Numbers and Models, Standard and Nonstandard.

Dedicated to the memory of Abraham Robinson.

4 April 2010

The following is a somewhat extended manuscript for a talk at the "Algebra Days", May 2008, in Antalya. I talked about my personal recollections of Abraham Robinson.

Contents

1 How I met Abraham Robinson and his infinitesimals

It was in the early months of 1963. I was visiting the California Institute of Technology on my sabbatical. Somehow during this visit I learned that one year ago Wim Luxemburg had given a lecture on A. Robinson's theory of infinitesimals and infinitely large numbers. Luxemburg was on leave but I got hold of his Lecture Notes [Lux62]. Although the topic was somewhat distant from my own work I got interested and, after thorough reading I wished to meet the person who had been able to put Leibniz' infinitesimals on a solid base and build the modern analysis upon it.

At that time Abraham Robinson was at the nearby University of California at Los Angeles, and I managed to meet him there. I remember an instructive discussion about his theory which opened my eyes for the wide range of possible applications; he also showed me his motivations and main ideas about it.

Abraham Robinson 1918-74

Perhaps I am allowed to insert some personal words explaining why I had been so excited about the new theory of infintesimals. This goes back to my school days in Königsberg, when I was 16. At that time the school syllabus required that we were to be instructed in Calculus or, as it was called in German, in *Diffentialrechnung.* Our Math teacher at that time was an elderly lady who had been retired already but was reactivated again for school work in order to fill the vacancy of our regular teacher; the latter had been drafted to the army. (It was war time: 1944.) I still remember the sight of her standing in front of the blackboard where she had drawn a wonderfully smooth parabola, inserting a *secant* and telling us that $\frac{\Delta y}{\Delta x}$ is its slope, until finally she convinced us that the slope of the *tangent* is $\frac{dy}{dx}$ where dx is infinitesimally small and dy accordingly.

My school in Königsberg[1]

This, I admit, impressed me deeply. Until then our school Math had consisted largely of Euclidean geometry, with so many problems of constructing triangles from some given data. This was o.k. but in the long run that stuff did not strike me as to be more than boring exercises. But now, with those infinitesimals, Math seemed to have more interesting things in stock than I had met so far. And I decided that I would study Mathematics if I survived the dangers of war which we knew we would be exposed to very soon. After all, I wanted to find out more about these wonderfully strange infinitesimals.

Well, I survived. And I managed to enter University and start with Mathematics. The first lecture I attended to was Calculus, with Professor Otto Haupt in Erlangen. There we were told to my disappointment that my Math teacher had not been up to date after all. We were warned to

[1]Wilhelmsgymasium. This is the same school where Hilbert in the year 1880 obtained his *Abitur*.

beware of infinitesimals since they do not exist, and in any case they lead to contradictions. Instead, although one writes $\frac{dy}{dx}$ then this does not really mean a quotient of two entities, but it should be interpreted as a symbolic notation only, namely the limit of the quotients $\frac{\Delta y}{\Delta x}$.

I survived this disappointment too. Later I learned that dy and dx can be interpreted, not as infinitesimals but as some entities of an abstract construction called *differential module*, and if that module is one-dimensional then the quotient $\frac{dy}{dx}$ would make sense and yield what we had learned anyhow. Certainly, this sounded nice but in fact it was only an abstract frame ignoring the natural idea of infinitesimally small numbers.

So when I learned about Robinson's infinitesimals, my early school day experiences came to my mind again and I wondered whether that lady teacher had not been so wrong after all.

The discussion with Abraham Robinson kindled my interest and I wished to know more about it. Some time later there arose the opportunity to invite him to visit us in Germany where he gave lectures on his ideas, first in Tübingen and later in Heidelberg, after I had moved there.

Before continuing with this let me briefly explain what I am talking about, i.e., Robinson's theory of nonstandard analysis.

2 What is Nonstandard Analysis?

2.1 A preliminary Axiom

Consider the hierarchy of numbers which we present to our students in their first year:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

Everything starts with the natural numbers $\mathbb{N}$ which, as Kronecker allegedly has maintained, are "created by God" (or whatever is considered to be equivalent to Him). The rest is constructed by mankind, i.e., by the minds of mathematicians. In each step, the structure in question is enlarged such as to admit greater flexibility with respect to some operations defined in the structure. In $\mathbb{Z}$ the operation of subtraction is defined such that $\mathbb{Z}$ becomes an additive group; in fact $\mathbb{Z}$ is a commutative ring without zero divisors. In $\mathbb{Q}$ the operation of division is defined such that $\mathbb{Q}$ becomes a field. Finally, in $\mathbb{R}$ every Cauchy sequence is convergent, such that $\mathbb{R}$ becomes a complete ordered field. In each step we tell our students that the respective enlargement exists and we explain how to construct it.

