1}, Y)) \xrightarrow{\sim} \prod_{0\leq i\leq n} \mathcal{S}et(\tilde{X}_n, Y_{n-1}).$$

By construction the lower horizontal map in the last diagram composed with d^* sends f to $(f_{n-1}d_i)_i$, while the composition with the right vertical map sends f to $(d_i f)_i$. Using the explicit description of $\mathcal{S}et$-limits of Proposition 2.38 the pullback is the set

$$\{(f_0, ..., f_{n-1}, \tilde{f}_n) \in s_{n-1}\mathcal{S}et((i_{n-1})^*X, (i_{n-1})^*Y) \times \mathcal{S}et(\tilde{X}_n, Y_n);\ d_i\tilde{f}_n = f_{n-1}d_i\ 0 \leq i \leq n\},$$

whose elements uniquely glue together to a natural transformation $s_n\mathcal{S}et((i_n)^*X, (i_n)^*Y)$. Indeed since by definition

$$X_n = \tilde{X}_n \cup \bigcup_{0\leq i\leq n-1} s_i X_{n-1},$$

we can define

$$f_n(x) := \begin{cases} s_i f_n(x), & x \in s_i X_{n-1}, \\ \tilde{f}_n(x), & x \in \tilde{X}_n, \end{cases}$$

which is well-defined, because $(f_0, ..., f_{n-1})$ is a natural transformation. The glueing process defines an inverse map to the map restriction maps induced by the upper left two maps of the diagram. This concludes the proof.

□

Corollary 2.77
For $X \in s\mathcal{S}et$ and $n \geq 0$, the counit $\varepsilon_X : \mathrm{sk}_n X \hookrightarrow X$ is injective.

So $\mathrm{sk}_n X \subset X$ is the simplicial subset generated by all nondegenerate simplices of dimension $\leq n$. Moreover:

(i) $(\mathrm{sk}_n X)_m = X_m$, *for all* $0 \leq m \leq n$.

(ii) $\operatorname*{colim}_{n\geq 0} \mathrm{sk}_n X = \bigcup_{n\geq 0} \mathrm{sk}_n X = X$.

(iii) $\mathrm{sk}_n X/\mathrm{sk}_{n-1} X \cong (\tilde{X}_n \times (\Delta^n/\partial\Delta^n))/(\tilde{X}_n \times *) = \bigvee_{x\in\tilde{X}} S^n$.

Corollary 2.78
The geometric realization $|X|$ of a simplicial set $X \in s\mathcal{S}et$ is a CW-complex.

Proof. The geometric realization functor is a left adjoint. So we get cocartesian squares

$$\begin{tikzcd} \tilde{X}_n \times |\partial\Delta^n| \arrow[r] \arrow[d, hook] & |\mathrm{sk}_{n-1} X| \arrow[d] \\ \tilde{X}_n \times |\Delta^n| \arrow[r] & |\mathrm{sk}_n X|. \end{tikzcd}$$

In other words $|\mathrm{sk}_n X|$ is obtained from $|\mathrm{sk}_{n-1} X|$ by glueing the set of nondegenerate simplices and

$$|X| = \operatorname*{colim}_{n\geq 0} |\mathrm{sk}_n X| = \bigcup_{n\geq 0} |\mathrm{sk}_n X|.$$

□

The following proposition shows, that the nerve functor behaves well with the two skeleton constructions.

Proposition 2.79

$$\mathrm{sk}_n BX \cong B\mathrm{sk}_X, \qquad X \in \mathcal{S}imp, \quad n \geq 0.$$

Proof. By naturality of the counit map $\mathrm{sk}_n = (i_n)_!(i_n)^* \xrightarrow{\varepsilon} \mathrm{id}_{s\mathcal{S}et}$ the inclusion $\mathrm{sk}_n X \xhookrightarrow{\iota_n} X_n$ induces a commutative square

$$\begin{tikzcd}
\mathrm{sk}_n B\mathrm{sk}_n X \arrow[r, "\varepsilon_{B\mathrm{sk}_n X}"] \arrow[d, "\mathrm{sk}_n B(\iota_n)"'] & B\mathrm{sk}_n X \arrow[d, "B(\iota_n)"] \\
\mathrm{sk}_n BX \arrow[r, "\varepsilon_{BX}"] & BX.
\end{tikzcd}$$

By definition of $\mathrm{sk}_n X$ and the nerve functor $\mathcal{S}imp_o \xrightarrow{B} s\mathcal{S}et$, we have an isomorphism

$$(i_n)^* B\mathrm{sk}_n X = \mathcal{S}imp_o(I(\underline{n}), \mathrm{sk}_n X) \xrightarrow{\sim} \mathcal{S}imp_o(I(\underline{n}), X) = (i_n)^* BX.$$

Applying the functor $(i_n)_!$ yields the left vertical map, which is therefore an isomorphism, too. Moreover for the nondegenerate simplices we have

$$\widetilde{B_m \mathrm{sk}_n} X = \{f \in \mathcal{S}imp_p(I(\underline{m}), \mathrm{sk}_n X);\ f \text{ injective}\} = \emptyset, \qquad m > n.$$

So in the diagram of Proposition 2.76, the left vertical map is an isomorphism. As the diagram is a pushout square, also the right vertical map is an isomorphism and we obtain isomorphisms

$$\varepsilon_{B\mathrm{sk}_n X} : \mathrm{sk}_n B\mathrm{sk}_n X \xrightarrow{\sim} \mathrm{sk}_{n+1} B\mathrm{sk}_n X \xrightarrow{\sim} \ldots \xrightarrow{\sim} B\mathrm{sk}_n X,$$

proving that also the upper horizontal map in the commutative square is an isomorphism.

□

Corollary 2.80
*For every $n \geq 0$, the simplicial standard n-simplex Δ^n is n-**skeletal**, i.e. $\mathrm{sk}_n \Delta^n \xrightarrow{\sim} \Delta^n$.*

Proof. Since $\mathrm{sk}_n I(\underline{n}) \xrightarrow{\sim} I(\underline{n})$, Proposition 2.79 yields a commutative square of isomorphisms

$$\begin{tikzcd}
\mathrm{sk}_n B\mathrm{sk}_n I(\underline{n}) \arrow[r, "\varepsilon_{B\mathrm{sk}_n I(\underline{n})}"] \arrow[d] & B\mathrm{sk}_n I(\underline{n}) \arrow[d] \\
\mathrm{sk}_n \Delta^n = \mathrm{sk}_n BI(\underline{n}) \arrow[r, "\varepsilon_{BI(\underline{n})}"] & BI(\underline{n}) = \Delta^n.
\end{tikzcd}$$

Alternatively, we can use that the category Δ/Δ^n has a final object $f := (\Delta^n \xrightarrow{\varepsilon = \mathrm{id}} \Delta^n)$, inducing an isomorphism ι_f and a factorization of the identity map

$$\mathrm{id} : \Delta^n \xrightarrow{\iota_f} \operatorname*{colim}_{\underline{m} \in \Delta^n} \Delta^m \xrightarrow{\varepsilon_{\Delta^n}} \Delta^n.$$

□

Remark 2.81
Let $X \in s\mathcal{S}et$ and $n \geq 0$.

Then using Corollary 2.77 we get

(i) $(\mathrm{cosk}_n X)_m = s\mathcal{S}et(\mathrm{sk}_n\Delta^m, X) = s\mathcal{S}et(\Delta^m, X) = X_m, \qquad 0 \leq m \leq n.$

(ii) $X \xrightarrow{\sim} \lim\limits_{n\geq 0} \mathrm{cosk}_n X.$

The map $X \xrightarrow{\eta_X} \mathrm{cosk}_n X$ *is* <u>*not*</u> *surjective in general. Defining the simplicial subset* $\mathrm{cosk}'_n X := \eta_X(X) \leq \mathrm{cosk}_n X$*, we get a tower*

$$X \twoheadrightarrow \dots \twoheadrightarrow \mathrm{cosk}'_1 X \twoheadrightarrow \mathrm{cosk}'_0 X,$$

satisfying (i) and (ii), that is closely related to the Postnikov-tower. It can be shown that for a pointed space $* \in X \in \mathcal{T}op$ *the fibre* F_n *of the map* $\mathrm{cosk}'_n S(X) \twoheadrightarrow \mathrm{cosk}'_{n-1} S(X)$ *satisfies*

$$\pi_m(|F_n|, *) \cong \begin{cases} \pi_n(X, *), & m = n, \\ 1, & m \neq n. \end{cases}$$

Similar properties hold for any fibrant simplicial set, that we will introduce later.

2.17 Small categories as simplicial sets

Remark 2.82
Every partially ordered set P *can be considered as a category* P *with objects* P *and homomorphisms*

$$P(x, y) := \begin{cases} \{\leq\}, & x \leq y, \\ \emptyset, & x \not\leq y. \end{cases}$$

Indeed there is only one way to define the composition maps, which are well-defined as '$\leq$*' is transitive. As it is reflexive, every object has an identity morphism.*

This construction defines a functor $\Delta \hookrightarrow \mathcal{C}at$*. By construction we have*

$$\mathcal{C}at(\underline{m}, \underline{n}) = \Delta(\underline{m}, \underline{n}), \qquad m, n \in \Delta,$$

which means that Δ *is a full subcategory of* $\mathcal{C}at$*.*

Definition 2.83
Let $C \in \mathcal{C}at$ *be an ordered simplicial complex.*

The ***nerve*** *of* C *is defined as the composite functor*

$$BC : \Delta^{\mathrm{op}} \longrightarrow \mathcal{C}at \xrightarrow{\mathcal{C}at(\text{-},X)} \mathcal{S}et, \qquad \underline{n} \longmapsto B_n X.$$

Remark 2.84 (i) *The nerve induces a functor* $\mathcal{C}at \xrightarrow{B} s\mathcal{S}et$*.*

As hom-functors preserve limits by Proposition 2.46, so does B *by Proposition 2.43.*

(ii) *Again the Yoneda embedding* $\Delta \longrightarrow s\mathcal{S}et$ *coincides with the composition*

$$\Delta \xrightarrow{I} \mathcal{C}at \xrightarrow{B} s\mathcal{S}et.$$

Remark 2.85
For $C \in \mathcal{C}at$ we have by construction

- $B_0 C = \mathcal{C}at(\underline{0}, C) = \mathcal{S}et(\underline{0}, U(C)) = \mathrm{Obj}(C)$,
- $B_1 C = \mathcal{C}at(\underline{1}, C) = \coprod_{x_0, x_1 \in C} C(x_0, x_1) =: \mathrm{Mor}(C)$.
- $B_n C = \coprod_{x_0, \ldots x_n \in C} C(x_0, x_1) \times \ldots \times C(x_{n-1}, x_n)$, *for all* $n \geq 0$.

Moreover the face and degeneracy maps are given by

- $d_i(f_1, \ldots, f_n) = \begin{cases} (f_2, \ldots, f_n), & i = 0, \\ (f_1, \ldots, f_i f_{i+1}, \ldots, f_n), & 0 < i < n, \\ (f_1, \ldots, f_{n-1}), & i = n. \end{cases}$
- $s_i(f_1, \ldots, f_n) = (f_1, \ldots, f_i, \mathrm{id}, f_{i+1}, \ldots, f_n), \qquad 0 \leq i \leq n.$

Remark 2.86
Every group $G \in \mathcal{G}rp$ (or more generally every monoid) gives rise to a category G with a single object $*$ and homomorphisms $G(*) = G$. Composition is defined by multiplication in G and the neutral element forms the identity morphism.

(i) *By definition of categories and groups, a homomorphisms $f \in \mathcal{G}rp(G, H)$ bijectively corresponds to a functor*

$$f : G \longrightarrow H, \quad * \longmapsto *, \quad G(*) = G \xrightarrow{f} H = H(*).$$

In particular we can view $\mathcal{G}rp$ as a full subcategory of $\mathcal{C}at$.

(ii) *In the context of group (co-)homology the nerve BG is better known as the* ***bar construction*** *of G. Using the comparison of the homotopy categories of $\mathcal{T}op$ and $s\mathcal{S}et$ we will see that*

$$H_*(G, \mathbb{Z}) = H_*(BG, \mathbb{Z}) \xrightarrow{\sim} H_*(|BG|, \mathbb{Z}),$$

where the first equality is by definition and the second isomorphism will be induced by the unit map $BG \xrightarrow{\eta_{BG}} S|BG|$.

Proposition 2.87
The nerve functor $\mathcal{C}at \xrightarrow{B} s\mathcal{S}et$ is ***fully faithful****, meaning that for all $C, D \in \mathcal{C}at$ it induces a bijection*

$$B : \mathcal{C}at(C, D) \xrightarrow{\sim} s\mathcal{S}et(BC, BD), \quad F \longmapsto B(F).$$

Proof. We define the maps

$$t^i : \underline{1} \longrightarrow \underline{n}, \qquad 0 \longmapsto i - 1, \quad 1 \longmapsto i, \qquad 1 \leq i \leq n,$$

and let $t_i := (t^i)^* \in \mathcal{S}et(X_n, X_1)$ denote the corresponding map, for $X \in s\mathcal{S}et$. Then for all $C \in \mathcal{C}at$ using Remark 2.85 we get

$$t_i(f_1, ..., f_n) = f_i, \qquad (f_1, ..., f_n) \in B_n C.$$

In particular, for all $X \in s\mathcal{S}et$ and $g \in s\mathcal{S}et(X, BC)$ we get

$$g_n(x) = (t_1 g_n, ..., t_n g_n)(x) = (g_1 t_1, ..., g_1 t_n)(x) = (g_1 t_1(x), ..., g_1 t_n(x)), \qquad x \in X_n, \quad n > 1.$$

So g is completely determined by g_0 and g_1. Moreover it follows that every $g \in s\mathcal{S}et(BC, BD)$ defines a functor $C \xrightarrow{G} D$ by setting

$$g_0 : \mathrm{Obj}(C) \longrightarrow \mathrm{Obj}(D), \qquad g_1 : \mathrm{Mor}(C) \longrightarrow \mathrm{Mor}(D).$$

Indeed:

- $G(f_1 \circ f_2) = g_1 d_1(f_1, f_2) = d_1 g_2(f_1, f_2) = d_1(g_1(f_1), g_1(f_2)) = G(f_1) \circ G(f_2)$,
- $G(\mathrm{id}_x) = g_1 s_0(x) = s_0 g_0(x) = \mathrm{id}_{G(x)}, \qquad x \in C.$

We have proven that there is a bijection

$$\begin{aligned} \mathcal{C}at(C, D) &\overset{\sim}{\rightleftarrows} s\mathcal{S}et(BC, BD), \\ F &\longmapsto BF, \\ G &\longleftarrow\!\shortmid\; g. \end{aligned}$$

$\square$

While in general it is often tedious to explicitly construct homotopies between two homomorphisms of simplicial sets, there is an easy description, when the simplicial sets are the nerves of categories. Like the categories $\mathcal{S}et$ and $s\mathcal{S}et$ also the category $\mathcal{C}at$ has an object of internal homomorphism.

Remark 2.88
There is a natural bijection

$$\mathcal{C}at(X \times Y, Z) = \mathcal{C}at(X, \mathcal{C}at(Y, Z)), \quad f \longmapsto [x \longmapsto f(x, \text{-})],$$

where we recall that $\mathcal{C}at(Y, Z)$ is the category of functors $Y \longrightarrow Z$ with natural transformations as homomorphisms.

Proposition 2.89
For $F_0, F_1 \in \mathcal{C}at(C, D)$ the set of simplicial homotopies $BF_1 \underset{h}{\simeq} BF_0$ bijectively corresponds to the set of natural transformations $\mathcal{C}at(C, D)(F_1, F_0)$.

In particular every adjunction $C(F(X), Y) = D(X, G(Y))$ induces a simplicial homotopy equivalence

$$BF : BD \overset{\simeq}{\rightleftarrows} BC : BG.$$

Proof. There is a commutative diagram

$$\begin{tikzcd}
s\mathcal{S}et(B\underline{0} \times BC, BD) \arrow[r, "\sim"] \arrow[d, "d_i"'] & s\mathcal{S}et(B(\underline{0} \times C), BD) \arrow[d, "d_i"] & \mathcal{C}at(\underline{0} \times C, D) = \mathcal{C}at(\underline{0}, \mathcal{C}at(C, D)) \arrow[l, "B"', "\sim"] \arrow[d, "d_i"] \\
s\mathcal{S}et(B\underline{1} \times BC, BD) \arrow[r, "\sim"] & s\mathcal{S}et(B(\underline{1} \times C), BD) & \mathcal{C}at(\underline{1} \times C, D) = \mathcal{C}at(\underline{1}, \mathcal{C}at(C, D)), \arrow[l, "B"', "\sim"]
\end{tikzcd}$$

where the vertical maps are induced by the map $\underline{0} \xrightarrow{d^i} \underline{1}$, for $i = 0, 1$. The left two horizontal maps are bijections, because B preserves products by Remark 2.84, the right two horizontal maps are the bijections of Proposition 2.87 and the right two horizontal maps are those of Remark 2.88. In particular using $B\underline{n} = \Delta^n$ we get a bijection

$$\begin{aligned}
\{h \in s\mathcal{S}et(\Delta^1 \times BD, BD);\ BF_1 \underset{h}{\simeq} BF_0\} &= \{h \in s\mathcal{S}et(\Delta^1 \times BC, BD);\ d_i(h) = F_i\} \\
&= \{t \in \mathcal{C}at(\underline{1}, \mathcal{C}at(C, D));\ d_i(t) = F_i\} \\
&= \mathcal{C}at(C, D)(F_1, F_0).
\end{aligned}$$

□

3 Abstract homotopy theory

3.1 Localizations of categories

Given a category $\mathcal{C} \in \mathcal{C}\mathrm{AT}$ and a class of morphisms in $\mathcal{C}$ preserving the structure "very well", but not as well as isomorphisms.

Example 3.1 (i) *$\mathcal{T}op$ and homotopy equivalences.*

(ii) *$s\mathcal{S}et$ and maps $X \xrightarrow{f} Y$, such that $|X| \xrightarrow{|f|} |Y|$ is a homotopy equivalence.*

(iii) *$\mathcal{T}op$ and **weak homotopy equivalences**, i.e. maps $X \xrightarrow{f} Y$ with*

$$\pi_0 f : \pi_0 X \xrightarrow{\sim} \pi_0 Y, \qquad \pi_n f : \pi_n(X, x) \xrightarrow{\sim} \pi_n(Y, f(x)), \qquad x \in X, \quad n > 0.$$

(iv) *Chain complexes and **quasi-isomorphisms**, i.e. chain maps $X \xrightarrow{f} Y$ with*

$$H_* f : H_* X \xrightarrow{\sim} H_* Y.$$

We want to study the objects in $\mathcal{C}$ modulo such equivalences.

Definition 3.2
Let $\mathcal{C} \in \mathcal{C}\mathrm{AT}$ and $S \subset \mathrm{Mor}(\mathcal{C})$ a subclass of morphisms.
*Then $\mathcal{C} \xrightarrow{\gamma} S^{-1}\mathcal{C}$ is called a **localization** of $\mathcal{C}$ at S, if the following holds.*

(i) *$\gamma(S) \subset \mathrm{Mor}(S^{-1}\mathcal{C})^\times$, i.e. γ sends S to isomorphisms.*

(ii) *It is universal with respect to property (i). This means, that for all $F \in \mathcal{C}\mathrm{AT}(\mathcal{C}, \mathcal{D})$ with $F(S) \subset \mathrm{Mor}(\mathcal{D})^\times$, there is a unique functor inducing a commutative diagram*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\gamma"'] & \mathcal{D} \\ S^{-1}\mathcal{C}. \arrow[ur, dashed, "\exists! \tilde{F}"'] & \end{tikzcd}$$

Example 3.3
For a commutative monoid $M \in \mathcal{M}on$ and a submonoid $S \leq M$, we define a monoid

$$S^{-1}M := (M \times S)/\sim, \qquad (m, s) \sim (m', s') \quad \Longleftrightarrow \quad \exists t \in S : \quad m + s' + t = m' + s + t.$$

Then $S^{-1}M$ is generated by the images of $M \times 0$ and $0 \times S$. The identity

$$(s,0) + (0,s) = (s,s) \sim (0,0), \qquad s \in S,$$

shows that $S^{-1}M$ is the monoid obtained from $M = M \times 0$ by adding formal inverses $s^{-1} = (0,s)$ for $s \in S$.

In particular the map $M \longrightarrow S^{-1}M$, sending m to $(m,0)$ is a localization for M considered as a category with one object.

Remark 3.4
In general localizations are very hard to construct and may not exist, if $\mathcal{C}$ is not small.

Definition 3.5
*A **category with weak equivalences** consists of a category $\mathcal{C} \in \mathcal{C}\mathrm{AT}$ and a subclass of morphisms $w\mathcal{C} \subset \mathrm{Mor}(\mathcal{C})$, so-called **weak equivalences (wes.)** (written "$\xrightarrow{\simeq}$"), such that:*

(i) $\mathrm{Mor}(\mathcal{C})^\times \subset w\mathcal{C}$.

(ii) *The **2-of-3 axiom** holds, i.e. for all $A, B, C \in \mathcal{C}$ and every commutative triangle*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "gf"'] & B \arrow[d, "g"] \\ & C, \end{tikzcd}$$

if 2 of the 3 morphisms are wes., so is the third.

*Its **homotopy category** for $\mathcal{C}$, if it exists, is defined as the localization $\mathrm{Ho}(\mathcal{C}) = (w\mathcal{C})^{-1}\mathcal{C}$.*

Remark 3.6 (i) *For every functor $F \in \mathcal{C}\mathrm{AT}(\mathcal{C}, \mathcal{D})$ the pair $(\mathcal{C}, F^{-1}\mathrm{Mor}(\mathcal{D})^\times)$ is a category with weak equivalences.*

(ii) *Let $\mathcal{C} \in \mathcal{C}\mathrm{AT}$ and suppose $S \subset \mathrm{Mor}(\mathcal{C})$ admits a localization $\mathcal{C} \xrightarrow{\gamma} S^{-1}\mathcal{C}$.*
Then $\overline{S} := \gamma^{-1}\mathrm{Mor}(S^{-1}\mathcal{C})^\times$ defines a category with weak equivalences $(\mathcal{C}, \overline{S})$ with

$$\mathrm{Ho}(\mathcal{C}) = \overline{S}^{-1}\mathcal{C} = S^{-1}\mathcal{C}.$$

3.2 Weak factorization systems

Definition 3.7
Let $\mathcal{C} \in \mathcal{C}\mathrm{AT}$ and $\ell \in \mathcal{C}(A,B), r \in \mathcal{C}(C,D)$.

If for any commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\ell"'] & C \arrow[d, "r"] \\ B \arrow[r] \arrow[ur, dashed, "\exists d"] & D \end{tikzcd}$$

there exists a diagonal d making the diagram commutative, then one says, that

(i) *ℓ has the* ***left lifting property (LLP)*** *with respect to r and*

(ii) *r has the* ***right lifting property (RLP)*** *with respect to ℓ.*

For a subclass $S \subset \mathrm{Mor}(\mathcal{C})$ define

(i) $LLP(S) := \{f \in \mathrm{Mor}(\mathcal{C});\ f$ *has the LLP w.r.t. all* $s \in S\}$,

(ii) $RLP(S) := \{f \in \mathrm{Mor}(\mathcal{C});\ f$ *has the RLP w.r.t. all* $s \in S\}$.

Definition 3.8
Let $\mathcal{C} \in \mathcal{C}\mathrm{AT}$ and $L, R \subset \mathrm{Mor}(\mathcal{C})$.
Then (L, R) is a ***weak factorization system*** *on $\mathcal{C}$, if*

(i) $\mathrm{Mor}(\mathcal{C}) = R \circ L$,

(ii) $L = LLP(R)$ *and* $R = RLP(L)$.

Example 3.9
The pair (Epi, Mono) *forms a weak factorization system on the category $\mathcal{S}et$:*

(i) *Every map* $X \xrightarrow{f} Y$ *can be factored as* $X \overset{f}{\twoheadrightarrow} f(X) \hookrightarrow Y$.

(ii) *Every lifting problem*

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "e"'] & C \arrow[d, hook, "m"] \\ B \arrow[r, "v"'] \arrow[ur, dashed, "\exists d"] & D \end{tikzcd}$$

has a solution d sending $b = e(a)$ to $u(a)$. This is well-defined, because $e(a) = e(a')$ implies

$$mu(a) = ve(a) = ve(a') = mu(a'),$$

and hence $u(a) = u(a')$, as m is injective. By construction $de = u$ and hence $mde = mu = ve$ implies also $md = e$, as e is surjective.

Note that in this case the diagonal d is unique. Similarly one shows that (Epi, Mono) *is also a weak factorization system on $\mathcal{G}rp$ or R-*Mod*, for $R \in \mathcal{R}ing$.*

Remark 3.10
On every category $\mathcal{C}$ there are two trivial weak factorization systems $(\mathrm{Mor}(\mathcal{C})^\times, \mathrm{Mor}(C))$ *and* $(\mathrm{Mor}(\mathcal{C}), \mathrm{Mor}(\mathcal{C})^\times)$*, where* $\mathrm{Mor}(\mathcal{C})^\times$ *is the class of isomorphisms in $\mathcal{C}$:*

(i) *Every map $f \in \mathcal{C}(X, Y)$ can be factored as $f \circ \mathrm{id}_X$ (resp. $\mathrm{id}_Y \circ f$).*

(ii) *Every lifting problem*

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "\ell"'] & C \arrow[d, "r"] \\ B \arrow[r, "v"'] \arrow[ur, dashed, "\exists d"] & D, \end{tikzcd}$$

can be solved by $d := u\ell^{-1}$, when ℓ is an isomorphism (resp. by $d := r^{-1}v$, when r is an isomorphism).

3.3 Model categories

Definition 3.11
Let $\mathcal{C}$ be a finitely complete and finitely cocomplete category, i.e. there are limits and colimits for functors $I \longrightarrow \mathcal{C}$, where $I \in \mathcal{C}at$ has only finitely many objects and morphisms.

*A **model structure** on $\mathcal{C}$ consists of three subclasses $w\mathcal{C}, \operatorname{fib}\mathcal{C}, \operatorname{cof}\mathcal{C} \subset \operatorname{Mor}(\mathcal{C})$, such that:*

(i) $(\mathcal{C}, w\mathcal{C})$ *is a category with weak equivalences.*

(ii) $(\operatorname{cof}\mathcal{C} \cap w\mathcal{C}, \operatorname{fib}\mathcal{C})$ *and* $(\operatorname{cof}\mathcal{C}, w\mathcal{C} \cap \operatorname{fib}\mathcal{C})$ *are weak factorization systems.*

*The tuple $(\mathcal{C}, w\mathcal{C}, \operatorname{fib}\mathcal{C}, \operatorname{cof}\mathcal{C})$ is also called a **model category**.*

We fix the following notation.

- *The morphisms in $\operatorname{fib}\mathcal{C}$ are called **fibrations** and written as "$\longrightarrow\!\!\!\!\triangleright$".*
- *The morphisms in $\operatorname{cof}\mathcal{C}$ are called **cofibrations** and written as "$\rightarrowtail$".*
- *A (co-)fibration is called **trivial**, if it is also a weak equivalence.*

We will show, that every model category has a homotopy category. One begins by formalizing the idea of a cylinder $[0,1] \times X$ and of homotopies $[0,1] \times X \xrightarrow{h} Y$ in the context of a model category. Then the proof is quite technical, but inspired by the constructions in the category of topological spaces.

Constructing model structures is often very hard. Given a right adjoint functor $\mathcal{C} \xrightarrow{G} \mathcal{D}$ mapping into a model category $\mathcal{D}$, it is often possible to construct an induced model structure on $\mathcal{C}$ with $w\mathcal{C} := G^{-1}w\mathcal{D}$ and $\operatorname{fib}\mathcal{C} := G^{-1}\operatorname{fib}\mathcal{D}$. Many important examples can be constructed in this way.

However the remaining problem is to construct at least one archetypical model category to begin with. This is the/a canonical model structure on the category of simplicial sets, for which weak equivalences are maps becoming homotopy equivalences after geometric realization. By the technique using right adjoint functors, we will get model structures for Example 3.1 (iii) and (iv).

We will proceed by first constructing the model structure on the category of simplicial sets. It will take us the next few lectures to achieve this goal.

3.4 Stability of the lifting property

Before coming to the construction of weak factorization systems, we will need some stabilty properties of the lifting property.

Definition 3.12
Let $\mathcal{C}$ be a category.

(i) *A* ***retract*** *of a morphism $f \in \mathcal{C}(A, B)$ is a retract of f in the category $\mathcal{C}\mathrm{AT}(\underline{1}, \mathcal{C})$ of morphisms in $\mathcal{C}$. This means a retract of f is a morphism $f' \in \mathcal{C}(A', B')$ inducing a commutative diagam*

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "f'"'] \arrow[rr, bend left, "\mathrm{id}_{A'}"] & A \arrow[r] \arrow[d, "f"] & A' \arrow[d, "f'"] \\ B' \arrow[r] \arrow[rr, bend right, "\mathrm{id}_{B'}"'] & B \arrow[r] & B'. \end{tikzcd}$$

(ii) *For $C \subset \mathrm{Mor}(\mathcal{C})$ we denote by $\mathrm{retr}(C)$ the class of retracts of morphisms in C.*

(iii) *A set of morphisms $C \subset \mathrm{Mor}(\mathcal{C})$ is* ***closed under retracts****, if $\mathrm{retr}(C) \subset C$.*

Lemma 3.13
Let $M \subset \mathrm{Mor}(\mathcal{C})$ be a class of morphisms in a category $\mathcal{C} \in \mathcal{C}\mathrm{AT}$.
Then the following holds:

(i) *$LLP(M)$ contains every isomorphism of $\mathcal{C}$.*

(ii) *$LLP(M)$ is closed under composition.*

(iii) *Let $Z_0 \xrightarrow{c_0} Z_1 \xrightarrow{c_1} \ldots$ be a tower of morphisms in $LLP(M)$.*
Then also $Z_0 \xrightarrow{\iota_0} \mathrm{colim}_{n\geq 0} Z_n$ lies in $LLP(M)$.

(iv) *$LLP(M)$ is closed under retracts.*

(v) *$LLP(M)$ is closed under pushouts along arbitrary morphisms in $\mathcal{C}$.*

(vi) *$LLP(M)$ is closed under arbitrary coproducts.*

By duality we have similar properties for $RLP(M)$.

Proof.

(i) A lifting problem

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "\ell"', "\wr"] & C \arrow[d, "m\in M"] \\ B \arrow[r, "v"'] \arrow[ur, dashed, "\exists d"] & D \end{tikzcd}$$

is solved by $d := u\ell^{-1}$.

(ii) Let $A, B, C \in \mathcal{C}$ and $A \xrightarrow{f} B \xrightarrow{g} C$ be two morphisms in $LLP(M)$. For any $(X \xrightarrow{m} Y) \in M$ admitting a commutative square

$$\begin{tikzcd} A \arrow[rr] \arrow[d, "f"'] & & X \arrow[dd, "m\in M"] \\ B \arrow[urr, dashed, "\exists d_f"] \arrow[d, "g"'] & & \\ C \arrow[rr] \arrow[uurr, dashed, "\exists d_g"'] & & Y \end{tikzcd}$$

there are morphisms d_f and d_g (depending on d_f) preserving the commutativity of the square, because $f, g \in LLP(M)$. Hence d_g is a diagonal for the outer square and hence $g \circ f \in LLP(M)$.

