The Riemann hypothesis in characteristic p, its origin and development

Part 4. Davenport-Hasse fields.

Peter Roquette (Heidelberg)

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Abstract

We report on the work of Davenport and Hasse on zeta functions of the so-called Davenport-Hasse function fields. Their zeros can be obtained by means of Gaussian sums. Thus the Riemann hypothesis was proved for a wide class of function fields of higher genus in the year 1934 already. We include a discussion of several other papers in the 1930s which came up in the context of this work. Besides Davenport and Hasse we will meet the names of Stickelberger, Artin, Witt, H. L. Schmid, Teichmüller, A. Weil, Chevalley and others.

Contents

1 Introduction

This is the fourth part of a larger work on the history of the Riemann hypothesis[1] for function fields. Parts 1, 2 and 3 have appeared earlier in these *Mitteilungen*.[2] This Part 4 is again written in such a way that it can be read independently from the other parts. As a result the reader might find some repetition of what I have already said in earlier parts; I am asking for your understanding.

In Part 3 we had reported on Hasse's work 1933-36 on the Riemann hypothesis in the case of *elliptic* function fields, including Deuring's subsequent work about their endomorphism rings. But parallel to the elliptic case there was much activity also for function fields of higher genus $g > 1$, with the aim of proving the Riemann hypothesis in general.

In this Part 4 we discuss the first step in this development, namely the work of Davenport and Hasse on certain special function fields of higher genus, which today are called the "Davenport-Hasse fields". These are function fields

$$F = K(x, y)$$

which are given by a defining equation of the form

$$x^m + y^n = 1 \qquad \text{or} \qquad y^p - y = x^m$$

where m, n are not divisible by the characteristic p. After suitable extension of the base field K it can be assumed that K contains the m-th and the n-th roots of unity.

The aim of Davenport and Hasse was not only to prove the Riemann hypothesis for these fields, but at the same time to give an *arithmetic characterization* of the roots of the corresponding zeta functions. This required a thorough investigation of the class field structure of the fields in question and their L-functions, as well as an arithmetic study of the Jacobi sums and Gaussian sums which appear in the process. All this is essentially contained in two papers, one by Hasse [Has34b] and the other jointly by Davenport and Hasse [DH34]. Both papers appeared 1934. We shall also discuss later work in the 1930s connected with these problems.

[1]Here and in the following, whenever we talk about the "Riemann hypothesis" we always mean the "analogue of the Riemann hypothesis for the zeta function of a function field F over a finite field of constants". This hypothesis proclaims that the zeros of the zeta function $\zeta_F(s)$ all have real part $\frac{1}{2}$.

[2]See [Roq02b, Roq04, Roq06].

It seems not without interest that precisely these detailed results about the arithmetic properties of the zeros turned out, in later years, to be the key to the arithmetic investigation of the so-called Hasse-Weil zeta functions over number fields. Thanks to the results of Davenport and Hasse those zeros could be identified as Hecke's *Größencharacters.* That happened in 1952.

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Again we shall use not only published material but various personal documents like letters, manuscripts and other papers of the protagonists. In this way we are able to get a closer look at the emergence of ideas in *status nascendi* and their subsequent development following the flow of information, until they were ready for publication. In short: we can observe the making of mathematics at first hand. We hope to be able to communicate to the reader our fascination of the story of one of the most seminal developments in the 20th century.

At the end of each section the reader will find a *Summary.* It may be profitable to first have a look at those summaries, in order to obtain an overview before going into the details.

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As we have pointed out in Part 3, in the theory of algebraic function fields there are three analogies prevalent:

1. to algebraic number theory,
2. to the analytic theory of Riemann surfaces,
3. to the algebraic geometry of curves.

Artin's thesis in 1921, where he formulated the Riemann hypothesis for hyperelliptic curves, was inspired by the analogy to number theory. This dominated also the later development when the algebraic theory of function fields was developed by Hasse, F. K. Schmidt and others; but in a number of cases the analogy to the theory of Riemann surfaces came into the viewpoint, e.g., when proving the Riemann-Roch theorem or the theorem of the residues.

In our time the language of algebraic geometry is prevalent. We say "language" because the "objects" of such investigation are mostly structures of commutative algebra. The geometric language appeals to the analogy to geometry, and it appears to be more flexible to deal with various phenomena in mathematics. In this framework a function field $F|K$ is considered to belong to a curve Γ as its field of algebraic functions. If the base field K is perfect then Γ can be assumed to be smooth and then it is uniquely determined (up to biregular correspondence). The prime divisors P of the function field are the closed points of the curve Γ; a finite field extension $E|F$ of function fields represents a covering $\Delta \to \Gamma$ of algebraic curves, etc.

Here, in this report, we will mostly use the language of function fields which Hasse and his collaborators used in their time. This appeals to the analogy to number theory and sometimes to complex analysis. Nowadays there is no problem to translate the language of function fields into the language of algebraic curves and back – although in the 1930s this was not yet standard.

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Remark: All unpublished documents which are cited can be found in the *Handschriftenabteilung* of Göttingen University Library, except when we explicitly mention another source. As a general rule, letters which were addressed to Hasse can be found in Göttingen, whereas letters which Hasse wrote to other people are preserved at other places (if preserved at all). Letters from Hasse to Mordell we have found in the archives of King's College, and those from Hasse to Davenport at Trinity College, both in Cambridge, England. Although quite a number of letters from the Hasse correspondence is preserved, the reader should be aware that, on the other hand, quite another number of letters seems to be lost. What we have found does not constitute a complete set of the Hasse correspondence.

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2 Prelude: Davenport's estimates

In Part 2, Section 3 we have reported in detail how Hasse became interested in the proof of the Riemann hypothesis; this happened through Harold Davenport. Let us briefly recall:

The young Davenport (23 years old) stayed in the summer semester 1931 with the Hasses in Marburg, and there developed a close friendship. To be sure, Davenport was not primarily interested in the Riemann hypothesis. In fact, when he arrived at Marburg he did not yet know much about Artin's thesis and the Riemann hypothesis, or about the algebraic theory of function fields. Originally Davenport was interested in estimating the number of solutions of congruences of the form

$$f(x,y) \equiv 0 \bmod p$$

where $f(x,y)$ is an absolutely irreducible polynomial with integer coefficients and p a prime number. If N denotes the number of solutions then it was conjectured that

$$|N - p\,| \leq C \cdot \sqrt{p} \qquad \text{for } p \to \infty \tag{1}$$

where C is a constant depending on the coefficients of f but not depending on p. In particular it would follow that $N > 0$ if p is sufficiently large. But the latter conclusion would be valid also if one just knew that $|N-p| \leq C \cdot p^{\gamma}$ with some $\gamma < 1$.

Davenport, and also his academic teacher Mordell, proved a number of results in this direction for various polynomials $f(x,y)$ and various $\gamma < 1$. Those results were considered as temporary in view of the full conjecture (1). Hasse became interested in this kind of problem but he was not impressed by the unsystematic methods which were applied, based on subtle computational finesse. He expressed his opinion that the use of abstract algebra and the structural approach would help to provide a better foundation to attack those problems, heading right away for the best estimate with $\gamma = \frac{1}{2}$. Davenport was not convinced and he challenged Hasse to do so.

In the fall of 1932, after the Zürich meeting of the International Mathematical Union (IMU) [3], Hasse took up Davenport's challenge. As a start he transformed the problem into the following setting.

Instead of working with congruences modulo p, i.e., with equations in $\mathbb{F}_p$, he worked over an arbitrary finite field

$$K = \mathbb{F}_q$$

where q is a power of p. Thus, given an absolutely irreducible polynomial $f(x,y)$ with coefficients in K, the problem now demands to give an estimate for the number $N = N_K$ of zeros of $f(x,y)$ in K.

In November 1932 Hasse visited Artin in Hamburg and gave a talk in the seminar, treating the Mordell-Davenport problem in this extended setting. In his discussion with Artin the latter mentioned the following observation which at that time was probably new to Hasse.

[3] There Hasse talked about his recent work on the arithmetic of simple algebras over number fields.

Consider the algebraic function field $F = K(x, y)$ defined by the absolutely irreducible equation $f(x, y) = 0$ over K. For each $r > 0$ let $K_r = \mathbb{F}_{q^r}$ denote the extension of K of degree r and $F_r = K_r(x, y)$ the corresponding constant field extension of F. Let N_r denote the number of K_r-rational solutions of $f(x, y) = 0$. Now:

Artin's observation. *In the situation explained above suppose that*

$$|N_r - q^r| \leq C \cdot \sqrt{q^r} \qquad \text{for } r \to \infty \tag{2}$$

with a constant $C > 0$ independent of r. Then the Riemann hypothesis holds for the zeta-function of $F|K$.

In Part 1, section 1 we have reported that we have found this statement in a letter of Artin of the year 1921, addressed to his academic teacher Herglotz. But Artin had never published this and hence we suppose that Hasse did not know it before Artin told him in November 1932.

After this discussion with Artin, Hasse knew that he had found what he was looking for, namely an arithmetic structure which stands behind the estimates of Davenport and Mordell and which can lead the way to significant results. That structure was the algebraic function field (of one variable) over a finite field of constants. Or in today's terminology: *the global field of prime characteristic.*

Immediately Hasse started to follow this lead, at first for the simplest nontrivial kind of function fields, those of genus 1 which are also called "elliptic" fields. Already in February 1933, two months after his visit with Artin in Hamburg, he succeeded to prove the Riemann hypothesis in the elliptic case. In Part 2 we have discussed the first version of Hasse's proof, and in Part 3 his final version. The latter was published in 1936 eventually.

But parallel to those investigations Hasse considered also the case of higher genus $g > 1$. As a starter he worked jointly with Davenport on those function fields which today are called "Davenport-Hasse fields".

3 The Davenport-Hasse fields

3.1 Davenport's letter January 1932

As said above, Davenport had stayed with the Hasses in Marburg in the summer semester of 1931. Already in January 1932 Davenport again visited

Hasse.[4] On that occasion he apparently had informed Hasse about his latest estimate concerning the number of solutions of congruences of the form

$$ax^m + by^n + c \equiv 0 \bmod p\,. \tag{3}$$

Hasse seems to have asked him for details, for as soon as Davenport was back in England he sent Hasse the full proof. The letter is not dated but Hasse wrote "January 1932" on the margin. It turned out that for this kind of congruence Davenport had alrady obtained the estimate (1), i.e., the best possible exponent $\gamma = \frac{1}{2}$.