In order to develop what nowadays is called "analysis" the construction usually stops with the real field $\mathbb{R}$; this is considered to be adequate and quite sufficient as a basis for (real) analysis. But it had not always been the case that way. Since Leibniz had used the natural idea of infinitesimals to build a systematic theory with it, many generations of mathematicians (including my lady teacher) had been taught in the Leibniz way. Prominent people like Euler, the Bernoullis, Lagrange and even Cauchy (to name only a few) did not hesitate to use them.

Gottfried Wilhelm Leibniz

The Leibniz idea for analysis, as interpreted by Robinson, is to work in a further enlargement:

$$\mathbb{R} \subset {}^*\mathbb{R}$$

such that the following Axiom is satisfied. In order to explain the main idea I will first state the Axiom in a preliminary form which, however, will not yet be sufficient. Later I will give the final, more general form.

Axiom (preliminary form).

(1.) ${}^*\mathbb{R}$ *is an ordered field extension of* $\mathbb{R}$.
(2.) ${}^*\mathbb{R}$ *contains infinitely large elements.*

An element $\omega \in {}^*\mathbb{R}$ is called "infinitely large" if $|\omega| > n$ for all $n \in \mathbb{N}$. Part (2.) says that the ordering of ${}^*\mathbb{R}$ does *not* satisfy the axiom of Archimedes.

Fields with the properties (1.) and (2.) were known for some time[2] but the attempts to build analysis on this basis were not quite satisfactory. Among all such fields one has to select those which in addition have more sophisticated properties. But for the moment let us stay with the Axiom in this preliminary form and see what we can do with it.

The elements of $\mathbb{R}$ are called *standard* real numbers, while the elements of ${}^*\mathbb{R}$ not in $\mathbb{R}$ are *nonstandard.* This terminology is taken from model theory but I find it not very suggestive in the present context. Sometimes the elements of ${}^*\mathbb{R}$ are called *hyperreal* numbers. Perhaps someone some time will find a more intuitive terminology.

The *finite elements* α in ${}^*\mathbb{R}$ are those which are not infinitely large, i.e. which satisfy Archimedes' axiom: $|\alpha| < n$ for some $n \in \mathbb{N}$ (depending on α). These finite elements form a subring $E \subset {}^*\mathbb{R}$. It contains all *infinitesimal elements* ε which are defined by the property that $|\varepsilon| < \frac{1}{n}$ for all $n \in \mathbb{N}$. It follows from the definition that the set of infinitesimals is an *ideal* $I \subset E$. We have:

$$\omega \text{ infinitely large} \qquad \Longleftrightarrow \qquad \omega^{-1} \text{ infinitesimal} \neq 0.$$

It is well known that this property characterizes E as a *valuation ring* in the sense of Krull [Kru32].

Theorem: *The finite elements E form a valuation ring of ${}^*\mathbb{R}$ with the infinitesimals I as its maximal ideal. The residue class field $E/I = \mathbb{R}$.*

Two finite elements α, β are said to be *infinitely close* to each other if $\alpha - \beta$ is infinitesimal, i.e., if they belong to the same residue class modulo the ideal I of infinitesimals. This is written as

$$\alpha \approx \beta\,.$$

The residue class of $\alpha \in E$ is called the "*monad*" of α; this terminology has been introduced by Robinson in reference to Leibniz' theory of monads. Every monad contains exactly one standard number $a \in \mathbb{R}$; this is called the *standard part* of α, and denoted by $st(\alpha)$. There results the standard part map

$$st : E \to \mathbb{R}$$

which in fact is nothing else than the residue class map of E modulo its maximal ideal I.

[2] See, e.g., [AS27].

In this situation let us consider the example of a parabola

$$y = x^2$$

which, as I have narrated above, had been used by my school teacher to introduce us to analysis. Suppose x is a standard number. If we add to x some infinitesimal $dx \neq 0$ then the ordinate of the corresponding point on the parabola will be

$$y + dy = (x + dx)^2 = x^2 + 2xdx + (dx)^2$$

which differs from y by

$$dy = 2xdx + (dx)^2 = (2x + dx)dx$$

so that the slope of the corresponding secant is

$$\frac{dy}{dx} = 2x + dx \approx 2x$$

since $dx \approx 0$ is infinitesimal. Hence:

If we step down from the hyperreal world into the real world again, by using the standard part operator, then the secant of two infinitely close points becomes the tangent, and the slope of this tangent is the standard part: $st\left(\frac{dy}{dx}\right) = 2x$.

I believe that such kind of argument had been used by my school teacher as narrated above. As we see, this is completely legitimate.