(iii) Given such a tower $(Z_n)_{n\geq 0}$. For any $(X \xrightarrow{m} Y) \in M$ and any commutative diagram

$$\begin{tikzcd} Z_0 \arrow[r, "u"] \arrow[d, "\iota_0"'] & X \arrow[d, "m\in M"] \\ \operatorname{colim}_{n\geq 0} Z_n \arrow[r, "v"] & Y \end{tikzcd}$$

we inductively construct maps $Z_n \xrightarrow{d_n} X$, for $n \geq 0$. Set $d_0 = u$ and for any $n \geq 0$ we get a diagonal d_{n+1}, because $c_n \in LLP(M)$.

$$\begin{tikzcd} Z_n \arrow[rr, "d_n"] \arrow[d, "c_n"'] & & X \arrow[d, "m"] \\ Z_{n+1} \arrow[r, "\iota_{n+1}"'] \arrow[urr, dashed, "\exists d_{n+1}"] & \operatorname{colim}_{n\geq 0} Z_n \arrow[r, "v"'] & Y \end{tikzcd}$$

Since by construction $d_{n+1}c_n = d_n$, for all $n \geq 0$, the universal property of colimits yields a map $d := \operatorname{colim}_{n\geq 0} d_n$ satisfying the required properties.

(iv) Consider a commutative diagram

$$\begin{tikzcd} A \arrow[r, "s_A"'] \arrow[d, "e"'] \arrow[rr, bend left, "\mathrm{id}_A"] & A' \arrow[r, "r_A"'] \arrow[d, "e'"] & A \arrow[r, "u"] \arrow[d, "e"] & X \arrow[d, "m\in M"] \\ B \arrow[r, "s_B"] \arrow[rr, bend right, "\mathrm{id}_B"'] & B' \arrow[r, "r_B"] & B \arrow[r, "v"] & Y, \end{tikzcd}$$

where $e' \in LLP(M)$. We want to show, that there is a diagonal $d \in \mathcal{C}(B, X)$, making the right square commute. Since $e' \in LLP(M)$, we have a diagonal d', such that

$$\begin{tikzcd} A' \arrow[r, "r_A"] \arrow[d, "e'"'] & A \arrow[r, "u"] & X \arrow[d, "m\in M"] \\ B' \arrow[r, "r_B"'] \arrow[urr, dashed, "\exists d'"] & B \arrow[r, "v"'] & Y \end{tikzcd}$$

commutes. Define $d = d's_B$ and compute

$$de = d's_Be = d'e's_A = ur_As_A = u, \qquad md = md's_B = vr_Bs_B = v,$$

hence d is a diagonal of the desired form.

(v) Let $A, B, C \in \mathcal{C}$ and $A \xrightarrow{f} C$ and $e \in LLP(M)$. Consider a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "e"'] & C \arrow[r, "u"] \arrow[d, "\iota_C"'] & X \arrow[d, "m \in M"] \\
B \arrow[r, "\iota_B"'] & B +_A C \arrow[r, "v"'] & Y.
\end{tikzcd}$$

Since $e \in LLP(M)$ there is a diagonal $B \xrightarrow{d'} X$ for the outer square, i.e. $d' \circ e = u \circ f$ and $r \circ d' = v \circ \iota_C$. Applying the universal property of pushouts to the first identity yields a diagonal for the right square $d := d' \cup u \in \mathcal{C}(B +_X C, X)$:

- $d\iota_C = u$ by construction,
- $md\iota_C = mu = v\iota_C$ and $md\iota_B = md' = v\iota_B$ and hence $md = v$ by the universal property of pushouts again.

(vi) Let $I \in \mathcal{S}et$ and $(A_i \xrightarrow{e_i} B_i) \in LLP(M)$ for any $i \in I$. Let e denote the coproduct of all e_i. Consider a commutative square (on the right)

$$\begin{tikzcd}
A_i \arrow[r, "\iota_i"] \arrow[d, "e_i"'] & \coprod_{i\in I} A_i \arrow[r] \arrow[d, "e"'] & X \arrow[d, "m \in M"] \\
B_i \arrow[r, "\iota_i"] & \coprod_{i\in I} B_i \arrow[r] & Y.
\end{tikzcd}$$

For any $i \in I$ there is a diagonal $B_i \xrightarrow{d_i} X$ for the outer square. Then $d = \coprod_{i\in I} d_i$ we have $d \circ e = u$ is a diagonal for the right square.

□

Corollary 3.14
Let $\mathcal{C} \in \mathcal{C}\mathrm{AT}$ and $L, R \subset \mathrm{Mor}(\mathcal{C})$.

Then (L, R) is a weak factorization system, if and only if the following holds:

(i) $\mathrm{Mor}(\mathcal{C}) = R \circ L$.

(ii) $L \subset LLP(R)$.

(iii) *L and R are closed under retracts.*

Proof. Suppose (L, R) is a weak factorization system. Then (i) and (ii) holds, because $L = LLP(R)$. Furthermore $LLP(R)$ resp. $RLP(L)$ are closed under retractions by (iii) of the preceding Lemma resp. its dual.

Vice versa assume the above hypotheses for (L, R). We have to show, that $L = LLP(R)$ and $R = RLP(L)$. Let $c \in \mathcal{C}(X, Y) \cap \in LLP(R)$ be arbitrary. Then c can be factored as $X \overset{\ell}{\rightarrowtail} Z \overset{r}{\twoheadrightarrow} Y$, where $\ell \in L$ and $r \in R$. Since $c \in LLP(R)$ we find a diagonal

$$\begin{tikzcd}
X \arrow[r, tail, "\ell"] \arrow[d, "c"'] & Z \arrow[d, two heads, "r"] \\
Y \arrow[r, equal] \arrow[ur, dashed, "d"] & Y,
\end{tikzcd}$$

showing that c is a retract of ℓ. Hence $LLP(R) \subset L \subset LLP(R)$ and so $L = LLP(R)$. Dually one gets $R = RLP(L)$.

□

Using the preceding Corollary we may deduce Quillen's original defintion of a closed model category.

Corollary 3.15
Let $\mathcal{C}$ be a category together with three classes of morphisms $w\mathcal{C}, \mathrm{fib}\,\mathcal{C}, \mathrm{cof}\,\mathcal{C} \subset \mathrm{Mor}(\mathcal{C})$.
Then $\mathcal{C}$ is a model category, if and only if the following holds[1].

(CM1) *$\mathcal{C}$ is closed under finite limits and colimits.*

(CM2) *$w\mathcal{C}$ satisfies the 2-of-3 axiom.*

(CM3) *The classes $w\mathcal{C}, \mathrm{fib}\,\mathcal{C}, \mathrm{cof}\,\mathcal{C}$ are closed under retracts.*

(CM4) $\mathrm{cof}\,\mathcal{C} \subset LLP(\mathrm{fib}\,\mathcal{C} \cap w\mathcal{C})$ *and* $\mathrm{cof}\,\mathcal{C} \cap w\mathcal{C} \subset LLP(\mathrm{fib}\,\mathcal{C})$.

(CM5) $\mathrm{Mor}\mathcal{C} = \mathrm{fib}\,\mathcal{C} \circ (\mathrm{cof}\,\mathcal{C} \cap w\mathcal{C}) = (\mathrm{fib}\,\mathcal{C} \cap w\mathcal{C}) \circ \mathrm{cof}\,\mathcal{C}$.

Proof. Axiom **(CM1)** is the general hypothesis in the Definition 3.11 of model categories. **(CM2)** is equivalent to $(\mathcal{C}, w\mathcal{C})$ being a category with weak equivalences. By Corollary 3.14 $(\mathrm{cof}\,\mathcal{C} \cap w\mathcal{C}, \mathrm{fib}\,\mathcal{C})$ and $(\mathrm{cof}\,\mathcal{C}, w\mathcal{C} \cap \mathrm{fib}\,\mathcal{C})$ are weak factorization systems, if and only if **(CM3)** - **(CM5)** hold. Indeed **(CM5)** and **(CM3)** imply that (trivial) (co-)fibrations are closed under retracts. Vice versa if trivial fibrations and cofibrations are closed under retracts, so are weak equivalences, since they can be factored into a trivial cofibration followed by a trivial cofibration.

□

3.5 Construction of weak factorization systems

Part of the construction of a model category is the construction of a weak factorization system. We will present Quillen's small object argument, which is probably the most powerful tool for building factorization systems.

Proposition 3.16
Let $\mathcal{C}$ be a category with finite coproducts.
Then there is a canonical weak factorization system $(\mathrm{retr}(I), R)$ on $\mathcal{C}$, where

(i) *I is the class of inclusions $X \xhookrightarrow{\iota_X} X + Y$, where $X, Y \in \mathcal{C}$,*

(ii) *R is the class of retractions.*

[1] This is what Quillen originally called a closed model category. He skipped **(CM3)** in his definition of a model category.

Proof. Every morphism $X \xrightarrow{f} Y$ can be factored as

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\iota_X"'] & & Y \\ & X+Y, \arrow[ur, "f\cup \mathrm{id}_Y"'] & \end{tikzcd}$$

and $f \cup \mathrm{id}_Y$ is a retraction with a section given by the inclusion ι_Y. Every lifting problem

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, hook, "\iota_X"'] & A \arrow[d, "r"'] \\ X+Y \arrow[r, "(ra)\cup b"'] \arrow[ur, dashed, "d"] & B, \arrow[u, bend right, "s"'] \end{tikzcd}$$

is solved by the morphism $d = a \cup (sb)$, because

$$d\iota_X = a, \qquad rd = (ra) \cup (rsb) = (ra) \cup b.$$

By Lemma 3.13 $LLP(R)$ is closed under retracts and thus also retracts of inclusions have the left lifting property to retractions.

Using Corollary 3.14 it remains to prove, that R is closed under retracts. Suppose we are given a commutative diagram

$$\begin{tikzcd} A' \arrow[r, "s_A"'] \arrow[rr, bend left, "\mathrm{id}_{A'}"] \arrow[d, "r'"'] & A \arrow[r, "r_A"'] \arrow[d, "r"] & A' \arrow[d, "r'"] \\ B' \arrow[r, "s_B"] \arrow[rr, bend right, "\mathrm{id}_{B'}"'] & B \arrow[r, "r_B"] & B', \end{tikzcd} \qquad \begin{tikzcd} A \arrow[d, "r"'] \\ B. \arrow[u, bend right, "s"'] \end{tikzcd}$$

Define $s' = r_A s s_B \in \mathcal{C}(B', A')$. Then we have

$$r's' = r'r_A s s_B = r_B r s s_B = r_B s_B = \mathrm{id}_{B'},$$

showing that r' is a retraction with section s'.

□

Example 3.17
In the category $\mathcal{S}et$ we have $(\mathrm{retr}(I), R) = (\mathrm{Mono}, \mathrm{Epi})$.

Indeed every inclusion is an injection and every injection $A \overset{m}{\hookrightarrow} B$ is isomorphic to the inclusion $A \xrightarrow{\iota_A} A + (B \backslash m(A))$, so $I = \mathrm{Mono}$. Using that $\mathrm{Mono} = RLP(\mathrm{Epi})$ *by Example 3.9, Lemma 3.13 implies that*

$$I = \mathrm{Mono} = \mathrm{retr}(\mathrm{Mono}) = \mathrm{retr}(I).$$

Moreover the axiom of choice is equivalent to $R = \mathrm{Epi}$.

So we have constructed two non-trivial weak factorization systems $(\mathrm{Epi}, \mathrm{Mono})$ *and* $(\mathrm{Mono}, \mathrm{Epi})$ *on $\mathcal{S}et$.*

Definition 3.18
Let $\mathcal{C} \in \mathcal{C}\mathrm{AT}$ and $C \subset \mathrm{Mor}(\mathcal{C})$.

*A C-**cell complex relative to** $X \in \mathcal{C}$ is an object $Z \in \mathcal{C}$ obtained by inductively glueing C-cells to X, i.e.*

(i) *$Z = \operatorname{colim}_{n\geq 0} Z_n$, for some $X = Z_0 \longrightarrow Z_1 \longrightarrow \dots$.*

(ii) *Z_{n+1} is a pushout*

$$\begin{tikzcd} \coprod_i D_i \arrow[r, "\coprod_i f_i"] \arrow[d, "\coprod_i \iota_i c_i"'] & Z_n \arrow[d] \\ \coprod_i E_i \arrow[r] & Z_{n+1}, \end{tikzcd} \qquad \begin{tikzcd} D_i \arrow[r, "f_i"] \arrow[d, "c_i \in C"] & Z_n \\ E_i, & \end{tikzcd}$$

Remark 3.19
The notion generalizes the construction of CW-complexes. More precisely in the category $\mathcal{T}op$, form the set

$$C := \{S^n \hookrightarrow D^n;\ n \geq 0\}.$$

Then a C-cell complex relative to $X \in \mathcal{T}op$ is a space Z obtained by inductively glueing cells to X.

In particular a C-cell complex relative to $\emptyset$ is a usual CW-complex.

Theorem 3.20 (Quillen's small object argument)
Let $\mathcal{C} \in \mathcal{C}\mathrm{AT}$ and $C \subset \mathrm{Mor}(\mathcal{C})$, such that

(i) *$\mathcal{C}$ is cocomplete.*

(ii) *C is a set.*

(iii) *For every $(D \to E) \in C$ its domain D is ω-**compact**[2] (also called **small**), i.e. $\mathcal{C}(D, \text{-})$ commutes with colimits over $(\mathbb{N}_0, \leq)$.*

Then every morphism $X \xrightarrow{g} Y$ has a factorization

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "c"'] & & Y \\ & Z, \arrow[ur, "f"'] & \end{tikzcd}$$

where $f \in RLP(C)$ and c is a C-cell complex relative to X.

Proof. Given a morphism $X \xrightarrow{g} Y$ in $\mathcal{C}$. For any $n \in \mathbb{N}_0$ we inductively construct a sequence

$$X = Z_0 \xrightarrow{c_0} Z_1 \xrightarrow{c_1} Z_2 \xrightarrow{c_2} \dots$$

[2]The term ω-compact refers to the smallest infinite ordinal ω. One can extend the notion of compactness to higher ordinals κ to allow more objects appearing as a domain of maps in C.

and morphisms $Z_n \xrightarrow{f_n} Y$ for any $n \geq 0$ commuting with the maps of this tower. This is done by attaching "cells" coming from maps in C to X. We will precise this idea in the following.

Let $Z_0 = X$ and $f_0 = g$. For a given Z_n let C/f_n denote the set of all commutative squares

$$\begin{tikzcd} D \arrow[r] \arrow[d, "C \ni c"'] & Z_n \arrow[d, "f_n"] \\ E \arrow[r] & Y. \end{tikzcd}$$

This is a set, because C is a set and $\mathcal{C}(D, Z_n)$ and $\mathcal{C}(E, Y)$ are sets, for all $(D \xrightarrow{c} E) \in C$. Define Z_{n+1} as the pushout

$$\begin{tikzcd} \coprod_{C/f_n} D \arrow[r] \arrow[d, "\coprod_{C/f_n} \iota_c c"'] & Z_n \arrow[d, "=:c_n"] \arrow[ddr, bend left, "f_n"] & \\ \coprod_{C/f_n} E \arrow[r] \arrow[drr, bend right] & Z_{n+1} \arrow[dr, dashed, "\exists! f_{n+1}"] & \\ & & Y. \end{tikzcd}$$

Using the universal property of colimits, we get a factorization

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "c := \iota_0"'] & & Y \\ & Z := \operatorname*{colim}_{n \geq 0} Z_n. \arrow[ur, "\operatorname{colim}_{n \geq 0} f_n =: f"'] & \end{tikzcd}$$

To show, that $f \in RLP(C)$ consider a commutative diagram

$$\begin{tikzcd} D \arrow[r, "u"] \arrow[d, "C \ni c"'] & Z \arrow[d, "f"] \\ E \arrow[r, "v"] & Y. \end{tikzcd}$$

Because D is ω-compact we have a natural bijection

$$\operatorname*{colim}_{n \geq 0} \mathcal{C}(D, Z_n) \xrightarrow{\sim} \mathcal{C}(D, \operatorname*{colim}_{n \geq 0} Z_n) = \mathcal{C}(D, Z).$$

This means the map $u \in \mathcal{C}(D, Z)$ comes from a map $u_n \in \mathcal{C}(D, Z_n)$ with $n \geq 0$, i.e. $u = \iota_n \circ u_n$. Hence $f_n \circ u_n = (f \circ \iota_n) \circ u_n = f \circ (\iota_n \circ u_n) = f \circ u = v \circ c$, meaning that $c \in C/f_n$ is in the indexing set for the construction of the pushout Z_{n+1}. So considering

$v' : E \overset{\iota_c}{\hookrightarrow} \coprod_{C/f_n} E \longrightarrow Z_{n+1}$ we obtain a commutative diagram

$$\begin{tikzcd}
D \arrow[r, "u_n"'] \arrow[rr, bend left, "u"] \arrow[dd, "c"'] & Z_n \arrow[r, "\iota_n"'] \arrow[d, "c_n"'] & Z \arrow[dd, "f"] \\
 & Z_{n+1} \arrow[ur, "\iota_{n+1}"'] \arrow[dr, "f_{n+1}"] & \\
E \arrow[ur, "v'"] \arrow[rr, "v"'] & & Y.
\end{tikzcd}$$

In particular the diagonal $d := \iota_{n+1} v'$ solves the lifting problem. □

Corollary 3.21
Let $\mathcal{C} \in \mathcal{C}\mathrm{AT}$ and $C \subset \mathrm{Mor}(\mathcal{C})$ a set of morphisms admitting the small object argument. Then

(i) $(LLP(RLP(C)), RLP(C))$ *is a weak factorization system in* $\mathcal{C}$.

(ii) $LLP(RLP(C))$ *is precisely the class of retracts of relative* C *cell complexes in* $\mathcal{C}$.

Proof.

(i) Clearly $C \subset LLP(RLP(C))$. By Lemma 3.13 $LLP(RLP(C))$ contains every identity morphism and is closed under coproducts, pushouts and directed limits. This implies $LLP(RLP(C))$ contains every relative C-cell complex. Hence the small object argument shows, that $\mathrm{Mor}(C) = RLP(C) \circ LLP(RLP(C))$. By definition every $\ell \in LLP(RLP(C))$ has the LLP for any map in $RLP(C)$. But this is also equivalent to the fact, that every map in $RLP(C)$ has the RLP for any map $LLP(RLP(C))$.

(ii) Since $C \subset LLP(RLP(C))$ and $LLP(RLP(C))$ by Lemma 3.13 is closed under pushouts, filtered colimits and retractions, we have, that every retraction of a relative C-cell complex lies in $LLP(RLP(C))$.

The other way round let $(X \xrightarrow{f} Y) \in LLP(RLP(C))$. By the small object argument we find a factorization

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "c"'] & & Y \\
 & Y', \arrow[ur, "r"'] &
\end{tikzcd}$$

where c is a relative C-cell complex and $r \in RLP(C)$. Since $c \in C \subset LLP(RLP(C))$ we find a diagonal in the diagram

$$\begin{tikzcd}
X \arrow[r, "c"] \arrow[d, "f"'] & Y' \arrow[d, "r"] \\
Y \arrow[r, equal] \arrow[ur, dashed, "\exists d"] & Y,
\end{tikzcd}$$

showing that f is a retraction of the cell complex c.

□

Proposition 3.22 (i) *Every finite set F is ω-compact in $\mathcal{S}et$.*

(ii) *Every finite simplicial set $F \in s\mathcal{S}et$ is ω-compact in $s\mathcal{S}et$.*

Proof. Note that the category $0 \longrightarrow 1 \longrightarrow ...$ is filtered.

(i) Given a sequence of sets

$$S_0 \longrightarrow S_1 \longrightarrow ...$$

that we consider a sequence of constant functors $D(F) \longrightarrow \mathcal{S}et$, by Proposition 2.48 we get the natural isomorphism

$$\begin{array}{ccc} \operatorname*{colim}_{n\geq 0} \prod_{x\in F} S_n & \xrightarrow{\sim} & \prod_{x\in F} \operatorname*{colim}_{n\geq 0} S_n \\ \| & & \| \\ \operatorname*{colim}_{n\geq 0} \mathcal{S}et(F, S_n) & \xrightarrow{\sim} & \mathcal{S}et(F, \operatorname*{colim}_{n\geq 0} S_n). \end{array}$$

(ii) Using Proposition 2.76 finiteness of F implies that there is an $n \geq 0$, such that $\varepsilon_F : \mathrm{sk}_n F \xrightarrow{\sim} F$. Moreover all the sets $F_0, ..., F_n$ are finite. Then there is a chain of isomorphisms

$$\begin{aligned} s\mathcal{S}et(F, Y) &\xrightarrow{\sim} s\mathcal{S}et(\mathrm{sk}_n F, Y) = s\mathcal{S}et(\operatorname*{colim}_{\underline{m}\in(i_n)/(i_n)^*F} \Delta^m, Y) \\ &\xrightarrow{\sim} \lim_{\underline{m}\in(i_n)/(i_n)^*F} s\mathcal{S}et(\Delta^m, Y) \xrightarrow{\sim} \lim_{\underline{m}\in(i_n)/(i_n)^*F} Y_m, \end{aligned}$$

where the first isomorphism is induced by the isomorphism ε_F, the second holds by Definition 2.74 of the skeleton, the third map is an isomorphism as hom-functors preserve limits by Proposition 2.46 and the fourth one is Yoneda's isomorphism. So the object on the right is the limit of the functor

$$\underline{m} \in (i_n)/(i_n)^*F \longrightarrow \Delta^{\mathrm{op}} \xrightarrow{Y} \mathcal{S}et$$

and the assertion again follows from Proposition 2.48, because the category $(i_n)/(i_n)^*F$ is finite.

□

Remark 3.23
However a compact topological space does not need to be ω-compact in $\mathcal{T}op$.

3.6 Lifting properties and adjunctions

Remark 3.24
Given an adjunction between two categories $\mathcal{C}$ and $\mathcal{D}$

$$\mathcal{C}(F(X), Y) = \mathcal{D}(X, G(Y)).$$

Let $\ell \in \mathcal{D}(D, D')$ and $r \in \mathcal{C}(C, C')$.
Then the following lifting propblems are equivalent

$$\begin{tikzcd} F(D) \arrow[r, "a"] \arrow[d, "F(\ell)"'] & C \arrow[d, "r"] \\ F(D') \arrow[r, "b"'] \arrow[ur, dashed, "d"] & C', \end{tikzcd} \qquad \begin{tikzcd} D \arrow[r, "a'"] \arrow[d, "\ell"'] & G(C) \arrow[d, "G(r)"] \\ D' \arrow[r, "b'"'] \arrow[ur, dashed, "d'"] & G(C'), \end{tikzcd}$$

where a', b' and d' corresponds to a, b and d under the adjunction bijection.
In particular the two axioms for structured adjunctions are equivalent.

Proposition 3.25
Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be categories with functors

$$\mathcal{B}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{F} \mathcal{A}, \qquad \mathcal{A} \times \mathcal{B} \xrightarrow{\otimes} \mathcal{C}, \qquad \mathcal{A}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{G} \mathcal{B}.$$

and natural bijections

$$\mathcal{A}(A, F(B, C)) = \mathcal{C}(A \otimes B, C) = \mathcal{B}(B, G(A, C)), \qquad A \in \mathcal{A},\ B \in \mathcal{B},\ C \in \mathcal{C}.$$

Let $a \in \mathcal{A}(A, A'), b \in \mathcal{B}(B, B')$ and $c \in \mathcal{C}(C, C')$.
Then the following three lifting problems are equivalent.

$$\begin{tikzcd} A \arrow[r, "u_1'"] \arrow[d, "a"'] & F(B', C) \arrow[d, "(b^*, c_*)"] \\ A' \arrow[r, "(u_2', u_3')"'] \arrow[ur, dashed, "\exists d'"] & F(B, C) \times_{F(B, C')} F(B', C'), \end{tikzcd} \qquad \begin{tikzcd} (A \otimes B') +_{(A \otimes B)} (A' \otimes B) \arrow[r, "u_1 \cup u_2"] \arrow[d, "(a \otimes \mathrm{id}) \cup (\mathrm{id} \otimes b)"'] & C \arrow[d, "c"] \\ A' \otimes B' \arrow[r, "u_3"'] \arrow[ur, dashed, "\exists d"] & C'. \end{tikzcd}$$

$$\begin{tikzcd} B \arrow[r, "u_2''"] \arrow[d, "b"'] & G(A', C) \arrow[d, "(a^*, c_*)"] \\ B' \arrow[r, "(u_1'', u_3'')"'] \arrow[ur, dashed, "\exists d''"] & G(A, C) \times_{G(A, C')} G(A', C'). \end{tikzcd}$$

where

- u_1', u_2', u_3' *correspond to* u_1, u_2, u_3 *under the first bijection and*
- u_1'', u_2'', u_3'' *correspond to* u_1, u_2, u_3 *under the second bijection.*

In other words, we have a chain of equivalences

$$
\begin{aligned}
& & a \in LLP(b^*, c_*) &\iff (b^*, c_*) \in RLP(a) \\
&\iff & (a \otimes \mathrm{id}) \cup (\mathrm{id} \otimes b) \in LLP(c) &\iff c \in RLP((a \otimes \mathrm{id}) \cup (\mathrm{id} \otimes b)) \\
& & \iff (a^*, c_*) \in RLP(b) &\iff b \in LLP(a^*, c_*)
\end{aligned}
$$

Proof. This holds, because

- a diagonal d' corresponds to a diagonal d under the first bijection and
- a diagonal d'' corresponds to a diagonal d under the second bijection.

Equivalently every solution of one of the lifting problems induces a solution for the other two problems.

□

Corollary 3.26
For $\mathcal{C} \in \mathcal{C}\mathrm{AT}$ with arbitrary coproducts and products, there are natural bijections

$$\mathcal{S}et(S, \mathcal{C}(X, Y)) = \mathcal{C}({}^S X, Y) = \mathcal{C}(X, Y^S), \qquad S \in \mathcal{S}et,\ X, Y \in \mathcal{C},$$

where ${}^S X := \coprod_{s \in S} X$ and $X^S := \prod_{s \in S} X$.
Suppose moreover that $\mathcal{C}$ carries a weak factorization system (L, R).
Then for $c \in \mathcal{C}(A, B) \cap L, m \in \mathcal{S}et(S, T) \cap \mathrm{Mono}$ and $f \in \mathcal{C}(C, D) \cap R$ we have:

(i) $\mathcal{C}(B, C) \overset{(c^*, f_*)}{\twoheadrightarrow} \mathcal{C}(A, C) \times_{\mathcal{C}(A,D)} \mathcal{C}(B, D)$,

(ii) $({}^S B +_{{}^S A} {}^T A \xrightarrow{{}^m\mathrm{id} \cup {}^{\mathrm{id}} c} {}^T B) \in L$,

(iii) $(C^T \xrightarrow{(m^*, f_*)} C^S \times_{D^S} D^T) \in R$.

Proof. The natural bijections follow from the universal property of (co-)products.

(i) For all such c and f surjectivity of the map (c^*, f_*) is equivalent to the existence of a diagonal d in every commutative square

$$
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "c"'] & C \arrow[d, "f"] \\
B \arrow[r, "v"'] \arrow[ur, dashed, "\exists d"] & D.
\end{tikzcd}
$$

Indeed each such commutative square bijectively corresponds to an element

$$(u, v) \in \mathcal{C}(A, C) \times_{\mathcal{C}(A,D)} \mathcal{C}(B, D)$$

by the explicit construction of limits in the category $\mathcal{S}et$. Hence (u, v) has a preimage

$$(c^*, f_*) : d \longmapsto (dc, fd),$$

if and only if there is a solution for the lifting problem.

(ii) Using that (Mono, Epi) is a weak factorization system on $\mathcal{S}et$ and (L, R) one on $\mathcal{C}$, Proposition 3.25 applied to the given natural bijection yields

$$\begin{aligned}
& (c^*, f_*) \in \mathrm{Epi} = RLP(\mathrm{Mono}), \quad \text{for all } c \in L \text{ and } f \in R, \\
\iff & (c^*, f_*) \in RLP(m), \quad \text{for all } c \in L, f \in R \text{ and } m \in \mathrm{Mono}(\mathcal{S}et), \\
\iff & ({}^m\mathrm{id} \cup {}^{\mathrm{id}}c) \in LLP(f), \quad \text{for all } c \in L, f \in R \text{ and } m \in \mathrm{Mono}(\mathcal{S}et), \\
\iff & ({}^m\mathrm{id} \cup {}^{\mathrm{id}}c) \in LLP(R) = L, \quad \text{for all } c \in L, f \in R \text{ and } m \in \mathrm{Mono}(\mathcal{S}et).
\end{aligned}$$

This shows that (ii) is equivalent to (i).

(iii) Similarly one checks that (iii) is equivalent to (i).

□

3.7 Towards the standard model structure on simplicial sets

Definition 3.27
Let $f \in s\mathcal{S}et(X, Y)$.

(i) *f is called a* ***cofibration****, if it is a monomorphism.*

(ii) *f is called a* ***fibration****, if it has the LLP with respect to all maps*

$$d^i \sqcup c := (d^i \times \mathrm{id}) \cup (\mathrm{id} \times c) : \Delta^0 \times B +_{\Delta^0 \times A} \Delta^1 \times A \longrightarrow \Delta^1 \times B, \qquad A \overset{c}{\rightarrowtail} B, \quad i = 0, 1.$$

(iii) *f is called a* ***weak equivalence****, if it induces bijections*

$$f^* : \pi_0 \underline{s\mathcal{S}et}(Y, T) \xrightarrow{\sim} \pi_0 \underline{s\mathcal{S}et}(X, T), \qquad T \longrightarrow *.$$

(iv) *An* ***anodyne extension*** *is an element in $LLP(\mathrm{fib}\, s\mathcal{S}et)$.*

Remark 3.28
The definition of weak equivalences might seem a bit awkward. However:

(i) *Every simplicial homotopy equivalence is a weak equivalence.*

(ii) *We will later see that $f \in s\mathcal{S}et(X, Y)$ is a weak equivalence, if and only if its geometric realization $|f|$ is a homotopy equivalence.*

Theorem 3.29
There are two weak factorization systems on the category $s\mathcal{S}et$

$$(\mathrm{cof}\, s\mathcal{S}et, RLP(\mathrm{cof}\, s\mathcal{S}et)), \qquad (LLP(\mathrm{fib}\, s\mathcal{S}et), \mathrm{fib}\, s\mathcal{S}et).$$

Moreover the following holds.