Davenport's proof seems to have caught Hasse's interest, for right away he copied it into his mathematical diary. There is an entry in Hasse's diary of January 1932 with the title:

> Davenport's Beweis der Lösbarkeit von $ax^m + by^n + c \equiv 0 \mod p$ und Bestimmung der Anzahl der Lösungen.
>
> *Davenport's proof of the solvability of* $ax^m + by^n + c \equiv 0 \mod p$ *and determination of the number of solutions.*

We shall see that Davenport's letter contains the nucleus of the joint work of Davenport and Hasse [DH34]. Therefore, as the proof is short and beautiful let us read Davenport's letter:

> *My dear Helmut,*
>
> *I promised to send you my treatment of the congruence*
>
> $$ax^m + by^n + c \equiv 0 \pmod{p}\,. \tag{4}$$
>
> *Let* $\chi_1, \ldots, \chi_{m-1}$ *be the non–principal characters for which* $\chi^m = 1$*, the principal character.*[5] *It is easily seen that*
>
> $$1 + \chi_1(t) + \cdots + \chi_{m-1}(t)$$
>
> *is precisely the number of solutions of* $x^m \equiv t$. *Hence the number of solutions of* (4) *is*

[4] As Hasse wrote to Mordell: "*We have invited Davenport for the second half of his Christmas vacation — or rather he has invited himself with our readily given consent ...*"

[5] Davenport has in mind the characters of the multiplicative group of $\mathbb{F}_p$. He did not explicitly mention that n and m are supposed to divide $p-1$, which is natural in this situation. (For otherwise, m, n could be replaced by their greatest common divisor with $p-1$ without changing the number N of solutions.) If $t \equiv 0 \bmod p$ he puts $\chi_r(t) = 0$ and similarly $\psi_s(t) = 0$. Moreover, a, b, c are supposed to be $\not\equiv 0 \bmod p$.

$$N = \sum_t \Big\{1+\chi_1(t)+\cdots+\chi_{m-1}(t)\Big\}\Big\{1+\psi_1\Big(-\frac{at+c}{b}\Big)+\cdots+\psi_{n-1}\Big(-\frac{at+c}{b}\Big)\Big\}$$

where $\psi_1, \ldots, \psi_{n-1}$ *are the non-principal characters for which* $\psi^n = 1$. *Hence*

$$N \;=\; p + \sum_{r=1}^{m-1}\sum_{s=1}^{n-1}\sum_t \;\chi_r(t)\psi_s(-\frac{at+c}{b})\,.$$

The sums in t *can be easily expressed in terms of generalized Gaussian sums*

$$\tau(\chi) \;=\; \sum_\nu \chi(\nu)e(\nu)\,, \qquad e(x) \;=\; e^{\frac{2\pi i x}{p}}\,.$$

These have the property $\overline{\chi}(u)\,\tau(\chi) \;=\; \sum_\nu \chi(\nu)e(u\nu)$. *Hence*

$$\begin{aligned}\sum_t \chi(t)\psi(at+c) \;&=\; \frac{1}{\tau(\overline{\psi})}\sum_{t,\nu}\chi(t)e((at+c)\nu)\overline{\psi}(\nu) \qquad (5)\\ &=\; \frac{\tau(\chi)}{\tau(\overline{\psi})}\sum_\nu \overline{\chi}(a\nu)\,\overline{\psi}(\nu)\,e(c\nu)\\ &=\; \frac{\tau(\chi)\,\tau(\overline{\chi}\,\overline{\psi})}{\tau(\overline{\psi})}\;\overline{\chi}(a)\,\chi\psi(c)\,.\end{aligned}$$

Therefore

$$\begin{aligned}N \;&=\; p + \sum_{r=1}^{m-1}\sum_{s=1}^{n-1}\frac{\tau(\chi_r)\tau(\overline{\chi}_r\overline{\psi}_s)}{\tau(\overline{\psi}_s)}\,\chi_r\left(\frac{c}{a}\right)\,\psi_s\left(-\frac{c}{b}\right)\\ &=\; p + \vartheta\sqrt{p}\,(m-1)(n-1) \quad \textit{since} \;\; |\tau| = \sqrt{p}\,, \;\; |\vartheta| \le 1\\ &>\; 0 \quad \textit{if} \quad p > (m-1)^2(n-1)^2\,.\end{aligned}$$

Quite trivial ! ......

Davenport continues with a discussion of certain Kloosterman sums which we will discuss later. (See section 6.1).

Davenport's computation yields

$$|N - p\,| \le C\cdot\sqrt{p} \qquad \text{with } \; C = (m-1)(n-1)\,. \tag{6}$$

We do not know why Hasse had considered this proof of such importance that he immediately took these lines down into his mathematical diary. Perhaps he did so because *Gaussian sums* are appearing. Hasse had always shown great interest in the arithmetic properties of Gaussian sums in various situations and he has a number of publications dealing with them.

The estimate (6) is also contained in a paper by Mordell, but with another constant C and not using Gaussian sums [Mor33]. Mordell had visited Hasse in January 1932 and on that occasion given him a copy of his manuscript.[6] Hence Hasse knew about it. But Hasse did not copy Mordell's proof into his diary. He just referred to it at the end of the entry containing Davenport's proof, as follows:

> Bemerkung. Siehe auch den Mordellschen Beweis, der zwar ein nicht ganz so scharfes Resultat liefert, aber dafür frei von Gaussschen Summen, somit völlig elementar ist.
>
> *Remark. See also Mordell's proof which works without Gaussian sums, hence is completely elementary, although not quite providing such a sharp result.*

We see that Hasse in his diary did not prefer Mordell's elementary proof but favored Davenport's proof using Gaussian sums. Hasse knew (or felt) that this approach may lead to generalization.

As said in the foregoing section, in the last months of 1932 Hasse succeeded to generalize most of Davenport's and Mordell's estimates to the case of arbitrary finite fields instead of just the prime fields $\mathbb{F}_p$. We do not know whether the estimate (6) was already among those but it seems very likely.

For, Davenport's computation works in the same way over any finite field $K = \mathbb{F}_q$, provided n, m divide $q - 1$. In this case χ, ψ range over the characters of the multiplicative group $\mathbb{F}_q^\times$ and the exponential $e(x)$ appearing in Davenport's definition of the Gaussian sum has to be defined as

$$e(x) = e^{\frac{2\pi i S(x)}{p}} \tag{7}$$

where $x \in \mathbb{F}_q$ and $S : \mathbb{F}_q \to \mathbb{F}_p$ denotes the trace function ("*Spur*" in German). Thus we have the following statement:

Let K be any finite field and q the number of its elements. Assume m and n divide $q - 1$. Then the number N of solutions in K of the equation

$$ax^m + by^n + c = 0 \quad \text{with} \quad a, b, c \in K^\times \tag{8}$$

[6] We have mentioned this already in Part 2, section 3.3. There the reader will find more information about the mathematical activities of Davenport and Mordell concerning diophantine congruences.

satisfies the estimate (6) *with* p *replaced by* q.

But then the same holds also over every finite extension $K_r = \mathbb{F}_{q^r}$ and so we have (6) also for q^r. It follows from Artin's observation (page 8) that the Riemann hypothesis holds for any function field $F = K(x, y)$ over K which is defined by an equation of the form (8).

As said in the foregoing section, Hasse had learned about Artin's observation when he visited Artin in November 1932.

We see that already in 1932 Hasse had proved the Riemann hypothesis for a large class of function fields of genus $g > 1$. *He used Davenport's method of estimating by Gaussian sums and then applied Artin's observation.*

Remark: In his computations Davenport had assumed χ and ψ to be different from the principal character. Nevertheless $\chi\psi$ may be principal. In this case $\tau(\overline{\chi}\,\overline{\psi}) = 0$ according to his definition. Consequently, Davenport could have omitted in his computations the pairs r, s with $\chi_r\psi_s = 1$. There are $d - 1$ such pairs where $d = \gcd(m, n)$. This would have given him an estimate with the better constant $C = (m-1)(n-1)-(d-1)$ instead of (6). But it is known that this number C is twice the genus g of the function field $F = K(x, y)$ given by the equation (8). Thus Davenport's computation could be refined to yield

$$|N - q| \le 2g \cdot q^{1/2}$$

which is precisely what is expected from the discussion of the zeta function.

3.2 The Davenport-Hasse fields.

But Hasse wished to get more. The Riemann hypothesis refers only to the real part of the zeros of the zeta-function $\zeta_F(s)$ of $F|K$. But what about the zeros themselves? Already on March 17, 1932 Davenport asked Hasse:

> *What do you think the form of the ordinates of the zeros of Artin's ζ-function will be?*

It is convenient to consider ζ_F as a function of $t = q^{-s}$.[7] Then the Riemann hypothesis says that its roots have absolute value $q^{-\frac{1}{2}}$. Accordingly

[7] We use the notation $\zeta(s)$ or $\zeta(t)$ according to whether we regard ζ as a function of s or of t. We know that this kind of notation (which in former times was not unusual) is today regarded not to be admissible. But we believe that in our situation this cannot lead to misunderstanding and so we ask the reader to tolerate this notation for the sake of simplicity.

Davenport's question is to be interpreted as to the angles of these complex numbers. It turned out that these roots can be expressed by Gaussian sums, and so they are algebraic numbers. Davenport's question can therefore be extended to ask for the prime decomposition of these numbers, i.e., for their arthmetic characterization. From the correspondence of Hasse with Davenport about this question there arose their joint paper [DH34].

For simplicity they assumed that $a = b = -c = 1$ in formula (8). (Otherwise the formulas would become unnecessarily burdened by the parameters a, b, c. It is an easy exercise to amend the following formulas by inserting these parameters in suitable places. In [DH34] it is said that (8) is "*insignificantly generalized*" against (9).) So we have now the situation

$$F = K(x, y) \qquad \text{with } \; x^m + y^n = 1. \tag{9}$$

Moreover, parallel to (9) the paper [DH34] deals also with another type of function fields, namely

$$F = K(x, y) \qquad \text{with } \; y^p - y = x^m \tag{10}$$

where, again, $m \mid q - 1$ and p is the characteristic.[8] We put

$$z = x^m = \begin{cases} 1 - y^n & \text{in case (9)} \\ y^p - y & \text{in case (10)} \end{cases} \tag{11}$$

and see that F is composed of two linearly disjoint cyclic extensions of the rational field $K(z)$, of degree m and n (resp. p).

$$\begin{tikzcd}
 & F = K(x,y) \arrow[dl, no head] \arrow[dr, no head] & \\
K(x) \arrow[dr, no head, "x^m = z"'] & & K(y) \arrow[dl, no head, "{y^n = 1 - z \atop (\text{or } y^p - y = z)}"] \\
 & K(z) &
\end{tikzcd} \tag{12}$$

[8]This had been done in order to deal not only with Gaussian sums but also with the so-called Kloosterman sums. We shall discuss these later.

As already said in the introduction, today any such field F is called a "Davenport-Hasse field". If the geometric language is used then the terminology is "Davenport-Hasse curve".

In the Davenport-Hasse paper [DH34] F is considered as class field over $K(z)$. In the years 1933-34, when that paper was composed, class field theory for function fields was not yet completely developed. Therefore Hasse wrote a separate paper on the arithmetic of function fields which included class field theory [Has34b]. That paper was meant to contain the prerequisites for determining the zeros of the zeta-function of the field F above.

Accordingly let us first have a brief look at this preparatory paper by Hasse, before discussing the Hasse-Davenport paper.

3.3 Cyclic extension of function fields

Hasse wrote his paper [Has34b] "*within a few days*", as we learn from a letter to Davenport dated May 15, 1934. He tried to write it in such a way that it could also serve as a basis for other work, beyond its application to the Davenport-Hasse fields.

In the first part of his paper he considered function fields over an arbitrary perfect field of constants K, not necessarily finite.

Hasse studied cyclic extensions $F|F_0$ of such function fields, and in particular two types: On the one hand he assumed that the degree $n = [F : F_0]$ is not divisible by the characteristic p and that the n-th roots of unity are contained in the base field. In today's terminology: $F|F_0$ is a cyclic *Kummer extension.* On the other hand Hasse considered the case $n = p$ when $F|F_0$ is an *Artin-Schreier extension.*

Actually, he also wished to cover the case when $n = p^r$ is an arbitrary power of the characteristic but then, he said, one should first generalize the Artin-Schreier theory to these fields. Today we know that the proper generalization uses Witt vectors. But that was discovered later only and was not yet available in the year 1934 [Wit36].[9]

[9] In a footnote Hasse mentions a recent paper by Albert [Alb34] which contained the description of cyclic extensions of p-power degree by iterating the Artin-Schreier construction. Artin and Schreier had already done the first iteration, obtaining cyclic extensions of degree p^2, and Albert continued this process by induction. But Hasse points out that Albert's stepwise approach is not suitable for his purpose. What was needed, instead, were explicit formulas for the ray class characters of the whole extension. Such formulas were obtained later only, in the year 1941 by H. L. Schmid [Sch41a].