It is apparent that in the same way one can differentiate any power x^n instead of x^2, and also polynomials and quotients of polynomials, i.e., rational functions, with coefficients in $\mathbb{R}$. All the well known algebraic rules for derivations can be obtained in this way. However, analysis does not deal with rational functions only. What can be done to include more functions?

2.2 The Axiom in its final form

As described by the preliminary Axiom, ${}^*\mathbb{R}$ is an ordered field. This can be expressed by saying that ${}^*\mathbb{R}$ is a *model* of the theory of ordered fields. The theory of ordered fields contains in its vocabulary the function symbols "+" for addition, and "·" for multiplication, as well as the relation symbol "<" for the ordering. The axioms of ordered fields are formulated in this language. If we add to the vocabulary constants for all real numbers $r \in \mathbb{R}$ and to the theory all statements which are true in $\mathbb{R}$ then the models of this theory are precisely the ordered field extensions of $\mathbb{R}$.

If we wish to talk about functions and relations which are not expressible in this language, then we have to use a language with a more extended vocabulary. In order not to miss anything which may be of interest let us include into our language symbols for *all* relations in $\mathbb{R}$.[3] The theory of $\mathbb{R}$ consists of all statements in this language which hold in $\mathbb{R}$. Thus, if we generalize the first part of the above Axiom as:

${}^*\mathbb{R}$ *is a model of the Theory of* $\mathbb{R}$*,*

then this will allow us to talk in ${}^*\mathbb{R}$ about every function and relation which is defined in $\mathbb{R}$.

In order to generalize the second part of the Axiom we have to refer not only to the relation "$<$" of the ordering, but to *every* relation of similar kind. More precisely: Consider a 2-place relation $\varphi(x, y)$ defined in $\mathbb{R}$. Such a relation is said to be *concurrent* if, given finitely many elements $a_1, \ldots, a_n \in \mathbb{R}$ in the left domain of φ, there exists $b \in \mathbb{R}$ in its right domain such that $\varphi(a_i, b)$ holds for $i = 1, \ldots, n$.[4] Such element b may be called a "bound" for $a_1, \ldots, a_n$ with respect to the relation φ.

Axiom (final form).

(1.) ${}^*\mathbb{R}$ *is a model of the Theory of* $\mathbb{R}$*.*
(2.) *Every concurrent relation* φ *over* $\mathbb{R}$ *admits a universal bound* $\beta \in {}^*\mathbb{R}$*, i.e., such that* $\varphi(a, \beta)$ *holds simultaneously for all* $a \in \mathbb{R}$ *which are contained in the left domain of* φ*.*

It is clear that this form of the Axiom is a generalization of its preliminary form, and a far reaching generalization at that. It was Abraham Robinson who had observed that Leibniz, when he worked with infinitesimals, seemed tacitly to use something which is equivalent to that Axiom.

Of course, the essential point is that indeed there exists a structure ${}^*\mathbb{R}$ satisfying this Axiom. This is guaranteed by general results of model theory. The most popular construction is by means of ultrapowers.

There is some ambiguity which has to be cleared. The Axiom refers to the "Theory of $\mathbb{R}$", and this refers to a given language as described above, its

[3] Functions can be viewed as 2-place relations and thus are included. Subsets may be defined as the range of their characteristic functions and hence are included too.

[4] The relation φ may not be defined on the whole of $\mathbb{R}$. The left domain of φ consists of those $a \in \mathbb{R}$ for which there exists a $b \in \mathbb{R}$ such that $\varphi(a, b)$ holds. The right domain is defined similarly. For the ordering relation "$<$" the left and the right domain coincides with $\mathbb{R}$.

vocabulary including symbols for all relations over $\mathbb{R}$. On first sight one would think of relations (and functions) between individuals, i.e., elements of $\mathbb{R}$. This would lead to the first order language (Lower Predicate Calculus), where quantification is allowed over individuals only. But in many mathematical investigations it is necessary (or at least convenient) to enlarge the language such as to contain also symbols for sets of functions, relations between sets of functions etc., and quantification should be allowed over entities of any given type. For instance, if we wish to state the induction axiom for the set $\mathbb{N}$ of natural numbers, we may say that:

"*Every non-empty subset of* $\mathbb{N}$ *contains a smallest element*"

and this statement contains a quantifier for subsets.

In order to include such statements too Robinson works with the *Higher Order Language* containing symbols also for higher entities, i.e., relations between sets, functions of relations between sets, etc. In other words:

We interpret the above Axiom as referring to the full structure over $\mathbb{R}$ *and accordingly work with the corresponding higher order language.*

This implies, among other things, that in ${}^*\mathbb{R}$ we have to distinguish between *internal* and *external* entities. Quantification ranges over internal quantities of any given type. Here we do not wish to go into details but refer, e.g., to the beautiful introduction which Abraham Robinson himself has given in his book on Nonstandard Analysis [Rob66]. See also the first section in [RR75].