(i) $\operatorname{cof} s\mathcal{S}et = LLP(RLP(\{\partial\Delta^n = \mathrm{sk}_{n-1}\Delta^n \xhookrightarrow{\varepsilon} \Delta^n;\ n \geq 0\}))$,

(ii) $\operatorname{fib} s\mathcal{S}et = RLP(\{\Delta^0 \times \Delta^n +_{\Delta^0\times\partial\Delta^n} \Delta^1 \times \partial\Delta^n \xrightarrow{d^i\sqcup\varepsilon} \Delta^1 \times \Delta^n;\ n \geq 0,\ i = 0,1\})$,

Proof. The domains of the maps

$$\partial\Delta^n \xhookrightarrow{\varepsilon} \Delta^n, \qquad \Delta^0 \times \Delta^n +_{\Delta^0\times\partial\Delta^n} \Delta^1 \times \partial\Delta^n \xrightarrow{d^i\sqcup\iota} \Delta^1 \times \Delta^n, \qquad n \geq 0,\ i = 0,1$$

are finite (i.e. have only finitely many nondegenerate simplices) and hence ω-compact by Proposition 3.22. So by Corollary 3.21 we get two induced weak factorization systems. As one of the structure class of morphisms in a weak factorization system determines the other one, the equalities (i) and (ii) imply that these are the weak factorization systems we are looking for.

(i) Using that $(\mathrm{Mono}, \mathcal{S}et)$ is a weak factorization system on $\mathcal{S}et$ by Example 3.17 and that colimits in $s\mathcal{S}et$ are constructed dimensionwise by Proposition 2.43, it follows that $\operatorname{cof} s\mathcal{S}et = \mathrm{Mono} s\mathcal{S}et$ is closed under pushouts, sequential colimits and retracts by Lemma 3.13. In particular $\operatorname{cof} s\mathcal{S}et$ also contains all retracts of relative $\operatorname{cof} s\mathcal{S}et$-cell complexes. Now the map $\partial\Delta^n \hookrightarrow \Delta^n$ is injective and thus a cofibration, for all $n \geq 0$. This shows the inclusion

$$L := LLP(RLP(\{\partial\Delta^n \xhookrightarrow{\varepsilon} \Delta^n;\ n \geq 0\})) \subset \operatorname{cof} s\mathcal{S}et,$$

because L consists of retracts of relative cell complexes by Corollary 3.21.

For the other inclusion let $c \in s\mathcal{S}et(A, B) \cap \operatorname{cof} s\mathcal{S}et$. Note first that c_n restricts to an injection $\tilde{A}_n \xhookrightarrow{\tilde{c}_n} \tilde{B}_n$. Indeed supposing $c_n(a) \notin \tilde{B}_n$, we have $c_n(a) = s_i(b)$, for some $b \in B_{n-1}$ and $0 \leq i \leq n-1$. Then

$$c_n(s_i d_i a) = s_i d_i c_n(a) = s_i d_i s_i(b) = s_i(b) = c_n(a)$$

and injectivity of c_n implies $a = s_i d_i a \notin \tilde{A}_n$.

Next we claim that there is a cocartesian square

$$\begin{tikzcd}
{}^{\tilde{A}_n}\Delta^n +_{{}^{\tilde{A}_n}\partial\Delta^n} {}^{\tilde{B}_n}\partial\Delta^n \arrow[r] \arrow[d, "({}^{c}\mathrm{id})\cup({}^{\mathrm{id}}\varepsilon)"'] & \mathrm{sk}_{n-1}B +_{\mathrm{sk}_{n-1}A} A \arrow[d] \\
{}^{\tilde{B}_n}\Delta^n \arrow[r] & \mathrm{sk}_n B +_{\mathrm{sk}_n A} A,
\end{tikzcd}$$

for all $n \geq -1$, where for the case $n = -1$ we define $\mathrm{sk}_{-1}X := \emptyset$ for any simplicial set $X \in s\mathcal{S}et$.

To prove the claim we consider the commutative diagram

$$\begin{tikzcd}
\mathrm{sk}_{n-1}B & \mathrm{sk}_{n-1}A \arrow[l] \arrow[r] & A \\
{}^{\tilde{B}_n}\partial\Delta^n \arrow[u] \arrow[d] & {}^{\tilde{A}_n}\partial\Delta^n \arrow[l] \arrow[r] \arrow[u] \arrow[d] & {}^{\tilde{A}_n}\Delta^n \arrow[u] \arrow[d, equal] \\
{}^{\tilde{B}_n}\Delta^n & {}^{\tilde{A}_n}\Delta^n \arrow[l] \arrow[r, equal] & {}^{\tilde{A}_n}\Delta^n,
\end{tikzcd}$$

where the vertical maps are those of Proposition 2.76, the left horizontal maps are induced by $\tilde{c}_n$ and the right horizontal maps are the canonical inclusions.

- By Proposition 2.76, taking colimits vertically yields a diagram

$$\mathrm{sk}_n B \longleftarrow \mathrm{sk}_n A \longrightarrow A,$$

of which again we may take the colimit of.

- Similarly taking colimits horizontally yields a diagram

$$^{\tilde{B}_n}\Delta^n \longleftarrow {}^{\tilde{A}_n}\Delta^n +_{\tilde{A}_n\partial\Delta^n} {}^{\tilde{B}_n}\partial\Delta^n \overset{({}^c\mathrm{id})\cup({}^{\mathrm{id}}\varepsilon)}{\longrightarrow} \mathrm{sk}_{n-1}B +_{\mathrm{sk}_{n-1}A} A,$$

of which we may take the colimit of.

Now if I is the category $2 \leftarrow 1 \rightarrow 0$, then the big diagram above can be considered as a functor in

$$\mathcal{C}\mathrm{AT}(I \times I, s\mathcal{S}et) = \mathcal{C}\mathrm{AT}(I, \mathcal{C}\mathrm{AT}(I, s\mathcal{S}et)).$$

By the dual version of Remark 2.42 (ii) taking the colimit is a left adjoint and so preserves colimits by Corollary 2.47. So the construction of colimits in functor categories given in Proposition 2.43 tells us, that we end up with the same object, no matter if we take the colimit vertically or horizontally first. This means

$$\mathrm{colim}\left({}^{\tilde{B}_n}\Delta^n \longleftarrow {}^{\tilde{A}_n}\Delta^n +_{\tilde{A}_n\partial\Delta^n} {}^{\tilde{B}_n}\partial\Delta^n \overset{({}^c\mathrm{id})\cup({}^{\mathrm{id}}\varepsilon)}{\longrightarrow} \mathrm{sk}_{n-1}B +_{\mathrm{sk}_{n-1}A} A \right) \cong \mathrm{sk}_n B +_{\mathrm{sk}_n A} A,$$

which is an equivalent formulation of the claim.

Now as c_n is injective and L is part of a weak factorization system, Corollary 3.26 implies that the left vertical map in the cocartesian square is in L. Hence also the right vertical map is in L, which by Lemma 3.13 (v) is closed under pushouts. Next using that pushouts commute with colimits by Remark 2.42 (ii) again, we see that

$$\mathrm{sk}_{-1}B +_{\mathrm{sk}_{-1}A} A \overset{\iota_0}{\longrightarrow} \operatorname*{colim}_{n\geq 0}(\mathrm{sk}_n B +_{\mathrm{sk}_n A} A) \overset{\sim}{\longrightarrow} (\operatorname*{colim}_{n\geq 0} \mathrm{sk}_n B) +_{(\operatorname*{colim}_{n\geq 0} \mathrm{sk}_n A)} A = B +_A A,$$

lies in L, because L is closed under sequential colimits by Lemma 3.13 (iii). But this map is isomorphic to c, which completes the proof that $\mathrm{cof}\, s\mathcal{S}et \subset L$.

(ii) Let $f \in s\mathcal{S}et(C, D)$. Then applying Proposition 3.25 to the natural bijection induced by the internal homomorphisms given in Proposition 2.54, statement (i) implies the equivalence

$$\begin{aligned}
& f \in \mathrm{fib}\, s\mathcal{S}et \\
\iff\; & f \in RLP(\Delta^0 \times B +_{\Delta^0 \times A} \Delta^1 \times A \overset{(d^i\times\mathrm{id})\cup(\mathrm{id}\times c)}{\longrightarrow} \Delta^1 \times B), \qquad \forall A \overset{c}{\rightarrowtail} B,\ i = 0,1 \\
\iff\; & \left(\underline{s\mathcal{S}et}(\Delta^1, C) \overset{((d^i)^*, f_*)}{\longrightarrow} \underline{s\mathcal{S}et}(\Delta^0, C) \times_{\underline{s\mathcal{S}et}(\Delta^0, D)} \underline{s\mathcal{S}et}(\Delta^1, D) \right) \in RLP(\mathrm{cof}\, s\mathcal{S}et) \\
\iff\; & ((d^i)^*, f_*) \in RLP(\{\partial\Delta^n \hookrightarrow \Delta^n;\ n \geq 0\}), \qquad \forall i = 0,1 \\
\iff\; & f \in RLP(\{\Delta^0 \times \Delta^n +_{\Delta^0 \times \partial\Delta^n} \Delta^1 \times \partial\Delta^n \longrightarrow \Delta^1 \times \Delta^n;\ n \geq 0,\ i = 0,1\}).
\end{aligned}$$

□

Theorem 3.30
For $c \in s\mathcal{S}et(A,B)$ and $c' \in s\mathcal{S}et(A',B')$, consider the map

$$c \sqcup c' := (c \times \mathrm{id}) \cup (\mathrm{id} \cup c') : A \times B' +_{A \times A'} B \times A' \longrightarrow B \times B'.$$

(i) *If $c, c' \in \operatorname{cof} s\mathcal{S}et$, then $c \sqcup c' \in \operatorname{cof} s\mathcal{S}et$.*

(ii) *If moreover $c \in LLP(\operatorname{fib} s\mathcal{S}et)$ or $c' \in LLP(\operatorname{fib} s\mathcal{S}et)$, then also $c \sqcup c' \in LLP(\operatorname{fib} s\mathcal{S}et)$.*

Proof.

(i) As colimits and hence pushouts are computed dimensionwise in $s\mathcal{S}et$, it suffices to check that $c, c' \in \mathrm{Mono}(\mathcal{S}et)$ implies $c'' \in \mathrm{Mono}(\mathcal{S}et)$. This is Corollary 3.26 applied to the weak factorization system $(\mathrm{Mono}, \mathrm{Epi})$ on $\mathcal{S}et$.

(ii) First note that by applying Proposition 3.25 to the natural bijections

$$s\mathcal{S}et(A, \underline{s\mathcal{S}et}(B,C)) = s\mathcal{S}et(A \times B, C) = s\mathcal{S}et(B, \underline{s\mathcal{S}et}(A,C)), \tag{3.1}$$

constructed in Proposition 2.54 and using the Definition 3.27 of a fibration, we get equivalences

$$\begin{aligned}
& c \sqcup c' \in LLP(\operatorname{fib} s\mathcal{S}et) && \forall\, c \in \operatorname{cof} s\mathcal{S}et,\ c' \in LLP(\operatorname{fib} s\mathcal{S}et) \\
\iff\ & c \sqcup c' \in LLP(f) && \forall\, c \in \operatorname{cof} s\mathcal{S}et,\ c' \in LLP(\operatorname{fib} s\mathcal{S}et),\ f \in \operatorname{fib} s\mathcal{S}et \\
\iff\ & (c^*, f_*) \in RLP(c') && \forall\, c \in \operatorname{cof} s\mathcal{S}et,\ c' \in LLP(\operatorname{fib} s\mathcal{S}et),\ f \in \operatorname{fib} s\mathcal{S}et \\
\iff\ & (c^*, f_*) \in \operatorname{fib} s\mathcal{S}et && \forall\, c \in \operatorname{cof} s\mathcal{S}et,\ f \in \operatorname{fib} s\mathcal{S}et \\
\iff\ & (c^*, f_*) \in RLP(d^i \sqcup c'') && \forall\, c, c'' \in \operatorname{cof} s\mathcal{S}et,\ f \in \operatorname{fib} s\mathcal{S}et,\ i = 0,1 \\
\iff\ & f \in RLP((d^i \sqcup c'') \sqcup c) && \forall\, c, c'' \in \operatorname{cof} s\mathcal{S}et,\ f \in \operatorname{fib} s\mathcal{S}et,\ i = 0,1.
\end{aligned}$$

Once we have shown that $(d^i \sqcup c'') \sqcup c \cong d^i \sqcup (c'' \sqcup c)$, the last statement is equivalent to

$$f \in RLP(d^i \sqcup (c'' \sqcup c)) \qquad \forall\, c, c'' \in \operatorname{cof} s\mathcal{S}et,\ f \in \operatorname{fib} s\mathcal{S}et,\ i = 0,1,$$

which is true by Definition 3.27 of a fibration and since $c'' \sqcup c \in \operatorname{cof} s\mathcal{S}et$ by (i).

So it remains to prove that $(d^i \sqcup c'') \sqcup c \cong d^i \sqcup (c'' \sqcup c)$. We consider the commutative diagram

$$\begin{tikzcd}
\Delta^0 \times B'' \times B & \Delta^0 \times B'' \times A \arrow[l] \arrow[r, equal] & \Delta^0 \times B'' \times A \\
\Delta^0 \times A'' \times B \arrow[u] \arrow[d] & \Delta^0 \times A'' \times A \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \Delta^0 \times B'' \times A \arrow[u, equal] \arrow[d] \\
\Delta^1 \times A'' \times B & \Delta^1 \times A'' \times A \arrow[l] \arrow[r] & \Delta^1 \times B'' \times A,
\end{tikzcd}$$

- Taking colimits vertically and using that (- $\times X$) by Proposition 2.54 as a left adjoint preserves colimits we obtain the diagram

  $$(\Delta^0 \times B'' +_{\Delta^0\times A''} \Delta^1 \times A'')\times B \longleftarrow (\Delta^0 \times A'' +_{\Delta^0\times A''} \Delta^1 \times A'')\times A \longrightarrow (\Delta^1 \times B'')\times A,$$

  whose colimit is the domain of $(d^i \sqcup c'') \sqcup c$.
- Similarly taking colimits horizontally yields the diagram

  $$\Delta^0 \times (B'' \times B) \leftarrow \Delta^0 \times (A'' \times B +_{A''\times A} B'' \times A) \longrightarrow \Delta^1 \times (A'' \times B +_{A''\times A} B'' \times A),$$

  whose colimit is the domain of $d^i \sqcup (c'' \sqcup c)$.

As in the proof of Theorem 3.29, we see that both colimits coincide and thus $(d^i \sqcup c'') \sqcup c \cong d^i \sqcup (c'' \sqcup c)$ as desired.

□

Corollary 3.31
For $c \in s\mathcal{S}et(A, B)$ and $f \in s\mathcal{S}et(C, D)$, consider the map

$$(c^*, f_*) : \underline{s\mathcal{S}et}(B, C) \longrightarrow \underline{s\mathcal{S}et}(A, C) \times_{\underline{s\mathcal{S}et}(A,D)} \underline{s\mathcal{S}et}(B, D).$$

Then the following holds.

(i) *If $c \in \operatorname{cof} s\mathcal{S}et$ and $f \in \operatorname{fib} s\mathcal{S}et$, then $(c^*, f_*) \in \operatorname{fib} s\mathcal{S}et$.*

(ii) *If moreover $c \in LLP(\operatorname{fib} s\mathcal{S}et)$ or $f \in RLP(\operatorname{cof} s\mathcal{S}et)$, then $(c^*, f_*) \in RLP(\operatorname{cof} s\mathcal{S}et)$.*

Proof. By Proposition 3.25 (i) applied to (3.1) assertion (i) is equivalent to Theorem 3.30 (ii). The first case of (ii) is equivalent to Theorem 3.30 (ii) and the second one to Theorem 3.30 (i).

□

Corollary 3.32

$$\operatorname{fib} s\mathcal{S}et \supset RLP(\operatorname{cof} s\mathcal{S}et), \qquad LLP(\operatorname{fib} s\mathcal{S}et) \subset \operatorname{cof} s\mathcal{S}et.$$

Proof. As $\Delta^0 \overset{d^i}{\rightarrowtail} \Delta^1$ is injective and hence a cofibration by Definition 3.27, for every cofibration $C \overset{c}{\rightarrowtail} D$ also the map $d^i \sqcup c$ is a cofibration by Theorem 3.30. In other words $d^i \sqcup c \in \operatorname{cof} s\mathcal{S}et = LLP(RLP(\operatorname{cof} s\mathcal{S}et))$ by Theorem 3.29 (i) or equivalently $\operatorname{fib} s\mathcal{S}et \supset RLP(\operatorname{cof} s\mathcal{S}et)$ or $LLP(\operatorname{fib} s\mathcal{S}et) \subset \operatorname{cof} s\mathcal{S}et$.

□

Proposition 3.33
For every $g \in s\mathcal{S}et(X, Y)$ the following holds.

(i) *g strong deformation retraction and fibration $\iff$ $g \in RLP(\operatorname{cof} s\mathcal{S}et)$.*

(ii) *g strong deformation section* $\quad\Rightarrow\quad$ $g \in LLP(\mathrm{fib}\, s\mathcal{S}et)$.

The other implication holds, if moreover $X, Y \longrightarrow\!\!\triangleright *$.

Proof.

(i) Suppose $X \xrightarrow{g} Y$ is a fibration and a strong deformation retraction, i.e. there is an $s \in s\mathcal{S}et(Y, X)$ with $gs = \mathrm{id}_Y$ and $\mathrm{id}_X \underset{h}{\simeq} sg$ with $gh = g\pi_X$. To every lifting problem on the left, we get a commutative square on the right

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, tail, "c"'] & X \arrow[d, "g"] \\ B \arrow[ur, dashed, "\exists d"] \arrow[r, "v"'] & Y, \end{tikzcd} \qquad \begin{tikzcd} \Delta^0 \times B +_{\Delta^0 \times A} \Delta^1 \times A \arrow[r, "(sv)\cup(h(\mathrm{id}\times u))"] \arrow[d, "d^0 \sqcup c"'] & X \arrow[d, "g"] \\ \Delta^1 \times B \arrow[ur, dashed, "\exists h'"] \arrow[r, "v\pi_B"'] & Y. \end{tikzcd}$$

Indeed the upper horizontal map is well-defined, because

$$gh(\mathrm{id} \times u)d^0 = g\pi_X(\mathrm{id}, u) = gu = sv.$$

Moreover the right square commutes, because

$$g(sv) = v, \qquad gh(\mathrm{id} \times u) = g\pi_X(\mathrm{id} \times u) = gu\pi_A = vc\pi_A = v\pi_Y(\mathrm{id} \times c).$$

As c is a cofibration, by Definition 3.27 of a fibration, there is a diagonal h' and $d := h'd^1$ solves the original lifting problem:

$$\begin{aligned} gd &= gh'd^1 = v\pi_Y d^1 = v, \\ dc &= h'd^1c = h'(\mathrm{id} \times c)d^1 = h(\mathrm{id} \times u)d^1 = hd^1u = u. \end{aligned}$$

Vice versa for every $g \in s\mathcal{S}et(X, Y) \cap RLP(\mathrm{cof}\, s\mathcal{S}et)$ there are liftings

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d, tail] & X \arrow[d, "g"] \\ Y \arrow[ur, dashed, "\exists s"] \arrow[r, equal] & Y, \end{tikzcd} \qquad \begin{tikzcd} \partial\Delta^1 \times X \arrow[r, "\mathrm{id}\cup(sg)"] \arrow[d, tail] & X \arrow[d, "g"] \\ \Delta^1 \times X \arrow[ur, dashed, "\exists h"] \arrow[r, "g\pi_X"'] & Y, \end{tikzcd}$$

proving that $gs = \mathrm{id}_Y$ and $\mathrm{id}_C \underset{h}{\simeq} sg$ with $gh = g\pi_X = \pi_Y(\mathrm{id} \times g)$. So g is a strong deformation retraction. Moreover $g \in \mathrm{fib}\, s\mathcal{S}et$ by Theorem 3.30.

(ii) Suppose $X \xrightarrow{g} Y$ is a strong deformation section, i.e. there is an $r \in s\mathcal{S}et(Y, X)$ with $rg = \mathrm{id}_X$ and $\mathrm{id}_Y \underset{h}{\simeq} gr$ with $h(\mathrm{id} \times g) = g\pi_Y$. Then in particular g is a monomorphism by Remark 2.32 and hence a cofibration by Definition 3.27. Then like in (i), to every lifting problem on the left, we get a commutative square on the right

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, tail, "g"'] & C \arrow[d, "f"] \\ Y \arrow[ur, dashed, "\exists d"] \arrow[r, "v"'] & D, \end{tikzcd} \qquad \begin{tikzcd} \Delta^0 \times Y +_{\Delta^0 \times X} \Delta^1 \times X \arrow[r, "(ur)\cup(u\pi_X)"] \arrow[d, "d^0 \sqcup g"'] & C \arrow[d, "f"] \\ \Delta^1 \times Y \arrow[ur, dashed, "\exists h'"] \arrow[r, "vh"'] & D. \end{tikzcd}$$

Then $(ur) \cup (u\pi_X)$ is well-defined, because

$$u\pi_X d^0 = u = urg.$$

Moreover the square commutes, because

$$f(ur) = vgr = vhd^0, \qquad f(u\pi_X) = vg\pi_X = vh(\mathrm{id} \times g).$$

As g is a cofibration, by Definition 3.27 of a fibration, there is a diagonal h' and $d := h'd^1$ solves the original lifting problem:

$$\begin{aligned} fd &= fh'd^1 = vhd^1 = v, \\ dg &= h'd^1 g = h'(\mathrm{id} \times g)d^1 = u\pi_X d^1 = u. \end{aligned}$$

The other implication is completely dual to (i).

□

Remark 3.34
By Proposition 3.33 and Remark 3.28 we have

$$RLP(\mathrm{cof}\, s\mathcal{S}et) \subset \mathrm{fib}\, s\mathcal{S}et \cap ws\mathcal{S}et.$$

The hardest part in the construction of the model structure is to see that this is infact an equality. As a consequence we will also get the equality $LLP(\mathrm{fib}\, s\mathcal{S}et) = \mathrm{cof}\, s\mathcal{S}et \cap ws\mathcal{S}et$, which will conclude the proof that $s\mathcal{S}et$ is a model category.

For checking that a given map is a fibration, the following characterization is often the most convenient one.

Proposition 3.35

$$\mathrm{fib}\, s\mathcal{S}et = RLP\{\Lambda^n_k := \bigcup_{\substack{0 \le i \le n, \\ i \ne k}} d^i \Delta^{n-1} \hookrightarrow \Delta^n;\ 0 \le k \le n\}.$$

Proof. First note, that $\Lambda^n_k = BA^n_k$, where $A^n_k \le I(\underline{n})$ is the ordered simplicial subcomplex with same vertices and simplices

$$S(A^n_k) = \{s \in SI(\underline{n});\ i \notin s \text{ for some } i \ne k\}.$$

Moreover given an ordered simplicial complex $X \in \mathcal{S}imp_o$ with an ordered subcomplex $U \overset{c}{\hookrightarrow} X$, we define the ordered subcomplex

$$U(i) := (d^i I(\underline{0}) \times X) \cup (I(\underline{1}) \times U) \le I(\underline{1}) \times X, \qquad i = 0, 1.$$

Then we have

$$Bd^i \sqcup Bc : \Delta^0 \times X +_{\Delta^0 \times BU} \Delta^1 \times BX = BU(i) \hookrightarrow B(I(\underline{1}) \times X) = \Delta^1 \times BX.$$

Indeed as $Bd^i \sqcup Bc$ is injective by Theorem 3.30 the left object is isomorphic to its image in $\Delta^1 \times BX$, which is exactly $BU(i)$ by construction.

Moreover by definition of the product complex the simplices of $U(i)$ are the nonempty subsets

$$t := \{(0, x_1), ..., (0, x_j), (1, x_{j+1}), ..., (1, x_m)\} \subset I(\underline{1}) \times X, \qquad 0 \leq j \leq m,$$

such that

- in the case $i = 0$, we have $\pi_X(t) \in S(X)$ and $\pi_X(t) \in S(U)$, whenever $j < m$.
- in the case $i = 1$, we have $\pi_X(t) \in S(X)$ and $\pi_X(t) \in S(U)$, whenever $j > 0$.

Next we define commutative diagrams of ordered simplicial complexes

$$\begin{tikzcd}
A^n_k \arrow[r, hook, "s|"] \arrow[d, hook] \arrow[rr, bend left, "\mathrm{id}"] & A^n_k(i) \arrow[r, two heads, "r|"] \arrow[d, hook] & A^n_k \arrow[d, hook] \\
I(\underline{n}) \arrow[r, hook, "s"] \arrow[rr, bend right, "\mathrm{id}"'] & I(\underline{1}) \times I(\underline{n}) \arrow[r, two heads, "r"] & I(\underline{n}).
\end{tikzcd}$$

- For $0 \leq k < n$, we let $i = 1$ and define s and r by

$$s(x) := (1, x), \qquad r(a, x) := \begin{cases} \min\{x, k\}, & a = 0, \\ y, & a = 1. \end{cases}$$

  To see that $r(A^n_k(1)) \subset A^n_k$, note that
  - for $j < m$, we have $r(t) \subset \pi_X(t) \cup \{k\} \in S(A^n_k)$, since $\pi_X(t) \in S(A^n_k)$.
  - for $j = m$, we have $r(t) \subset \underline{k}$ and hence $n \notin r(t) \in S(A^n_k)$.

- Similarly, for $0 < k \leq n$, we let $i = 0$ and define s and r by

$$s(x) := (0, x), \qquad r(a, x) := \begin{cases} x, & a = 0, \\ \max\{x, k\}, & a = 1. \end{cases}$$

  To see that $r(A^n_k(0)) \subset A^n_k$, note that
  - for $j > 0$, we have $r(t) \subset \pi_X(t) \cup \{k\} \in S(A^n_k)$, since $\pi_X(t) \in S(A^n_k)$.
  - for $j = 0$, we have $r(t) \subset \{k, ..., n\}$ and hence $0 \notin r(t) \in S(A^n_k)$.

Applying the nerve of the diagram yields $(\Lambda^n_k \hookrightarrow \Delta^n) \in LLP(\mathrm{fib}\, s\mathcal{S}et)$, for all $0 \leq k \leq n$, as the latter is closed under retracts by Lemma 3.13. Equivalently we have

$$RLP(\{\Lambda^n_k \hookrightarrow \Delta^n;\ 0 \leq k \leq n\}) \supset RLP(\mathrm{fib}\, s\mathcal{S}et).$$

For the other inclusion consider the inclusion $U := \partial I(\underline{n}) \overset{c}{\hookrightarrow} I(\underline{n})$. Then every simplex in $S(I(\underline{1}) \times I(\underline{n}))$ not contained in $S(U(1))$ is of the form

$$t_k := \{(0,0), ..., (0,k-1), (1,k), ...(1,n)\}, \quad t'_k := t_k \cup \{(0,k)\}, \qquad 0 < k \leq n.$$

Defining $\partial I(\underline{n}) \leq U(1,k) \leq I(\underline{1}) \times I(\underline{n})$ as the intermediate subcomplex with

$$S(U(1,k)) := S(U(1)) \cup \{t_1, t'_1, ..., t_k, t'_k\}, \qquad 0 \leq k \leq n,$$

we get isomorphisms of ordered simplicial complexes

$$a_j : A_k^{n+1} \xrightarrow{\sim} U(1,k-1) \cap t'_k, \quad j \longmapsto \begin{cases} (0,j), & j < k, \\ (1,j), & j \geq k. \end{cases}$$

So $U(1,k)$ is obtained by glueing $I(t'_k) \cong I(\underline{n})$ to $U(1,k-1)$ along A_k^n. In other words there is a cocartesian square

$$\begin{tikzcd} \Lambda_k^n \arrow[r, "Ba_k"] \arrow[d, hook] & BU(1,k-1) \arrow[d] \\ \Delta^n \arrow[r] & BU(1,k). \end{tikzcd}$$

It follows that

$$(d^1 \sqcup Bc) : BU(1) = BU(1,0) \hookrightarrow ... \hookrightarrow BU(1,n) = B(I(\underline{1}) \times I(\underline{n}))$$

is contained in $LLP(RLP(\{\Lambda_k^n := \bigcup_{\substack{0 \leq i \leq n, \\ i \neq k}} d^i \Delta^{n-1} \hookrightarrow \Delta^n;\ 0 \leq k \leq n\})$, which by Lemma 3.13 is closed under pushouts and composition.

By similar arguments we see that it also contains $d^0 \sqcup Bc$, which by using (ii) proves that

$$RLP(\{\Lambda_k^n \hookrightarrow \Delta^n;\ 0 \leq k \leq n\}) \subset RLP(\mathrm{fib}\, s\mathcal{S}et).$$

□

3.8 Absolute weak equivalences

For the key step in the construction of the model structure we will need another class of morphisms, which behave very well under many operations. Infact it will turn out that absolute weak equivalences are precisely the weak equivalences. However we will not know this until the construction of the model structure, for which the two notions are needed for.

Definition 3.36
*An **absolute weak equivalence** is an $a \in s\mathcal{S}et(A,B)$, such that for every commutative square as on the left, there is a diagonal in the induced right one*

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "a"'] & X \arrow[d, "f"] \\ B \arrow[r, "v"'] & Y, \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "u"] \arrow[d, "a':=d^1"'] & & X \arrow[d, "f"] \\ M(a) = B +_A (\Delta^1 \times A) \arrow[urr, dashed, "\exists d \cup h"] \arrow[rr, "v \cup (fu\pi_A)"'] & & Y. \end{tikzcd}$$

In other words, there is a diagonal $d \in s\mathcal{S}et(B,X)$ in the square

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "a"'] \arrow[dr, phantom, "\simeq_h"] & X \arrow[d, "f"] \\ B \arrow[ur, dashed, "\exists d"] \arrow[r, "v"'] & Y, \end{tikzcd}$$

such that the lower triangle strictly commutes and the upper triangle commutes up to a simplicial homotopy $u \underset{h}{\simeq} da$ satisfying $fh = fu\pi_A$.

Proposition 3.37
Every absolute weak equivalence is a weak equivalence.

Proof. Let $a \in s\mathcal{S}et(A,B)$ be an absolute weak equivalence and $T \xrightarrow{t} *$.

- For every $u \in s\mathcal{S}et(A,T)$, the left square below commutes. So by definition there is a diagonal in the right square

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "a"'] & T \arrow[d, "t"] \\ B \arrow[r, "v"'] & *, \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "u"] \arrow[d, "a'"'] & T \arrow[d, "t"] \\ M(a) \arrow[ur, dashed, "d \cup h"] \arrow[r, "v \cup (fu\pi_A)"'] & *. \end{tikzcd}$$

  It follows that $u \underset{h}{\simeq} da$, which proves surjectivity

$$a^* = (\text{-} \circ a) : \pi_0 \underline{s\mathcal{S}et}(B,T) \twoheadrightarrow \pi_0 \underline{s\mathcal{S}et}(A,T).$$

- For all $v_0, v_1 \in s\mathcal{S}et(B,T)$ with $v_1 a \underset{h}{\simeq} v_0 a$, we let $h' \in s\mathcal{S}et(A, \underline{s\mathcal{S}et}(\Delta^1, T))$ be the map corresponding to h under the natural bijection of Proposition 2.54

$$s\mathcal{S}et(\Delta^1 \times A, T) = s\mathcal{S}et(A, \underline{s\mathcal{S}et}(\Delta^1, T)).$$

  Then the left square below commutes. So by definition there is a diagonal in the right square

$$\begin{tikzcd} A \arrow[r, "h'"] \arrow[d, "a"'] & \underline{s\mathcal{S}et}(\Delta^1, T) \arrow[d] \\ B \arrow[r, "v:=(v_0,v_1)"'] & T \times T = \underline{s\mathcal{S}et}(\partial\Delta^1, T) \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "h'"] \arrow[d, "a'"'] & \underline{s\mathcal{S}et}(\Delta^1, T) \arrow[d] \\ M(a) \arrow[ur, dashed, "k' \cup H"] \arrow[r, "v \cup (fu\pi_A)"'] & \underline{s\mathcal{S}et}(\partial\Delta^1, T). \end{tikzcd}$$

  It follows that $v_1 \underset{k}{\simeq} v_0$, where $k \in s\mathcal{S}et(\Delta^1 \times B, T)$ is the map corresponding to k'.