For those two types of cyclic extensions of function fields Hasse discussed the arithmetic structure, including ramification and splitting of prime divisors, conductor, different and discriminant and the behavior of differentials, including the genus change.

If $n \not\equiv 0 \bmod p$ then this was more or less the transition of the situation for number fields to function fields. But in the case $n = p$ Hasse entered completely new ground. It was known since the paper of Artin-Schreier [AS27] that such cyclic extensions $F|F_0$ are generated by an equation of the form

$$F = F_0(y) \qquad \text{with} \;\; y^p - y \in F_0 \,.$$

But until Hasse's paper nothing was known about the arithmetic of those fields. We can observe that and when Hasse established the relevant results. On July 23, 1933 he wrote to Davenport:

> *I suppose that $y^p - y = f_3(x)$ has genus $(p-1)$ ...*

Here, $f_3(x)$ denotes a polynomial of degree 3 with distinct roots; if $p = 2$ then this implies that the corresponding function field is elliptic. One day later Hasse wrote:

> *For $y^p - y = f_3(x)$ the genus is really $p - 1$. Further I can explicitly give the characters for any $y^p - y = f(x)$ (polynomial)...*

Here, the "characters" are the ray class characters of the cyclic extension $F|K(x)$ when considered as class field (if K is finite). Still another day later we read:

> *I have got much more general results on*
>
> $$y^p - y \;=\; f(x)$$
>
> *than I first thought. As a matter of fact, I have determined the genus, and with it the number of zeros of the corresponding L-function for every $f(x)$ (integral or fractional)...*

Here, $f(x)$ denotes a rational function in $K(x)$. Hasse had discovered the genus formula

$$2g \;=\; (p-1)(m+r-2) \qquad\qquad (m = m_1 + \cdots + m_r)\,, \qquad (13)$$

where r is the number of poles P_i of $f(x)$ and m_i the order of the pole P_i (provided this order is not divisible by p which can be achieved by a suitable transformation).

Today this and other facts on the arithmetics of Artin-Schreier extensions are well known and can be found in textbooks.[10] We see here how and when they were discovered.

But the main highlight of Hasse's preparatory paper [Has34b] is its second part where he developed class field theory when the field of constants K is finite; this included Artin's reciprocity law. It is true that class field theory for functions fields had been partly covered by F. K. Schmidt already (see [Sch31b]). But that paper covered only the case when the field degree is not divisible by p, and it did not contain Artin's reciprocity law.[11] Hasse presented here a remarkable proof in the general case, using the Local-Global Principle for central simple algebras.

Two years ago, in his paper dedicated to Emmy Noether in 1932, Hasse had presented a proof of Artin's reciprocity law in the case of number fields using the Local-Global Principle for algebras [Has33]. One of the main facts which he had to use there was that every central simple algebra over a number field is split by some cyclotomic extension. Now here, in the function field case, this is also true but it is much more elementary since cyclotomic extensions are given by extensions of the field of constants. Therefore Hasse could use Tsen's theorem which says that a function field over an algebraically closed field of constants has a trivial Brauer group.[12] Although Hasse considered only cyclic extensions of special type (as explained above), he pointed out that it is straightforward to extend the arguments to arbitrary abelian extensions of function fields (over finite fields of constants), in order to obtain Artin's reciprocity law in full generality.[13]

[10]See, e.g., Stichtenoth's book [Sti09].

[11]F. K. Schmidt's paper was published in a relatively unknown journal. We get the impression that he considered this as kind of pre-publication, to be completed later. If so, then the intended later publication never appeared.

[12]Tsen's theorem had just been published [Tse33]. More about Tsen and his theorem can be found in [Lor99].

[13]We note in passing that the Local-Global Principle for function fields was also established in Witt's thesis [Wit34], published in the same yer 1934. Witt used the analytic theory of algebras which he had established in the function field case. – But there remained the solution of two important problems without which class field theory could not be considered complete, namely the *Existence Theorem* (which guarantees the existence of class fields with prescribed finite ray class group) and the *Functional Equation* for L-functions with ray class character. Both problems were solved in the next years by Ernst Witt. For the Existence Theorem see [Wit35]. For the functional equation see section 5.4.

Of particular importance for the application to the Riemann hypothesis is the explicit description of the L-series $L(\chi, s)$ for the ray class characters $\chi \neq \mathbf{1}$ of cyclic extensions $F|F_0$. Hasse does it for arbitrary F_0 (the field of constants K being finite), but in the case of a rational field $F_0 = K(z)$ he adds an elementary presentation. If the field degree is relatively prime to p then again, this is a straightforward transfer of the corresponding situation in number fields but if the degree is p, i.e., in the case of an Artin-Schreier extension, this was completely new. In the next section we shall exhibit those computations in the case of the fields in the diagram (12). This may give the reader an idea of the general formulas in Hasse's paper [Has34b].

3.4 The zeros and Gaussian sums

3.4.1 L-functions and the Riemann hypothesis

Here we shall give an account of how the authors proceed in [Dav33] to compute the zeros of the zeta functions of Davenport-Hasse function fields.

Consider the situation of the field diagram (12) above.

According to class field theory, the Galois group of the abelian extension $F|K(z)$ is isomorphic to a certain ray class group in $K(z)$. The isomorphism between those groups is obtained by Artin's reciprocity law. It is more convenient to consider the duals of those groups, i.e., their character groups. The ray class character group belonging to the extension F is defined modulo the conductor $\mathfrak{f}_F$. This is a divisor of $K(z)$ composed of those prime divisors which are ramified in F.

Two cases have to be distinguished: the cases (9) and (10). In case (9) we have $[F : K(z)] = mn \not\equiv 0 \bmod p$. Therefore the ramification of $F|K(z)$ is tame. There are precisely three primes of $K(z)$ which are ramified in F, namely the primes P_0, P_1, P_∞ which belong to the specializations $z \mapsto 0, 1, \infty$ respectively. P_0, P_∞ are ramified in $K(x)$, and P_1, P_∞ are ramified in $K(y)$. Because of tame ramification, each of these primes occurs in the conductor with multiplicity 1. Hence the conductor $\mathfrak{f}_F = P_0 P_1 P_\infty$ in this case.

In case (10) we have still $m \not\equiv 0 \bmod p$ but $n = p$. There are only two primes which are ramified in F, namely P_0 and P_∞. Both are tamely ramified in $K(x)$, whereas in $K(y)$ only the prime P_∞ ramifies. This is wild ramification, and the conductor $\mathfrak{f}_{K(y)} = P_\infty^2$. The conductor of F is $\mathfrak{f}_F = P_0 P_\infty^2$.

Thus in both cases the conductor of F is of degree 3.

The L-function for any ray class character φ belonging to $F|K(z)$ is defined (in the variable $t = q^{-s}$) as follows:

$$L(\varphi, t) = \sum_A \varphi(A) t^{\deg A} = \sum_{0 \le \nu < \infty} \left(\sum_{\deg A = \nu} \varphi(A) \right) t^\nu \tag{14}$$

where A ranges over all integral divisors of $K(z)$. Here, φ is considered as a “proper” (*eigentlich*) character, which means that φ is defined modulo its own conductor $\mathfrak{f}_\varphi$ (which is a divisor of $\mathfrak{f}_F$), and φ takes the value 0 for divisors A which are not relatively prime to $\mathfrak{f}_\varphi$. If $\varphi = \mathbf{1}$ (the trivial character) then

$$L(\mathbf{1}, t) = \zeta_{K(z)}(t) = \frac{1}{(1-t)(1-qt)}$$

is the zeta-function of the rational function field. Otherwise $L(\varphi, t)$ is a polynomial, in consequence of the Riemann-Roch theorem. The degree of this polynomial is

$$\deg L(\varphi, t) = -2 + \deg \mathfrak{f}_\varphi . \qquad ^{14} \tag{15}$$

As a consequence of the product decomposition of the zeta- and L-functions we have

$$\zeta_F(t) = \zeta_{K(z)}(t) \cdot \prod_{\varphi \ne \mathbf{1}} L(\varphi, t) . \tag{16}$$

The problem to determine the zeros of $\zeta_F(t)$ is equivalent to finding the zeros of the L-series $L(\varphi, t)$ for the ray class characters $\varphi \ne \mathbf{1}$ of $F|K(z)$.

We have $\deg \mathfrak{f}_\varphi \ge 2$. If $\deg \mathfrak{f}_\varphi = 2$ then from (15) we see that the polynomial $L(\varphi, t)$ is constant: $L(\varphi, t) = 1$. Thus on the right hand side of (16) there appear properly only those characters $\varphi \ne \mathbf{1}$ whose conductor is of degree > 2, hence $\mathfrak{f}_\varphi = \mathfrak{f}_F$ is of degree 3. Then $L(\varphi, t)$ is a polynomial of degree 1:

$$L(\varphi, t) = 1 + c_\varphi t \qquad \text{with} \quad c_\varphi = \sum_{\deg P = 1} \varphi(P) \tag{17}$$

where P ranges over all prime divisors of $K(z)$ of degree 1. There is only one zero of $L(\varphi, t)$, given by $t \mapsto -c_\varphi^{-1}$.

Hence there arises the problem to describe those ray class characters φ whose conductor $\mathfrak{f}_\varphi$ is of degree 3, and then to evaluate the sum c_φ in (17).

[14] The term -2 appears since the ground field $K(z)$ is rational in our situation, hence its genus $g_0 = 0$. For an arbitrary function field F_0 as ground field, there would appear the term $2g_0 - 2$ when g_0 is the genus of F_0.

This is the starting point of the Davenport-Hasse paper [DH34].

Remember the diagram (12). The extension $F|K(z)$ is composed of the linearly disjoint fields $K(x)$ and $K(y)$. Hence every ray class character φ belonging to F is uniquely a product $\varphi = \chi\psi$ of a ray class character χ belonging to $K(x)$ and a ray class character ψ belonging to $K(y)$. If $\deg \mathfrak{f}_\varphi = 3$ then necessarily $\chi \neq \mathbf{1}$ (otherwise $\varphi = \psi$ would belong to $K(y)$ and its conductor would be of degree 2). Similarly $\psi \neq \mathbf{1}$.

Every prime P of $K(z)$ of degree 1 is represented by a polynomial $z - a$ with $a \in K$ or by $\frac{1}{z}$; let us write P_a or P_∞ respectively. Then (17) can be written as

$$c_{\chi\psi} = \sum_{a \in K \cup \infty} \chi(P_a)\psi(P_a)\,. \tag{18}$$

On the right hand side the terms belonging to the primes of the conductor can be omitted.

Let us first describe the characters χ belonging to $K(x)|K(z)$. Remember the defining relation $x^m = z$. This leads to the m-th power residue symbol $\left\{\frac{z}{A}\right\}_m$ which is defined for primes $P \neq P_0, P_\infty$ by

$$\left\{\frac{z}{P}\right\}_m \equiv z^{\frac{|P|-1}{m}} \bmod P\,. \tag{19}$$

where $|P|$ denotes the number of elements in the residue field of P. Explicitly, if $|P| = q^f$ then $z^{\frac{|P|-1}{m}} = z^{(1+q+\cdots+q^{f-1})\frac{q-1}{m}} \equiv N_P(z)^{\frac{q-1}{m}} \bmod P$ where N_P denotes the norm from the residue field of P to the base field K. Hence

$$\left\{\frac{z}{P}\right\}_m \equiv N_P(z)^{\frac{q-1}{m}} \bmod P\,. \tag{20}$$

More precisely, $\left\{\frac{z}{P}\right\}_m$ is the unique element in $K^\times$ which satisfies this congruence. In particular, if $A = P_a$ is a prime divisor of degree 1 we have

$$\left\{\frac{z}{P_a}\right\}_m = a^{\frac{q-1}{m}}\,. \tag{21}$$

The extended power residue symbol $\left\{\frac{z}{A}\right\}_m$ for divisors A relatively prime to P_0 and P_∞ is defined by linearity from (20). Artin's reciprocity law implies that the map $A \mapsto \left\{\frac{z}{A}\right\}_m$ is a homomorphism and defines an isomorphism of the ray class group belonging to $K(x)|K(z)$ onto the cyclic subgroup of order m of $K^\times$. In other words: This is a ray class character of order m, with values in $K^\times$.