Robinson introduced the terminology **enlargement** for a structure satisfying the Axiom. As said above, such an enlargement can be obtained by ultrapower construction. It is not unique. In the following we choose one such enlargement and regard it as a fixed universe during our discussion.

2.3 Some excercises

Having learned all this from Abraham Robinson, my immediate reaction was what probably every newcomer would have done: I wished to put this method of reasoning to a test in simple exemplary situations. I do not have time here for a long discussion although much could be said to convince the reader of the enormous potential of the new way of reasoning which Robinson's theory of enlargements offers to us. Here let me be content with a few examples.

Let f be a standard function and consider an element $x \in \mathbb{R}$ in its domain of definition. According to the part (1) of the Axiom, f extends uniquely to a function on ${}^*\mathbb{R}$.

Continuity: *f is continuous in x if and only if*

$$x' \approx x \Longrightarrow f(x') \approx f(x)\,. \tag{1}$$

Of course, it is assumed that x' is contained in the domain of f, so that $f(x')$ is defined. If the domain of the original f is open then $f(x')$ is defined for *every* x' in the monad of x.

The above statement can be used as *definition* of continuity of a function. Note that the usual quantifier prefix $\forall\,\varepsilon\,\exists\,\delta\ldots$ is missing.

I have chosen this example because I found precisely this definition in an old textbook. This was the German "*Kiepert, Differential- und Integralrechnung*" of which the first edition had appeared in 1863. It had been very popular, and it got at least 12 editions, the 12th appearing in 1912 [Kie12]. The text there reads as follows (in English translation):

> *If some function is given by $y = f(x)$ then, in general, infinitely small changes of x will give rise to infinitely small changes of y. For all values of x where this is the case, the function is called continuous.*

We see that this is precisely the definition (1).

Euler's "Analyse des infiniment-petits"[5]

[5]This frontispiece of Euler's book was used by Abraham Robbinson in his book on Nonstandard Analysis [Rob66].

Derivative: Let dx be an infinitesimal. Define dy by $y + dy = f(x + dx)$. Then the derivative $f'(x) \in \mathbb{R}$ is defined by

$$f'(x) \approx \frac{dy}{dx}\,. \tag{2}$$

More precisely: it is required that this is a valid definition, i.e., the quotient $\frac{dy}{dx}$ should be finite and its monad should be independent of the choice of the infinitesimal dx. If this is the case then f is called differentiable at x and $f'(x)$ is defined as the standard part of $\frac{dy}{dx}$.

I have chosen this example since it is the definition presented by my school teacher mentioned above. It is well possible that she had been trained using Kiepert's textbook.

Integration: Suppose the function $f(x)$ is defined in the closed interval $[a, b]$ with $a, b \in \mathbb{R}$. Let n be a natural number and divide $[a, b]$ into n subintervals $[x_{i-1}, x_i]$ of equal length. We take n infinitely large; then the length $dx = \frac{b-a}{n}$ of each subinterval is infinitesimal. Now the integral is defined by:

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} f(x_i)dx\,. \tag{3}$$

More precisely: It is required that this is a valid definition, i.e., the sum on the right hand side should be a finite element in ${}^*\mathbb{R}$ and its monad should be independent of the choice of the infinite number n. If this is the case then f is called (Riemann) integrable over $[a, b]$ and the integral $\int_a^b f(x)dx$ is defined as the standard part of that sum.

Maybe some explanation about infinite natural numbers is in order. ${}^*\mathbb{R}$ is an enlargement of $\mathbb{R}$, and therefore every subset of $\mathbb{R}$ has an interpretation in ${}^*\mathbb{R}$. So does $\mathbb{N}$. This new subset of ${}^*\mathbb{R}$ is denoted by ${}^*\mathbb{N}$. (In fact, ${}^*\mathbb{N}$ is an enlargement of $\mathbb{N}$.) Using part (2.) of the Axiom, it follows that there exists $n \in {}^*\mathbb{N}$ which is larger than every number in $\mathbb{N}$, i.e., n is infinite. As to the sum on the right hand side of (3), it is to be interpreted as follows: For every finite $n \in \mathbb{N}$ the sum $s_n = \sum_{i=1}^{n} f(x_i)dx$ has finitely many terms, and so s_n is well defined in $\mathbb{R}$. The function $n \mapsto s_n$ from $\mathbb{N}$ to $\mathbb{R}$ has an interpretation in the enlargement, i.e., it extends to a function from ${}^*\mathbb{N}$ to ${}^*\mathbb{R}$. Thus s_n is defined for every $n \in {}^*\mathbb{N}$. Note that s_n for infinite n is *not* an infinite series in the usual sense. It is to be regarded as the nonstandard interpretation of a sum whose number of terms is a natural number.