□

The subsequent proposition shows that the class of absolute weak equivalences satisfies some weakened 2-of-3 property.

Proposition 3.38

For two maps of simplicial sets $A \xrightarrow{a} B \xrightarrow{b} C$ the following implications hold.

(i) *a, ba absolute weak equivalences $\Rightarrow$ b absolute weak equivalence.*

(ii) *a, b absolute weak equivalences $\Rightarrow$ ba absolute weak equivalence.*

Proof.

(i) For every commutative diagram as on the left, there is a diagonal in the right square, as ba is an absolute weak equivalence by assumption.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[dr, "ba"'] & B \arrow[r, "u"] \arrow[d, "b"] & X \arrow[d, "f"] \\ & C \arrow[r, "v"'] & Y. \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "u"] \arrow[d, "(ba)'"'] & X \arrow[d, "f"] \\ M(ba) \arrow[r, "v\cup(fu\pi_A)"'] \arrow[ur, dashed, "\exists d\cup h"] & Y. \end{tikzcd}$$

In particular we have $dba \underset{h}{\simeq} ua$. Like in the proof of Proposition 3.37 (ii) we construct a simplicial homotopy $db \underset{k}{\simeq} u$. To that aim consider the commutative square

$$\begin{tikzcd} A \arrow[r, "h'"] \arrow[d, "a"'] & \underline{s\mathcal{S}et}(\Delta^1, X) \arrow[d, "(\varepsilon^*, f_*)"] \\ B \arrow[r, "V:=((db,u),s_0v)"'] & \underline{s\mathcal{S}et}(\partial\Delta^1, X) \times_{\underline{s\mathcal{S}et}(\partial\Delta^1, Y)} \underline{s\mathcal{S}et}(\Delta^1, Y), \end{tikzcd}$$

where the right vertical map is a fibration by Corollary 3.31 and the map h' corresponds to the homotopy h under the natural bijection of Proposition 2.54. So using that also a is an absolute weak equivalence we obtain a diagonal in the square

$$\begin{tikzcd} A \arrow[r, "h'"] \arrow[d, "a'"'] & \underline{s\mathcal{S}et}(\Delta^1, X) \arrow[d, "(\varepsilon^*, f_*)"] \\ M(a) \arrow[r, "V\cup(fu\pi_A)"'] \arrow[ur, dashed, "k'\cup H"] & \underline{s\mathcal{S}et}(\partial\Delta^1, X) \times_{\underline{s\mathcal{S}et}(\partial\Delta^1, Y)} \underline{s\mathcal{S}et}(\Delta^1, Y), \end{tikzcd}$$

and we define k as the simplicial homotopy corresponding to k'. By construction $d \cup k$ defines a diagonal in the square

$$\begin{tikzcd} B \arrow[r, "u"] \arrow[d, "b'"'] & X \arrow[d, "f"] \\ M(b) \arrow[r, "v"'] \arrow[ur, dashed, "d\cup k"] & Y, \end{tikzcd}$$

which finishes the proof, that b is an absolute weak equivalence.

(ii) For every commutative square as on the left, there are diagonals in the middle and on the right

$$
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "a"'] & X \arrow[dd, "f"] \\
B \arrow[d, "b"'] & \\
C \arrow[r, "v"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "a'"'] & X \arrow[d, "f"] \\
M(a) \arrow[r, "(vb)\cup(fu\pi_A)"'] \arrow[ur, dashed, "\exists d\cup h"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "d"] \arrow[d, "b'"'] & X \arrow[d, "f"] \\
M(b) \arrow[r, "v\cup(fu\pi_A)"'] \arrow[ur, dashed, "\exists d'\cup h'"] & Y.
\end{tikzcd}
$$

In particular we have $\underset{h}{uda} \underset{h'(\mathrm{id}\times a)}{\simeq} d'ba$ and like in the proof for transitivity of the simplicial homotopy relation of Proposition 3.46, we obtain a composite simplicial homotopy $u \underset{k}{\simeq} db'a$ defining a diagonal in the square

$$
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "(ab)'"'] & X \arrow[d, "f"] \\
M(ba) \arrow[r, "v"'] \arrow[ur, dashed, "d\cup k"] & Y.
\end{tikzcd}
$$

□

Proposition 3.39
Absolute weak equivalences are stable under pushouts along cofibrations.

Proof. Given a commutative square as on the left

$$
\begin{tikzcd}
A \arrow[r, tail, "c"] \arrow[d, "a"'] & C \arrow[r, "u"] \arrow[d, "\iota_C"] & X \arrow[d, "f"] \\
B \arrow[r] & B +_A C \arrow[r, "v\cup w"'] & Y,
\end{tikzcd}
$$

where a is an absolute weak equivalence, we have to prove that also ι_C is an absolute weak equivalence. So given a commutative square as on the right, we use that a is an absolute weak equivalence to obtain a diagonal in the commutative square

$$
\begin{tikzcd}
A \arrow[r, "uc"] \arrow[d, "a'"'] & X \arrow[d, "f"] \\
M(a) \arrow[r, "v\cup(fuc\pi_A)"'] \arrow[ur, dashed, "\exists d\cup h"] & Y.
\end{tikzcd}
$$

By construction of h we get a well-defined map $(uc) \cup h$ rendering the diagram below commutative

$$
\begin{tikzcd}
M(c) \arrow[r, "(uc)\cup h"] \arrow[d, "d^1\sqcup c"'] & X \arrow[d, "f"] \\
\Delta^1 \times C \arrow[r, "w\pi_C"'] \arrow[ur, dashed, "\exists k"] & Y.
\end{tikzcd}
$$

As c is a cofibration, we have $d^1 \sqcup c \in LLP(\mathrm{fib}\, s\mathcal{S}et)$ by Theorem 3.30 (ii). Hence there is a diagonal k as depicted. As the first two small squares below are cocartesian by definition

$$
\begin{tikzcd}
A \arrow[r, "c"] \arrow[d, "a"'] & C \arrow[r, "d^1"] \arrow[d, "\iota_C"'] & \Delta^1 \times C \arrow[d] \\
B \arrow[r, "\iota_B"] & B +_A C \arrow[r, "(\iota_C)'"] & M(\iota_C),
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "d^1"] \arrow[d, "a"'] & \Delta^1 \times A \arrow[r, "\mathrm{id}\times c"] \arrow[d] & \Delta^1 \times C \arrow[d] \\
B \arrow[r, "a'"] & M(a) \arrow[r, "\iota_B+(\mathrm{id}\times c)"] & M(\iota_C),
\end{tikzcd}
$$

so is their composite, which coincides with the composite of the other two squares. As also the third small square is cocartesian, the same holds for the fourth small square. So by construction $D := (d \cup h) \cup k$ defines a diagonal in the diagram below

$$
\begin{tikzcd}
C \arrow[r, "u"] \arrow[d, "(\iota_C)'"'] & X \arrow[d, "f"] \\
M(\iota_C) \arrow[r, "(v\cup w)\cup(fu\pi_{B+_A C})"'] \arrow[ur, "D"] & Y,
\end{tikzcd}
$$

which concludes the proof that ι_C is an absolute weak equivalence.

□

From the following technically complicated Lemma we will derive more useful properties of absolute weak equivalences.

Lemma 3.40
Given a commutative diagram

$$
\begin{tikzcd}
A \arrow[r, tail, "c"] \arrow[d, "a"'] & B \arrow[d, "b"] \\
A' \arrow[r, tail, "c'"] & B',
\end{tikzcd}
$$

where a and b are absolute weak equivalences.

Then for every commutative square as on the left and every diagonal $d \cup h$, there is is a diagonal $d'' \cup h''$ rendering the right diagram commutative

$$
\begin{tikzcd}
B \arrow[r, "u"] \arrow[d, "b"'] & X \arrow[d, "f"] \\
B' \arrow[r, "v"'] & Y,
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "c"] \arrow[d, "a'"'] & B \arrow[r, "u"] \arrow[d, "b'"'] & X \arrow[d, "f"] \\
M(a) \arrow[r, "C:=c'+(\mathrm{id}\times c)"'] \arrow[urr, "d\cup h"] & M(b) \arrow[r, "v\cup(fu\pi_B)"'] \arrow[ur, dashed, "\exists d''\cup h''"'] & Y,
\end{tikzcd}
$$

Proof. As b is an absolute weak equivalence, there is a diagonal in

$$
\begin{tikzcd}
B \arrow[r, "u"] \arrow[d, "b'"'] & X \arrow[d, "f"] \\
M(b) \arrow[r, "v\cup(fu\pi_B)"'] \arrow[ur, dashed, "\exists d'\cup h'"] & Y.
\end{tikzcd}
$$

However this does not guarantee that $(d' \cup h') \circ C = d \cup h$. The strategy is to construct a deformation $d'' \cup h''$ of $d' \cup h'$ restricting to $d \cup h$ along C. The construction of $d'' \cup h''$ is quite technical and requires to apply the lifting property 5 more times. For $i = 0, 1$ we define the maps

$$\iota_i := \varepsilon \sqcup d^i : U_i := \partial\Delta^1 \times \Delta^1 +_{\partial\Delta^1 \times \Delta^0} \Delta^1 \times \Delta^0 \hookrightarrow \Delta^1 \times \Delta^1,$$

which for $i = 0$ e.g. corresponds to the inclusion

$$\begin{tikzcd} (0,1) & (1,1) & & (0,1) \arrow[r] & (1,1) \\ (0,0) \arrow[u] \arrow[r] & (1,0) \arrow[u] \arrow[r, phantom, "\hookrightarrow"] & {} & (0,0) \arrow[u] \arrow[r] \arrow[ur, phantom, "\text{“solid”}"] & (1,0). \arrow[u] \end{tikzcd} \tag{3.2}$$

In particular ι_i maps U_i isomorphically onto the simplicial subset

$$(d^1\Delta^0 \times \Delta^1) \cup (d^0\Delta^0 \times \Delta^1) \cup (\Delta^1 \times d^i\Delta^1) \cong \Delta^1 \cup \Delta^1 \cup \Delta^1. \tag{3.3}$$

Recall that by Theorem 3.30 (ii) we have $\iota_i \in LLP(\mathrm{fib}\, s\mathcal{S}et)$.

1) In the commutative diagram

$$\begin{tikzcd} U_1 \times A \arrow[r, "h \cup (h'(\mathrm{id}\times c)) \cup (uc\pi_A)"] \arrow[d, "\iota_1 \times \mathrm{id}"'] & X \arrow[d, "f"] \\ \Delta^1 \times \Delta^1 \times A \arrow[r, "vc'a\pi_A = fuc\pi_A"'] \arrow[ur, dashed, "\exists H"] & Y, \end{tikzcd}$$

the upper horizontal map is defined using the description (3.3). Since $\iota_i \in LLP(\mathrm{fib}\, s\mathcal{S}et)$, we also have $\iota_i \times \mathrm{id}_A \in LLP(\mathrm{fib}\, s\mathcal{S}et)$ by Theorem 3.30 applied to the cofibration $\emptyset \rightarrowtail A$. So there is a diagonal H as depicted.

To clarify the situation we will introduce another representation for the lifting problem. Using the picture (3.2) we may represent the left vertical map of the diagram by the picture

$$\begin{tikzcd} A & A & & A \arrow[r, "A"] & A \\ A \arrow[u, "A"] \arrow[r, "A"'] & A \arrow[u, "A"'] \arrow[r, phantom, "\hookrightarrow"] & {} & A \arrow[u, "A"] \arrow[r, "A"'] \arrow[ur, phantom, "A"] & A. \arrow[u, "A"'] \end{tikzcd}$$

Moreover we may represent the upper horizontal map by the left diagram below and the lifting property translates as there is an extension H represented by the right diagram

$$\begin{tikzcd} da & d'bc = d'c'a & & da \arrow[r, dashed, "k:="] & d'c'a \\ uc \arrow[u, "h"] \arrow[r, "uc\pi_A"'] & uc \arrow[u, "h'(\mathrm{id}\times c)"'] \arrow[r, phantom, "\hookrightarrow"] & {} & uc \arrow[u, "h"] \arrow[r, "uc\pi_A"'] \arrow[ur, phantom, "\exists H"] & uc, \arrow[u, "h'(\mathrm{id}\times c)"'] \end{tikzcd}$$

2) Next we use that a is an absolute weak equivalence for the commutative square

$$\begin{tikzcd}
A \arrow[rr, "k"] \arrow[d, "a"'] & & \underline{s\mathcal{S}et}(\Delta^1, X) \arrow[d, "(\varepsilon^*, f_*)"] \\
A' \arrow[r, "{((d,d'c'),s_0vc')}"'] & \underline{s\mathcal{S}et}(\partial\Delta^1, X) \times_{\underline{s\mathcal{S}et}(\partial\Delta^1, Y)} \underline{s\mathcal{S}et}(\Delta^1, Y) &
\end{tikzcd}$$

to get a diagonal $k' \cup K$ in the corresponding square (use Remark 3.24 for Proposition 2.54)

$$\begin{tikzcd}
\partial\Delta^1 \times M(a) +_{\partial\Delta^1 \times A} \Delta^1 \times A \arrow[rr, "(d\cup(d'c'))\cup k"] \arrow[d, "a"'] & & X \arrow[d, "f"] \\
\Delta^1 \times M(a) \arrow[r, "\pi_{M(a)}"'] \arrow[rru, dashed, "k' \cup K"] & M(a) \arrow[r, "(vc')\cup(fk\pi_A)"'] & Y.
\end{tikzcd}$$

Translating into our notation the left vertical map corresponds to the picture

$$\begin{tikzcd}
A' & A' & & A' \arrow[r, "A'"] \arrow[dr, phantom, "A"] & A' \\
A \arrow[u, "A"] \arrow[r, "A"'] & A \arrow[u, "A"'] \arrow[r, hook] & {} & A \arrow[u, "A"] \arrow[r, "A"'] & A, \arrow[u, "A"']
\end{tikzcd}$$

and the lifting property can be described by the existence of an extension

$$\begin{tikzcd}
d & d'c' & & d \arrow[r, dashed, "\exists k'"] \arrow[dr, phantom, "\exists K"] & d'c' \\
da \arrow[u, "da\pi_A"] \arrow[r, "k"'] & d'c'a \arrow[u, "d'c'a\pi_A"'] \arrow[r, hook] & {} & da \arrow[u, "da\pi_A"] \arrow[r, "k"'] & d'c'a. \arrow[u, "d'c'a\pi_A"']
\end{tikzcd}$$

3) Using that c' is a cofibration and the Definition 3.27 of a fibration there is a diagonal in the commutative square

$$\begin{tikzcd}
\Delta^0 \times B' +_{\Delta^0 \times A'} \Delta^1 \times A' \arrow[r, "d' \cup k'"] \arrow[d, "d^1 \sqcup c'"'] & X \arrow[d, "f"] \\
\Delta^1 \times B' \arrow[r, "v\pi_{B'}"'] \arrow[ru, dashed, "\exists k''"] & Y.
\end{tikzcd}$$

Again the left vertical map is represented by

$$A' \xrightarrow{A'} B' \quad \hookrightarrow \quad B' \xrightarrow{B'} B',$$

and the lifting property by

$$d \xrightarrow{k'} d' \quad \hookrightarrow \quad d'' := k''d^1 \overset{\exists k''}{\dashrightarrow} d'.$$

4) Using that c is a cofibration, so is the following map by Theorem 3.30 again

$$\begin{tikzcd} B \arrow[r, "B"] \arrow[dr, phantom, "A"] & B \\ B \arrow[u, "B"] \arrow[r, "A"'] & B \arrow[u, "B"'] \end{tikzcd} \hookrightarrow \begin{tikzcd} B \arrow[r, "B"] \arrow[dr, phantom, "B"] & B \\ B \arrow[u, "B"] \arrow[r, "B"'] & B, \arrow[u, "B"'] \end{tikzcd}$$

and thus there is an extension

$$\begin{tikzcd} d''b \arrow[r, "k''(\mathrm{id}\times b)"] \arrow[dr, phantom, "K"] & d'b \\ d''b \arrow[u, "d''b\pi_B"] \arrow[r, "k"'] & d'b \arrow[u, "d'b\pi_B"'] \end{tikzcd} \hookrightarrow \begin{tikzcd} d''b \arrow[r, "k''(\mathrm{id}\times b)"] \arrow[dr, phantom, "\exists L"] & d'b \\ d''b \arrow[u, "d''b\pi_B"] \arrow[r, dashed, "\ell:="'] & d'b. \arrow[u, "db\pi_B"'] \end{tikzcd}$$

Here it should be noted that the left map is well-defined, because by construction

- $k''(\mathrm{id} \times b)(\mathrm{id} \times c) = k''(\mathrm{id} \times c')(\mathrm{id} \times a) = k'(\mathrm{id} \times a) = K(\mathrm{id} \times d^0)$,
- $d''b\pi_B(\mathrm{id} \times c) = d''bc\pi_A = d''c'a\pi_A = da\pi_A = K(d^1 \times \mathrm{id})$,
- $d'b\pi_B(\mathrm{id} \times c) = d'bc\pi_A = d'c'a\pi_A = K(d^0 \times \mathrm{id})$.

5) Finally, using that c is a cofibration again, so is the following map by Theorem 3.30 again

$$\begin{tikzcd} B \arrow[r, "B"] \arrow[dr, phantom, "A"] & B \\ B \arrow[u, "A"] \arrow[r, "B"'] & B \arrow[u, "B"'] \end{tikzcd} \hookrightarrow \begin{tikzcd} B \arrow[r, "B"] \arrow[dr, phantom, "B"] & B \\ B \arrow[u, "B"] \arrow[r, "B"'] & B, \arrow[u, "B"'] \end{tikzcd}$$

and thus there is an extension

$$\begin{tikzcd} d''b \arrow[r, "\ell"] \arrow[dr, phantom, "H"] & d'b \\ u \arrow[u, "d''b\pi_B"] \arrow[r, "u\pi_B"'] & u \arrow[u, "h'"'] \end{tikzcd} \hookrightarrow \begin{tikzcd} d''b \arrow[r, "\ell"] \arrow[dr, phantom, "\exists M"] & d'b \\ u \arrow[u, dashed, "h'':="] \arrow[r, "u\pi_B"'] & u. \arrow[u, "h'"'] \end{tikzcd}$$

Again the left map is well-defined, because by construction we have

- $\ell(\mathrm{id} \times c) = k = H(\mathrm{id} \times d^0)$,
- $u\pi_B(\mathrm{id} \times c) = H(\mathrm{id} \times d^1)$,
- $h'(\mathrm{id} \times c) = J(d^0 \times \mathrm{id})$.

By construction $d'' \cup h''$ has the desired property.

□

The next two properties are essential for showing that a certain natural map between two functors targetting to $s\mathcal{S}et$ is an absolute weak equivalence.

Theorem 3.41 (i) *For every commutative diagram of simplicial sets*

$$\begin{tikzcd} Y \arrow[d, "y"'] & X \arrow[l, tail, "c"'] \arrow[r, "g"] \arrow[d, "x"'] & Z \arrow[d, "z"'] \\ Y' & X' \arrow[l, tail, "c'"'] \arrow[r, "g'"] & Z' \end{tikzcd}$$

if x, y, z are absolute weak equivalences, then taking the colimit horizontally induces an absolute weak equivalence

$$y +_x z : Y +_X Z \longrightarrow Y +_{X'} Z'.$$

(ii) *For every map of sequences of cofibrations of simplicial sets*

$$\begin{tikzcd} X_0 \arrow[r, tail, "c_0"] \arrow[d, "x_0"'] & X_1 \arrow[r, tail, "c_1"] \arrow[d, "x_1"'] & \ldots \\ X_0' \arrow[r, tail, "c_0'"] & X_1' \arrow[r, tail, "c_1'"] & \ldots \end{tikzcd}$$

if x_n is an absolute weak equivalence, for all $n \geq 0$, then taking the colimit horizontally induces an absolute weak equivalence

$$x := \operatorname*{colim}_{n\geq 0} x_n : X := \operatorname*{colim}_{n\geq 0} X_n \longrightarrow \operatorname*{colim}_{n\geq 0} X_n' =: X'.$$

Proof.

(i) For every commutative square as on the left, there is a diagonal D_Z on the right, since z is an absolute weak equivalence

$$\begin{tikzcd} Y +_X Z \arrow[r, "t\cup u"] \arrow[d, "y+_x z"'] & C \arrow[d, "f"] \\ Y' +_{X'} Z' \arrow[r, "v\cup w"'] & D, \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "g"] \arrow[d, "x'"'] & Z \arrow[r, "u"] \arrow[d, "z'"'] & C \arrow[d, "f"] \\ M(x) \arrow[r, "G:=g'+(\mathrm{id}\times g)"'] & M(z) \arrow[r, "w\cup(fu\pi_Z)"'] \arrow[ur, dashed, "\exists D_Z"] & D. \end{tikzcd}$$

Setting $D_X := D_Z G$, we also get a diagonal for the outer square on the right. By Lemma 3.40 we also obtain a diagonal D_Y in the diagram

$$\begin{tikzcd} X \arrow[r, "c"] \arrow[d, "x'"'] & Y \arrow[r, "u"] \arrow[d, "y'"'] & C \arrow[d, "f"] \\ M(x) \arrow[r, "c'+(\mathrm{id}\times c)"'] \arrow[urr, "D_X"] & M(y) \arrow[r, "v\cup(fu\pi_Y)"'] \arrow[ur, dashed, "\exists D_Y"'] & D, \end{tikzcd}$$

So by construction $D_Y \cup D_Z$ defines a diagonal for the square

$$\begin{tikzcd} Y +_X Z \arrow[r, "t\cup u"] \arrow[d, "(y+_x z)'"'] & C \arrow[d, "f"] \\ M(y +_x z) \arrow[r, "(v\cup w)\cup(f(t\cup u)\pi_{Y+_X Z})"'] \arrow[ur, "D_Y \cup D_Z"] & D, \end{tikzcd}$$

which proves that $y +_x z$ is an absolute weak equivalence.

(ii) For every commutative square as on the left, there is a diagonal D_0 on the right, since x_0 is an absolute weak equivalence

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "x"'] & C \arrow[d, "f"] \\
X' \arrow[r, "v"'] & D,
\end{tikzcd}
\qquad
\begin{tikzcd}
X_0 \arrow[r, "u\iota_0"] \arrow[d, "x'_0"'] & C \arrow[d, "f"] \\
M(x_0) \arrow[r, "(v\iota_0)\cup(fu\iota_0\pi_{X_0})"'] \arrow[ur, dashed, "\exists D_0"] & D.
\end{tikzcd}$$

By induction on $n \geq 0$, Lemma 3.40 provides a diagonal D_{n+1} in the diagram

$$\begin{tikzcd}
X_n \arrow[r, "c_n"] \arrow[d, "x'_n"'] & X_{n+1} \arrow[r, "u\iota_{n+1}"] \arrow[d, "x'_{n+1}"] & C \arrow[d, "f"] \\
M(x_n) \arrow[r, "c'_n+(\mathrm{id}\times c_n)"'] \arrow[urr, "D_n"] & M(x_{n+1}) \arrow[r, "(v\iota_{n+1})\cup(fu\iota_{n+1}\pi_{X_{n+1}})"'] \arrow[ur, dashed, "\exists D_{n+1}"] & D,
\end{tikzcd}$$

Then by construction $\operatorname{colim}_{n\geq 0} D_n$ defines a diagonal in the diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "x'"'] & C \arrow[d, "f"] \\
M(x) \arrow[r, "v\cup(fu\pi_X)"'] \arrow[ur, "\operatorname{colim}_n D_n"] & D,
\end{tikzcd}$$

proving that x is an absolute weak equivalence.

□

Proposition 3.42
The class $w_a s\mathcal{S}et$ of absolute weak equivalences has the following properties.

(i) *$w_a s\mathcal{S}et$ is closed under pushouts along cofibrations.*

(ii) *$w_a s\mathcal{S}et$ is closed under arbitrary coproducts.*

Absolute weak equivalences are stable under pushouts along cofibrations.

Proof.

(i) Given a sequence of maps $C \overset{c}{\longleftarrowtail} A \overset{a}{\longrightarrow} B$, where a is an absolute weak equivalence. Then applying Theorem 3.41 (i) to the commutative diagram

$$\begin{tikzcd}
C \arrow[d, equal] & A \arrow[l, tail, "c"'] \arrow[r, equal] \arrow[d, equal] & A \arrow[d, "a"] \\
C & A \arrow[l, tail, "c"] \arrow[r, "a"] & B
\end{tikzcd}$$

yields that also $C \overset{\iota_C}{\longrightarrow} B +_A C$ is an absolute weak equivalence.

(ii) Given a family of absolute weak equivalences $(A_i \xrightarrow{a_i} B_i)_{i\in I}$ indexed by some set $I \in \mathcal{S}et$. Setting

$$a := \coprod_{i\in I} a_i : A := \coprod_i A_i \longrightarrow \coprod_i B_i =: B,$$

and using that coproducts commute pushouts, we get a natural isomorphism

$$\begin{tikzcd} \coprod_i A_i \arrow[r, equal] \arrow[d, "\coprod_i a_i'"'] & A \arrow[d, "a'"] \\ \coprod_i M(a_i) \arrow[r, "\sim"] & M(a). \end{tikzcd}$$

So for every commutative square as on the left, there are diagonals in the middle square, which glue to a diagonal in the right square

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "a"'] & C \arrow[d, "f"] \\ B \arrow[r, "v"'] & D \end{tikzcd} \qquad \begin{tikzcd} A_i \arrow[r, "u\iota_i"] \arrow[d, "a_i'"'] & C \arrow[d, "f"] \\ M(a_i) \arrow[r, "(v\iota_i)\cup(fu\iota_i\pi_{A_i})"'] \arrow[ur, dashed, "\exists D_i"] & D \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "u"] \arrow[d, "a"'] & C \arrow[d, "f"] \\ M(a) \arrow[r, "(v\iota_i)\cup(fu\iota_i\pi_{A_i})"'] \arrow[ur, "D:=\coprod_i D_i"] & D \end{tikzcd}$$

□

Proposition 3.43
Let $w_a s\mathcal{S}et$ denote the class of absolute weak equivalences. Then

$$LLP(\text{fib}\, s\mathcal{S}et) = \text{cof}\, s\mathcal{S}et \cap w_a s\mathcal{S}et.$$

Proof. Let $c \in s\mathcal{S}et(A, B) \cap LLP(\text{fib}\, s\mathcal{S}et)$. Then for every fibration $f \in s\mathcal{S}et(C, D)$ there is a diagonal in the left square, providing a diagonal in the right square

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, tail, "c"'] & C \arrow[d, "f"] \\ B \arrow[r, "v"'] \arrow[ur, dashed, "\exists d"] & D, \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "u"] \arrow[d, "c'"'] & C \arrow[d, "f"] \\ M(c) \arrow[r, "v\cup(fu\pi_X)"'] \arrow[ur, dashed, "d\cup(u\pi_A)"] & D, \end{tikzcd}$$

proving that c is an absolute weak equivalence. Moreover $c \in \text{cof}\, s\mathcal{S}et$ by Corollary 3.32, so we get the inclusion

$$LLP(\text{fib}\, s\mathcal{S}et) \subset \text{cof}\, s\mathcal{S}et \cap w_a s\mathcal{S}et. \tag{3.4}$$

Vice versa suppose $X \overset{c}{\rightarrowtail} Y$ is an absolute weak equivalence. As $(LLP(\text{fib}\, s\mathcal{S}et), \text{fib}\, s\mathcal{S}et)$ is a weak factorization system, we may factor c as $X \xrightarrow{c'} X' \xrightarrow{f} Y$, where $c' \in LLP(\text{fib}\, s\mathcal{S}et)$ and $f \in \text{fib}\, s\mathcal{S}et$. Proposition 3.38 implies that f is an absolute weak equivalence, because c' is one by (3.4) and $fc' = c$ is one by assumption. Solving the lifting problem

$$\begin{tikzcd} X' \arrow[r, equal] \arrow[d, "f"'] & X' \arrow[d, "f"] \\ Y \arrow[r, equal] \arrow[ur, dashed, "\underset{h}{\simeq}", "\exists s"'] & Y \end{tikzcd}$$

proves that f is a strong deformation retraction and hence $f \in RLP(\mathrm{cof}\, s\mathcal{S}et)$ by Proposition 3.33. So we can also find a diagonal in

$$\begin{tikzcd} X \arrow[r, tail, "c'"] \arrow[d, tail, "c"'] & X' \arrow[d, "f"] \\ Y \arrow[ur, dashed, "\exists d"] \arrow[r, equal] & Y, \end{tikzcd}$$

which shows that c is a retract of c' and hence $c \in LLP(\mathrm{fib}\, s\mathcal{S}et)$, because the latter is closed under retracts by Lemma 3.13.

□

If we knew that every absolute weak equivalence was a weak equivalence Proposition 3.43 would comparably easily imply that $s\mathcal{S}et$ is a model category with the three classes of maps.

Corollary 3.44
For $X \xrightarrow{f} Y \xrightarrow{g} Z$ the following implication holds.

$$f, gf \in LLP(\mathrm{fib}\, s\mathcal{S}et),\ g \in \mathrm{cof}\, s\mathcal{S}et \quad \Rightarrow \quad g \in LLP(\mathrm{fib}\, s\mathcal{S}et).$$

Proof. By Proposition 3.43 the maps f and gf are absolute weak equivalences, hence g is an absolute weak equivalence by Propostion 3.38. So $g \in \mathrm{cof}\, s\mathcal{S}et \cap w_a s\mathcal{S}et = LLP(\mathrm{fib}\, s\mathcal{S}et)$ by using Proposition 3.43 again.

□

Combined with Proposition 3.43 the subsequent lemma will be used later to show that the endofunctor of barycentric subdivision on the category $s\mathcal{S}et$ preserves anodyne extensions.

Lemma 3.45
Let $C, D \in \mathcal{C}at$ be small categories with a terminal object. Then:

(i) *The nerve of every functor $\underline{0} \xrightarrow{f} C$ is an anodyne extension.*

(ii) *The nerve of every functor $C \xrightarrow{f} D$ is an absolute weak equivalence.*

Proof.

(i) Let $\underline{0} \xrightarrow{g} C$ be the functor sending 0 to the terminal object $* \in C$. Moreover let $C \xrightarrow{r} \underline{0}$ be the unique functor sending every object to 0. Then $rg = \mathrm{id}_{\underline{0}}$ and by the universal property of the terminal object there are unique C-morphisms

$$x \xrightarrow{\eta_x} * = gr(x), \qquad x \in C.$$

In other words the functors r and g form an adjunction

$$\underline{0}(r(x), y) = C(x, g(y)).$$

So Proposition 2.89 implies that Bg and Br form a simplicial homotopy equivalence. As $\underline{0}$ consists of a single object and morphism, it follows that Bg is infact a strong deformation section and thus $Bg \in LLP(\mathrm{fib}\, s\mathcal{S}et)$ by Proposition 3.43. This proves the assertion, for $f = g$.