But the characters in the sense of L-series are supposed to have complex roots of unity as their values. Hence we choose an isomorphism between $K^\times$

and the group U_{q-1} of complex $(q-1)$-th roots of unity and transfer the power residue symbol $\left\{\frac{z}{A}\right\}_m$ to a symbol with values in U_{q-1}. It is convenient to choose a prime divisor $\mathfrak{p}$ of $\mathbb{Q}(\sqrt[q-1]{1})$ dividing p and then take the isomorphism $U_{q-1} \to K^\times$ given by the residue map modulo $\mathfrak{p}$. Hasse denotes the resulting map by $\left(\frac{z}{A}\right)_m$. It is defined by

$$\left(\frac{z}{A}\right)_m \equiv \left\{\frac{z}{A}\right\}_m \bmod \mathfrak{p}\,. \tag{22}$$

More precisely: $\left(\frac{z}{A}\right)_m$ is the unique complex m-th root of unity whose residue class mod $\mathfrak{p}$ is $\left\{\frac{z}{A}\right\}_m$, the latter symbol being defined by the Artin reciprocity map of the cyclic extension $K(x)|K(z)$.[15]

If $A = P_a$ is a prime divisor of degree 1 we have from (21):

$$\left(\frac{z}{P_a}\right)_m \equiv a^{\frac{q-1}{m}} \bmod \mathfrak{p}\,. \tag{23}$$

From this we conclude that the ray class character $A \mapsto \left(\frac{z}{A}\right)_m$ is of order m.

The group of all ray class characters for $K(x)|K(z)$ is cyclic of order m, hence every other ray class character is a power of the above and we have

$$\left(\frac{z}{P_a}\right)_m^\mu \equiv a^{\frac{q-1}{m}\mu} \bmod \mathfrak{p} \qquad (a \in K^\times)$$

for some μ, unique modulo m.

We conclude:

Every ray class character χ of $K(x)|K(z)$ determines a character of the multiplicative group $K^\times$ of the same order as χ. For simplicity let us denote that character of $K^\times$ again with χ; then we have

$$\chi(P_a) = \chi(a) \equiv a^{\frac{q-1}{m}\mu} \bmod \mathfrak{p} \qquad (\text{for } a \in K^\times) \tag{24}$$

for some unique μ with $0 \le \mu < m$.

We similarly deal with the ray class characters ψ belonging to $K(y)|K(z)$. But here we have to distinguish between the two cases (9) and (10). Let us first discuss the case (9).

[15] This is the notation used by Hasse in [DH34].

3.4.2 The case $n \not\equiv 0 \bmod p$

In that case the defining relation of $K(y)|K(z)$ is $y^n = 1 - z$. Accordingly we now consider the n-th power residue symbol $\left\{\frac{1-z}{A}\right\}_n$ defined for prime divisors $P \neq P_1, P_\infty$ of $K(x)$ by

$$\left\{\frac{1-z}{P}\right\}_n \equiv N_P(1-z)^{\frac{q-1}{n}} \bmod P$$

and then extended linearly in the denominator. Again Artin's reciprocity law implies that the map $A \mapsto \left(\frac{1-z}{A}\right)_n$ yields an isomorphism of the ray class group of $K(y)$ with the subgroup of order n of $K^\times$. Again, this symbol is lifted along the prime $\mathfrak{p}$ of $\mathbb{Q}(\sqrt[q-1]{1})$, resulting to the symbol $\left(\frac{1-z}{A}\right)_n$.

> *In this way every ray class character ψ of $K(y)|K(x)$ determines a character of $K^\times$ of the same order, again denoted by ψ, and we now have similarly as in* (24)
>
> $$\psi(P_a) = \psi(1-a) \equiv (1-a)^{\frac{q-1}{n}\nu} \bmod \mathfrak{p} \qquad (1 - a \in K^\times) \qquad (25)$$
>
> *for some unique ν with $0 \leq \nu < n$.*

We may now write (18) in the form

$$c_{\chi\psi} = \sum_{a \in K} \chi(a)\psi(1-a) \qquad (26)$$

where now χ, ψ are characters of $K^\times$ of orders dividing m, n respectively, with values in the group of complex roots of unity. (We have to interpret $\chi(0) = \psi(0) = 0$.)

Recall that the conductor $\mathfrak{f}_\varphi$ of the ray class character $\varphi = \chi\psi$ is supposed to be of degree 3, and that this requires the conditions $\chi \neq \mathbf{1}$ and $\psi \neq \mathbf{1}$. But these conditions are in general not sufficient to guarantee that $\deg \mathfrak{f}_\varphi = 3$. One has also to observe another condition, namely $\chi\psi \neq \mathbf{1}$ as characters of $K^\times$. For, if $\psi = \chi^{-1}$ then (26) yields $c_{\chi\psi} = \sum_a \chi(\frac{a}{1-a}) = 0$.[16] In this case the ray class character $\chi\psi$ has order dividing $d = \gcd(m, n)$ and belongs to the extension

$$K(u) \subset F \qquad \text{with} \quad u = \frac{x^{m/d}}{y^{n/d}} \quad \text{hence} \quad u^d = \frac{z}{1-z}.$$

[16] Note that the map $a \mapsto \frac{a}{1-a}$ is a bijection of $K \cup \infty$ to itself.

There are only two primes of $K(z)$ which ramify in $K(u)$, namely P_0 and P_1. The prime P_∞ is unramified in $K(u)$. Hence the ray class characters belonging to $K(u)$ have trivial L-series.

There are $(m-1)(n-1)-(d-1)$ pairs of characters χ, ψ of $K^\times$ of order dividing m, n respectively, satisfyingthe conditions $\chi, \psi, \chi\psi \neq \mathbf{1}$. On the other hand the genus g of the field F is given by

$$2g = (m-1)(n-1) - (d-1)\,. \tag{27}$$

But $2g$ is also the number of zeros of the zeta-function of F. We conclude that the $2g$ expressions

$$-\,\pi(\chi,\psi) := \sum_{a\in K} \chi(a)\psi(1-a) \qquad \text{with } \chi,\psi,\chi\psi \neq \mathbf{1} \tag{28}$$

coincide with the numbers c_φ appearing in (18).[17] Hence the zeros of $\zeta(t)$ are given by $t^{-1} \mapsto \pi(\chi,\psi)$. The sums (28) are called *Jacobian sums.*

Next we observe that the $\pi(\chi,\psi)$ are the same as the expressions which appear on the left hand side of (5) in Davenport's letter which we have cited in section 3.1, except that the computation is now done in $K = \mathbb{F}_q$ instead of $\mathbb{F}_p$. Following Davenport's computation we obtain a representation of the $\pi(\chi,\psi)$ by means of Gaussian sums. In their paper [DH34] Davenport and Hasse streamlined this computation and obtained:

Theorem in case $n \not\equiv 0 \bmod p$. *Let $K = \mathbb{F}_q$ be the finite field with q elements and $F = K(x,y)$ the function field with the defining relation $x^m + y^n = 1$. It is assumed that m and n divide $q-1$. Then the zeros of the zeta function ζ_F (in the variable $t = q^{-s}$) are in one-to-one correspondence with the pairs (χ,ψ) of characters of $K^\times$ of order dividing m resp. n, with the specification that $\chi,\psi,\chi\psi \neq \mathbf{1}$. For any such pair the corresponding zero is given by assigning t^{-1} to the algebraic integer*

$$\pi(\chi,\psi) = \frac{\tau(\chi)\tau(\psi)}{\tau(\chi\psi)} \tag{29}$$

with the following definition of (generalized) Gaussian sums:

$$\tau(\chi) = -\sum_{a\in K^\times} \chi(a)e(a) \qquad \textit{with } \; e(a) = e^{\frac{2\pi i S(a)}{p}} \tag{30}$$

where $S : K \to \mathbb{F}_p$ denotes the trace function.

[17] The minus sign in (28) has been inserted by Hasse for normalization purposes. Same for the minus sign in formula (30) below.

Indeed, this is a remarkable result. It shows that the factor systems of Gaussian sums, which appeared in Davenport's letter during his calculation for his estimate (see page 10) are in fact identical with the zeros of the zeta-function. The Riemann hypothesis is an immediate consequence of the theorem since it is easily verified that $|\tau(\chi)|^2 = q$, i.e., $|\tau(\chi)| = \sqrt{q}$. This yields a direct proof of the Riemann hypothesis with the help of class field theory, without recourse to Artin's observation (page 8).

3.4.3 The case $n = p$

The case $n = p$ is of different kind. Here, Davenport and Hasse entered completely new ground. Whereas the case $n \not\equiv 0 \bmod p$ could be handled analogously to the similar situation in number fields, in case $n = p$ the Artin-Schreier extension $K(y)|K(z)$ has no counter part with number fields. Hasse had to define an additive analogue to the power residue symbol. For arbitrary Artin-Schreier extensions of function fields he had done this in his preliminary paper [Has34b]. In the present situation where the ground field $K(z)$ is rational, this looks as follows.

The generating equation for $K(y)|K(z)$ is $y^p - y = z$. This time the corresponding residue symbol, which we call "$\wp$-residue symbol", is of additive kind.[18] If $P \neq P_\infty$ is a prime divisor of $K(z)$ then the (additive) $\wp$-residue symbol is defined by

$$\left\{\frac{z}{P}\right\}_\wp \equiv S_P(z) \bmod P\,.$$

where S_P denotes the trace (*Spur*) of the residue field modulo P to the prime field $\mathbb{F}_p$. More precisely, $\left\{\frac{z}{P}\right\}_\wp$ is the unique element in $\mathbb{F}_p$ which satisfies the congruence relation. For any divisor A relatively prime to P_∞ the extended symbol $\left\{\frac{z}{A}\right\}_\wp$ is defined by linearity in the denominator. Artin's reciprocity law implies that the map $A \mapsto \left\{\frac{z}{A}\right\}_\wp$ is a homomorphism and defines an isomorphism of the ray class group belonging to $K(y)|K(z)$ onto the additive group $\mathbb{F}_p^+$, which is cyclic of order p. In other words: This is a ray class character of order p, with values in $\mathbb{F}_p^+$.

But again, the characters in the sense of L-series are supposed to have complex roots of unity as their values. Hence we have to choose an isomorphism between $\mathbb{F}_p^+$ and the group U_p of complex p-th roots of unity and transfer the power residue symbol $\left\{\frac{z}{A}\right\}_p$ to a symbol with values in U_p. For this purpose Hasse chooses the operator $\exp(X) = e^{\frac{2\pi i}{p}X}$. In this way he

[18]In this connection the symbol $\wp$ is used to denote the additive operator $\wp(X) = X^p - X$.

defines the symbol:

$$\left(\frac{z}{A}\right)_{\wp} = \exp\left\{\frac{z}{A}\right\}_{\wp}$$

If $A = P_a$ is a prime of degree 1 then we obtain

$$\left(\frac{z}{P_a}\right)_{\wp} = e(a)$$

where we use the notation $e(a)$ as on page 22 in the definition of the Gaussian sums. This shows in particular that the ray class character $A \mapsto \left(\frac{z}{A}\right)_{\wp}$ is of order p.