The definition (3) of the integral explains Leibniz' idea that the integral is essentially a sum (up to infinitesimals). This idea had led him to introduce the integral sign $\int$ as a variant of the letter $\mathcal{S}$ which he used for sums (instead of Σ which is used today).

I have been inspired to choose example (3) because of its relation to Archimedes' method of measuring the area of a plane region. This method consists of cutting the area into parallel strips of, say, length ℓ and infinitesimal breadth ε; then $\ell \cdot \varepsilon$ is the (infinitesimal) area of the strip and the sum of all those areas will give the area of the whole region - up to infinitesimals. The Leibniz formula (3) does precisely this in the case of a positive function, when the region to be measured is that between the function graph and the x-axis.

That Archimedes' method can indeed be interpreted in this way (contrary to what is commonly attributed to him) is well documented by the Archimedes Codex which has been recently discovered and deciphered; see the report [NN07] about what is called "the world's greatest palimpsest".

3 Robinson's visits

3.1 Tübingen

As said at the beginning I had met Abraham Robinson in Los Angeles in California during my sabbatical. In the summer term of 1963 I was back at my university in Tübingen. There I started a workshop where together with some students and colleagues, we read Robinson's papers and his book on model theory [Rob63] which had just appeared. We tried to understand his ideas for nonstandard analysis and to apply them to various situations. His book on nonstandard analysis [Rob66] had not yet been written.

University of Tübingen[6]

[6] This is the Main Building where Robinson delivered his 1966 lecture.

Some time later when I had heard that Robinson was in Germany, I was able to meet him and suggested that he spend a month or so in Tübingen as visiting professor, for a course on a topic from nonstandard analysis. He reacted favorably and so he visited us in Tübingen in the summer of 1966.[7]

I had advertised his lecture course to students and colleagues, and so he had a full auditorium. The aim of the course, two hours weekly, was to cover the fundamentals of model theory with particular emphasis on the application to analysis and algebra. This job was not easy since the students (and most colleagues) did not have a formal training in mathematical logic; so he had to start from scratch. He was not what may be called a brilliant lecturer who would be able to rouse a large audience regardless of the content of his talk. His way was quiet, with great patience when questions came up from the students, but strong when it came to convince the students about the impact of nonstandard applications. And this kept the attention of the large audience throughout his lecture.

In addition Robinson was available for discussion in our workshop. Just in time his book on nonstandard analysis [Rob66] had appeared; he presented to us some of the more sophisticated applications.

I recall my impression that his Tübingen visit could be considered as a success, and from what is reported in Dauben's biography it appears that Robinson thought so too.

3.2 Heidelberg

Next year, 1967, I moved from Tübingen to the University of Heidelberg. The general academic conditions in Heidelberg in those years were quite favorable. So it was not difficult to convince the faculty and the rector (president) that the visit of a distinguished scholar like Abraham Robinson would be of enormous importance for the development of a strong mathematics group in Heidelberg. And so in the following year, 1968, I was able to extend a cordial invitiation to Abraham Robinson to visit us again, this time in Heidelberg. And he came, this time not from UCLA but from Yale where he had moved in the meantime.

Again he delivered a course on model theory and applications. To a certain extent this job was kind of a repetition of his Tübingen lecture; again

[7] I am relying here on the extensive Robinson biography by Dauben [Dau95] where this year is recorded for Robinson's Tübingen visit. Unfortunately I did not save our letters or other documents from that time and so I have to look at Dauben's book for help in the matter of dates.

he had a large audience. But there was a difference. For in his seminar, on a smaller scale, he found an audience which was highly motivated. On the one hand, there was a group of gifted students and postdocs who had also switched from Tübigen to Heidelberg and who had already attended Robinson's Tübingen lecture. On the other hand, in Heidelberg there had been regular courses on Mathematical Logic (by Gert Müller who held a position as "associate professor"), and so there had been opportunities for the students to acquire knowledge in this field, in particular in model theory.

The Mathematics Institute in Heidelberg

But the essential new feature of Robinson's Heidelberg visit was that he talked not only on nonstandard *analysis* but also on nonstandard *algebra* and *arithmetic*; in the seminar he was able to expound his ideas in more detail. This found a respondent audience. His impact on the work of these young people in the seminar was remarkable. And so it came about that he more or less regularly visited us in Heidelberg during the following years, continuing his seminar talks and working with those who responded to his ideas.

In the next two sections I will give some kind of overview on the work resulting of his influence on the Heidelberg group, which was apparent even after his untimely death in April 1974.