If $f \neq g$ we define the functor

$$h : \underline{1} \longrightarrow C, \quad 0 \longmapsto f(0), \quad 1 \longmapsto *,$$

sending the map $0 \longrightarrow 1$ to the unique map $\eta_{f(0)}$. Taking nerves we get a commutative diagram

$$\begin{tikzcd}
\Delta^0 \arrow[r, equal] \arrow[d, "d^1"'] & B\underline{0} \arrow[d, "Bd^1"'] \arrow[dr, bend left, "Bf"] & \\
\Delta^1 \arrow[r, equal] & B(\underline{1}) \arrow[r, "Bh"] & BC. \\
\Delta^0 \arrow[r, equal] \arrow[u, "d^0"] & B\underline{0} \arrow[u, "Bd^0"] \arrow[ur, bend right, "Bg"'] &
\end{tikzcd}$$

Since $f \neq g$, the map Bh is injective and hence a cofibration by Definition 3.27. So $Bg, d^0 \in LLP(\mathrm{fib}\, s\mathcal{S}et)$ implies $Bh \in LLP(\mathrm{fib}\, s\mathcal{S}et)$ by Corollary 3.44. Moreover $d^1, Bh \in LLP(\mathrm{fib}\, s\mathcal{S}et)$ implies $Bf \in LLP(\mathrm{fib}\, s\mathcal{S}et)$ by Lemma 3.13.

(ii) Choose an arbitrary functor $\underline{0} \xrightarrow{g} C$, e.g. the functor g constructed in the proof of (i). Then the functors g and fg satisfy the hypothesis in (i) and so $Bg, Bf \circ Bg \in LLP(\mathrm{fib}\, s\mathcal{S}et) \subset w_a s\mathcal{S}et$ by Propostion 3.43. So the 2-of-3 property for absolute weak equivalences proven in Proposition 3.38 implies that also Bf is an absolute weak equivalence.

□

3.9 Maps with fibrant codomain

In general simplicial homotopy does not define an equivalence relation on the set of homomorphisms between two simplicial sets. But it does once the target simplicial set is fibrant, as the following proposition demonstrates.

Proposition 3.46
Let $Y \overset{t}{\twoheadrightarrow} T$ and $e, f, g \in s\mathcal{S}et(X, Y)$ with $s = te = tf = tg$.

(i) $f \underset{f\pi_X}{\simeq} f$,

(ii) $f \underset{h}{\simeq} g, \quad th = s\pi_X \quad \Rightarrow \quad g \underset{h'}{\simeq} f, \quad sh' = s\pi_X,$

(iii) $e \underset{h}{\simeq} f \underset{k}{\simeq} g, \quad sh = sk = s\pi_X \quad \Rightarrow \quad e \underset{\ell}{\simeq} g, \quad s\ell = s\pi_X.$

In particular simplicial homotopy "$\simeq$" defines an equivalence relation on $s\mathcal{S}et(X, Y)$, if $Y \twoheadrightarrow *$.

Proof. (i) holds by construction, because $f\pi_X d^i = f$, for $i = 0, 1$. For the rest of the proof we will again use the anodyne extension

$$\iota_i := \varepsilon \sqcup d^i : U_i := \partial\Delta^1 \times \Delta^1 +_{\partial\Delta^1\times\Delta^0} \Delta^1 \times \Delta^0 \hookrightarrow \Delta^1 \times \Delta^1$$

and the description

$$\iota_i : U_i \xrightarrow{\sim} (d^1\Delta^0 \times \Delta^1) \cup (d^0\Delta^0 \times \Delta^1) \cup (\Delta^1 \times d^i\Delta^1) \cong \Delta^1 \cup \Delta^1 \cup \Delta^1$$

like in the proof of Lemma 3.40.

Under the hypothesis of (ii) there is a solution for the lifting problem

$$\begin{tikzcd}
U_i \times X \arrow[rr, "(g\pi_X)\cup h\cup(g\pi_X)"] \arrow[d, "\iota_0\times\mathrm{id}"'] & & Y \arrow[d, two heads, "t"] \\
\Delta^1 \times \Delta^1 \times X \arrow[rr, "s\pi_X"'] \arrow[urr, dashed, "\exists H"] & & T,
\end{tikzcd}$$

which in the notation of Lemma 3.40 corresponds to the picture

$$\begin{tikzcd}
g \arrow[r, "g\pi_X"] & g & & g \arrow[rr, "g\pi_X"] & \arrow[d, phantom, "\exists H"] & g \\
& & \arrow[r, hook] & & & \\
g \arrow[uu, "g\pi_X"] & f \arrow[uu, "h"'] & & g \arrow[uu, "g\pi_X"] \arrow[rr, dashed, "h':="'] & & f, \arrow[uu, "h"']
\end{tikzcd}$$

Applying Theorem 3.30 for the cofibration $\emptyset \rightarrowtail X$ implies that $\iota_0 \times \mathrm{id} \in LLP(\mathrm{fib}\, s\mathcal{S}et)$ and hence there is a diagonal H. By construction we get a simplicial homotopy $g \underset{h'}{\simeq} f$.

Similarly for $e \underset{h}{\simeq} f \underset{k}{\simeq} g$ there is a diagonal in the commutative square

$$\begin{tikzcd}
U_i \times X \arrow[rr, "h\cup(g\pi_X)\cup k"] \arrow[d, "\iota_0\times\mathrm{id}"'] & & Y \arrow[d, two heads, "t"] \\
\Delta^1 \times \Delta^1 \times X \arrow[rr, "s\pi_X"'] \arrow[urr, dashed, "\exists H"] & & T.
\end{tikzcd}$$

which in the notation of Lemma 3.40 corresponds to the picture

$$\begin{tikzcd}
f \arrow[r, "k"] & g & & f \arrow[rr, "k"] & \arrow[d, phantom, "\exists H"] & g \\
& & \arrow[r, hook] & & & \\
e \arrow[uu, "h"] & g \arrow[uu, "g\pi_X"'] & & e \arrow[uu, "h"] \arrow[rr, dashed, "\ell:="'] & & g, \arrow[uu, "g\pi_X"']
\end{tikzcd}$$

and by construction $e \underset{\ell}{\simeq} g$.

$\square$

Transitivity and symmetric of the simplicial homotopy relation is not the only property that holds, when the target is fibrant.

Proposition 3.47
For $X \xrightarrow{f} Y \longrightarrow *$ the following are equivalent.

(i) *f is a weak equivalence.*

(ii) *f is a homotopy equivalence.*

(iii) *f is a strong deformation retraction.*

Proof. We begin by assuming (i), i.e. f is a weak equivalence.

- Since $Y \longrightarrow *$, the map $\pi_0 \underline{s\mathcal{S}et}(X, Y) \twoheadrightarrow \underline{s\mathcal{S}et}(Y, Y)$ is surjective. So there is a $Y \xrightarrow{g} X$ with $\mathrm{id}_X \underset{h}{\simeq} gf$.
- Since $X \longrightarrow *$, the map $\pi_0 \underline{s\mathcal{S}et}(X, X) \hookrightarrow \underline{s\mathcal{S}et}(Y, X)$ is injective. So $f \underset{fh}{\simeq} fgf$ implies $\mathrm{id}_Y \underset{k}{\simeq} fg$.

This proves, that f is a homotopy equivalence.

Next we assume (ii), i.e. that there is a map $Y \xrightarrow{g} X$ and simplicial homotopies $\mathrm{id}_X \underset{h}{\simeq} gf$ and $\mathrm{id}_Y \underset{k}{\simeq} fg$. Using that $\emptyset \rightarrowtail Y$ is a cofibration there is a solution for the lifting problem

$$\begin{tikzcd} \Delta^0 \times Y \arrow[r, "g"] \arrow[d, "d^0 \times \mathrm{id}"'] & X \arrow[d, "f"] \\ \Delta^1 \times Y \arrow[r, "k"'] \arrow[ur, dashed, "\exists k'"] & Y. \end{tikzcd}$$

Setting $s := k'd^1$ we get

$$fs = fk'd^1 = kd^1 = \mathrm{id}_Y, \qquad \mathrm{id}_X \underset{h}{\simeq} gf, \quad sf \underset{k'(\mathrm{id} \times f)}{\simeq} gf.$$

So by Proposition 3.46 there is a simplicial homotopy $\mathrm{id}_X \underset{h'}{\simeq} sf$. Using that $\emptyset \rightarrowtail X$ is a cofibration there is a solution for the lifting problem

$$\begin{tikzcd} U_0 \times X \arrow[r, "h' \cup (sfh') \cup (sf\pi_X)"] \arrow[d, "\iota_0 \times \mathrm{id}"'] & X \arrow[d, "f"] \\ \Delta^1 \times \Delta^1 \times X \arrow[r, "f\pi_X(\mathrm{id} \times h')"'] \arrow[ur, dashed, "\exists H"] & Y, \end{tikzcd}$$

which in the notation of Lemma 3.40 corresponds to the picture

$$\begin{tikzcd} sf \arrow[r, "sf\pi_X"] & sf & & sf \arrow[r, "sf\pi_X"] \arrow[dr, phantom, "\exists H"] & sf \\ \mathrm{id}_X \arrow[u, "h'"] & sf \arrow[u, "sfh'"'] \arrow[r, phantom, ""] & {} \arrow[r, hook] & \mathrm{id}_Y \arrow[u, "h'"] \arrow[r, dashed, "h'':="'] & sf. \arrow[u, "sfh'"'] \end{tikzcd}$$

By construction we have

$$\mathrm{id}_Y \underset{h''}{\simeq} sf, \qquad fh' = fH(\mathrm{id} \times d^1) = f\pi_X(\mathrm{id} \times (h'd^1)) = f\pi_X,$$

so f is a strong deformation retraction.

Finally every simplicial homotopy equivalence is a weak equivalence by Remark 3.28, which shows that (iii) implies (i).

□

If we knew the preceding proposition for arbitrary fibrations (without assuming that the codomain is fibrant), this would immediately give the desired model structure on $s\mathcal{S}et$. Indeed it yields the following characterization for anodyne extensions with fibrant codomain, which we (for general codomains) could only verify (yet) for absolute weak equivalences instead of weak equivalences in Proposition 3.33. The key idea in the verification of the model structure on $s\mathcal{S}et$ is to find a well-behaved method of replacing a map by one having a fibrant codomain.

Corollary 3.48
The following implications are valid.

(i) *If $c \in \mathrm{cof}\, s\mathcal{S}et \cap ws\mathcal{S}et$ has a fibrant codomain, then $c \in LLP(\mathrm{fib}\, s\mathcal{S}et)$.*

(ii) *if $f \in \mathrm{fib}\, s\mathcal{S}et$ has a fibrant codomain, then $f \in RLP(\mathrm{cof}\, s\mathcal{S}et \cap ws\mathcal{S}et)$.*

Proof.

(i) The proof is similar to that of Proposition 3.43. Using that $(LLP(\mathrm{fib}\, s\mathcal{S}et), \mathrm{fib}\, s\mathcal{S}et)$ is a weak factorization system, we may factor c as $X \overset{c'}{\rightarrowtail} X' \overset{f}{\twoheadrightarrow} Y$, where $c' \in LLP(\mathrm{fib}\, s\mathcal{S}et)$ and $f \in \mathrm{fib}\, s\mathcal{S}et$. Then $f \in ws\mathcal{S}et$ by the 2-of-3 axiom, because $fc' = c \in ws\mathcal{S}et$ by assumption and $c' \in w_a s\mathcal{S}et \subset ws\mathcal{S}et$ by Proposition 3.43 and Proposition 3.37. So f is a strong deformation retraction by Proposition 3.47 and hence $f \in RLP(\mathrm{cof}\, s\mathcal{S}et)$ by Proposition 3.33. Again we can find a diagonal in

$$\begin{tikzcd} X \arrow[r, tail, "c'"] \arrow[d, tail, "c"'] & X' \arrow[d, two heads, "f"] \\ Y \arrow[r, equal] \arrow[ur, dashed, "\exists d"] & Y, \end{tikzcd}$$

which shows that c is a retract of c' and hence $c \in LLP(\mathrm{fib}\, s\mathcal{S}et)$, because the latter is closed under retracts by Lemma 3.13.

(ii) Given a lifting problem

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, tail, "c"', "\simeq"] & X \arrow[d, two heads, "f"] \\ B \arrow[r, "v"] \arrow[ur, dashed, "\exists d"] & Y \arrow[r, two heads] & *, \end{tikzcd}$$

we may factor $B \longrightarrow *$ as $B \xrightarrow{c'} B' \longrightarrow *$ with $c' \in LLP(\mathrm{fib}\, s\mathcal{S}et)$. So there is a diagonal in the square

$$\begin{tikzcd} B \arrow[r, "v"] \arrow[d, "c'"'] & Y \arrow[d, ->>] \\ B' \arrow[r] \arrow[ur, dashed, "\exists v'"] & *. \end{tikzcd}$$

Moreover $c'c \in \mathrm{cof}\, s\mathcal{S}et \cap ws\mathcal{S}et$ and hence $c'c \in LLP(\mathrm{fib}\, s\mathcal{S}et)$ by (i), because $B' \longrightarrow *$. So we can find a diagonal in the square

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "c'c"'] & X \arrow[d, ->>, "f"] \\ B' \arrow[r, "v'"] \arrow[ur, dashed, "\exists d'"] & Y. \end{tikzcd}$$

Then $d := d'c'$ solves the first lifting problem, because by construction

$$fd = fd'c' = v'c' = v, \qquad dc = d'c'c = u.$$

□

We also need the following "fibrerd version" of the dual of Proposition 3.47.

Proposition 3.49
*Given a t-**fibered homotopy equivalence***

$$\begin{tikzcd} X \arrow[rr, tail, "c"] \arrow[dr, ->>, "tc"'] & & Y \arrow[dl, ->>, "t"] \\ & T & \end{tikzcd}$$

meaning that there is a map $Y \xrightarrow{d} X$ satisfying

$$tcd = t, \qquad \mathrm{id}_X \underset{h}{\simeq} dc, \quad tch = tc\pi_X, \qquad \mathrm{id}_Y \underset{k}{\simeq} cd, \quad tk = t\pi_Y.$$

Then c is a strong deformation section.

Proof. Ignoring the fact that in general we have $T \neq *$, the arguments of the proof are exactly the same as the proof of the implication (ii) $\Rightarrow$ (iii) in Proposition 3.47. First there is a solution for the lifting problem

$$\begin{tikzcd} \Delta^0 \times Y +_{\Delta^0 \times X} \Delta^1 \times X \arrow[r, "d \cup k"] \arrow[d, "d^0 \sqcup c"'] & X \arrow[d, ->>, "tc"] \\ \Delta^1 \times Y \arrow[r, "t\pi_Y"'] \arrow[ur, dashed, "\exists k'"] & T. \end{tikzcd}$$

Setting $r := k'd^1$ we get

$$rc = k'd^1c = k'(\mathrm{id} \times c)d^1 = kd^1 = \mathrm{id}_X, \qquad \mathrm{id}_Y \underset{h}{\simeq} cd, \quad cr \underset{ck'}{\simeq} cd.$$

Using the hypothesis on h and k Proposition 3.46 yields a homotopy

$$\mathrm{id}_Y \underset{h'}{\simeq} cd, \qquad th' = t\pi_Y.$$

Moreover there is a diagonal H in

$$\begin{tikzcd}
U_0 \times Y +_{U_0 \times X} \Delta^1 \times \Delta^1 \times X \arrow[r, "(h' \cup (h'(\mathrm{id} \times (cr))) \cup cr\pi_Y) \cup (\pi_Y(\mathrm{id} \times h')(\mathrm{id} \times \mathrm{id} \times c))"] \arrow[d, "\iota_0 \sqcup c"'] & Y \arrow[d, ->>, "t"] \\
\Delta^1 \times \Delta^1 \times Y \arrow[r, "t\pi_Y"'] \arrow[ur, dashed, "\exists H"] & T
\end{tikzcd}$$

corresponding to the picture

$$\begin{tikzcd}
cr \arrow[r, "cr\pi_X"] & cr & & cr \arrow[r, "cr\pi_Y"] \arrow[dr, phantom, "\exists H"] & cr \\
\mathrm{id}_Y \arrow[u, "h'"] & cr \arrow[u, "crh'"'] \arrow[r, hook] & {} & \mathrm{id}_Y \arrow[u, "h'"] \arrow[r, dashed, "h'':="'] & cr. \arrow[u, "crh'"']
\end{tikzcd}$$

By construction we have

$$\mathrm{id}_X \underset{h''}{\simeq} cr, \qquad h''(\mathrm{id} \times c) = c\pi_X,$$

meaning that c is a strong deformation section.

□

3.10 Verifying the model structure on simplicial sets using Kan's functor Ex^∞

As mentioned before, for the key step in the construction of the model structure on $s\mathcal{S}et$, we need a well-behaved functorial fibrant replacement $X \xrightarrow{\simeq} Q(X) \twoheadrightarrow *$, for $X \in s\mathcal{S}et$. One example of such a functor is Kan's functor Ex^∞, which is closely related to barycentric subdivision.

Definition 3.50
The ***barycentric subdivision*** $\mathrm{sd}_B X$ *of a simplicial set* $X \in s\mathcal{S}et$ *is defined as*

$$\mathrm{sd}_B X := \operatorname*{colim}_{\underline{m} \in \Delta/X} B\mathrm{sd}_B I(\underline{m}),$$

where $B\mathrm{sd}_B I(\underline{n})$ *is the nerve of the barycentric subdivision of the simplicial complex* $I(\underline{n})$.

Remark 3.51
Using Yoneda's Lemma 2.11, we get an adjunction

$$s\mathcal{S}et(\mathrm{sd}_B X, Y) = s\mathcal{S}et(X, \mathrm{Ex} Y),$$

where $\mathrm{Ex} Y := s\mathcal{S}et(\mathrm{sd}_B \Delta^\bullet, Y)$.

As the following proposition shows, this extends the notion of the barycentric subdivision for simplicial complexes of Definition 1.13.

Proposition 3.52
There is a natural isomorphism

$$f_C := \operatorname*{colim}_{\underline{m}\in\Delta/BC} B\mathrm{sd}_B(\varepsilon_{\underline{m}}) : \mathrm{sd}_B BC = \operatorname*{colim}_{\underline{m}\in\Delta/BC} B\mathrm{sd}_B I(\underline{m}) \xrightarrow{\sim} B(\mathrm{sd}_B C), \quad C \in \mathcal{S}imp_o.$$

Proof. Suppose first, that $n := \sharp C < \infty$. Then $C = s_1 \cup ... \cup s_m$ with $s_i \in S(C)$ maximal. We will prove that f_C is an isomorphism by induction on $m \geq 1$.

- For $m = 1$ there is a (unique) isomorphism $g : I(\underline{n}) \xrightarrow{\sim} C$. In particular $g \in \Delta/BC$ forms a final object, so

  $$\iota_g : B\mathrm{sd}_B I(\underline{n}) \xrightarrow{\sim} \operatorname*{colim}_{\underline{m}\in\Delta/BC} B\mathrm{sd}_B(\varepsilon_{\underline{m}}) = \mathrm{sd}_B BC.$$

  As also $f_C \iota_g = B\mathrm{sd}_B(g)$ is an isomorphism, it follows that f_C is an isomorphism.

- For $m > 1$ define the ordered subcomplex $C' := s_1 \cup ... \cup s_{m-1} \leq C$. Then there is are canonical cartesian squares of ordered simplicial complexes

  $$\begin{tikzcd} C' \cap s_m \arrow[r, hook] \arrow[d, hook] & s_m \arrow[d, hook] \\ C' \arrow[r, hook] & C \end{tikzcd} \qquad \begin{tikzcd} \mathrm{sd}_B C' \cap s_m \arrow[r, hook] \arrow[d, hook] & \mathrm{sd}_B s_m \arrow[d, hook] \\ \mathrm{sd}_B C' \arrow[r, hook] & \mathrm{sd}_B C. \end{tikzcd}$$

  As the nerve functor $\mathcal{S}imp_o \xrightarrow{B} s\mathcal{S}et$ preserves limits, we get induced cartesian squares of simplicial sets

  $$\begin{tikzcd} B(C' \cap s_m) \arrow[r, hook] \arrow[d, hook] & Bs_m \arrow[d, hook] \\ BC' \arrow[r, hook] & BC. \end{tikzcd} \qquad \begin{tikzcd} B\mathrm{sd}_B(C' \cap s_m) \arrow[r, hook] \arrow[d, hook] & B\mathrm{sd}_B s_m \arrow[d, hook] \\ B\mathrm{sd}_B C' \arrow[r, hook] & B\mathrm{sd}_B C. \end{tikzcd}$$

  Using $C = C' \cup s_m$ and hence $\mathrm{sd}_B C = \mathrm{sd}_B C' \cup \mathrm{sd}_B s_m$, we get $BC = BC' \cup Bs_m$ and $B\mathrm{sd}_B C = B\mathrm{sd}_B C' \cup B\mathrm{sd}_B s_m$. It follows that the two squares are also cocartesian. Using that $s\mathcal{S}et \xrightarrow{\mathrm{sd}_B} s\mathcal{S}et$ is a left adjoint by Remark 3.51, it commutes with colimits by Corollary 2.47, so also sd_B applied to the left square is a pushout square. By the induction hypothesis for $C' = s_1 \cup ... \cup s_{m-1}$ and $C' \cap s_m = (s_1 \cap s_m) \cup ... \cup (s_{m-1} \cap s_m)$, the vertical maps in the diagram

  $$\begin{tikzcd} \mathrm{sd}_B BC' \arrow[d, "f_{C'}"', "\wr"] & \mathrm{sd}_B B(C' \cap s_m) \arrow[l] \arrow[r] \arrow[d, "f_{C' \cap s_m}"', "\wr"] & \mathrm{sd}_B Bs_m \arrow[d, "f_{s_m}"', "\wr"] \\ B\mathrm{sd}_B C' & B\mathrm{sd}_B(C' \cap s_m) \arrow[l] \arrow[r] & B\mathrm{sd}_B s_m \end{tikzcd}$$

  are isomorphisms. So taking the colimit horizontally yields the isomorphism $f_C : \mathrm{sd}_B BC \xrightarrow{\sim} B\mathrm{sd}_B C$.

For $\sharp C = \infty$ there is a comutative square

$$\begin{tikzcd} \operatorname*{colim}_{\substack{F \le C,\\ \text{finite}}} \mathrm{sd}_B BF \arrow[r, "\operatorname{colim}_F f_F", "\sim"'] \arrow[d, "\wr"'] & \operatorname*{colim}_{\substack{F \le C,\\ \text{finite}}} B\mathrm{sd}_B F \arrow[d, "\wr"'] \\ \mathrm{sd}_B BC \arrow[r, "f_C"] & B\mathrm{sd}_B C, \end{tikzcd}$$

where the vertical maps are isomorphisms, because sd_B and B commute with the filtered colimit, which is just a union in this case. As the square commutes, f_C must be an isomorphism.

□

Corollary 3.53
The natural homeomorphism of Theorem 1.17

$$h_C : |\mathrm{sd}_B C| \xrightarrow{\sim} |C|, \qquad C \in \mathcal{S}imp_o$$

extends to a natural homeomorphism

$$\begin{tikzcd} {|\mathrm{sd}_B C|} \arrow[rr, "h_C", "\sim"'] \arrow[d, equal] & & {|C|} \\ \operatorname*{colim}_{\underline{m} \in \Delta/C} |B\mathrm{sd}_B I(\underline{n})| \arrow[r, "\operatorname{colim} h_{I(\underline{m})}", "\sim"'] & \operatorname*{colim}_{\underline{m} \in \Delta/C} |BI(\underline{n})|, \arrow[ur, "\wr"'] \end{tikzcd}$$

where the isomorphism on the right is induced by the canonical one from the co-Yoneda Lemma 2.51.

Having developed the language of category, it is easy to see that the barycentric subdivision functor on the category of simplicial complexes factors over the category of partially ordered sets:

Remark 3.54
Every partially ordered set $(P, \le)$ induces an ordered simplicial complex $T(P)$ with vertices P and simplices

$$ST(P) = \{s \subset P;\ 0 < \sharp s < \infty,\ \text{“}\le\text{” restricts to a total order on } s\}.$$

Then by construction, we have

(i) $BP = \mathcal{C}at(\text{-}, P) = \mathcal{S}imp(\text{-}, T(P)) = BT(P)$,

(ii) $\mathrm{sd}_B X = TS(X)$, *for all* $X \in \mathcal{S}imp$.

In particular we have

$$\mathrm{sd}_B X := \operatorname*{colim}_{\underline{m} \in \Delta/X} B\mathrm{sd}_B I(\underline{n}) = \operatorname*{colim}_{\underline{m} \in \Delta/X} BSI(\underline{n}),$$

where $BSI(\underline{n})$ is the nerve of the partially ordered set of simplices in $I(\underline{n})$.

Using that partially ordered sets can be considered as categories, for which we can more easily construct homotopy equivalences by using Proposition 2.89, the preceding characterization of the subdivision functor is quite useful for actual proofs.

Proposition 3.55
The following holds.

(i) $\mathrm{sd}_B \mathrm{cof}\, s\mathcal{S}et \subset \mathrm{cof}\, s\mathcal{S}et$, $\quad \mathrm{Ex}(RLP(\mathrm{cof}\, s\mathcal{S}et)) \subset RLP(\mathrm{cof}\, s\mathcal{S}et)$,

(ii) $\mathrm{Ex}(\mathrm{fib}\, s\mathcal{S}et) \subset \mathrm{fib}\, s\mathcal{S}et$.

Proof.

(i) There are equivalences

$$\begin{aligned}
&\mathrm{sd}_B c \in \mathrm{cof}\, s\mathcal{S}et, && \forall\, c \in \mathrm{cof}\, s\mathcal{S}et \\
\iff\ &\mathrm{sd}_B c \in LLP(f), && \forall\, c \in \mathrm{cof}\, s\mathcal{S}et,\ f \in RLP(\mathrm{cof}\, s\mathcal{S}et) \\
\iff\ &\mathrm{Ex} f \in RLP(c), && \forall\, c \in \mathrm{cof}\, s\mathcal{S}et,\ f \in RLP(\mathrm{cof}\, s\mathcal{S}et) \\
\iff\ &\mathrm{Ex} f \in RLP(\partial\Delta^n \overset{\varepsilon}{\hookrightarrow} \Delta^n), && \forall\, f \in RLP(\mathrm{cof}\, s\mathcal{S}et), \quad n \geq 0 \\
\iff\ &\mathrm{sd}_B(\partial\Delta^n \overset{\varepsilon}{\hookrightarrow} \Delta^n) \in LLP(f), && \forall\, f \in RLP(\mathrm{cof}\, s\mathcal{S}et), \quad n \geq 0 \\
\iff\ &\mathrm{sd}_B(\partial\Delta^n \overset{\varepsilon}{\hookrightarrow} \Delta^n) \in \mathrm{cof}\, s\mathcal{S}et, && \forall\, n \geq 0,
\end{aligned}$$

where the first equivalence holds, because $(\mathrm{cof}\, s\mathcal{S}et, RLP(\mathrm{cof}\, s\mathcal{S}et))$ is a weak factorization system by Theorem 3.29 and hence $\mathrm{cof}\, s\mathcal{S}et = LLP(RLP(\mathrm{cof}\, s\mathcal{S}et))$. The second equivalence follows from Remark 3.24 applied to the adjunction $(\mathrm{sd}_B, \mathrm{Ex})$ of Remark 3.51. The thirs equivalence holds, because

$$RLP(\mathrm{cof}\, s\mathcal{S}et) = RLP\{\partial\Delta^n \overset{\varepsilon}{\hookrightarrow} \Delta^n;\ n \geq 0\}$$

by Theorem 3.29 again. The last two equivalences are similar.

For checking the last assertion, by using the commutative diagram

$$\begin{tikzcd}
\mathrm{sd}_B\partial\Delta^n \arrow[r, equal] \arrow[d, "\mathrm{sd}_B\varepsilon"'] & \mathrm{sd}_B B\partial I(\underline{n}) \arrow[r, "f_{\partial I(\underline{n})}", "\sim"'] \arrow[d, "\mathrm{sd}_B\varepsilon"'] & B\mathrm{sd}_B\partial I(\underline{n}) \arrow[d, "B\mathrm{sd}_B(\varepsilon)"'] \\
\mathrm{sd}_B\Delta^n \arrow[r, equal] & \mathrm{sd}_B BI(\underline{n}) \arrow[r, "f_{I(\underline{n})}", "\sim"'] & B\mathrm{sd}_B I(\underline{n})
\end{tikzcd}$$

it suffices to verify that the right vertical map is injective. But this is true since $\partial I(\underline{n}) \hookrightarrow I(\underline{n})$ and hence $\mathrm{sd}_B I(\underline{n}) \hookrightarrow \mathrm{sd}_B I(\underline{n})$ is injective.

(ii) By Theorem 3.29 we have

$$\mathrm{fib}\, s\mathcal{S}et = RLP\{\Delta^0 \times \Delta^n +_{\Delta^0 \times \partial\Delta^n} \Delta^1 \times \partial\Delta^n \overset{d^i \sqcup \varepsilon}{\hookrightarrow} \Delta^1 \times \Delta^n;\ n \geq 0,\ i = 0, 1\}.$$

So by the same arguments as in (i), it suffices to check that

$$\mathrm{sd}_B(d^i \sqcup \varepsilon) \in LLP(\mathrm{fib}\, s\mathcal{S}et), \qquad \forall\, n \geq 0,\ i = 0, 1.$$

However this is much more difficult than the case in (i) and will be obtained in several steps.

a) First we prove that $\mathrm{sd}_B BI(\underline{m} \overset{j}{\hookrightarrow} \underline{n}) \in LLP(\mathrm{fib}\, s\mathcal{S}et)$, for every injection $\underline{m} \overset{j}{\hookrightarrow} \underline{n}$. Using Proposition 3.52 and Remark 3.54 we have a natural isomorphism

$$\mathrm{sd}_B BI(\underline{m}) \xrightarrow{\sim} B\mathrm{sd}_B I(\underline{m}) = BTSI(\underline{m}) = BSI(\underline{m}), \qquad m \geq 0,$$

and the partially ordered set $SI(\underline{m})$ considered as a category has a terminal object $\underline{m}$. So Lemma 3.45 implies that $\mathrm{sd}_B BI(j) \in w_a s\mathcal{S}et$. Since $\mathrm{sd}_B BI(j) \in \mathrm{cof}\, s\mathcal{S}et$ by (i), Proposition 3.43 implies that $\mathrm{sd}_B BI(j) \in LLP(\mathrm{fib}\, s\mathcal{S}et)$.

b) Next we prove that $\mathrm{sd}_B B(I(\underline{0}) \times C \overset{d^i\times\mathrm{id}}{\longrightarrow} I(\underline{1}) \times C) \in LLP(\mathrm{fib}\, s\mathcal{S}et)$, where $i = 0, 1$ and C is an ordered simplicial complex with $n := \sharp C < \infty$. Then $C = s_1 \cup ... \cup s_m$ with $s_i \in S(C)$ maximal. We prove that $d^i \times \mathrm{id}$ is an absolute weak equivalence by induction on $m \geq 1$.