The group of ray class characters of $K(y)|K(z)$ is cyclic of order p, hence every other ray class character is a power $\left(\frac{z}{P_a}\right)_{\wp}^{\kappa}$ of the above, and we have

$$\left(\frac{z}{P_a}\right)_{\wp}^{\kappa} = e^{\kappa}(a)$$

for some κ. We conclude:

> **Theorem in case $\boldsymbol{n = p}$**. *Let $K = \mathbb{F}_q$ be the finite field of characteristic p with q elements and $F = K(x, y)$ the function field with the defining relation $y^p - y = x^m$. It is assumed that m divides $q-1$. Then the zeros of the zeta-function ζ_F (in the variable $t = q^{-s}$) are in one-to-one correspondence with the pairs (χ, ψ) where $\chi \neq \mathbf{1}$ is a character of $K^\times$ of order dividing m and $\psi \neq \mathbf{1}$ is a character of K^+ of order p.[19] We may write $\psi = e^\kappa$ for some $\kappa = 1, 2, \ldots, p-1$. For any such pair the corresponding zero is given by assigning t^{-1} to the (generalized) Gaussian sum*
>
> $$\tau_\kappa(\chi) = -\sum_{a \in K} \chi(a) e^\kappa(a)\,. \tag{31}$$

Again, the Riemann hypothesis in case $n = p$ is an immediate consequence of the theorem since $|\tau_\kappa(\chi)| = |\tau(\chi)| = \sqrt{q}$.

It seems remarkable that in case $n = p$ the Gaussian sums themselves appear as the roots of the zeta-function, whereas in case $n \not\equiv 0 \bmod p$ we had seen that *their factor systems* turn out to be the roots. In this respect the result in case $n = p$ looks simpler than if $n \not\equiv 0 \bmod p$. Accordingly in the Davenport-Hasse paper the case $n = p$ is discussed first, and only

[19] Observe that the condition $\chi\psi \neq 1$ is not necessary here since the order of χ divides m and hence is relatively prime to the order p of ψ.

thereafter the case $n \not\equiv 0 \bmod p$ is treated. We have changed the order of discussion since this reflects the historic line. After all, this work started with Davenport's letter (see section 3.1), and that belongs to the case $n \not\equiv 0 \bmod p$.

3.5 *Summary*

In early January 1932 Davenport wrote a letter to Hasse containing an estimate for the number of solutions of the generalized Fermat congruence $ax^m + by^n \equiv c \bmod p$ *where p is a prime number. It turned out that this number is approximately p with an error term of order of magnitude* $\sqrt{p}$. *This constituted the best possible result which was to be expected. Later Hasse observed that the result can be generalized over an arbitrary finite field* $\mathbb{F}_q$ *with the error term estimated by* $\sqrt{q}$. *After a visit to Artin in Hamburg in November 1932 Hasse learned that this observation yields a proof of the Riemann hypothesis for the corresponding function field in characteristic p.*

Moreover, the zeros of the corresponding zeta function (if considered as function of the variable $t = q^{-s}$*) can be described as factor systems of Gaussian sums. This led to a joint paper of Davenport and Hasse which appeared 1934. In the same paper the authors dealt with function fields defined by an Artin-Schreier equation of the form* $y^p - y = x^m$. *The zeros of the zeta function of such a field are Gaussian sums. Today those fields are called Davenport-Hasse fields.*

In their proofs the authors used class field theory for global fields of characteristi p, including L-functions for the corresponding ray class characters. At that time this theory had not yet been fully developed in the literature, hence Hasse published another paper containing full proofs of the relevant class field theory, including Artin's reciprocity law.

4 Gaussian sums

4.1 Arithmetic description

We have said above that Davenport and Hasse wanted not only to prove the Riemann hypothesis for their function fields, but they also wished to determine explicitly the zeros of the zeta functions. On first sight the two theorems above appear to answer this question (see pages 22 and 24). However this is not the end of the story. In the second part of their joint paper [DH34] the authors endeavor to give an "arithmetic characterization" of these zeros.

This started already in February 1932 when Davenport asked Hasse (in a letter dated on the 25th):

> *Now what I should like to know is whether these τ's are known in terms of, say, the decomposition of p in the fields of the m'th and n'th roots of unity – or should it be the decomposition of p in the fields of $\sqrt{m}$ or $\sqrt{n}$? I am very ignorant of all this; can you help me at all, or give me some references ?*

When Davenport speaks of "these τ's" then he has in mind the Gaussian sums $\tau(\chi)$ as above.

We do not know Hasse's direct reply to this. Originally Davenport had in mind the *number of solutions* of the congruence $ax^m + by^m \equiv c \bmod p$ which he tried to express in terms of the τ's. But in the course of their further work, when it became clear that even the roots of the zeta function themselves can be expressed in terms of the Gaussian sums, then the problem of their arithmetic characterization became more important. To be sure, this problem does not really belong to the theory of function fields; it belongs to classical algebraic number theory.

At first Hasse and Davenport planned to write a separate paper about this. That paper, which they called their "snappy" paper, seems to have been essentially completed in early 1934. Among the Hasse papers in Göttingen we have found a manuscript written by Davenport on Gaussian sums. Although this is undated we have reason to assume that it was written for the paper in question. In any case Hasse wrote to Davenport on February 12, 1934:

> *I received your manuscript on Gaussian sums. . . . I have not been able to give it more than a very superficial glance. It seems perfectly alright, though. I will look at it more carefully to-morrow.*

But from the same date there is a letter of Davenport to Hasse; both letters seem to have crossed paths. There we read:

> *I regret to say that the second part of our paper (prime ideal decomposition of the gen*[*eralized*] *Gaussian sums) was done 39 years ago by Stickelberger. . . Mordell reminded me of this paper, which is cited in Hilbert. . . . Sorry to be a bearer of this ill news.*

Here, Davenport referred to Stickelberger's paper [Sti90] of 1890.[20] Hasse

[20] Thus Stickelberger's results had appeared 44 years before Davenport's letter, and not 39 years as he wrote.

replied on February 17, 1934:

> *My dear Harold, it is rather a pity that Stickelberger already proved what took us so many hours. But it is better this has turned up now than later.*

In the same letter Hasse wrote:

> *What I propose is to abandon the original plan of a "snappy" paper on Gaussian sums, but to give our new and simpler proof of Stickelberger's result in our planned paper on $y^n = 1 - x^m$ and $y^p - y = x^m$ as an appendix.*

We do not know Davenport's reply but we know Hasse's next letter of February 22, 1934 where he wrote:

> *You are quite right with your criticism of our proof in favour of the old. But on the other hand ours is more concise. Moreover the old proof and the whole matter seems to have slipped from the minds of our generation, presumably owing to Hilbert's inconceivable not giving it in his Zahlbericht.*

The last sentence puts into evidence the role which Hilbert's *Zahlbericht* of 1897 had played for the next generations of mathematicians. That *Zahlbericht* had become the main source for studying algebraic number theory and it was "inconceivable" that Hilbert would not have included every relevant result from the past. Well, Hilbert did mention Stickelberger's paper in his list of references, but he did not mention it in the text. Hilbert presented Stickelberger's result in the case when $q = p$ is a prime only (in § 112), not saying that the result generalizes to higher powers $q = p^k$. But just this case was now in the focus of interest of Davenport and Hasse.

Finally, after an extended discussion between the two authors it was decided that a presentation of Stickelberger's old proof was to be included as an appendix in their paper [DH34], but in the main body the new proof of the authors was given.

But what is the content of Stickelberger's theorem?

4.2 Stickelberger

By its very definition, $\tau(\chi)$ is an algebraic integer in the cyclotomic field $\mathbb{Q}(\sqrt[q-1]{1}\,, \sqrt[p]{1})$; see (30). The problem is to determine the prime decomposition of $\tau(\chi)$. In view of the relation

$$\tau(\chi)\tau(\overline{\chi}) = \chi(-1)q \tag{32}$$

(where $\overline{\chi}$ is the complex conjugate of χ) we see that only those prime divisors of $\mathbb{Q}(\sqrt[q-1]{1}\,,\sqrt[p]{1})$ appear in $\tau(\chi)$ which are divisors of q, hence of p.

The following diagram shows the structure of the field.

$$\begin{tikzcd}[column sep=small]
 & \mathbb{Q}(\sqrt[q-1]{1}\,,\sqrt[p]{1})\,,\mathfrak{P} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{Q}(\sqrt[q-1]{1})\,,\mathfrak{p} \arrow[dr, dash] & & \mathbb{Q}(\sqrt[p]{1})\,,\Pi \arrow[dl, dash, "p-1"] \\
 & \mathbb{Q}\,,p &
\end{tikzcd} \tag{33}$$

Here, $\mathfrak{P}$ denotes a prime divisor of $\mathbb{Q}(\sqrt[q-1]{1}\,,\sqrt[p]{1})$ dividing p. Observe that p is totally ramified in $\mathbb{Q}(\sqrt[p]{1})$. A prime element of its extension is

$$\Pi = e^{\frac{2\pi i}{p}} - 1\,.$$

On the other side, p is unramified in $\mathbb{Q}(\sqrt[q-1]{1})$, and therefore Π is unramified in $\mathbb{Q}(\sqrt[q-1]{1}\,,\sqrt[p]{1})$. Hence Π is a prime element also for $\mathfrak{P}$. If s denotes the multiplicity of $\mathfrak{P}$ in $\tau(\chi)$ then we have

$$\tau(\chi) \equiv c \cdot \Pi^s \bmod \mathfrak{P}^{s+1}$$

where $c \not\equiv 0 \bmod \mathfrak{P}$ is uniquely determined modulo $\mathfrak{P}$. Using Hasse's notation of multiplicative congruence this can be written as

$$\tau(\chi) \equiv c \cdot \Pi^s \bmod^{\times} \mathfrak{P}\,.$$

which means that the quotient of the two sides is $\equiv 1 \bmod \mathfrak{P}$. Stickelberger had determined the exponent s and the coefficient c in terms of χ and $\mathfrak{P}$, as follows.

In our situation χ is a nontrivial character of $\mathbb{F}_q^{\times}$ of order dividing m, hence it is of the form

$$\chi(a) \equiv a^{\frac{q-1}{m}\mu} \bmod \mathfrak{p} \qquad \text{(for } a \in K^{\times}) \tag{34}$$

for some unique μ with $0 < \mu < m$. (Compare with (24).) We write $q = p^f$ and consider the p-adic expansion of μ:

$$\mu = c_0 + c_1 p + \cdots c_{f-1} p^{f-1}$$

where $0 \leq c_i \leq p-1$ and not all $c_i = p-1$. Let

$$s(\mu) = c_0 + c_1 + \cdots + c_{p-1}$$

be the p-adic digit sum of μ, and

$$\gamma(\mu) = c_0! \cdots c_{f-1}!$$

With this notation one can formulate the theorem for whose proof Davenport and Hasse had spent so much time but then found out that it had been proved in 1890 already:

Stickelberger's theorem:[21]

$$\tau(\chi) \equiv \frac{\Pi^{s(\mu)}}{\gamma(\mu)} \bmod^{\times} \mathfrak{P} \,. \tag{35}$$

It turns out that the algebraic number $\tau(\chi)$ is uniquely determined by the relations (35) and (32), except if $p = 2$ when it is unique up to a minus sign only.

As said above already, this result about the arithmetic characterization of Gaussian sums does not properly belong to the theory of function fields but to algebraic number theory. The question arises *why* Davenport and Hasse had included this into their paper [DH34] whose main purpose was, after all, to prove the Riemann hypothesis for the function fields in question, i.e., for the Davenport-Hasse fields. The authors do not explain their motivation. They only say:

> Es ist zu erwarten, dass auch im Falle eines beliebigen algebraischen Funktionenkörpers mit endlichem Konstantenkörper die Nullstellen der zugehörigen Kongruenzzetafunktion sich arithmetisch charakterisieren lassen, und zwar im Zusammenhang mit der Klassenkörpertheorie durch Teilwerte Abelscher Funktionen.
>
> *It is to be expected that also in the case of an arbitrary algebraic function field with finite field of constants, the zeros of the corresponding zeta function can be arithmetically characterized, namely in connection with class field theory by division values of abelian functions.*

[21] [Sti90], page 354.