Robinson's influence was also helpful in another project. In view of Robinson's striking applications of model theory to mathematics proper, I became convinced that a chair devoted to mathematical logic could be of help to mathematicians in their daily work, in particular if the chair was occupied by someone from model theory. Therefore I tried to obtain help from the university administration and the ministry of education for establishing such a chair in the mathematics faculty. I had started this project in Tübingen already but after I moved to Heidelberg this would have to be a chair for the Heidelberg faculty. Indeed, after some time such a chair was installed (in those times such thing was still possible). This was in 1973. It was clear to me that Robinson's encouragement and judgement had been of great help in this matter. When I asked him whether he would accept an offer to Heidelberg for this chair then he did not say "no" but from the way he reacted it seemed to me that he really meant "no". After all, Heidelberg seemed to be no match for Yale at that time. In any case, in a few months after that the problem was not existent any more. But it should be remembered that this chair, which is still in existence, had been installed with the strong help of Abraham Robinson.

During his repeated visits to Heidelberg we came to know Abraham Robinson not only as a mathematician and scholar but also as a friend. He lived around the corner from our house and on his way to town he regularly stepped in for a coffee and conversation with us. (If I say "we" and "us" in this context then I include my wife Erika.) He was a man with a wide horizon and far reaching interests. If he talked about Leibniz then one could feel not only his knowledge about Leibniz' life and work but also his sympathy for that remarkable man. There was only one thing about which he strongly disagreed with Leibniz, namely Leibniz' insistence that "our world is the best of all possible worlds".

Abby liked to talk to people, and sometimes we had the impression that he knew more about our neighbors than we did. He was keenly interested in the local history. When we took him on tour to show him the country and its places then it often turned out that he knew more about it than we did, and he gave us a lecture on the history of those places.

In the course of those years there developed a friendship of rare quality. Abby belongs to the few close friends whom I have met in my life. I have learned much from him, not only in Mathematics but also in questions of attitude towards the problems of life.

4 Nonstandard Algebra

Looking at the Axiom in its final form (in section 2.2) it is apparent that this Axiom has little to do with the special properties of the real number field $\mathbb{R}$. It makes sense for every mathematical structure. And so there is not only nonstandard analysis, but nonstandard mathematics at large. Abraham Robinson was well aware of this; he has applied his method, partly in collaboration with others, to various mathematical problems ranging from topology, Hilbert spaces, Lie groups, complex analysis, differential algebra, quantum theory to mathematical economics.

There were also investigations in the direction of algebra and number theory. As said above, Abraham Robinson reported on this in his Heidelberg seminar lectures. One of his first topics was his nonstandard interpretation of Hilbert's irreducibility theorem (jointly with Gilmore in [GR55]). This paper of Robinson has been said to mark a "watershed" in the development of model theory (in the same line with another paper of Robinson's, of the same year 1955, on Artin's solutiom of Hilbert's 17th problem [Rob55]).

Hilbert had published his irreducibility theorem in 1892 [Hil92]. Suppose that $f(X,Y)$ is an irreducible polynomial in 2 (or more) variables then, Hilbert showed, there are infinitely many specializations $X \mapsto t$ such that $f(t,Y)$ remains irreducible. The coefficients of f are taken from the rational field $\mathbb{Q}$ and the specialized variable t is also assumed to be in $\mathbb{Q}$. Since then there had been numerous proofs of this theorem, also over other base fields K, e.g., number fields. Hasse had the idea to study arbitrary fields over which Hilbert's irreducibility theorem may hold, and his Ph.D. student Wolfgang Franz started the theory of such fields which today are called *Hilbert fields* [Fra31]. This was the point where Abraham Robinson stepped in. He stated a nonstandard characterization of Hilbert fields.

As a follow-up of our discussions with Robinson we were able to amend his result of [GR55] by presenting a new, "metamathematical" proof of Hilbert's irreducibility theorem in the number field and the function field cases. It turned out that Hilbert's irreducibility is, in fact, equivalent to the well known theorem of Bertini in algebraic geometry [Roq75]. Further investigations by R. Weissauer showed that every field with a set of valuations satisfying the product formula is Hilbertian. This covered all classical fields which were known to be Hilbertian. Moreover, Weissauer found quite a number of new and interesting Hilbertian fields, e.g., formal power series fields in more than one variable [Wei82].

Weissauer's paper is a good example of the usefulness of Robinson's enlargements. On the one hand, it can be shown that any result which has been proved using the notion and the properties of enlargements can also be obtained without this. On the other hand, the use of enlargements provides the mathematician with new methods and it opens up new analogies to other problems which sometimes help to understand the situation. Abraham Robinson used to say that his method may reduce a "dynamical" to a "statical" situation. For instance, an infinite sequence $t_1, t_2, t_3 \ldots$ which preserves the irreducibility of the polynomial $f(X,Y)$ under the specialization $X \mapsto t_i$ leads to a *nonstandard* t which renders $f(t,Y)$ irreducible.