- For $m = 1$ there is an isomorphism $g : I(\underline{n}) \xrightarrow{\sim} C$, so we may assume $C = I(\underline{n})$. We only prove the case $i = 0$ and proceed similarly as in the proof of Proposition 3.35 by constructing intermediate subcomplexes

$$I(\underline{0}) \times I(\underline{n}) = U_0 \hookrightarrow U_1 \hookrightarrow ... \hookrightarrow U_n = I(\underline{1}) \times I(\underline{n}).$$

More precisely we define

$$U_{k+1} := U_k \cup t_k, \quad t_k = \{(0,0), ..., (0,k), (1,k), ..., (1,n)\}, \quad 1 \leq k \leq n.$$

Then $U_k \cap t_k = t'_k = \{(0,0), ..., (0,k-1), (1,k), ..., (1,n)\} \cong I(\underline{n})$ and we get a cartesian square as on the left

$$\begin{tikzcd} I(\underline{n}) \arrow[r] \arrow[d, "d^k"'] & U_k \arrow[d] \\ I(\underline{n+1}) \arrow[r] & U_{k+1}, \end{tikzcd} \qquad \begin{tikzcd} BI(\underline{n}) \arrow[r] \arrow[d, "d^k"'] & BU_k \arrow[d, "=:j_{k+1}"] \\ BI(\underline{n+1}) \arrow[r] & BU_{k+1}. \end{tikzcd}$$

Hence also the right square is cartesian and by definition of U_{k+1} also cocartesian. The right square stays cocartesian after applying the left adjoint functor sd_B. Since $\mathrm{sd}_B d^k \in LLP(\mathrm{fib}\, s\mathcal{S}et)$ by b), Lemma 3.13 implies that also $(BU_k \longrightarrow BU_{k+1}) \in LLP(\mathrm{fib}\, s\mathcal{S}et)$ and thus $\mathrm{sd}_B(d^i \times \mathrm{id}) = \mathrm{sd}_B j_n \circ ... \circ \mathrm{sd}_B j_1 \in LLP(\mathrm{fib}\, s\mathcal{S}et)$.

- For $m > 1$ we consider the ordered subcomplex $C' := s_1 \cup ... \cup s_{m-1} \leq C$ again. Like in the proof of Proposition 3.52 we get a commutative diagram

$$\begin{tikzcd} \mathrm{sd}_B B(I(\underline{0}) \times C') \arrow[d, "\mathrm{sd}_B B(d^i\times\mathrm{id})"'] & \mathrm{sd}_B B(I(\underline{0}) \times (C' \cap s_m)) \arrow[l, tail] \arrow[r, tail] \arrow[d, "\mathrm{sd}_B B(d^i\times\mathrm{id})"'] & B(I(\underline{1}) \times s_m) \arrow[d, "\mathrm{sd}_B B(d^i\times\mathrm{id})"'] \\ \mathrm{sd}_B B(I(\underline{1}) \times C') & B(I(\underline{1}) \times (C' \cap s_m)) \arrow[l, tail] \arrow[r, tail] & B(I(\underline{1}) \times s_m), \end{tikzcd}$$

in which the vertical maps are absolute weak equivalences by the induction hypothesis. As all the horizontal maps are cofibrations by (i), Theorem 3.41

(i) implies that the horizontal colimit $\mathrm{sd}_B B(I(\underline{0})\times C) \overset{\mathrm{sd}_B B(d^i\times \mathrm{id}}{\longrightarrow} B(I(\underline{1})\times C)$ also is an absolute weak equivalence. As it is injective and thus a cofibration, Proposition 3.43 implies that is in $LLP(\mathrm{fib}\, s\mathcal{S}et)$.

c) Now in the commutative diagram

$$\begin{tikzcd}
\mathrm{sd}_B(\Delta^0 \times \partial\Delta^n) \arrow[r, "\mathrm{sd}_B(d^i\times\mathrm{id})"] \arrow[d, "\mathrm{sd}_B(\mathrm{id}\times\varepsilon)"'] & \mathrm{sd}_B(\Delta^1 \times \partial\Delta^n) \arrow[d] \arrow[dr, bend left, "\mathrm{sd}_B(\mathrm{id}\times\varepsilon)"] & \\
\mathrm{sd}_B(\Delta^0 \times \Delta^n) \arrow[r] \arrow[rr, bend right, "\mathrm{sd}_B(d^i\times\mathrm{id})"'] & \mathrm{sd}_B(\Delta^0 \times \Delta^n +_{\Delta^0\times\partial\Delta^n} \Delta^1 \times \partial\Delta^n) \arrow[r, dashed, "d^i\sqcup\varepsilon"] & \mathrm{sd}_B(\Delta^1 \times \Delta^n)
\end{tikzcd}$$

every map is injective by Theorem 3.30 and (i). Using the isomorphisms

$$\Delta^m \times \partial\Delta^n = B(I(\underline{m}) \times \partial I(\underline{n})), \qquad \Delta^m \times \Delta^n = B(I(\underline{m}) \times I(\underline{n})), \qquad m = 0, 1,$$

we have shown that the upper horizontal map and the lower curved map is in $LLP(\mathrm{fib}\, s\mathcal{S}et)$. As the square is cocartesian, Lemma 3.13 implies that the lower horizontal map is in $LLP(\mathrm{fib}\, s\mathcal{S}et)$. So Corollary 3.44 implies that $d^i\sqcup\varepsilon \in LLP(\mathrm{fib}\, s\mathcal{S}et)$.

□

For our purposes much more important than the map h is the following map a.

Proposition 3.56
The natural so-called ***last vertex map*** *of partially ordered sets*

$$V : SI(\underline{m}) \twoheadrightarrow \underline{m}, \quad s \longmapsto \max s, \qquad m \geq 0$$

induces a natural surjective absolute weak equivalence

$$\begin{tikzcd}
\mathrm{sd}_B X \arrow[rr, two heads, "a_X", "\simeq"'] \arrow[d, equal] & & X \\
\operatorname*{colim}_{\underline{m}\in\Delta/X} B\mathrm{sd}_B I(\underline{n}) \arrow[r, equal] & \operatorname*{colim}_{\underline{m}\in\Delta/X} BSI(\underline{n}) \arrow[r, "\operatorname{colim}_{\underline{m}} BV", "\sim"'] & \operatorname*{colim}_{\underline{m}\in\Delta/X} B(\underline{n}). \arrow[u, "\wr"']
\end{tikzcd}$$

Again the isomorphism on the right is induced by the canonical one from the co-Yoneda Lemma 2.51.

Proof. The map V is a retraction, an order-preserving section is given by

$$W : \underline{m} \longrightarrow SI(\underline{m}), \quad k \longmapsto \underline{k}.$$

It follows that also BV is a retraction and thus an epimorphism. As epimorphisms are stable under colimits, it follows that a_X is an epimorphism.

It remains to prove that a_X is an absolute weak equivalence, which is a lenghty induction using our theory of absolute weak equivalence established in section 3.8. First we assume $X = BC$, where C is an ordered simplicial complex with $n := \sharp C < \infty$. Then $C = s_1 \cup \ldots \cup s_m$ with $s_i \in S(C)$ maximal. We prove that a_{BC} is an absolute weak equivalence by induction on $m \geq 1$.

- For $m = 1$ there is an isomorphism $g : I(\underline{n}) \xrightarrow{\sim} C$. Using Remark 3.54 we get a commutative diagram

$$\begin{tikzcd} BSI(\underline{n}) \arrow[r, equal] \arrow[d, "BV"'] & B\mathrm{sd}_B I(\underline{n}) & \mathrm{sd}_B BI(\underline{n}) \arrow[l, "f_{I(\underline{n})}"', "\sim"] \arrow[r, "\mathrm{sd}_B B(g)", "\sim"'] \arrow[d, "a_{BI(\underline{n})}"'] & \mathrm{sd}_B BC \arrow[d, "a_{BC}"] \\ B(\underline{n}) \arrow[rr, equal] & & BI(\underline{n}) \arrow[r, "B(g)", "\sim"'] & BC, \end{tikzcd}$$

  so it suffices to prove that the left vertical map is an absolute weak equivalence. But considering the partially ordered sets $SI(\underline{n})$ and $\underline{n}$ as categories, the objects $\underline{n} \in SI(\underline{n})$ and $n \in \underline{n}$ are terminal. So Lemma 3.45 implies that BV is an anodyne extension.

- For $m > 1$ we consider the ordered subcomplex $C' := s_1 \cup \ldots \cup s_{m-1} \leq C$ again. Like in the proof of Proposition 3.52 we get a commutative diagram

$$\begin{tikzcd} \mathrm{sd}_B BC' \arrow[d, "a_{BC'}"'] & \mathrm{sd}_B B(C' \cap s_m) \arrow[l, tail] \arrow[r, tail] \arrow[d, "a_{B(C'\cap s_m)}"'] & Bs_m \arrow[d, "a_{Bs_m}"'] \\ BC' & B(C' \cap s_m) \arrow[l, tail] \arrow[r, tail] & Bs_m, \end{tikzcd}$$

  in which the vertical maps are absolute weak equivalences by the induction hypothesis. Moreover taking the colimit horizontally yields the map $\mathrm{sd}_B BC \xrightarrow{a_{BC}} BC$. As all the horizontal maps are injective and hence cofibrations, Theorem 3.41 (i) implies that also a_{BC} is an absolute weak equivalence. This concludes the proof for the induction step.

Now let $X \in s\mathcal{S}et$ be an arbitrary simplicial set. By induction on $n \geq 0$ we will show that $\mathrm{sd}_B \mathrm{sk}_n X \xrightarrow{a_{\mathrm{sk}_n X}} \mathrm{sk}_n X$ is an absolute weak equivalence.

- For $n = 0$ the simplicial set $\mathrm{sk}_0 X$ is a set of points. As $\mathrm{sd}_B BI(\underline{0}) \xrightarrow{\sim} B\mathrm{sd}_B I(\underline{0}) = BI(\underline{0})$ it follows that $\mathrm{sd}_B \mathrm{sk}_0 X \xrightarrow{a_{\mathrm{sk}_0 X}} \mathrm{sk}_0 X$ is an isomorphism.

- For $n > 0$, recall that by Proposition 2.76 we have cocartesian squares

$$\begin{tikzcd} {}^{\tilde{X}_n} B\partial I(\underline{n}) = \tilde{X}_n \times \partial\Delta^n \arrow[r] \arrow[d, hook] & \mathrm{sk}_{n-1} X \arrow[d] \\ {}^{\tilde{X}_n} BI(\underline{n}) = \tilde{X}_n \times \Delta^n \arrow[r] & \mathrm{sk}_n X. \end{tikzcd}$$

Using that sd_B is a left adjoint and therefore commutes with pushouts and coproduct, taking colimits horizontally in the commutative diagram

$$\begin{tikzcd}
\coprod^{\tilde{X}_n} \mathrm{sd}_B BI(\underline{n}) \arrow[d, "\coprod^{\tilde{X}_n} a_{BI(\underline{n})}"'] & \coprod^{\tilde{X}_n} \mathrm{sd}_B B\partial I(\underline{n}) \arrow[l, tail] \arrow[r] \arrow[d, "\coprod^{\tilde{X}_n} a_{B\partial I(\underline{n})}"'] & \mathrm{sd}_B \mathrm{sk}_{n-1} X \arrow[d, "a_{\mathrm{sk}_{n-1}X}"'] \\
\coprod^{\tilde{X}_n} BI(\underline{n}) & \coprod^{\tilde{X}_n} B\partial I(\underline{n}) \arrow[l, tail] \arrow[r] & \mathrm{sk}_{n-1} X
\end{tikzcd}$$

yields the map $\mathrm{sd}_B \mathrm{sk}_n X \xrightarrow{a_{\mathrm{sk}_n X}} \mathrm{sk}_n X$. Using that absolute weak equivalences are closed under arbitrary coproducts by Proposition 3.42, the left two vertical maps are absolute weak equivalences, because $\partial I(\underline{n}) \leq I(\underline{n})$ are finite simplicial complexes. It follows that also $a_{\mathrm{sk}_n X}$ is an absolute weak equivalence by Theorem 3.41, which proves the induction step.

Finally we apply Theorem 3.41 to the map of sequences

$$\begin{tikzcd}
\mathrm{sd}_B \mathrm{sk}_0 X \arrow[r, tail] \arrow[d, "a_{\mathrm{sk}_0 X}"'] & \mathrm{sk}_1 X \arrow[r, tail] \arrow[d, "a_{\mathrm{sk}_1 X}"'] & \mathrm{sk}_2 X \arrow[r, tail] \arrow[d, "a_{\mathrm{sk}_2 X}"'] & \ldots \\
\mathrm{sk}_0 X \arrow[r, tail] & \mathrm{sk}_1 X \arrow[r, tail] & \mathrm{sk}_2 X \arrow[r, tail] & \ldots
\end{tikzcd}$$

to see that also $\mathrm{sd}_B X \xrightarrow{a_X} X$ is an absolute weak equivalence.

□

Infact we are more interested in the functor Ex right adjoint to sd_B. An colimit over its iteration will provide the desired fibrant replacement functor Ex^∞, which was introduced by Daniel Kan during his foundational studies of simplicial sets.

Definition 3.57
For every $X \in s\mathcal{S}et$, we define

$$\mathrm{Ex}^\infty X := \operatorname{colim}(X \xrightarrow{b_X} \mathrm{Ex}X \xrightarrow{b_{\mathrm{Ex}X}} \mathrm{Ex}^2 X \xrightarrow{b_{\mathrm{Ex}^2 X}} \ldots),$$

where $X \xrightarrow{b_X} \mathrm{Ex}X$ is the map corresponding to $\mathrm{sd}_B \xrightarrow{a_X} X$ under the adjunction 3.51.

Proposition 3.58
For every $X \in s\mathcal{S}et$, we have $\mathrm{Ex}^\infty X \longrightarrow\!\!\!\!\triangleright \, *$.

Proof. Using the description of Proposition 3.35

$$\mathrm{fib}\, s\mathcal{S}et = RLP\{\Lambda^n_k := \bigcup_{\substack{0 \leq i \leq n, \\ i \neq k}} d^i \Delta^{n-1} \hookrightarrow \Delta^n;\ 0 \leq k \leq n\}$$

we need to prove that there is a solution for every lifting problem

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "u"] \arrow[d, hook] & \mathrm{Ex}^\infty X \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dashed, "\exists d"] & *.
\end{tikzcd}$$

As Λ^n_k is a finite simplicial set, Proposition 3.22 implies that u factors as

$$\Lambda^n_k \xrightarrow{v} \mathrm{Ex}^m X \xrightarrow{\iota_m} \mathrm{Ex}^\infty X,$$

for some $m \geq 1$. So setting $Y := \mathrm{Ex}^m X$ it suffices to construct a map d in every diagram

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "v"] \arrow[d, hook] & \mathrm{Ex}Y \arrow[r, "b_{\mathrm{Ex}Y}"] & \mathrm{Ex}^2 Y \arrow[d] \\
\Delta^n \arrow[rru, dashed, "\exists d"] \arrow[rr] & & *
\end{tikzcd}$$

By applying Remark 3.24 twice and using that $\mathrm{Ex}* = s\mathcal{S}et(\mathrm{sd}_B\Delta^\bullet, *) = *$, we can equivalently solve one of the lifting problems below

$$\begin{tikzcd}
\mathrm{sd}_B\Lambda^n_k \arrow[r, "v'"] \arrow[d, hook] & Y \arrow[r, "b_Y"] & \mathrm{Ex}Y \arrow[d] \\
\mathrm{sd}_B\Delta^n \arrow[rru, dashed, "\exists d"] \arrow[rr] & & *
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{sd}^2_B\Lambda^n_k \arrow[r, "\mathrm{sd}_B a_{\Lambda^n_k}"] \arrow[d, hook] & \mathrm{sd}_B\Lambda^n_k \arrow[r, "v'"] & Y \arrow[d] \\
\mathrm{sd}^2_B\Delta^n \arrow[ru, dashed, "\exists d"] \arrow[rru, dashed, "\exists d'"'] \arrow[rr] & & *
\end{tikzcd}$$

Recall that in the proof of Proposition 3.35 we introduced the ordered subcomplex $A^n_k \leq I(\underline{n})$, which after applying the nerve functor yields the simplicial subset $\Lambda^n_k \leq \Delta^n$. Now by construction of a and f there is a commutative square

$$\begin{tikzcd}
BSI(\underline{n}) \arrow[r, equal] \arrow[d, "BV"'] & BTSI(\underline{n}) \arrow[r, equal] \arrow[d, "BV"'] & B\mathrm{sd}_B I(\underline{n}) & \mathrm{sd}_B BI(\underline{n}) \arrow[l, "f_{I(\underline{n})}"', "\sim"] \arrow[r, equal] \arrow[d, "a_{BI(\underline{n})}"'] & \mathrm{sd}_B\Delta^n \arrow[d, "a_{\Delta^n}"'] \\
B(\underline{n}) \arrow[r, equal] & BI(\underline{n}) \arrow[rr, equal] & & BI(\underline{n}) \arrow[r, equal] & \Delta^n.
\end{tikzcd}$$

So by naturality of a and f the map $TSI(\underline{n}) \xrightarrow{V} I(\underline{n})$ restricts to a map $TSI(A^n_k) \xrightarrow{V} A^n_k$ and there are natural isomorphisms

$$\begin{tikzcd}
\mathrm{sd}_B\Delta^n \arrow[d, "f_{I(\underline{n})}"', "\wr"] & \mathrm{sd}_B\Lambda^n_k \arrow[l, hook'] \arrow[r, "a_{\Lambda^n_k}"] \arrow[d, "f_{A^n_k}"', "\wr"] & \Lambda^n_k \arrow[d, equal] \\
BTSI(\underline{n}) & BTS(A^n_k) \arrow[l, hook'] \arrow[r, "BV"] & B(A^n_k).
\end{tikzcd}$$

So applying sd_B and using f again it suffices to solve the lifting problem of simplicial complexes (the map d need and will not be order-preserving!)

$$\begin{tikzcd}
TS(A^n_k) \arrow[r, "V"] \arrow[d, hook] & A^n_k \\
TSI(\underline{n}). \arrow[ru, dashed, "\exists d"']
\end{tikzcd}$$

But this is possible by setting

$$d : TSI(\underline{n}) \longrightarrow A^n_k, \quad s \longmapsto \begin{cases} \max s, & s \in S(A^n_k), \\ \{k\}, & s \notin S(A^n_k). \end{cases}$$

Indeed for $s := \{s_0 \subsetneq ... \subsetneq s_m\} \in STSI(\underline{n})$, let j be maximal with $s_j \notin S(A^n_k)$. Then there is an $0 \leq i \leq n,\ i \neq k$ with $i \notin s_j$ and hence

$$i \notin \{\max s_0, ..., \max s_j, \{k\}\} \supset d(s) \in S(A^n_k).$$

$\square$

Finally by using all the good properties of the functor Ex^∞ we can prove the difficult inclusion $\mathrm{fib}\, s\mathcal{S}et \subset RLP(\mathrm{cof}\, s\mathcal{S}et \cap ws\mathcal{S}et)$ in the construction of the model structure.

Proposition 3.59
For every fibration $f : X \longrightarrow Y$ the following holds.

(i) $\mathrm{Ex}^\infty f \in \mathrm{fib}\, s\mathcal{S}et$.

(ii) $X \overset{(b_X,f)}{\longrightarrow} \mathrm{Ex}X \times_{\mathrm{Ex}Y} Y$ *is a strong deformation section.*

(iii) $(X \overset{(\iota_X,f)}{\longrightarrow} \mathrm{Ex}^\infty X \times_{\mathrm{Ex}^\infty Y} Y) \in LLP(\mathrm{fib}\, s\mathcal{S}et)$.

(iv) $f \in RLP(\mathrm{cof}\, s\mathcal{S}et \cap ws\mathcal{S}et)$.

Proof.

(i) Using Theorem 3.29 we need to solve every lifting problem

$$\begin{tikzcd}
\Delta^0 \times \Delta^n +_{\Delta^0 \times \partial\Delta^n} \Delta^1 \times \partial\Delta^n \arrow[r, "u"] \arrow[d, "d^i \sqcup \varepsilon"'] & \mathrm{Ex}^m X \arrow[d, "\mathrm{Ex}^m f"] \\
\Delta^1 \times \Delta^n \arrow[r, "v"'] \arrow[ur, dashed, "\exists d"] & \mathrm{Ex}^m Y,
\end{tikzcd}$$

for $i = 0, 1$ and $m = \infty$. As the two simplicial sets on the left are finite and hence ω-compact by Proposition 3.22, the map u and v factor over $\mathrm{Ex}^m X$ and $\mathrm{Ex}^m Y$ respectively, for some $m < \infty$. So we may assume that $m < \infty$. But then there is a solution d, because $\mathrm{Ex}^m f$ is a fibration by m-fold application of Proposition 3.55.

(ii) Using that the natural map a epimorphic by Proposition 3.56, it follows that

$$b_X : X \cong s\mathcal{S}et(\Delta^\bullet, X) \overset{(a_{\Delta^\bullet})^*}{\longrightarrow} s\mathcal{S}et(\mathrm{sd}_B\Delta^\bullet, X) = \mathrm{Ex}X$$

is injective and hence a cofibration by Definition 3.27. It follows that also the map

$$c := (b_X, f) : X \hookrightarrow \mathrm{Ex}X \times_{\mathrm{Ex}Y} =: X'$$

is injective, because $\pi_Y(b_X, f) = b_X$. So in the commutative diagram

$$\begin{tikzcd}
\mathrm{sd}_B X \arrow[r, "a_X"] \arrow[d, tail, "\mathrm{sd}_B c"'] & X \arrow[d] \arrow[ddr, bend left, "c"] & \\
\mathrm{sd}_B X' \arrow[r] \arrow[drr, bend right, "a_{X'}"'] & \mathrm{sd}_B X' +_{\mathrm{sd}_B X} X \arrow[dr, dashed, "a_{X'} \cup c"'] & \\
 & & X',
\end{tikzcd}$$

the maps c and $\mathrm{sd}_B c$ are cofibrations by Definition 3.27 and Proposition 3.55 respectively. As a_X and $a_{X'}$ are absolute weak equivalences by Proposition 3.56, also $a_{X'} \cup c$ is an absolute weak equivalence by the 2-of-3 property for absolute weak equivalences proven in Proposition 3.38.

Now the commutative diagram on the left induces a commutative square in the middle, for which there is a diagonal and homotopies h, k as depicted below

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "c"'] & X \arrow[d, "(b_X{,}f)"] \\ X' \arrow[r, "(\pi_{\mathrm{Ex}X}{,}\pi_Y)", "="'] & \mathrm{Ex}X \times_{\mathrm{Ex}Y} Y \end{tikzcd} \qquad \begin{tikzcd} \mathrm{sd}_B X' +_{\mathrm{sd}_B X} X \arrow[r, "\pi'_{\mathrm{Ex}X} \cup \mathrm{id}"] \arrow[d, "a_{X'} \cup c"'] & X \arrow[d, "f"] \\ X' \arrow[r, "\pi_Y"'] \arrow[ur, dashed, "\overset{\simeq}{h \cup k}", "\exists d"'] & Y, \end{tikzcd}$$

where $\pi'_{\mathrm{Ex}X}$ is the map corresponding to $\pi_{\mathrm{Ex}X}$ under the adjunction bijection

$$s\mathcal{S}et(\mathrm{sd}_B X', X) = s\mathcal{S}et(X', \mathrm{Ex}X)$$

of Remark 3.51. Defining $k' \in s\mathcal{S}et(\Delta^1, X', \mathrm{Ex}X)$ as the map similarly corresponding to

$$\mathrm{sd}_B(\Delta^1 \times X') \overset{(a_{\Delta^1}, \mathrm{sd}_B(\pi_{X'}))}{\longrightarrow} \Delta^1 \times \mathrm{sd}_B X' \overset{k}{\longrightarrow} X,$$

we can define $k'' := (k', \pi_Y) : \Delta^1 \times X' \longrightarrow \mathrm{Ex}X \times_{\mathrm{Ex}Y} Y = X'$ and get

$$\mathrm{id}_X \underset{h}{\simeq} dc, \quad fh = f\pi_X, \qquad \mathrm{id}_{X'} \underset{k''}{\simeq} cd, \quad fdk'' = \pi_Y k'' = \pi_Y.$$

As f is a fibration and hence also $\mathrm{Ex}f$ by Proposition 3.55, so is also the left vertical map in the cartesian square

$$\begin{tikzcd} \mathrm{Ex}X \times_{\mathrm{Ex}Y} Y \arrow[r, "\pi_{\mathrm{Ex}X}"] \arrow[d, "\pi_Y"'] & \mathrm{Ex}X \arrow[d, "\mathrm{Ex}f"] \\ Y \arrow[r, "b_Y"] & \mathrm{Ex}Y. \end{tikzcd}$$

because $\mathrm{fib}\, s\mathcal{S}et$ is stable under pullbacks by the dual of Lemma 3.13. In particular the map

$$X \overset{c=(b_X,f)}{\longrightarrow} X' = \mathrm{Ex}X \times_{\mathrm{Ex}Y} Y \overset{\pi_Y}{\longrightarrow} *$$

is a π_Y-fibrered homotopy equivalence and hence $(b_X, f) \in LLP(\mathrm{fib}\, s\mathcal{S}et)$ by Proposition 3.49.

(iii) For $m \geq 0$ the lower horizontal map is a fibration by m-fold appliation of Proposition 3.55

$$\begin{tikzcd} \mathrm{Ex}^m X \times_{\mathrm{Ex}^m Y} Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_{\mathrm{Ex}^m X}"'] & Y \arrow[d] \\ \mathrm{Ex}^m X \arrow[r, "\mathrm{Ex}^m f"] & \mathrm{Ex}^m Y. \end{tikzcd}$$

As $\mathrm{fib}\, s\mathcal{S}et$ is stable under pullbacks by the dual of Lemma 3.13, it follows that also the upper horizontal map is a fibration. So applying (ii) yields that the left vertical map in the commutative square below is a strong deformation retraction and hence in $LLP(\mathrm{fib}\, s\mathcal{S}et)$ by Proposition 3.33

$$\begin{tikzcd} \mathrm{Ex}^m X \times_{\mathrm{Ex}^m Y} Y \arrow[r, "b_{\mathrm{Ex}^m X} \times \mathrm{id}"] \arrow[d, "(b{,}\pi_Y)"'] & \mathrm{Ex}^{m+1} Y \times_{\mathrm{Ex}^{m+1} Y} Y \\ \mathrm{Ex}(\mathrm{Ex}^m X \times_{\mathrm{Ex}^m Y} Y) \times_{\mathrm{Ex}Y} Y \arrow[r, "(\mathrm{Ex}(\pi_{\mathrm{Ex}^m X}){,}\mathrm{Ex}(\pi_Y))\times \mathrm{id}"] & (\mathrm{Ex}^{m+1} X \times_{\mathrm{Ex}^{m+1} Y} \mathrm{Ex}Y) \times_{\mathrm{Ex}Y} Y. \arrow[u, "(\pi_{\mathrm{Ex}^{m+1}X}{,}\pi_Y)"'] \end{tikzcd}$$

- As Ex is a right adjoint and therefore commutes with limits, the map $(\mathrm{Ex}(\pi_{\mathrm{Ex}^m X}), \mathrm{Ex}(\pi_Y))$ and hence the lower horizontal map is an isomorphism.
- The right vertical map is an isomorphism, because the two small squares

$$\begin{tikzcd} (\mathrm{Ex}^{m+1} X \times_{\mathrm{Ex}^{m+1} Y} \mathrm{Ex}Y) \times_{\mathrm{Ex}Y} Y \arrow[r] \arrow[d] & \mathrm{Ex}^{m+1} X \times_{\mathrm{Ex}^{m+1} Y} \mathrm{Ex}Y \arrow[r] \arrow[d, "\pi_{\mathrm{Ex}Y}"'] & \mathrm{Ex}^{m+1} X \arrow[d, "\mathrm{Ex}^{m+1} f"'] \\ Y \arrow[r, "b_Y"] & \mathrm{Ex}Y \arrow[r, "b_{\mathrm{Ex}^m Y} \circ \ldots \circ b_{\mathrm{Ex}Y}"] & \mathrm{Ex}^{m+1} Y \end{tikzcd}$$

  are cartesian and hence also the outer square is cartesian.

It follows that the upper horizontal map is in $LLP(\mathrm{fib}\, s\mathcal{S}et)$ and by Lemma 3.13 the same holds for the map

$$(\iota_0, f) : X \xrightarrow{\iota_0} \mathrm{colim}\,(X \overset{(b_X, f)}{\longrightarrow} \mathrm{Ex}X \times_{\mathrm{Ex}Y} Y \overset{b_{\mathrm{Ex}X} \times \mathrm{id}}{\longrightarrow} \ldots) \overset{\sim}{\longrightarrow} \mathrm{Ex}^\infty X \times_{\mathrm{Ex}^\infty Y} Y.$$

Here the last map is an isomorphism, because filtered colimits commute with finite limits and hence pullbacks by Proposition 2.48.

(iv) In the commutative diagram

$$\begin{tikzcd} X \arrow[rr, equal] \arrow[d, "(\iota_0{,}f)"'] & & X \arrow[d, "f"] \\ \mathrm{Ex}^\infty X \times_{\mathrm{Ex}^\infty Y} Y \arrow[rr, "\pi_Y"] \arrow[d, "\pi_{\mathrm{Ex}^\infty X}"'] \arrow[urr, dashed, "\exists d"] & & Y \arrow[d, "\iota_0"] \\ \mathrm{Ex}^\infty X \arrow[rr, "\mathrm{Ex}^\infty f"] & & \mathrm{Ex}^\infty Y \arrow[r] & * \end{tikzcd}$$

the left vertical map is in $LLP(\mathrm{fib}\, s\mathcal{S}et)$. So there is a diagonal d, which proves that f is a retract of π_Y. The lower left horizontal map is a fibration by (i), while the lower right horizontal map is a fibration by Proposition 3.58. So Corollary 3.48 implies that $\mathrm{Ex}^\infty f \in RLP(\mathrm{cof}\, s\mathcal{S}et \cap w\mathcal{S}et)$. It follows that also π_Y and its retract f is in $RLP(\mathrm{cof}\, s\mathcal{S}et \cap w\mathcal{S}et)$, which is stable under pullbacks and retracts by the dual of Lemma 3.13.

□

Remark 3.60
By the structure of the proof of Proposition 3.59 (iv), we see that one can prove the inclusion $\mathrm{fib}\, s\mathcal{S}et \subset RLP(\mathrm{cof}\, s\mathcal{S}et \cap ws\mathcal{S}et)$ *by using any natural transformation* $X \xrightarrow{\eta_X} Q(X)$ *instead of* $X \xrightarrow{\iota_0} \mathrm{Ex}^\infty X$ *having the following properties.*

(i) $Q(X) \longrightarrow\!\!\!\triangleright\, *$, *for all* $X \in s\mathcal{S}et$,

(ii) $Q(\operatorname{fib} s\mathcal{S}et) \subset \operatorname{fib} s\mathcal{S}et$,

(iii) $(X \xrightarrow{(\eta_X, f)} Q(X) \times_{Q(Y)} Y) \in LLP(\operatorname{fib} s\mathcal{S}et)$.