Certainly this text had been inserted by Hasse. He refers to his experience with the Riemann hypothesis for elliptic function fields. There the "class field theory by division values of abelian functions" is known under the name of "complex multiplication". Hasse's work on the Riemann hypothesis in the elliptic case had shown a close connection with the theory of complex multiplication. In the elliptic case the two roots of the zeta function are contained in the endomorphism ring and can be arithmetically identified as quadratic numbers.[22] It appears that Hasse had envisaged quite generally a generalization of complex multiplication and the role of the Riemann hypothesis there, in the direction of what later became known under the name of CM-fields. But his remarks here (and elsewhere) are quite vague and do not give any indication that he had already definite ideas how to approach this problem.[23]

There are two cases where the Davenport-Hasse fields are elliptic, namely the fields generated by the equations

$$y^2 = 1 - x^3 \quad (q \equiv 1 \bmod 6) \qquad \text{and} \qquad y^2 = 1 - x^4 \quad (q \equiv 1 \bmod 4)\,.$$

In these cases the Davenport-Hasse paper gives an explicit expression for the two zeros of the respective zeta function.

4.3 Relations

The Davenport-Hasse paper [DH34] contains a section with two important relations between Gaussian sums.

Consider the following situation:

$$\begin{array}{rl} K & \text{a finite field,} \\ \chi & \text{a nontrivial character of } K^\times, \\ \tau(\chi) & \text{the corresponding Gaussian sum,} \\ K_r & \text{the extension of } K \text{ of degree } r, \\ N_r & \text{the norm function } K_r \to K, \\ \chi_r & \text{the character of } K_r^\times \text{ induced by } \chi, \text{ i.e., } \chi_r = \chi \circ N_r, \\ \tau(\chi_r) & \text{the corresponding Gaussian sum over } K_r. \end{array}$$

[22] See Part 3, [Roq06].

[23] Many years later Yamada has taken up this idea and determined to some extent the arithmetic structure of the endomorphism algebra of the Jacobian of the Davenport-Hasse curves. See [Yam68]. An essential ingredient of Yamada's work is the arithmetic characterization of the roots of the zeta function by Stickelberger's theorem.

The first of the two relations reads:

$$\tau(\chi_r) = \tau(\chi)^r\,. \tag{36}$$

For the second relation consider:

- m an integer dividing the order of $K^\times$,
- χ ranges over the nontrivial characters of $K^\times$ with $\chi^m = 1$,
- ψ another character of $K^\times$, so that $\psi^m \neq 1$,
- π (χ, ψ) the corresponding Jacobian sum as defined in (28).

Then:

$$\prod_{\substack{\chi^m = \mathbf{1} \\ \chi \neq \mathbf{1}}} \pi(\chi, \psi) = \psi^m(m) \cdot \frac{\tau(\psi)^m}{\tau(\psi^m)}\,. \tag{37}$$

These two relations (36), (37) refer to Gaussian and Jacobian sums over finite fields, and *prima facie* they have nothing to do with function fields. However, the two theorems above in section 3.4.2 and 3.4.3 show that, indeed, these are statements about the zeros of zeta functions of Davenport-Hasse fields. And it was in this connection that Davenport and Hasse discovered these relations.

The first relation (36) connects to the case $n = p$ of the Davenport-Hasse fields (see page 24). In this case the Gaussian sums $\tau(\chi)$ represent zeros of the zeta function of ζ_F for the function field $F = K(x, y)$ with $y^p - y = x^m$ where m is the order of χ. The relation describes the behavior of the zeros of ζ_F when F is replaced by the constant extension $F_r = FK_r$. Quite generally, it was known that the zeros of ζ_{F_r} (as a function of $t' = q^{-rs}$) are the r-th powers of the zeros of ζ_F (as a function of $t = q^{-s}$). This general fact, valid for arbitrary function fields over finite fields of constants, had been included by Hasse in his survey on zeta functions (thereby citing Artin) [Has34c]. In the case of Davenport-Hasse fields for $n = p$, the relation (36) describes, more precisely, which zero of ζ_{F_r} is obtained when a given $\tau(\chi)$ of ζ_F is raised to its r-th power. Its proof in the Davenport-Hasse paper rests on the same fact as the proof of the general statement in Hasse's survey, namely the decomposition law of prime divisors in a constant extension – except that now, in order to obtain (36), this has to be applied to each individual L-series instead of the whole zeta function.[24]

[24]Note that the complete set of zeros in the case $n = p$ consists of the $\tau_\kappa(\chi)$ for $\kappa = 1, 2, \ldots p-1$. But since $\tau_\kappa(\chi) = \chi(\kappa)^{-1}\tau(\chi)$ it suffices to consider the $\tau(\chi)$.

The second relation (37) refers to the case $n \not\equiv 0 \bmod p$ of the Davenport-Hasse fields (see page 22). In this case the Jacobian sums $\pi(\chi, \psi)$ represent the zeros of the zeta function ζ_F; to this end F had been regarded as a class field over $K(z)$ and the products $\varphi = \chi\psi$ were considered as the corresponding ray class characters. But F can also be regarded as class field over $K(x)$, cyclic of degree m. According to class field theory, its ray class characters are obtained from the ray class characters ψ of $K(y)|K(x)$ by means of the norm function, i.e. these are the functions $\psi^*(A) = \psi(N_{K(x)|K(z)}A)$ where $N_{K(x)|K(z)}$ is the norm function and A ranges over the divisors of $K(x)$ relatively prime to the conductor. This has consequences for the L-function: The L-function $L(\psi^*, t)$ for $F|K(x)$ appears as the product

$$L(\psi^*, t) = \prod_{\chi \neq 1} L(\chi\psi, t)$$

where on the right hand side the L-functions are meant for $F|K(z)$. The discussion of this formula leads to the relation (37).

In the Davenport-Hasse paper we read:

> Wir kamen auf den einfachsten Fall dieser Relationen, nämlich den Fall $m = 2$, durch Rechnungen im Zusammenhang mit Theorem 5 in H. Davenport [Dav33].
>
> *We discovered the most simple case of these relations*[25]*, namely the case $n = 2$, while doing some computations in connection with Theorem 5 in the paper of H. Davenport* [Dav33].

We have checked the paper [Dav33] which investigated the so-called Klosterman sums and strives to obtain estimates for their order of magnitude. But we did not find the relation (37) (in case $n = 2$) explicitly written down. It appears that the relation (37) was discovered *after* Davenport had completed his paper. In fact, that paper was "received by the editors" on July 15, 1932 already. Note that the editor of Crelle's Journal was Hasse, and that Hasse usually scrutinized every paper which he received. Thus we may reconstruct the situation as follows:

In the second half of 1932 Hasse checked Davenport's paper, in particular the computations therein. Thereafter he communicated with Davenport and on this occasion the relation (37) was found. This seems to have happened during the summer semester 1933 when Davenport, who had obtained a

[25] This refers to the relations (37).

stipend, studied in Göttingen.[26] During the weekends and holidays he used to stay with the Hasses in nearby Marburg. This was the time when the Davenport-Hasse paper originated. Most of the discussions about the paper took part in Marburg but there are also a few letters between Göttingen and Marburg by which we can follow the gradual discovery of the relation (37).

On Wednesday June 21, 1933 Davenport wrote from Göttingen:

> *My dear Helmut, The relation we spent so much time looking for is incredibly simple:*
>
> $$\frac{\tau(\psi)\tau(\chi\psi)\tau(\chi^2\psi)\ldots\tau(\chi^{m-1}\psi)}{\tau(\psi^m)} = \varepsilon\cdot p^{\frac{m-1}{2}}. \tag{38}$$
>
> *Here χ,ψ are characters of order m,n resp., where $(m,n)=1$, and ε is an mn-th root of unity depending on ψ and m; $\varepsilon=\varepsilon_m(\psi)$.*

Davenport continued the letter by giving a proof of this relation. At the end he wrote:

> P.S. *I wonder if I shall receive a letter from you in the morning with roughly the same contents!*

In fact, on the same date Hasse had sent a letter from Marburg:

> *My dear Harold, your relation is alright. It generalises at once to*
>
> $$\frac{\tau(\psi)\,\tau(\chi\psi)\cdots\tau(\chi^{m-1}\psi)}{\tau(\psi^m)\,\tau(\chi)\cdots\tau(\chi^{m-1})}\sim 1\,, \tag{39}$$
>
> *where m, n are any numbers prime to each other.*

Again two days later Hasse sent another letter with, he said, a much simpler proof which is valid for arbitrary m,n dividing $p-1$, whereas in their earlier proofs they had to assume that m,n are relatively prime. But so far these computations were done for $q=p$ only, i.e., for Gaussian sums in the prime field $\mathbb{F}_p$. In the last mentioned letter Hasse expressed his wish to deal with the case of arbitrary prime power $q=p^\nu$. Apparently this took some time, for on October 28, 1933 Hasse wrote to Davenport:

[26] In this way Davenport witnessed personally the liquidation of Göttingen as a mathematical center as a consequence of the antisemitic policy of the Nazi government. During this semester in Göttingen began the friendship between Davenport and Heilbronn which later continued in England when Heilbronn had to emigrate.

> *... I am hard at work. Progress comes very slowly indeed. Next week term will prevent me from taking further steps in this matter. I am afraid the thing will not be finished until then.*

And some days later on November 5:

> *Although I should most heartily welcome the publication of a joined paper from both of us after so long a time of our acquaintanceship, I will not press you in the least to finishing the thing now. I can also understand that pursuing new questions is often far more alluring than polishing off old matter.*

So it took another half a year until Davenport sent Hasse his manuscript on Gaussian sums which, however, had to be rewritten since the old paper of Stickelberger had been discovered; we have reported this in the foregoing section.

As to the relations (36) and (37), there were two different proofs given in the Davenport-Hasse paper: one as reported above, using class field theory of the respective function fields, and a second one based on Stickelberger's theorem (see page 29). Since the paper contains two different proofs of Stickelberger's theorem (one new proof by Davenport and Hasse, and also the old proof by Stickelberger) we see that altogether the paper contains three proofs of the relations for Gaussian sums.

But the authors seemed not yet to be entirely satified. In a letter of July 7, 1935 Hasse reports to Davenport about formula (36):

> *My Seminar on Gaussian sums has had one outcome at least: a research student of mine, H. L. Schmid, has found an elementary proof for the relation* $\tau(\chi_r) = \tau(\chi)^r$.

Here, "elementary" means that it does not refer to the theory of function fields and proceeds by algebraic manipulation within finite fields. Schmid's paper appeared 1936 in Crelle's Journal [Sch36a]. Schmid tried also to give an elementary proof of (37) but there was some obstacle which he could not remove, so his proof of (37) was presented conditionally, with a certain formula left open.

Another elementary proof of (36) was given by Davenport in [Dav39]; see page 43.

4.4 *Summary*

After the zeros of the zeta functions of the Davenport-Hasse fields had been represented by Gaussian sums, Davenport and Hasse wished to describe these roots by arithmetic properties within the relevant cyclotomic field. Their motivation was to identify the roots within the anticipated algebra of endomorphisms of the Jacobian, whose algebraic theory was, however, not yet developed. The arithmetic characterization of Gaussian sums required the description of their prime decomposition, together with certain congruences. After the authors had completed their proof it turned out that the result had already been published in the year 1890 by Ludwig Stickelberger. So in their joint paper they included two proofs, their new proof and the old one by Stickelberger. As a byproduct of their computations two important relations between Gaussian sums were obtained. In the later development elementary proofs of these relations were given, one by Hasse's student H. L. Schmid and the other one by Davenport.