For another topic of algebra, remember that group theory had been started by Galois in order to study the roots of algebraic equations. Today the notion of Galois group belongs to the basics of algebra. But there arose the need to study simultaneously infinitely many algebraic equations; this led Krull in 1928 to the discovery of the topological structure of infinite Galois groups [Kru28], and this developed into the theory of profinite groups. Robinson has pointed out that profinite Galois groups can be naturally understood within the enlargement, connected to the "finite" groups in the sense that their order is an infinite large natural number $n \in {}^*\mathbb{N}$. The corresponding profinite groups in the standard world are obtained from these nonstandard "finite" groups by a similar process as the derivative $f'(x)$ is obtained from the nonstandard differential quotient $\frac{dy}{dx}$ in the manner as explained above. Hence again:

If we step down from the nonstandard world into the standard world again, then Krull's Galois theory of infinite algebraic extensions appears as an immediate consequence of the Galois-Steinitz theory for finite algebraic field extensions.

There arises the interesting question which fields K are uniquely determined (up to elementary equivalence) by their full profinite Galois groups G_K. See [Pop88], [Koe95].

The description of the structure of $G_{\mathbb{Q}}$ as profinite group is at the focus of current arithmetical research.

5 Nonstandard Arithmetic

Remember Hensel's p-adic number fields which Hensel had conceived at around 1900 and which today have become standard tools in algebraic number theory and beyond. In the course of time it became necessary to consider all p-adic completions at once; this has led to the introduction of adeles and ideles in the sense of Chevalley which play a fundamental role, e.g., in class field theory. Now, Abraham Robinson has pointed out that his notion of enlargement comprises all those constructions at the same time. His enlargements are indeed the most universal "completions" in as much as *every* concurrent relation admits a bound. The classical notions of p-adics, adeles and ideles, pro-finite groups etc. are obtained from his enlargement by a universal transfer principle, similar to obtaining the derivative $f'(x)$ as the standard part of the differential quotient $\frac{dy}{dx}$ as explained above.

In the ensuing discussions with Abraham Robinson we wished to test his method in some more situations of fundamental importance. The *Siegel-Mahler theorem* seemed to us a good example to begin with. Finally in November 1973 he invited me to Yale with the aim of discussing in more detail the possibility of a nonstandard proof of this theorem.

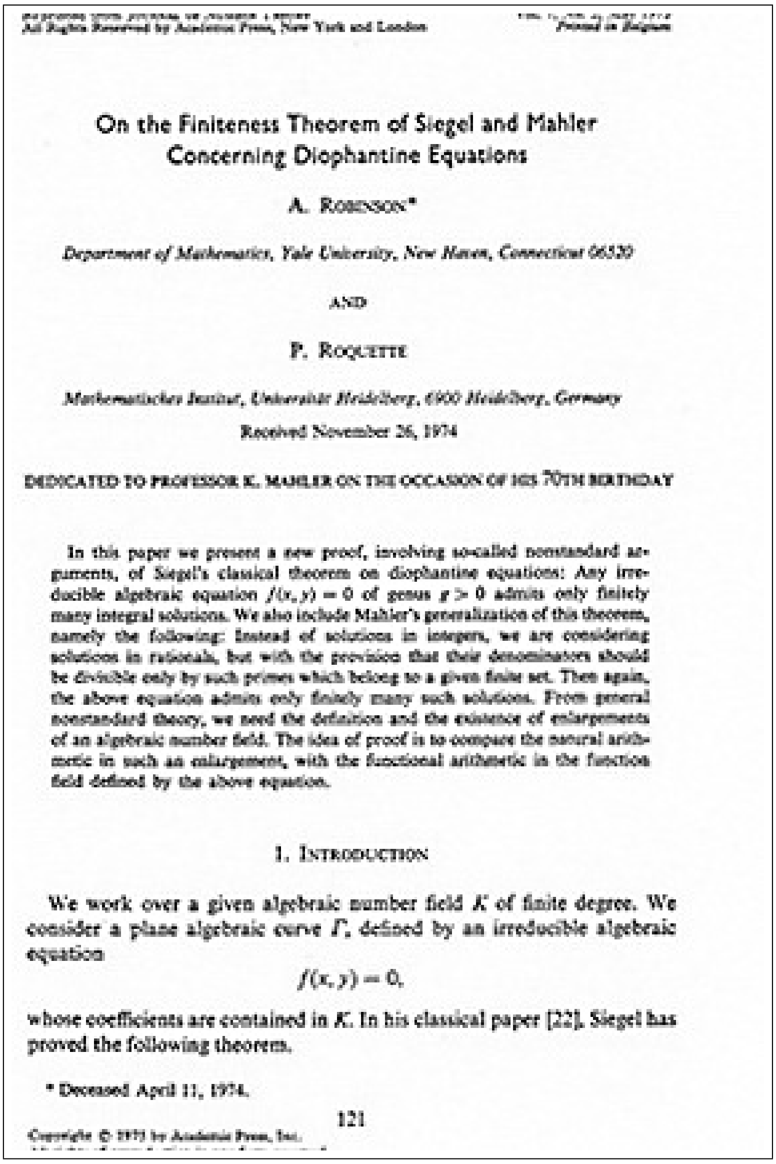
Let $\Gamma : f(x,y) = 0$ be an irreducible curve defined over a number field K of finite degree. If Γ is of genus $g > 0$ then Siegel's theorem says that Γ admits only finitely many points whose coordinates are integers in K. Mahler had generalized this by proving that for any finite set S of primes of K there are only finitely many points in Γ whose coordinates are S-integers in K. The S-integers are those numbers in K whose denominator consists of primes in S only.