By using different techniques one can show that also the unit $X \xrightarrow{\eta_X} S|X|$ *of the adjunction* $\mathcal{T}op(|X|, Y) = s\mathcal{S}et(X, S(Y))$ *has all these properties.*

After having shown the difficult inclusion $\operatorname{fib} s\mathcal{S}et \subset RLP(\operatorname{cof} s\mathcal{S}et \cap ws\mathcal{S}et)$ in Proposition 3.59 (iv), it is now comparably easy to deduce the model structure:

Theorem 3.61
The category $s\mathcal{S}et$ *is a model category with weak equivalences, fibrations and cofibrations defined as in Definition 3.27.*

Proof. For every $T \longrightarrow\!\!\!\triangleright\, *$ Remark 3.6 implies that the inverse image W_T of $\operatorname{Mor}(\mathcal{S}et^{\mathrm{op}})^\times$ under the functor

$$s\mathcal{S}et \longrightarrow \pi_0\underline{s\mathcal{S}et} \xrightarrow{\pi_0\underline{s\mathcal{S}et}(-,T)} \mathcal{S}et^{\mathrm{op}}$$

satisfies the 2-of-3 axiom. Hence the same holds for the class

$$ws\mathcal{S}et = \bigcap_{T \longrightarrow\!\!\!\triangleright *} W_T.$$

Recall that by Theorem 3.29 there are two weak factorization systems

$$(\operatorname{cof} s\mathcal{S}et, RLP(\operatorname{cof} s\mathcal{S}et)), \qquad (LLP(\operatorname{fib} s\mathcal{S}et), \operatorname{fib} s\mathcal{S}et),$$

and it remains to prove that

$$RLP(\operatorname{cof} s\mathcal{S}et) = \operatorname{fib} s\mathcal{S}et \cap ws\mathcal{S}et, \qquad LLP(\operatorname{fib} s\mathcal{S}et) = \operatorname{cof} s\mathcal{S}et \cap ws\mathcal{S}et.$$

- We begin with the second equality. By Proposition 3.59 we have

  $$\operatorname{fib} s\mathcal{S}et \subset RLP(\operatorname{cof} s\mathcal{S}et \cap ws\mathcal{S}et).$$

  Combining this with Proposition 3.43 and Proposition 3.37 yields

  $$\begin{aligned} LLP(\operatorname{fib} s\mathcal{S}et) &= \operatorname{cof} s\mathcal{S}et \cap w_a s\mathcal{S}et \subset \operatorname{cof} s\mathcal{S}et \cap ws\mathcal{S}et \\ &\subset LLP(RLP(\operatorname{cof} s\mathcal{S}et \cap ws\mathcal{S}et) \subset LLP(\operatorname{fib} s\mathcal{S}et), \end{aligned}$$

  so all these classes are equal.

- For the first equality note that in Remark 3.34 we have already checked that

  $$RLP(\operatorname{cof} s\mathcal{S}et) \subset \operatorname{fib} s\mathcal{S}et \cap ws\mathcal{S}et. \tag{3.5}$$

  So let $f \in \operatorname{fib} s\mathcal{S}et \cap ws\mathcal{S}et$. As $(\operatorname{cof} s\mathcal{S}et, RLP(\operatorname{cof} s\mathcal{S}et))$ is a weak factorization system, the map f can be factored as $X \overset{c}{\rightarrowtail} Z \xrightarrow{g} Y$, where $c \in \operatorname{cof} s\mathcal{S}et$

and $g \in RLP(\mathrm{cof}\, s\mathcal{S}et)$. As $f = gc$ and g are weak equivalences by assumption and (3.5) respectively, also c is a weak equivalence by the 2-of-3 property. So $c \in \mathrm{cof}\, s\mathcal{S}et \cap ws\mathcal{S}et = LLP(\mathrm{fib}\, s\mathcal{S}et)$ by (i) and hence there is a diagonal d in the commutative square

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, tail, "c"'] & X \arrow[d, two heads, "f"] \\ Z \arrow[r, "g"] \arrow[ur, dashed, "\exists d"] & Y. \end{tikzcd}$$

In particular f is a retract of g and thus also $f \in RLP(\mathrm{cof}\, s\mathcal{S}et)$, as this class of maps is closed under retracts by the dual of Lemma 3.13.

□

3.11 Homotopies in model categories

The construction of the homotopy category of a model category is obtained by using the terms of abstract homotopies and homotopy equivalences.

Definition 3.62
Let $\mathcal{C}$ be a model category, $\emptyset$ an initial and $*$ a terminal object in $\mathcal{C}$.

(i) *A **cylinder object** for $X \in \mathcal{C}$ is an object $I \cdot X \in \mathcal{C}$ together with morphisms*

$$\begin{tikzcd} X + X \arrow[d, tail, "i = i_0 \cup i_1"'] \arrow[dr, "\mathrm{id}_X \cup \mathrm{id}_X =: \nabla"] & \\ I \cdot X \arrow[r, "\simeq"'] & X. \end{tikzcd}$$

Every object has a cylinder object given by any factorization of the fold map ∇ provided by the model structrue on $\mathcal{C}$.

(ii) *Two morphisms $f, g \in \mathcal{C}(X, Y)$ are called **left homotopic** via h, short $f \,{}_h\!\simeq g$, if there is a map $h \in \mathcal{C}(I \cdot X, Y)$, for some cylinder object $X + X \overset{i}{\rightarrowtail} I \cdot X \xrightarrow{\simeq} X$, giving rise to a commutative diagram*

$$\begin{tikzcd} & X + X \arrow[dl, "f \cup g"'] \arrow[d, tail, "i"] \arrow[dr, "\nabla"] & \\ Y & I \cdot X \arrow[l, "h"] \arrow[r, "\simeq"'] & X. \end{tikzcd}$$

(iii) *Dually a **path object** for $Y \in \mathcal{C}$ is an object $Y^I \in \mathcal{C}$ together with morphisms*

$$\begin{tikzcd} Y \arrow[r, "\simeq"] \arrow[dr, "\Delta"'] & Y^I \arrow[d, two heads, "p"] \\ & Y \times Y. \end{tikzcd}$$

Every object has a path object given by any factorization of the diagonal map Δ provided by the model structrue on $\mathcal{C}$.

(iv) *Two morphisms $f, g \in \mathcal{C}(X, Y)$ are called* **right homotopic** *via h, short $f \simeq_h g$, if there is a map $h \in \mathcal{C}(X, Y^I)$ for some path object $Y \xrightarrow{\simeq} Y^I \xrightarrow{p} Y \times Y$, giving rise to a commutative diagram*

$$\begin{tikzcd} Y \arrow[r, "\simeq"] \arrow[dr, "\Delta"'] & Y^I \arrow[d, "p"] & X \arrow[l, "h"'] \arrow[dl, "{(f,g)}"] \\ & Y \times Y. & \end{tikzcd}$$

Example 3.63
In the model category $s\mathcal{S}et$ the following holds.

(i) *For every $X \in s\mathcal{S}et$ there is a canonical cylinder object, given by*

$$\begin{tikzcd} X + X = \partial\Delta^1 \times X \arrow[d, tail, "i=d^0 \cup d^1"'] \arrow[dr, "\mathrm{id}_X \cup \mathrm{id}_X =: \nabla"] & \\ \Delta^1 \times X \arrow[r, "\simeq", "\pi_X"'] & X. \end{tikzcd}$$

Indeed i is injective and hence a cofibration, while π_X is a strong deformation retraction and thus a weak equivalence.

(ii) *For every $Y \in s\mathcal{S}et$ with $Y \longrightarrow *$ there is a canonical path object, given by*

$$\begin{tikzcd} Y = \underline{s\mathcal{S}et}(\Delta^0, Y) \arrow[r, "s_0", "\simeq"'] \arrow[dr, "\Delta"'] & \underline{s\mathcal{S}et}(\Delta^1, Y) \arrow[d, "{p:=(d_0,d_1)}"] \\ & \underline{s\mathcal{S}et}(\partial\Delta^1, Y) = Y \times Y. \end{tikzcd}$$

Indeed $p = (d^0 \cup d^1)^*$ is a fibration by Corollary 3.31, since $Y \longrightarrow *$, while s_0 is a strong deformation section and thus a weak equivalence.

In contrast to the simplicial theory developed before, the term of homotopies is now slightly more relaxed, meaning that the cylinder/path objects may vary. This freedom allows the proof of the following Lemma, which we know does not hold in this generality for the term "simplicial homotopy".

Lemma 3.64
Let $\emptyset \rightarrowtail X$ with a cylinder object

$$X + X \overset{i_0 \cup i_1}{\rightarrowtail} I \cdot X \xrightarrow{r_X} X.$$

Then the following holds.

(i) $i_j : X \xrightarrow{\iota_j} X + X \rightarrowtail I \cdot X$ *is a trivial cofibration, for $j = 0, 1$.*

(ii) $X + X \overset{i_1 \cup i_0}{\rightarrowtail} I \cdot X \xrightarrow{\simeq} X$ *is a cylinder object for* X.

To distinguish it from $I \cdot X$ *we denote it by* $I' \cdot X = I \cdot X$ *and call it* ***inverse cylinder object***.

(iii) *Let* $X + X \overset{i_0 \cup i_0}{\rightarrowtail} J \cdot X \xrightarrow{r_X} X$ *be another cylinder object for* X.

Then there is a ***composed cylinder object*** $X + X \xrightarrow{(\iota_0 \circ i_0) \cup (\iota_1 \circ i_1)} (I * J) \cdot X \xrightarrow{r_X \cup r_X} X$ *defined by the pushout square*

$$\begin{tikzcd}
 & & X \arrow[d, "i_1"] & \\
 & X \arrow[r, "i_0"] \arrow[d, "i_1"'] & I \cdot X \arrow[d, "\iota_I"] \arrow[ddr, bend left, "r_X"] & \\
X \arrow[r, "i_0"] & J \cdot X \arrow[r, "\iota_J"] \arrow[drr, bend right, "r_X"'] & (I * J) \cdot X \arrow[dr, dashed, "r_X \cup r_X"] & \\
 & & & X.
\end{tikzcd}$$

(iv) *Left homotopy is an equivalence relation on* $\mathcal{C}(X, Y)$ *for every* $Y \in \mathcal{C}$.

Dual assertions hold for path objects and right homotopies.

Proof.

(i) We have a pushout diagram

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & X \arrow[d, "\iota_0"] \\
X \arrow[r, "\iota_1"] & X + X.
\end{tikzcd}$$

Because X is cofibrant and cofibrations are preserved by pushouts by Lemma 3.13, ι_0, ι_1 is a cofibration. As compositions preserve cofibrations i_0, i_1 are cofibrations, too.

For $j = 0, 1$ the map i_j is a weak equivalence, because of the 2-of-3 axiom for weak equivalences.

$$\begin{tikzcd}
X + X \arrow[r, tail] & I \cdot X \arrow[d, "\simeq"] \\
X \arrow[u, "\iota_j"] \arrow[ur, "i_j"] \arrow[r, "\mathrm{id}_X", "\simeq"'] & X.
\end{tikzcd}$$

(ii) This is by symmetry of the axiom for cylinder objects.

(iii) By (i) the morphism $X \xrightarrow{i_0} I \cdot X$ s a trivial cofibration. Since trivial cofibrations are stable under pushouts, the morphism $J \cdot X \xrightarrow{\iota_J} (I * J) \cdot X$ is also a trivial cofibration. It follows from the 2-of-3 axiom, that $(I * J) \cdot X \xrightarrow{r_X \cup r_X} X$ is a weak equivalence, because $J \cdot X \xrightarrow{r_X} X$ is a weak equivalence.

We have

$$X + X \overset{i_0+i_1}{\rightarrowtail} (J \cdot X) + (I \cdot X) \overset{\iota_J \cup \iota_I}{\rightarrowtail} X,$$

where the first morphism is a cofibration, because i_0 and i_1 are cofibrations by (i) and cofibrations are stable under coproducts. The second morphism is a cofibration, because we have a pushout diagram

$$\begin{tikzcd}
X + X \arrow[r, "i_1+i_0"] \arrow[d, tail, "i_0 \cup i_1"'] & (I \cdot X) + (J \cdot X) \arrow[d, "\iota_I \cup \iota_J"] \\
J(X) \arrow[r, "\iota_J"] & (I * J) \cdot X,
\end{tikzcd}$$

where the left vertical morphism is a cofibration by definition and hence the right vertical morphism is a cofibration, because cofibrations are stable under pushouts.

(iv) Let $f \in \mathcal{C}(X,Y)$. Let $r_X : I \cdot X \overset{\simeq}{\longrightarrow} X$ be the canonical map. Then there is the **constant homotopy** $f \,{}_{fr_X}\!\simeq f$.

$$\begin{tikzcd}
X + X \arrow[d, tail] \arrow[dr, "\nabla"] \arrow[drr, bend left, "f \cup f"] & & \\
I \cdot X \arrow[r, "r_X"] & X \arrow[r, "f"] & Y.
\end{tikzcd}$$

Now let $f, g \in \mathcal{C}(X,Y)$ with $f \,{}_h\!\simeq g$, for some $h \in \mathcal{C}(I \cdot X, Y)$. Then $I \cdot X$ with $i_j = i_{1-j}$, for $j = 0,1$, is also a cylinder object and the same morphism $h' = h$ defines an **inverse homotopy** $g \,{}_{h'}\!\simeq f$ with respect to its inverse cylinder object.

Finally if $e, f, g \in \mathcal{C}(X,Y)$ with $e \,{}_k\!\simeq f \,{}_h\!\simeq g$, where $h \in \mathcal{C}(I \cdot X, Y)$ and $k \in \mathcal{C}(J \cdot X, Y)$. Then there is a canonical **composition of homotopies** $h * k$, such that $e \,{}_{h*k}\!\simeq g$ with respect to the cylinder object $(I * J) \cdot X$ of (iii). More precisely $h * k$ is defined as the natural pushout morphism in the diagram

$$\begin{tikzcd}
 & & X \arrow[d, tail, "i_1", "\simeq"'] & \\
 & X \arrow[r, tail, "i_0"] \arrow[d, tail, "i_1"', "\simeq"] & I \cdot X \arrow[d, tail, "\iota_1", "\simeq"'] \arrow[ddr, bend left, "h"] & \\
X \arrow[r, tail, "i_0", "\simeq"'] & J \cdot X \arrow[r, tail, "\iota_0", "\simeq"'] \arrow[drr, bend right, "k"'] & (I * J) \cdot X \arrow[dr, dashed, "h*k"] & \\
 & & & Y.
\end{tikzcd}$$

Note that the outer paths commute, since by hypothesis $hi_1 = f = ki_0$.

□

It was crucial in the construction of the model structure on simplicial sets, that we have a "Pushout-Product" property as given Theorem 3.30. It enabled us to deform homotopies in the desired way. Unfortunately we do not have a similar property in an arbitrary model category. Instead deformations of homotopies and other construction of higher homotopies are established by the interplay of cylinder and path objects. This will be demonstrated in the following proposition.

Proposition 3.65
Let $\emptyset \rightarrowtail X \xrightarrow{f,g} Y$ *and* $Y \xrightarrow{\simeq} Y^I \xrightarrow{p} Y \times Y$ *be a fixed path object.*
If f, g *are left homotopic, then* f, g *are right homotopic w.r.t.* Y^I.

Proof. Let $i_j : X \overset{\iota_j}{\hookrightarrow} X + X \rightarrowtail I \cdot X$ denote the inclusions for $j = 0, 1$. Since X is cofibrant, i_0 is a trivial cofibration. Consider the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, tail, "i_0"', "\simeq"] & Y \arrow[r, "\simeq"] & Y^I \arrow[d, "p"] \\ I \cdot X \arrow[r, "{(can, \mathrm{id})}"'] \arrow[urr, dashed, "\exists K"] & X \times (I \cdot X) \arrow[r, "f \times h"'] & Y \times Y. \end{tikzcd}$$

The upper right composition is equal to $\Delta \circ f$, the lower left one is $(f \times h) \circ (\mathrm{id}_X \times i_0) \circ \Delta = (f \times f) \circ \Delta = \Delta \circ f$. Hence the square commutes and the diagonal exists. Because $h \circ i_1 = g$ we obtain a right homotopy

$$\begin{tikzcd} Y \arrow[r, "\simeq"] \arrow[dr, "\Delta"'] & Y^I \arrow[d, "p"] & X \arrow[l, "K \circ i_1"'] \arrow[dl, "{(f,g)}"] \\ & Y \times Y. & \end{tikzcd}$$

□

Corollary 3.66
Let $f, g \in \mathcal{C}(X, Y)$, *where* X *is cofibrant and* Y *is fibrant. Then the following is equivalent:*

(i) f, g *are left homotopic,*

(ii) f, g *are left homotopic w.r.t. a fixed cylinder* $I \cdot X$,

(iii) f, g *are right homotopic,*

(iv) f, g *are right homotopic w.r.t. a fixed path object* Y^I.

Lemma 3.67
Let $\mathcal{C}$ *be a model category and consider morphisms* $A \xrightarrow{a} X \overset{f}{\underset{g}{\rightrightarrows}} Y \xrightarrow{b} B$.

(i) $f \,{}_h\!\simeq g \quad \Rightarrow \quad bf \,{}_{bh}\!\simeq bg$.

(ii) $f \,{}_{h}\!\simeq g \quad \Rightarrow \quad fa \,{}_{h_a}\!\simeq ga,$ *for some* h_a*, if* Y *is fibrant or for* $h_a = h(I \cdot a)$ *if we have a map* $I \cdot a$ *rending the following diagram commutative*

$$\begin{tikzcd}
A + A \arrow[r, tail, "i"] \arrow[d, "a+a"'] & I \cdot A \arrow[r, "\simeq"] \arrow[d, dashed, "\exists I\cdot a"'] & A \arrow[d, "a"'] \\
X + X \arrow[r, tail, "i"] & I \cdot X \arrow[r, "\simeq"] & X.
\end{tikzcd}$$

(iii) $f \simeq_h g \quad \Rightarrow \quad fa \simeq_{ha} ga.$

(iv) $f \simeq_h g \quad \Rightarrow \quad bf \simeq_{h_b} bg,$ *for some* h_b*, if* X *is cofibrant or for* $h_b = b^I h$ *if we have a map* $Y^I \xrightarrow{b^I} B^I$ *rendering the following diagram commutative*

$$\begin{tikzcd}
Y \arrow[r, "\simeq"] \arrow[d, "b"'] & Y^I \arrow[r, two heads, "p"] \arrow[d, dashed, "\exists b^I"'] & Y \times Y \arrow[d, "b\times b"'] \\
B \arrow[r, "\simeq"] & B^I \arrow[r, two heads, "p"] & B \times B
\end{tikzcd}$$

Proof.

(i) By assumption there is a cylinder $I \cdot X$ and an h, such that $f \,{}_{h}\!\simeq g$. Then $bf \,{}_{bh}\!\simeq bg$ is a left homotopy:

$$\begin{tikzcd}
 & X + X \arrow[dl, "\mathrm{id}_X \cup \mathrm{id}_X = \nabla"'] \arrow[d, tail] \arrow[dr, "f\cup g"] \arrow[drr, bend left, "(bf)\cup(bg)"] & & \\
X & I \cdot X \arrow[l, "\simeq"'] \arrow[r, "h"] & Y \arrow[r, "b"] & B.
\end{tikzcd}$$

(ii) Consider a homotopy

$$\begin{tikzcd}
 & X + X \arrow[dl, "\mathrm{id}_X \cup \mathrm{id}_X = \nabla"'] \arrow[d, tail] \arrow[dr, "f\cup g"] & \\
X & I \cdot X \arrow[l, "r_X"', "\simeq"] \arrow[r, "h"] & Y.
\end{tikzcd}$$

Then r_X can be factored as on the left and we find a diagonal in the right

$$\begin{tikzcd}
I \cdot X \arrow[rr, "r_X", "\simeq"'] \arrow[dr, tail, "\simeq", "c"'] & & X \\
 & I \cdot X'. \arrow[ur, two heads, "\simeq", "r'_X"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
I \cdot X \arrow[r, "h"] \arrow[d, tail, "c"', "\simeq"] & Y \arrow[d, two heads] \\
I \cdot X' \arrow[r] \arrow[ur, dashed, "h'"] & *.
\end{tikzcd}$$

By construction we have $f \,{}_{h'}\!\simeq g$, because $X + X \overset{\simeq}{\rightarrowtail} I \cdot X \overset{\simeq}{\rightarrowtail} I \cdot X'$ is another cylinder object for X. Furthermore we find a diagonal

$$\begin{tikzcd}
A + A \arrow[r, "a+a"] \arrow[d, tail] & X + X \arrow[r, tail] & I \cdot X' \arrow[d, two heads, "\simeq"', "r'_X"] \\
I \cdot A \arrow[r] \arrow[urr, dashed, "I\cdot a"] & A \arrow[r, "a"] & X.
\end{tikzcd}$$

Then $h_a = h(I \cdot a)$ is the desired homotopy

$$\begin{tikzcd}
 & A+A \arrow[r, "a+a"] \arrow[dl, "\mathrm{id}_A \cup \mathrm{id}_A = \nabla"'] \arrow[d, tail] & X+X \arrow[d, tail] \arrow[dr, "f \cup g"] & \\
A & I \cdot A \arrow[l, "\simeq"'] \arrow[r, "I \cdot a"] & I \cdot X' \arrow[r, "h'"] & Y.
\end{tikzcd}$$

The last two statements are dual.

□

3.12 The homotopy category of a model category

As the title suggests, the goal of this section is the construction of the derived category of an arbitrary model category. Before we do that let us record the following abstract Whitehead Theorem, which partly has strong similarity to Proposition 3.33 in the context of simplicial sets. It is an abstract version of the classical Whitehead Theorem for topological spaces, stating that each map between CW-complexes inducing isomorphisms on homotopy groups is infact a homotopy equivalence.

Interestingly in the abstract setting of an arbitrary model category its proof is much easier than the other implication that every homotopy equivalence is a weak equivalence, which in the context of topological spaces or simplicial sets is trivial. The proof of the latter will be given in the subsequent section.

Theorem 3.68 (Abstract Whitehead Theorem)
Let $\mathcal{C}$ be a model category.

(i) *Let $f : Z \overset{\simeq}{\twoheadrightarrow} Y$ with Y cofibrant and $I \cdot Z$ is a fixed cylinder object for Z.*
Then f is a strong deformation retraction with respect to $I \cdot Z$.

(ii) *Let $c : X \overset{\simeq}{\rightarrowtail} Z$ with X fibrant and Z^I is a fixed cylinder object for Z.*
Then c is a strong deformation section with respect to Z^I.

(iii) *Every weak equivalence between bifibrant objects is a homotopy equivalence.*

Proof.

(i) Using that $\mathrm{fib}\,\mathcal{C} \cap w\mathcal{C} = RLP(\mathrm{cof}\,\mathcal{C})$ the proof is the same as Proposition 3.33 in the context of simplicial sets. Because f is a trivial fibration and Y is cofibrant, we can find a section s for f and an h, such that the following diagrams commute

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d, tail] & Z \arrow[d, two heads, "f", "\simeq"'] \\
Y \arrow[r, equal] \arrow[ur, dashed, "\exists s"] & Y,
\end{tikzcd}
\qquad
\begin{tikzcd}
Z+Z \arrow[r, "(sf)\cup \mathrm{id}_Z"] \arrow[d, tail] \arrow[dd, bend right=60, "\nabla"'] & Z \arrow[d, two heads, "f", "\simeq"'] \\
I \cdot Z \arrow[r] \arrow[ur, dashed, "\exists h"] \arrow[d, "\simeq"] & Y. \\
Z \arrow[ur, "f"'] &
\end{tikzcd}$$

Hence $fs = \mathrm{id}_Y$ and $sf \mathrel{{}_h{\simeq}} \mathrm{id}_Z$ with fh being trivial, meaning that the lower two triangles commute. This means f is a strong deformation retraction.

(ii) This is dual to (i).

(iii) Let $X, Y \in \mathcal{C}$ bifibrant and $a : X \xrightarrow{\simeq} Y$ a weak equivalence. Then a has a factorization

$$\begin{tikzcd} X \arrow[rr, "a", "\simeq"'] \arrow[dr, tail, "c"'] & & Y \\ & Z, \arrow[ur, two heads, "\simeq", "f"'] & \end{tikzcd}$$

where f is a trivial fibration and c is a cofibration. By the 2-of-3 axiom c is trivial, too. Since X is cofibrant, Y fibrant and fibrations and cofibrations are closed under composition Z is bifibrant. By (i) and (ii) we find a section s for f with $sf \mathrel{{}_h{\simeq}} \mathrm{id}_Z$, for some left homotopy h, and a retraction r for c with $cr \simeq_k \mathrm{id}_Z$ for some right homotopy k. Using Lemma 3.67, we see that $crs \simeq_{ks} s$. Define $b := rs$. By Corollary 3.66 there is a k', such that $crs \mathrel{{}_{k'}{\simeq}} s$ and again by the preceding Lemma we get $ab = fcrs \mathrel{{}_{fk'}{\simeq}} fs = \mathrm{id}_Y$. Similarly we find a h' such that $ba = rsfc \mathrel{{}_{h'c}{\simeq}} rc = \mathrm{id}_X$. Thus a and b define a homotopy equivalence.

□

The homotopy category of a model category will be constructed via the quotient category of the subcategory of objects $\emptyset \rightarrowtail X \twoheadrightarrow *$. For this purpose we need the notion of a bifibrant replacement of an object.

Definition 3.69
Let $\mathcal{C}$ be a model category and $X \in \mathcal{C}$.

(i) *A* ***fibrant replacement*** *of X is an object $F(X)$ in a factorization*

$$X \xrightarrow{\simeq} F(X) \twoheadrightarrow *.$$

For $f \in \mathcal{C}(X, Y)$ and a chosen fibrant replacement $F(Y)$ of Y, we let $F(X)$ and $F(f)$ be given by a factorization

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, tail, "\simeq"'] & Y \arrow[d, "\simeq"] & \\ F(X) \arrow[r, two heads, "F(f)"] & F(Y) \arrow[r, two heads] & *. \end{tikzcd}$$

(ii) *A* ***cofibrant replacement*** *of X is an object $C(X)$ in a factorization*

$$\emptyset \rightarrowtail C(X) \xrightarrow{\simeq} X.$$

For $c \in \mathcal{C}(X,Y)$ and a chosen cofibrant replacement $C(X)$ of X, we let $C(Y)$ and $C(c)$ be given by a factorization

$$\begin{tikzcd}
\emptyset \arrow[r] & C(X) \arrow[r, rightarrowtail, "C(c)"] \arrow[d, "\simeq"'] & C(Y) \arrow[d, two heads, "\simeq"] \\
& X \arrow[r, "c"] & Y.
\end{tikzcd}$$

(iii) *A* ***bifibrant replacement*** *of X is a cofibrant replacement $CF(X)$ of a fibrant replacement $F(X)$ of X.*

Lemma 3.70
Let $\mathcal{C}$ be a model category, $f_0, f_1 \in \mathcal{C}(X,Y)$ with liftings

$$\begin{tikzcd}
\emptyset \arrow[rr, rightarrowtail] \arrow[d, rightarrowtail] & & C(Y) \arrow[d, two heads, "\simeq"', "q_Y"] \\
C(X) \arrow[r, two heads, "\simeq", "q_X"'] \arrow[urr, dashed, "C(f_i)"] & X \arrow[r, "f_i"'] & Y.
\end{tikzcd} \qquad i = 0, 1.$$

If Y is fibrant, then $f_0 \simeq_h f_1$ implies $C(f_0) {}_k\!\simeq C(f_1)$, for some right homotopy k.

Proof. By Lemma 3.67 $f_0 \simeq_h f_1$ implies $q_Y C(f_0) = f_0 q_X \simeq_{hq_X} f_1 q_X = q_Y C(f_1)$, and thus $q_Y C(f_0) {}_\ell\!\simeq q_Y C(f_1)$ by the dual of Proposition 3.65, since Y is fibrant. This homotopy can be lifted as

$$\begin{tikzcd}
C(X) + CF(X) \arrow[r, "C(f_0) \cup C(f_1)"] \arrow[d, rightarrowtail, "i_0 \cup i_1"'] & CY \arrow[d, two heads, "\simeq"', "q_Y"] \\
I \cdot C(X) \arrow[r, "\ell"'] \arrow[ur, dashed, "k"] & Y.
\end{tikzcd}$$

□

Theorem 3.71
Let $\mathcal{C}$ be a model category.

Then $\mathcal{C}$ has a homotopy category $\mathrm{Ho}(\mathcal{C})$, whose objects are the same as $\mathcal{C}$ and

$$\mathrm{Ho}(\mathcal{C})(X,Y) := \mathcal{C}(CF(X), CF(Y))/\simeq, \qquad \textit{for all } X, Y \in \mathcal{C},$$

for chosen bifibrant replacements $CF(X)$ of X and $CF(Y)$ of Y.

Proof. For every $X \in \mathcal{C}$, there is a bifibrant replacement, given by factorizations

$$\begin{tikzcd}
& & X \arrow[d, rightarrowtail, "\simeq"] \\
\emptyset \arrow[r, rightarrowtail] & CF(X) \arrow[r, two heads, "\simeq"] & F(X) \arrow[d, two heads] \\
& & *,
\end{tikzcd}$$

where we chose $F(X) = X$, if X is fibrant, and $CF(X) = F(X)$, if X and hence $F(X)$ is cofibrant.

Let $X, Y \in \mathcal{C}$ and $f \in \mathcal{C}(X, Y)$. Then the lifting property induces a morphism $CF(f) \in \mathcal{C}(CF(X), CF(Y))$ between some chosen bifibrant replacements of X and Y:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, tail, "\simeq"'] & Y \arrow[r, tail, "\simeq"] & F(Y) \arrow[d, two heads] \\
F(X) \arrow[rr, two heads] \arrow[urr, dashed, "F(f)"] & & *.
\end{tikzcd}
\qquad
\begin{tikzcd}
\emptyset \arrow[rr, tail] \arrow[d] & & CF(Y) \arrow[d, two heads, "\simeq"] \\
CF(X) \arrow[r, two heads, "\simeq"'] \arrow[urr, dashed, "CF(f)"] & F(X) \arrow[r, "F(f)"'] & F(Y).
\end{tikzcd}$$

We have to check that $CF(f)$ is well-defined up to homotopy. Suppose $f_0, f_1 \in \mathcal{C}(F(X), F(Y))$ are two liftings of f, meaning that the left squares commute. Then there exists a homotopy h as in the right square

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, tail, "\simeq"'] & Y \arrow[d, tail, "\simeq"] \\
F(X) \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_1"'] & F(Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, tail, "\simeq"'] & Y \arrow[r, tail, "\simeq"] & F(Y) \arrow[r, "\simeq"] & F(Y)^I \arrow[d, two heads, "(p_0,p_1)"] \\
F(X) \arrow[rrr, "(f_0,f_1)"'] \arrow[urrr, dashed, "h"] & & & F(Y) \times F(Y).
\end{tikzcd}$$

It follows that $f_0 \simeq_h f_1$ and thus $C(f_0)\,{}_k\!\simeq C(f_1)$ by the preceding Lemma, since $F(Y)$ is fibrant. Similarly one proves that the lifting $CF(f)$ of the map $F(f)$ is unique up to homotopy.