5 Functional equation for L-functions

5.1 Weil's question

We have seen that the relevant L-functions for the Davenport-Hasse fields are polynomials of degree 1. This fact was essential for the proof by Davenport-Hasse about the realization of the zeros by Gaussian sums. Are there other function fields where this happens? In the Davenport-Hasse paper this question is not discussed. Years later, in 1939, the question turned up in the correspondence of Hasse with André Weil. In a letter of February 9, 1939 Weil wrote:[27]

> . . . Eine Bemerkung noch, die Ihnen wahrscheinlich schon bekannt ist: aus der Funktionalgleichung ergibt sich die Riemannsche Vermutung für sämtliche L-Reihen, die vom 1.ten Grade sind, also auch schon für diejenigen Zetafunktionen in Funktionenkörpern, die in (abelsche) L–Reihen 1.ten Grades zerfallen; wenn ich mich nicht irre, sind das genau diejenigen Zetafunktionen, für welche Sie in Ihrer mit Davenport gemeinsam geschriebenen Arbeit die Riemannsche Vermutung bewiesen haben, und das ist wohl der eigentliche Grund dafür, dass der Beweis gelingt.

[27] André Weil used German language in his letters to Hasse.

...Still another remark which may probably be known to you already: the functional equation implies the Riemann hypothesis for all those L-series which are of degree 1, and hence also for those zeta-functions of function fields which split into (abelian) L-series of 1st degree. If I am not mistaken these are precisely those zeta-functions for which you have proved the Riemann hypothesis in your joint paper with Davenport. Apparently this is the true reason for the success of your proof.

Hasse confirmed this in his reply of March 7 1939:

Sie haben recht: Die Zetafunktionen, die in L–Reihen ersten Grades zerfallen, sind wesentlich dieselben, die ich in meiner Arbeit mit Davenport behandelt habe ...

You are right: The zeta-functions which split into L-series of first degree are essentially those which I have treated in my paper with Davenport ...

When Hasse says "essentially" then he means that the primes P_0, P_1, P_∞ of $K(z)$ could be replaced by any other three primes of degree 1. But this amounts to an automorphism of $K(z)$, which then can be extended to an isomorphism of the field F. The only property of F essential in this context is that the characters of the extension $F|K(z)$ have conductor of degre ≤ 3.

We shall see below that and how the Riemann hypothesis for the Davenport-Hasse fields may be deduced from the functional equation for the L-functions.

But Weil's comment was not only directed to simplifying the proof of the Riemann hypothesis for Davenport-Hasse fields. He tried to use Hasse's method in a more general setting which would then lead to a proof of the Riemann hypothesis for arbitrary function fields. To this end he tried to interpret the linear factors of the polynomials $L(\chi, t)$ as kind of mock L-functions satisfying a functional equation. He explained this in his above mentioned letter as follows:

Ich vermute aber, dass sämtliche Linearfaktoren der Zetafunktionen in Funktionenkörpern als nicht–abelsche L–Reihen im Artinschen Sinne betrachtet werden können; daraus würde sich der allgemeine Beweis der Riemannschen Vermutung in Funktionenkörpern ergeben. Allerdings bin ich noch nicht so weit, dass ich die nicht-abelschen L-Reihen im Grundkörper darstellen kann; es hängt alles von der Weiterführung meiner Liouvilleschen Arbeit ab. Wenn es mir gelingt, die in Angriff genommene Frage zu lösen, werde ich Ihnen gewiss davon schreiben.

But I suspect that all linear factors of the zeta functions in function fields can be regarded as L-series in the sense of Artin; this would imply the general proof of the Riemann hypothesis in function fields. However I am not yet able to represent the non-abelian L-series in the base field; all this depends on the continuation of my Liouville paper. If I will be able to solve this pending question then I will certainly write to you.

When Weil mentions "non-abelien L-series" he refers to Artin's L-series for Galois extensions of number fields [Art23, Art30]. However Artin had given his theory of Galois L-series for number fields only; in order to use it in the present situation it would have to be transferred to function fields. At the time of Weil's letter this had not yet been done explicitly in the literature but Weil seems to take this for granted. This was admissible since the main prerequisite for Artin's theory was the validity of Artin's reciprocity law, and that had been proved in the function field case by Hasse in [Has34a].

Weil's "Liouville paper" is the one which had appeared in the "Journal de Liouville" [Wei38a]. There he generalizes the Riemann-Roch theorem to matrices over function fields. Although in this paper Weil discusses complex functions where the field of constants is $\mathbb{C}$, he had been able in the meantime to transfer his theory to function fields over arbitrary fields of constants, in particular over finite fields. He had informed Hasse about this in earlier letters so that he could assume that Hasse was familiar with it. In modern language one could describe Weil's results as belonging to the theory of sheaves over a smooth algebraic curve.[28]

We do not know whether this idea of Weil for the proof of the Riemann hypothesis has been followed in the literature.[29] The final proof of the Riemann hypothesis works along different lines. Nevertheless the functional equation for L-series has played an important part in the development of the theory of algebraic function fields in the 1930s, in connection with the quest for the Riemann hypothesis. Therefore it seems appropriate to report here on it.

5.2 The functional equation

But what does the functional equation look like?

Let $F|K$ be an algebraic function field whose field of constants K is finite

[28]Already Witt in his thesis [Wit34] had obtained some results in this direction, and Weil had duly mentioned this in his letters to Hasse.

[29]But note that Weil in [Wei48b] has developed the theory of Artin's L-functions in characteristic p, and in particular proved that they are polynomials (in the variable $t = q^{-s}$).

with q elements. Let χ be a ray class character of F with conductor $\mathfrak{f}_\chi$. We have already given the definition of its L-series (in the variable $t = q^{-s}$) as

$$
\begin{aligned}
L(\chi, t) &= \sum_A \chi(A) t^{\deg A} \\
&= \sum_{0 \le \nu < \infty} c_\nu(\chi) t^\nu \\
\text{with } \; c_\nu(\chi) &= \sum_{\deg A = \nu} \chi(A) \,.
\end{aligned}
$$

where A ranges over the integral divisors of F. If A is not relatively prime to the conductor $\mathfrak{f}_\chi$ then $\chi(A) = 0$. If $\chi \neq \mathbf{1}$ then $L(\chi, t)$ is a polynomial in t of degree

$$d = 2g - 2 + f_\chi \tag{40}$$

where g denotes the genus of F and $f_\chi = \deg \mathfrak{f}_\chi$ the degree of the conductor.

The functional equation governs the behavior of $L(\chi, t)$ under the substitution $t \mapsto q^{-1}t^{-1}$ (which coresponds to the substitution $s \mapsto 1 - s$). Performing this substitution and multiplying the result with $q^t t^d$ we obtain again a polynomial of degree d:

$$q^d t^d L(\chi, q^{-1}t^{-1}) = \sum_{0 \le \nu \le d} c_{d-\nu}(\chi)\, q^\nu\, t^\nu$$

Functional equation for L-functions:

$$q^d t^d L(\chi, q^{-1}t^{-1}) = \varepsilon(\chi) \cdot q^{d/2} \cdot L(\overline{\chi}, t) \tag{41}$$

where $\overline{\chi} = \chi^{-1}$ is the complex conjugate character of χ and $|\varepsilon(\chi)| = 1$. Comparing coefficients we can write this as:

$$c_\nu(\chi) q^{d-\nu} = \varepsilon \cdot q^{d/2} \cdot c_{d-\nu}(\overline{\chi}) \qquad\qquad (0 \le \nu \le d)$$

If $d = 1$ then we obtain:

$$c_1(\chi) = \varepsilon \cdot q^{1/2} \tag{42}$$

and hence $|c_1(\chi)| = q^{1/2}$ which is the Riemann hypothesis for $L(\chi, t)$.

However, this kind of argument does not lead to a proper simplification of the Davenport-Hasse proof which we have reported in section 3.4.1. For, the mere proof of the Riemann hypothesis for the Davenport-Hasse fields had already been achieved in the wake of Davenport's letter 1932 which we have shown in section 3.1. In their joint paper [DH34] the authors went for more, namely to obtain an explicit description of the zeros of the L-series, not only of their absolute value. This would also have been possible by explicitly determining ε in (42) but then one would have to do exactly the same computations which Davenport and Hasse did and which we have reported above.

André Weil was not the first who was interested in the functional equation of L-series in connection with the Riemann hypothesis. Already in Hasse's 1934 paper [Has34b] we find the formula (41) for the functional equation stated with the comment:

> Von besonderem Interesse wäre es ... nachzuweisen, dass die $L(s, \chi)$ der Funktionalgleichung genügen. Doch ist mir das bisher noch nicht gelungen.
>
> *It would be of particular interest ... to verify that the $L(s, \chi)$ satisfy the functional equation. But so far I did not succeed.*

Although Hasse and Davenport in their paper were finally able to do without the functional equation, nevertheless they continued to be interested in it, and other people did so too. There was a general feeling that the functional equation for the $L(\chi, s)$ was closely connected to the Riemann hypothesis. In fact, it had been remarked by Witt that the functional equation for the L-functions is an easy consequence of the Riemann hypothesis for the zeta function. (We read this in a letter from Hasse to Davenport dated May 27, 1934, and also in Hasse's paper [Has34a].)

5.3 Davenport 1934

In the 1930s several proofs of the functional equation were obtained. The first who was actively working on it seems to have been Hasse's correspondence partner Davenport. In the Davenport-Hasse correspondence we find numerous letters where the functional equation of L-series is discussed. This discussion started in May 1934. At that time the Hasse-Davenport paper [DH34] as well as Hasse's preparatory paper [Has34a] had just been completed and sent to publication. It seems that the proof of the functional equation was considered as an unfinished leftover from the work on those papers, and now the authors were trying to fill this gap. From the correspondence it appears that this time Davenport was the active part.

The following excerpts from the Hasse-Davenport letters should give the reader an idea about the intensity of their search for a proof of the functional equation.

01 May 1934 D→H : *I have not got my proof of the functional equation into suitable form for writing up yet. It is simply elementary algebra, and that is something I never was good at.*

02 May 1934 H→D : *I am looking forward to your proof of the functional equation of the L–functions arising from character sums. I hope you will succeed in mastering the exponential sums, too. I wonder, whether the method can be carried through for cyclic equations over arbitrary algebraic function fields, or is by its nature restricted to the rational function field.*

23 May 1934 D→H : *I have got the proof of the functional equation for* $y^n = f(x)$ *into a simple form – direct calculation with polynomials. You may not like the look of it, but it could easily be translated into more elegant languages, I should think. I have been intending to write it up + send it you, but there have been so many distractions.*

27 May 1934 H→D : *Witt has remarked that the functional equation of the congruence L-functions is quite generally a simple consequence from the Riemann Hypothesis. I should very much like to know your proof without R.H.*

22 Oct 1934 H→D : *Witt has made headway towards the functional equation.*

24 Oct 1934 D→H : *I have now got out the proof of the functional eqn. for congruence L-functions, + will send it you soon.*

26 Oct 1934 D→H : *Here is a rough MS on the functional eqn. It is all really very simple, though concealed by a mass of suffixes. The replacement of* $\mathbb{F}_p$ *by a general* $\mathbb{F}_{p^f}$ *is naturally entirely trivial. The other restriction made about the h's cannot be important. Yours in haste . . .*

27 Oct 1934 D→H : *I hope to send you an M.S. on the functional eqn. for the L-functions arising from exponential sums in a day or two. Have you noticed the following amusing consequence of the functional equation: Any L-function of degree 3 has at least one zero on* $\sigma = \frac{1}{2}$.