Nonstandard methods seem to be useful to distinguish between finite and infinite. We work in a fixed enlargement ${}^*K$ of K, with the properties as statetd in the Axiom. If Γ would admit infinitely many S-integral points in K then it would also admit a nonstandard S-integral point in ${}^*K$. Such a point (x,y) is a *generic* point of Γ over K and hence $F = K(x,y)$ is the function field of Γ over K. By construction F is embedded into ${}^*K$:

$$F \subset {}^*K.$$

Now, both these fields carry a natural arithmetic structure: F as an algebraic function field over K and ${}^*K$ as a nonstandard model of the number field K. What is the relation between the arithmetic in F and in K? In our joint paper [RR75] we were able to prove the following

> **Theorem 1**: *If F is of genus $g > 0$ then every functional prime divisor P of F is induced by some nonstandard prime divisor $\mathfrak{p}$ of ${}^*K$.*



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On the Finiteness Theorem of Siegel and Mahler Concerning Diophantine Equations

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DEDICATED TO PROFESSOR K. MAHLER ON THE OCCASION OF HIS 70TH BIRTHDAY

In this paper we present a new proof, involving so-called nonstandard arguments, of Siegel's classical theorem on diophantine equations: Any irreducible algebraic equation $f(x, y) = 0$ of genus $g > 0$ admits only finitely many integral solutions. We also include Mahler's generalization of this theorem, namely the following: Instead of solutions in integers, we are considering solutions in rationals, but with the provision that their denominators should be divisible only by such primes which belong to a given finite set. Then again, the above equation admits only finitely many such solutions. From general nonstandard theory, we need the definition and the existence of enlargements of an algebraic number field. The idea of proof is to compare the natural arithmetic in such an enlargement, with the functional arithmetic in the function field defined by the above equation.

1. Introduction

We work over a given algebraic number field K of finite degree. We consider a plane algebraic curve Γ, defined by an irreducible algebraic equation

$$f(x, y) = 0,$$

whose coefficients are contained in K. In his classical paper [22], Siegel has proved the following theorem.

* Deceased April 11, 1974.

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From here it is only a small step to deduce the validity of the Siegel-Mahler theorem. Abby agreed to work out the proof of the theorem for elliptic curves, and I was to deal with curves of higher genus. Two weeks after I had left Yale he sent me his manuscript for the elliptic part. But he could not see any more my part for higher genus.

Actually, there is a famous conjecture of Mordell to the effect that a curve Γ of genus $g > 1$ over a number field K of finite degree has only finitely many K-rational points, even without specifying that they are S-integers. This conjecture has been proved by Faltings in 1983. In nonstandard terms it can be formulated as follows:

Theorem 2: *A function field $F|K$ of genus $g > 1$ cannot be embedded into the enlargement $^*K$.*

Clearly, this contains Theorem 1 in the case $g > 1$, which was my own contribution in the joint work with Robinson. But in 1973 Mordell's conjecture had not yet been proved and hence, at that time, the proof of Theorem 1 was necessary also for the case $g > 1$.

In 1973 I discussed with Robinson also a possible nonstandard proof of Mordell's conjecture. We planned first to develop the tools which we believed to be necessary for this project. However, due to Robinson's sudden death our plan could not be realized.

In later years Kani [Kan80b, Kan80a, Kan82] has studied systematically function fields which are embedded into the enlargement $^*K$. In my opinion, the tools and the results which have been obtained in his work are well capable to give a nonstandard proof of Mordell's conjecture (together with Roth's theorem on rational approximation of algebraic numbers [Rot55] which is a standard tool to deal with questions of this kind, including theorems 1 and 2). But this has not yet been worked out. It remains an open challenge.

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