It follows that $\mathrm{Ho}(\mathcal{C})$ is a well-defined category with the composition induced by the composition in $\mathcal{C}$. Moreover the assignment $f \longmapsto CF(f)$ defines a functor $\mathcal{C} \xrightarrow{\gamma} \mathrm{Ho}(\mathcal{C})$. Furthermore, if $f \in \mathcal{C}(X, Y)$ is a weak equivalence, then $CF(f)$ is a homotopy equivalence by the Whitehead Theorem and so $\gamma(f)$ is an isomorphism.

To check the universal property let $\mathcal{C} \xrightarrow{G} \mathcal{D}$ be a functor, which sends weak equivalences to isomorphisms. Suppose $f, g \in \mathcal{C}(X, Y)$ with X and Y bifibrant and $f\,{}_h\!\simeq g$, where $X + X \overset{i_0 \cup i_1}{\rightarrowtail} I \cdot X \xrightarrow{r_X} X$ is a cylinder object. Since r_X is a weak equivalence $G(r_X)$ is an isomorphism, so $G(r_X)G(i_0) = G(\mathrm{id}_X) = G(r_X)G(i_1)$ implies $G(i_0) = G(i_1)$. Hence

$$G(f) = G(hi_0) = G(h)G(i_0) = G(h)G(i_1) = G(hi_1) = G(g),$$

proving that G factors as $\mathcal{C} \xrightarrow{\gamma} \mathrm{Ho}(\mathcal{C}) \xrightarrow{G'} \mathcal{D}$. By definition of $\mathrm{Ho}(\mathcal{C})$ the functor G' is uniquely determined on objects. Since γ is full on bifibrant objects, G' is uniquely determined on the full subcategory of $\mathrm{Ho}(\mathcal{C})$ of objects, which are bifibrant in $\mathcal{C}$. But for every other object in $\mathrm{Ho}(\mathcal{C})$ there is a formal isomorphism to an object in this category, proving that G' is uniquely determined on $\mathrm{Ho}(\mathcal{C})$.

□

3.13 Characterization of weak equivalences

The goal of this section is to prove that in every model category the weak equivalences are exactly those morphisms, which become isomorphisms in the homotopy category. This is a often very useful characterization of weak equivalences.

Lemma 3.72
Let $\mathcal{C}$ be a model category and $f_0, f_1 \in \mathcal{C}(X, Y)$ with fibrant Y.
Then $f_0 \simeq_h f_1$ implies $\gamma(f_0) = \gamma(f_1)$.

Proof. By construction of the homotopy category $\mathrm{Ho}(\mathcal{C})$ we have to check that the bifibrant replacements of f_0 and f_1 are homotopic. As Y is fibrant, we have $Y \xrightarrow{\mathrm{id}_Y} Y = F(Y)$. It follows that the maps f_0 and f_1 can be lifted to fibrant replacments $\tilde{f}_0$ and $\tilde{f}_1$ as in the square on the left, and the homotopy h can be lifted to a homotopy $\tilde{h}$ as on the right.

$$
\begin{tikzcd}
X \arrow[r, "f_i"] \arrow[d, tail, "i_X"', "\simeq"] & Y \arrow[d, two heads] \\
F(X) \arrow[r] \arrow[ur, dashed, "\tilde{f}_i"] & *,
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "h"] \arrow[d, tail, "i_X"', "\simeq"] & Y^I \arrow[d, two heads, "(p_0,p_1)"] \\
F(X) \arrow[r, "(\tilde{f}_0,\tilde{f}_1)"'] \arrow[ur, dashed, "\tilde{h}"] & Y \times Y.
\end{tikzcd}
$$

Now Lemma 3.70 implies that $C(\tilde{f}_0)$ is homotopic to $C(\tilde{f}_1)$ and thus

$$\gamma(f_0) = \gamma(\tilde{f}_0 i_X) = \gamma(\tilde{f}_0)\gamma(i_X) = \gamma(\tilde{f}_1)\gamma(i_X) = \gamma(\tilde{f}_1 i_X) = \gamma(f_1).$$

□

Lemma 3.73
Let $\mathcal{C}$ be a model category. Let $a, b : X \longrightarrow Y$ with $a\,{}_h{\simeq}\, b$, for some $h \in \mathcal{C}(I \cdot X, Y)$.
Then $h' * h\,{}_H{\simeq}\, ar_X$, where $h' * h$ is the composed homotopy of h with its inverse h', obtained as the pushout morphism in the diagram

$$
\begin{tikzcd}
 & & X \arrow[d, tail, "\simeq"', "i_0"] & \\
 & X \arrow[r, tail, "i_1"] \arrow[d, tail, "i_1"', "\simeq"] & I \cdot X \arrow[d, tail, "\simeq"', "\iota_1"] \arrow[ddr, bend left, "h"] & \\
X \arrow[r, tail, "i_0", "\simeq"'] & I \cdot X \arrow[r, tail, "\iota_0", "\simeq"'] \arrow[drr, bend right, "h"'] & I' \cdot X \arrow[dr, dashed, "h' * h"] & \\
 & & & Y.
\end{tikzcd}
$$

The homotopy H is meant as a commutative diagram

$$
\begin{tikzcd}
 & (I' \cdot X) +_{(X+X)} (I' \cdot X) \arrow[dl, "r'_X \cup r'_X"'] \arrow[d, tail, "c"] \arrow[dr, "(h' * h) \cup a r'_X"] & \\
X & \tilde{X} \arrow[l, two heads, "f"', "\simeq"] \arrow[r, "H"] & Y,
\end{tikzcd}
$$

where $r'_X : I' \cdot X \xrightarrow{\simeq} X$ is the canonical map and $r'_X \cup r'_X = fc$ is an arbitrary factorization of the given type.

Proof. We choose a path object for Y as depicted in the left diagram and find a diagonal in the right square

$$\begin{tikzcd}
Y \arrow[r, tail, "s_Y", "\simeq"'] \arrow[dr, "\Delta=(\mathrm{id}_Y,\mathrm{id}_Y)"'] & Y^I \arrow[d, two heads, "(p_0,p_1)"] \\
& Y \times Y,
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "a"] \arrow[d, tail, "i_0"', "\simeq"] & Y \arrow[r, "s_Y"] & Y^I \arrow[d, two heads, "(p_0,p_1)"] \\
I \cdot X \arrow[urr, dashed, "k"] \arrow[rr, "(ar_X,h)"'] & & Y \times Y.
\end{tikzcd}$$

Let $k = k'$ be the inverse homotopy and define a morphism

$$\bar{k} = k \cup k' : I \cdot X' = I \cdot X +_X I \cdot X \longrightarrow Y^I,$$

which fits in the diagram

$$\begin{tikzcd}
X + X \arrow[r, "a \cup a"] \arrow[d, "i_0 \cup i_1"'] & Y \arrow[r, "s_Y"] & Y^I \arrow[d, "(p_0,p_1)"] \\
I' \cdot X \arrow[urr, "\bar{k}"] \arrow[rr, "(ar'_X, h' * h)"'] & & Y \times Y.
\end{tikzcd}$$

Now as c is a cofibration and p_1 a trivial fibration we get a diagonal in the square

$$\begin{tikzcd}
(I' \cdot X) +_{(X+X)} (I' \cdot X) \arrow[rr, "\bar{k} \cup (s_Y ar'_X)"] \arrow[d, tail, "c"'] & & Y^I \arrow[d, two heads, "\simeq"', "p_0"] \\
\tilde{X} \arrow[urr, dashed, "K"] \arrow[r, "f"'] & X \arrow[r, "a"'] & Y.
\end{tikzcd}$$

Then $H = p_1 K$ has the desired property, because by construction

$$Hc = p_1 K c = p_1(\bar{k} \cup (s_Y ar'_X)) = (p_1 \bar{k}) \cup (p_1 s_Y ar'_X) = (h' * h) \cup (ar'_X).$$

□

The subsequent proposition has strong similarity to the simplicial analog Proposition 3.47.

Proposition 3.74 (Quillen)
Let $\mathcal{C}$ be a model category and $X \xrightarrow{f} Y$, where X, Y are bifibrant.
Then the following is equivalent.

(i) *f is a weak equivalence.*

(ii) *f is a strong deformation retraction.*

(iii) *$\gamma(f)$ is an isomorphism in* $\mathrm{Ho}(\mathcal{C})$.

Proof. The implication (i) $\Rightarrow$ (ii) was proven in the abstract Whitehead Theorem 3.68. Vice versa let $tf \;{}_h\!\simeq \mathrm{id}_X$ and choose a path object $Y \xrightarrow{s_Y} Y^I \xrightarrow{(p_0,p_1)} Y \times Y$. Next choose a factorization

$$
\begin{tikzcd}
X \arrow[rr, "{(\Delta, s_Y f)}"] \arrow[dr, tail, "s_X"', "\simeq"] & & (X \times X)_{Y\times Y} Y^I, \\
& X^I \arrow[ur, "{((p_0,p_1),f^I)}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, dashed, "{(\Delta, s_Y f)}"] \arrow[ddr, bend right, "\Delta"'] & Y \arrow[dr, bend left, "s_Y"] & \\
& (X \times X) \times_{Y\times Y} Y^I \arrow[r] \arrow[d] & Y^I \arrow[d, "{(p_0,p_1)}"] \\
& X \times X \arrow[r, "f \times f"'] & Y \times Y.
\end{tikzcd}
$$

Since fibrations are stable under pullbacks and composition we have a fibration

$$X^I \xrightarrow{((p_0,p_1),f^I)} (X \times X) \times_{Y\times Y} Y^I \longrightarrow X \times X$$

and thus $X \overset{s_X}{\rightarrowtail} X^I \xrightarrow{(p_0,p_1)} X \times X$ is a path object for X. Moreover we find a diagonal

$$
\begin{tikzcd}
X \arrow[rr, "s_X"] \arrow[d, tail, "i_1"', "\simeq"] & & X^I \arrow[d, "{((p_0,p_1),f^I)}"] \\
I \cdot X \arrow[urr, dashed, "H"] \arrow[rr, "{((h,r_X),s_Y f r_X)}"'] & & (X \times X) \times_{Y\times Y} Y^I,
\end{tikzcd}
$$

because the square commutes, since

$$(f \times f)(h, r_X)i_1 = (fhi_1, fr_Xi_1) = (fr_Xi_1, fr_Xi_1) = (f, f) = \Delta f = (p_0, p_1)s_Y fr_Xi_1.$$

Let $k = Hi_0 : X \longrightarrow X^I$. Then by construction

$$p_0k = p_0Hi_0 = hi_0 = tf, \quad p_1k = p_1Hi_0 = p_1r_X = \mathrm{id}_X, \qquad f^Ik = f^IHi_0 = s_Y fr_Xi_0 = s_Y f.$$

In particular $tf \simeq_k \mathrm{id}_X$. Now suppose we have a lifting problem as on the left, then we find a diagonal as on the right

$$
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, tail, "c"'] & X \arrow[d, "f"] \\
B \arrow[ur, dashed, "d"] \arrow[r, "v"'] & Y,
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "ku"] \arrow[d, tail, "c"'] & X^I \arrow[d, "{(p_0,f^I)}"] \\
B \arrow[ur, dashed, "D"] \arrow[r, "{(tv,s_Y v)}"'] & X \times_Y Y^I.
\end{tikzcd}
$$

It follows that $d = p_1D$ solves the left problem, because

$$dc = p_1Dc = p_1ku = u, \qquad fd = fp_1D = p_1f^ID = p_1s_Y v = v.$$

Hence f has the right lifting property with respect to all cofibration, meaning that it is a trivial fibration and thus in particular a weak equivalence.

To see that (ii) implies (iii), it suffices to note, that for any homotopy equivalence f with homotopy inverse t, the morphism $\gamma(f)$ is an isomorphism with inverse $\gamma(t)$. The other implication is similar to the proof of the implication (ii) $\Rightarrow$ (iii) in Proposition 3.47. However the deformation of the homotopy is slightly more difficult, because we have to work with path objects.

Supposing $\gamma(f)$ is an isomorphism, then f is a homotopy equivalence with homotopy inverse g, i.e. we have a left homotopy $fg\,{}_h{\simeq}\,\mathrm{id}_Y$, for some $h \in \mathcal{C}(I \cdot Y, Y)$. By the homotopy lifting property of f, there is a diagonal

$$\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d, "i_0"', "\simeq"] & X \arrow[d, "f"] \\
I \cdot Y \arrow[r, "h"] \arrow[ur, dashed, "\tilde{h}"] & Y.
\end{tikzcd}$$

Note that i_0 is a cofibration by Lemma 3.64 (i), because Y is cofibrant. For $t = \tilde{h}i_1$ we have by construction $ft = f\tilde{h}i_1 = hi_1 = \mathrm{id}_Y$ and $g\,{}_{\tilde{h}}{\simeq}\,t$. By Lemma 3.67 (ii) tf is homotopic to gf, since X is fibrant. But gf is homotopic to id_X, so by composition we get a homotopy $\mathrm{id}_X\,{}_k{\simeq}\,tf$ with $k \in \mathcal{C}(I'' \cdot X, X)$. We compose it with the homotopy tfk', i.e.

$$\begin{tikzcd}
 & & X \arrow[d, tail, "\simeq"', "i_0"] & \\
 & X \arrow[r, tail, "i_1"] \arrow[d, tail, "i_1"', "\simeq"] & I'' \cdot X \arrow[d, tail, "\simeq"', "\iota_1"] \arrow[ddr, bend left, "tfk"] & \\
X \arrow[r, tail, "i_0", "\simeq"'] & I'' \cdot X \arrow[r, tail, "\iota_0", "\simeq"'] \arrow[drr, bend right, "k"'] & I' \cdot X \arrow[dr, dashed, "tfk' * k"'] & \\
 & & & Y.
\end{tikzcd}$$

We get two induced diagrams, where the right one comes from the homotopy $fk' * fk\,{}_H{\simeq}\,fr_X$ we get from the preceding Lemma.

$$\begin{tikzcd}
 & X \arrow[d, "f"] \\
I' \cdot X \arrow[ur, "tfk' * k"] \arrow[r, "fk' * fk"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (I' \cdot X) +_{(X+X)} (I' \cdot X) \arrow[dl, "r_X \cup r_X"'] \arrow[d, tail, "c"] \arrow[dr, "(fk' * fk) \cup fr_X"] & \\
X & \tilde{X} \arrow[l, "\simeq"] \arrow[r, "H"'] & Y.
\end{tikzcd}$$

From the right diagram we get a diagonal K in the diagram

$$\begin{tikzcd}
 & I' \cdot X \arrow[dl, bend right, "\iota_0"'] \arrow[r, "tfk' * k"] \arrow[d, tail, "\simeq"'] & X \arrow[d, "f"] \\
(I' \cdot X) +_{(X+X)} (I' \cdot X) \arrow[r, bend right, "c"'] & \tilde{X} \arrow[r, "H"'] \arrow[ur, dashed, "K"] & Y.
\end{tikzcd}$$

It follows, that $h = Kc\iota_1$ is the desired homotopy $tf\,{}_h{\simeq}\,\mathrm{id}_X$ fibred over f, because

$$h\iota_j = Kc\iota_1\iota_j = Kc\iota_0\iota_j = (tfk' * k)\iota_j = \begin{cases} ki_0 = \mathrm{id}_X, & j = 0, \\ tfki_0 = tf, & j = 1, \end{cases}$$

and furthermore we have $fh = fKc\iota_1 = Hc\iota_1 = fr_X$ by construction.

□

Using Proposition 3.74 we can deduce the following two useful corollaries characterizing weak equivalences.

Corollary 3.75
Let $\mathcal{C}$ be a model category and $X \xrightarrow{g} Y$, where X, Y are bifibrant.
Then the following is equivalent.

(i) *g is a weak equivalence.*

(ii) *g is a homotopy equivalence.*

(iii) *$\gamma(g)$ is an isomorphism in* $\mathrm{Ho}(\mathcal{C})$*.*

Proof. By the Whitehead Theorem (i) implies (ii). Furthermore (ii) implies (iii) by construction of the homotopy category in Theorem 3.71. Finally suppose (iii) holds. We can factor g as $X \overset{c}{\rightarrowtail} Z \overset{f}{\twoheadrightarrow} Y$, where c is a weak equivalence. Thus by the implications we have shown $\gamma(c)$ is an isomorphism in $\mathrm{Ho}(\mathcal{C})$. It follows that $\gamma(f) = \gamma(g)\gamma(c)^{-1}$ is an isomorphism and hence f is a weak equivalence by the preceding Proposition. Hence $g = fc$ is a weak equivalence, proving (i).

□

Corollary 3.76
Let $\mathcal{C}$ be a model category and $X \xrightarrow{g} Y$. Then the following is equivalent.

(i) *g is a weak equivalence.*

(ii) *$\gamma(g)$ is an isomorphism in* $\mathrm{Ho}(\mathcal{C})$*.*

Proof. A bifibrant replacement of g induces a commutative diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\simeq"'] & Y \arrow[d, "\simeq"] \\ F(X) \arrow[r, "F(g)"] & F(Y) \\ CF(X) \arrow[r, "CF(g)"] \arrow[u, "\simeq"] & CF(Y). \arrow[u, "\simeq"'] \end{tikzcd}$$

Using the preceding Corollary and the 2-of-3 axiom for weak equivalences resp. for morphisms in $\mathcal{C}$ being mapped to isomorphisms in $\mathrm{Ho}(\mathcal{C})$ by γ, the following holds.

$$\begin{array}{llll} & g \text{ is a weak equivalence} & \Longleftrightarrow & F(g) \text{ is a weak equivalence} \\ \Longleftrightarrow & CF(g) \text{ is a weak equivalence} & \Longleftrightarrow & \gamma CF(g) \text{ is an isomorphism} \\ \Longleftrightarrow & \gamma F(g) \text{ is an isomorphism} & \Longleftrightarrow & \gamma(g) \text{ is an isomorphism.} \end{array}$$

□

3.14 Derived functors and the comparison of model categories

Like in the context of chain complexes, we are able to define left and right derived functors for functors on a model category.

Definition 3.77
Let $\mathcal{C}$ be a model category, $\mathcal{D}$ an arbitrary category and $\mathcal{C} \xrightarrow{G} \mathcal{D}$.

(i) *The* ***left derived functor*** *LG is defined as the* <u>***right***</u> ***Kan extension*** *of G along $\mathcal{C} \xrightarrow{\gamma} \mathrm{Ho}(\mathcal{C})$, it it exists. That is a functor $\mathrm{Ho}(\mathcal{C}) \xrightarrow{LG} \mathcal{D}$ with a natural transformation $LG \circ \gamma \xrightarrow{\varepsilon} G$, such that LG is a terminal object in the comma category*

$$\gamma^*/G, \qquad \gamma^* := (\text{-} \circ \gamma) : \mathcal{C}\mathrm{AT}(\mathrm{Ho}(\mathcal{C}), \mathcal{D}) \xrightarrow{\text{-}\circ\gamma} \mathcal{C}\mathrm{AT}(\mathcal{C}, \mathcal{D}).$$

(ii) *The* ***right derived functor*** *$\mathbb{R}G$ is defined as the* <u>***left***</u> ***Kan extension*** *of G along $\mathcal{C} \xrightarrow{\gamma} \mathrm{Ho}(\mathcal{C})$, if it exists. That is a functor $\mathrm{Ho}(\mathcal{C}) \xrightarrow{RG} \mathcal{D}$ with a natural transformation $G \xrightarrow{\eta} RG \circ \gamma$, such that RG is an initial object in the comma category G/γ^*.*

A useful tool in many applications is the following Lemma by Ken Brown.

Lemma 3.78 (K. Brown)
In every model category $\mathcal{C}$ the following holds.

(i) *Every weak equivalence $w : X \xrightarrow{\simeq} Y$ between cofibrant $X, Y \in \mathcal{C}$ can be factored as*

$$\begin{tikzcd} X \arrow[rr, "w", "\simeq"'] \arrow[dr, tail, "\simeq"'] & & Y \arrow[dl, tail, bend left, "\simeq"] \\ & Z, \arrow[ur, two heads, "\simeq"'] & \end{tikzcd}$$

meaning that $Y \overset{\simeq}{\rightarrowtail} Z \overset{\simeq}{\twoheadrightarrow} Y$ is the identity on Y.

(ii) *Every weak equivalence $w : X \xrightarrow{\simeq} Y$ between fibrant $X, Y \in \mathcal{C}$ can be factored as*

$$\begin{tikzcd} X \arrow[rr, "w", "\simeq"'] \arrow[dr, tail, "\simeq"] & & Y \\ & Z, \arrow[ur, two heads, "\simeq"'] \arrow[ul, two heads, bend left, "\simeq"] & \end{tikzcd}$$

meaning that $X \overset{\simeq}{\rightarrowtail} Z \overset{\simeq}{\twoheadrightarrow} X$ is the identity on X.

Proof. Since X and Y are cofibrant and cofibrations are stable under pushouts, we have $X \rightarrowtail X + Y \leftarrowtail Y$. Take a factorization $X + Y \rightarrowtail Z \overset{\simeq}{\twoheadrightarrow} Y$ and let $X \rightarrowtail X + Y \rightarrowtail Z$

be the composition. This is a trivial cofibration by the 2-of-3 axiom. Moreover $Y \rightarrowtail X + Y \rightarrowtail Z$ is a section for $Z \xrightarrow{\simeq} Y$ and so is also a trivial cofibration.

Statement (ii) is dual to (i).

□

Theorem 3.79
Let $\mathcal{C}$ be a model category and $G \in \mathcal{C}\mathrm{AT}(\mathcal{C}, \mathcal{D})$.

(i) *Suppose G maps trivial cofibrations between cofibrants to isomorphisms.*
Then G exists and is given by

$$\varepsilon_G : LG(X) = GC(X) \longrightarrow G(X), \qquad \emptyset \rightarrowtail C(X) \xrightarrow{\simeq} X.$$

(ii) *Suppose G maps trivial fibrations between fibrants to isomorphisms.*
Then RG exists and is given by

$$\eta_G : G(X) \longrightarrow GF(X) = RG(X), \qquad X \xrightarrow{\simeq} F(X) \twoheadrightarrow *.$$

Proof. Let $\emptyset \rightarrowtail C(X) \xrightarrow{\simeq} X$ be a chosen cofibrant replacement, for every $X \in \mathcal{C}$, where we set $C(X) \xrightarrow{\mathrm{id}_X} X$, if $X \in \mathcal{C}$ is already cofibrant. Dually to as was seen in the proof of Theorem 3.71 any two liftings $f_0, f_1 \in \mathcal{C}(C(X), C(Y))$ of a morphism $f \in \mathcal{C}(X,Y)$ are left homotopic via some homotopy

$$\begin{tikzcd}
C(X)+C(X) \arrow[rrr, "f_0\cup f_1"] \arrow[d, "i_0\cup i_1"'] & & & C(Y) \arrow[d, two heads, "\simeq"] \\
I\cdot C(X) \arrow[r, "r_{C(X)}"'] \arrow[rrru, dashed, "h"] & C(X) \arrow[r, two heads, "\simeq"'] & X \arrow[r, "f"'] & Y,
\end{tikzcd}$$

where $C(X)+C(X) \overset{i_0\cup i_1}{\rightarrowtail} I\cdot C(X) \xrightarrow{\simeq} C(X)$ is a cylinder object for $C(X)$. Since $C(X)$ is cofibrant so is $\emptyset \rightarrowtail C(X) \overset{i_0}{\rightarrowtail} I\cdot C(X)$ by Lemma 3.64 (i). By K. Brown's Lemma G maps weak equivalences between cofibrant objects to isomorphisms and thus $G(r_{C(X)})$ is an isomorphism. Hence

$$G(r_{C(X)})G(i_0) = G(r_{C(X)}i_1) = G(\mathrm{id}_{C(X)}) = G(r_{C(X)}i_1) = G(r_{C(X)})G(i_1)$$

implies $G(i_0) = G(i_1)$ and thus

$$G(a) = G(hi_0) = G(h)G(i_0) = G(h)G(i_1) = G(hi_1) = G(b).$$

In particular we get a well-defined functor $\mathcal{C} \xrightarrow{GC} \mathcal{D}$. Since $f \in \mathcal{C}(X,Y)$ is a weak equivalence, if and only if a lifting $F(f) \in \mathcal{C}(F(X), F(Y))$ is one by the 2-of-3 axiom, the functor GC maps weak equivalences to isomorphisms and we get a unique factorization $GC : \mathcal{C} \xrightarrow{\gamma} \mathrm{Ho}(\mathcal{C}) \xrightarrow{LG} \mathcal{D}$ by the universal property of the homotopy category $\mathrm{Ho}(\mathcal{C})$. By construction $p_X : C(X) \xrightarrow{\simeq} X$ defines a natural transformation

$\varepsilon_G = G(p) : \gamma^* L(G) = GC \longrightarrow G$. We have to check the universal property of right Kan extensions, meaning that for all natural $\gamma^* G' \xrightarrow{f} G$ there is a natural f' rendering the diagram below commutative

$$\begin{tikzcd} \gamma^* G' \arrow[r, "f"] \arrow[d, dashed, "\gamma^*(f')"'] & G \\ \gamma^* LG \arrow[ur, "\varepsilon_G"'] & \end{tikzcd} \qquad \begin{tikzcd} G' \arrow[d, dashed, "\exists! f'"'] \\ LG. \end{tikzcd}$$

For $X \in \mathcal{C}$ consider the commutative diagram

$$\begin{tikzcd} \gamma^*(G')(C(X)) \arrow[r, "\gamma^* G'(p_X)", "\sim"'] \arrow[d, "f_{C(X)}"'] & \gamma^*(G')(X) \arrow[d, "f_X"] \\ \gamma^* LG(X) = GC(X) \arrow[r, "\varepsilon_G"] & G(X), \end{tikzcd}$$

where the upper horizontal morphism is an isomorphism, since γ maps weak equivalences to isomorphisms. It follows that $f' = G'(f_C)G'(p)^{-1}$ is the unique morphism we are looking for. Statement (ii) is dual to (i).

□

We will also introduce the notion of total derived functors between model categories, providing the most important tool for comparing different model categories and their homotopy categories.

Definition 3.80
Let $\mathcal{C}$ and $\mathcal{D}$ be model categories and $\mathcal{C} \xrightarrow{G} \mathcal{D}$.

(i) *The* ***total left derived functor*** *of G is defined as $\mathbb{L}G = L(\gamma G)$.*

(ii) *The* ***total right derived functor*** *of G is defined as $\mathbb{R}G = R(\gamma G)$.*

Theorem 3.81 (Quillen's adjoint functor theorem)
An adjunction between model categories $\mathcal{C}$ and $\mathcal{D}$

$$\mathcal{C}(E(X), Y) = \mathcal{D}(X, G(Y)),$$

subject to the following (by Remark 3.24 equivalent) hypotheses.

- *E preserves trivial cofibrations between cofibrants and cofibrations.*
- *G preserves trivial fibrations between fibrants and fibrations.*

Then the following holds.

(i) *The total derived funtors of E and G exist and induce an adjunction*

$$\mathrm{Ho}(\mathcal{C})(\mathbb{L}E(X), Y) = \mathrm{Ho}(\mathcal{D})(X, \mathbb{R}G(Y)).$$

(ii) $(\mathbb{L}E, \mathbb{R}G)$ *form an equivalence of categories, if and only if*

a) $X \xrightarrow{\eta_X} GE(X) \longrightarrow GFE(X)$ *is a weak equivalence, for all cofibrant* $X \in \mathcal{D}$*,*

b) $ECG(Y) \longrightarrow EG(Y) \xrightarrow{\varepsilon_Y} Y$ *is a weak equivalence, for all fibrant* $Y \in \mathcal{C}$*.*

Proof. The existence of $\mathbb{L}E$ and $\mathbb{R}G$ follows from Theorem 3.79 and its dual. Let F resp. C denote a chosen fibrant resp. cofibrant replacement in the particular model category. Let $X \in \mathcal{D}$ and $Y \in \mathcal{C}$. We have a chain of natural bijections

$$\begin{aligned}\mathrm{Ho}(\mathcal{C})(\mathbb{L}E(X), Y) &= \mathrm{Ho}(\mathcal{C})(\gamma EC(X), Y) = \mathcal{C}(CFEC(X), CF(Y))/\sim \\ &= \mathcal{C}(EC(X), F(Y))/\sim = \mathcal{D}(C(X), GF(Y))/\sim \\ &= \mathcal{D}(CF(X), CFGF(Y))/\sim = \mathrm{Ho}(\mathcal{D})(X, \gamma GF(Y))/\sim = \mathrm{Ho}(\mathcal{D})(X, \mathbb{R}G(Y)),\end{aligned}$$

where the first one is by definition and the second one by construction of the homotopy category in Theorem 3.71. The third one is induced by composition with the (co-)fibrant replacement morphisms using Corollary 3.66 and Lemma 3.67. The middle one is induced by the adjunction and the arguments for the rest are dual to those given before.

By construction of the adjunction between the homotopy categories we see that its unit resp. counit is an isomorphism, if and only if the conditions a) and b) hold.

□

Remark 3.82
By Brown's Lemma the hypotheses of Theorem 3.81 are satisfied, if one of the following equivalent conditions holds:

- E *preserves cofibrations and trivial cofibrations.*
- G *preserves fibrations and trivial fibrations.*

In applications it is often more convieniant to check these conditions.

Definition 3.83
Let $\mathcal{C}$ *and* $\mathcal{D}$ *be model categories.*

(i) *A* ***Quillen adjunction*** *between* $\mathcal{C}$ *and* $\mathcal{D}$ *is an adjunction*

$$\mathcal{C}(E(X), Y) = \mathcal{D}(X, G(Y))$$

subject to the following (by Remark 3.24 equivalent) hypotheses.

- E *preserves trivial cofibrations and cofibrations.*
- G *preserves trivial fibrations and fibrations.*

(ii) *A* ***Quillen equivalence*** *is a Quillen adjunction inducing an equivalence between the homotopy categories.*

Example 3.84
Using the geometric realization and singular nerve adjunction of Proposition 2.28

$$\mathcal{T}op(|X|, Y) = s\mathcal{S}et(X, S(Y))$$

one can also construct a model structure on the category of topological spaces, induced by the model structure on simplicial sets. More precisely for a map of topological spaces $f \in \mathcal{T}op(X, Y)$ *we define*

- f *is a weak equivalence, if* $S(f)$ *is a weak equivalence,*
- f *is a fibration, if* $S(f)$ *is a fibration,*
- f *is a cofibraiton, if* $f \in LLP(\mathrm{fib}\,\mathcal{T}op \cap w\mathcal{T}op)$.

Then by Theorem 3.81 and the subsequent Remark the total derived functors exist and form an adjunction

$$\mathrm{Ho}(\mathcal{T}op)(\mathbb{L}|X|, Y) = \mathrm{Ho}(X, \mathbb{R}S(Y)).$$

One can moreover prove that also |-| preserves fibrations, which is quite unusual for a left adjoint in a Quillen adjunction and therefore not so easy. It follows that the Quillen adjunction is infact a Quillen equivalence.

Moreover one can prove that a continuous map f *is a weak equivalence if and only if*

$$\pi_0 f : \pi_0 X \xrightarrow{\sim} \pi_0 Y, \qquad \pi_n f : \pi_n(X, x) \xrightarrow{\sim} \pi_n(Y, f(x)), \qquad x \in X,\ n > 0.$$