27 Oct 1934 H→D : *I am looking forward to your proof of the functional equation for the congruence L–series.*

30 Oct 1934 H→D : *I got your proof of the L–functional equation immediately after posting my last communication. I devoured it greedily. My heartiest congratulations on this extremely fine achievement. I find your proof absolutely oke, and more than this: a precious gem. (I hear your reply to this: don't overdo it; but I cannot help, its very simplicity and naturalness fascinated me.) I think I can do the general case (base–field* $\mathbb{F}_q(x, y)$ *algebraic instead the rational field* $E_q(x)$ *, order of* χ *arbitrary instead of prime to p) after the same lines. I put Witt before the question whether I should tell him your proof or not. He decided on not being told. Would you mind my trying to do the generalization indicated ?*

05 Nov 1934 D→H : *I am glad you approve of the M.S. Of course it will require considerable revision before it is fit for publication. I have not got the case* χ *of order p out satisfactorily yet: I can do it (in the case of a polynomial) by the obvious slogging-out method which I thought of over a year ago. But I hope to get this out, and to write a paper on the subject of "The functional eqns of the congruence L-functions" – but in some English Journal,*

27 Nov 1934 H→D : *I could not give another thought to the problem of generalising your functional equation for the polynomial L–series. Although I spent considerable energy on finding the algebraic principle lying behind your curious functional equation connected with a cubic–polynomial, I have not found anything that elucidates this rum thing.*

16 Apr 35 D→H : *Could you let me have back sometime the letter I wrote you (I suppose last autumn) containing a proof of the functional eqns of the exponential-sum L-functions.*

Whit Monday 1935 D→H : [30] *Don't forget to let me have my letter on the functional eqn. back sometime.*

27 March 1936 D→H : *I will make an effort with the functional equation paper. ϑ-series are a sound idea, of course, though I do not regard the ϑ-series proof for the ordinary ζ-fn. as being the "natural" proof.*

30 March 1936 H→D : *The ϑ-functions in Witt's proof of the functional equation are only formally analogous to the analytic ϑ-functions. They are finite series involving a character. Witt's proof, apart from this formal apparatus, may be described as generalizing the proof for Riemann-Roch's theorem to Strahlklassen instead of ordinary Divisorenklassen.*

30 Apr 1936 H→D : *Witt's proof for the* ***L****-functional equation. Manuscript. (No address.)*

08 May 1936 D→H : *Very many thanks for your letter, and account of Witt's method. I have not read this yet, as I prefer to write my MS first. Nothing prevents me from doing this except infinite laziness and total lack of interest for this kind of 'formal' mathematics, where one knows there can be nothing more amusing behind things than trivial identities.*

From this we learn the following: Already in May 1934 Davenport claimed to have a proof of the functional equation. But from Hasse's reply one day later we see that this was the functional equation for "character sums" only. This means that the coefficients of the corresponding L-function can be written as sums of products of characters of the finite base field[31]; this occurs for a ray class character whose order is not divisible by the characteristic p and, therefore, its ramification is tame. Thus in his first attempt Davenport did not deal with the most general case but was content with what he could do with "simple elementary algebra", as he wrote.

Hasse expressed his hope that Davenport could also deal with "exponential sums"; in these sums the terms contain characters combined with exponential functions[32], and they appear when Artin-Schreier extensions are considered. Moreover, Hasse wondered whether the same method can be used also in the more general case where the ground field is not necessarily a rational function field. But Davenport did not answer for a while

[30] In the year 1935 Whit Monday was June 10.

[31] Of similar kind as we have seen in the case of the Davenport-Hasse fields on the right hand side of formula (28). – The terminology "character sums" reflects the fact that Davenport was not a friend of abstract notions like function field and ray class character. His "characters" are characters of finite fields.

[32] Of similar kind as, in the case of Davenport-Hasse fields, on the right hand side of formula (31).

However, when Hasse wrote on Oct 22, 1934 that Witt had "*made headway*" towards the functional equation, then Davenport reacted immediately and sent Hasse at least a rough manuscript. And briefly thereafter, on Oct 30, Hasse could acknowledge the receipt of Davenport's full proof. Hasse found this to be an "extremely fine achievement". But still, this proof dealt with character sums only. On Nov 5 Davenport wrote that he could do also with characters of order p but this seemed not yet to be in a satisfactory shape.

Davenport hoped to be able to publish a paper on the functional equation "in an English journal" as he wrote on Nov 5, 1934. Finally his paper appeared not in an English journal but in the Acta Mathematica [Dav39]. And this was 5 years later only, in 1939. What was the cause of this long delay?

We know that Hasse had visited Davenport in England between February 25 and March 25, 1936. It appears that on this occasion Hasse had tried to push Davenport to publish his results on the functional equation. This would explain Davenport's announcement immediately after Hasse's visit, in a letter of March 27, that he "*will make an effort with the functional equation paper*". In the same letter Davenport mentions ϑ-functions. This indicates that Davenport knew at least some ideas of Witt's proof since in that proof certain polynomials occur which Witt had named ϑ-functions. Davenport may have asked Hasse to send him the full proof of Witt, for Hasse did that on April 30, 1936. (But Davenport did not read it immediately, as he wrote on May 8.) This fact, namely that Davenport had been informed about Witt's general proof, may have contributed to his "*total lack of interest*" which he admitted to Hasse on May 8, 1936. When combined with Davenport's what he called "*infinite laziness*", this seems to have been the cause of the long delay of the publication of his paper [Dav39]. The paper still deals with character sums only, no attempt to generalize his method is mentioned.

Today Davenport's paper [Dav39] seems to have been forgotten.

Let us briefly describe how L-functions look like in Davenport's setup [Dav39]. Consider the following situation:

K	$= \mathbb{F}_q$ is a finite field with $q = p^k$ elements,
f	finitely many different normalized irreducible polynomials $f_1, \ldots, f_r$ over K,
χ	finitely many non-principal characters $\chi_1, \ldots, \chi_r$ of $K^\times$ with the convention $\chi_i(0) = 0$,
(f_i, g)	$\in K$ the resultant of f_i and another normalized polynomial g over K.

$c_\nu = \sum_{\deg g=\nu} \chi_1\big((f,g)\big)\cdots\chi_r\big((f,g)\big)$, the sum being extended over all normalized polynomials g over K of degree ν.

Now Davenport defines the corresponding L-function as

$$L_f(\chi,t) = \sum_{\nu\geq 0} c_\nu t^\nu \,.$$

We see that Davenport's definition does not mention function fields at all, nor does he talk about ray class characters. He only mentions in passing that this is the L-function in $K(x)$ for the cyclic extension $K(x,y)$ with $y^n = f(x)$, where here $f(x)$ is interpreted as the product of the f_i (and $n \not\equiv 0 \bmod p$). In fact, the f_i are seen to be the ramified primes (and they are tamely ramified); in addition the infinite prime P_∞ of $K(x)$ is ramified[33] The degree of the conductor is $d = 1 + \deg f$. The functional equation stated by Davenport blends with the form (41) given by Hasse.[34] Davenport's proof in [Dav39] works by manipulating the terms appearing in the coefficients c_ν. Looking at the details of his arguments it turns out that the main idea is to use the reciprocity

$$(f,g) = (-1)^{\deg f \deg g}(g,f)$$

for the resultant.[35] Indeed this is beautiful, and we can understand Hasse's exclamation in his letter of October 30, 1934 that this is an "*extremely fine achievement*".

However, this argument cannot be straightforwardly generalized to cover the case of wild ramification.

By the way, Davenport's paper [Dav39] contains also a proof of the relation (36) between Gaussian sums; see section 4.3. Moreover, although he cannot prove the Riemann hypothesis, Davenport shows that the roots of the L-functions can be sufficiently estimated such that the result of Bilharz [Bil37] becomes true. Bilharz, a Ph.D. student of Hasse, had discussed the analogue of Artin's conjecture for primitive roots in the case of function fields over a finite constant field. He could prove the expected densities under the assumption of the Riemann hypothesis for function fields. Although this was

[33]Except if n divides $\deg f$. For reasons of simplicity we shall exempt this case in our discussion; it occurs if $\prod_i \chi_i = 1$).

[34]Davenport attributes the conjecture of the functional equation (41) to Hasse [Has34a].

[35]We observe that this reciprocity had been the basis for the reciprocity law for power residues in the rational function field $K(x)$, according to F. K. Schmidt [Sch28]. See also Part 1, section 4.4.2.

not yet proved at the time of Davenport's paper, he had observed that some suitable estimates of the roots of the zeta functions would suffice for Bilharz' arguments in [Bil37]. Davenport was able to establish these estimates with his method.

5.4 Witt 1936

On October 30, 1934 Hasse wrote to Davenport that Witt preferred not to be told the details of Davenport's proof of the functional equation. This indicates that indeed Witt had made "headway" with his proof (as Hasse had written on Oct 22) and was convinced that he could do it on his own. Finally in March 1936 Witt had completed his proof.

But Witt did not publish it. The reason was that he did not wish to endanger the thesis of a young Ph.D. student of Artin in Hamburg who also was working at the same time (1936) on a proof of the functional equation.[36] The name of the student was Weissinger. The topic of Weissinger's thesis had been suggested by Artin who apparently had also supplied the main ideas. Witt had been asked by Artin to abstain from publication for the time being.

What are the main ideas of Witt's proof? We have mentioned already that on April 30, 1936 Hasse had sent Davenport an outline of Witt's proof. The following text is copied from Hasse's letter. Hasse tries to explain everything to Davenport and therefore it may be suitable for us to read at least part of the letter verbatim:

> *...Let F be an algebraic function field with a finite field K of q elements as constant field.*
>
> *Let χ be any character of the group of the divisors of F, which is a congruence character, i. e.,*
>
> $$\chi(A) = 1 \qquad \textit{for} \quad A \sim 1 \ \textit{mod.}\, \mathfrak{f}$$
>
> *with a suitable $\mathfrak{f}$, and let $\mathfrak{f}$ be the exact Führer, i. e., the least divisor with this property, also $\mathfrak{f} \neq 1$ (the case $\mathfrak{f} = 1$ is trivial).*
>
> *($A \sim 1$ mod. $\mathfrak{f}$ means, that A is a principal divisor, i. e., corresponding to an element α of F, and that*
>
> $$\alpha \equiv c \ \textit{mod.}\, \mathfrak{f} \quad \textit{with a constant } c \neq 0\,.)$$

[36]Witt mentioned this in [Wit83].

Now let C be any divisor class (in the ordinary sense) of F, and

$$\vartheta(\chi, C) \; = \sum_{\substack{A \text{ in } C \\ A \text{ ganz}}} \chi(A) \tag{43}$$

the sum extended over all integral divisors A in the class C.[37] *Let $\deg C$ be the degree of all divisors of C, f the degree of $\mathfrak{f}$ and g the genus of K, hence $2g-2$ the degree of the differential class W. Then, putting $d = 2g - 2 + f$,*

$$q^{d-\deg C} \vartheta(\chi, C) = \vartheta(\chi, W\mathfrak{f}) \cdot \vartheta\left(\overline{\chi}, \frac{W\mathfrak{f}}{C}\right), \tag{44}$$

where $W\mathfrak{f}$ denotes the class generated by multiplying the differential divisors with $\mathfrak{f}$. This …gives the functional equation for $L(s, \chi)$ by the usual argument.

Hasse in his letter does not explain the "usual argument"; perhaps he had shown it to Davenport during his visit in March, and so he could assume that this was known to him. The argument is as follows:

Recall the definition (14) of the L-function $L(\chi, t)$ for a ray class character χ of an arbitrary function field F (with finite field of constants), not necessarily a rational field. For any divisor class C (in the ordinary sense) consider the partial sum belonging to those divisors A which are in the class C. This partial sum is $\vartheta(\chi, C)t^{\deg C}$ in the notation of Hasse's letter, and so we can write

$$L(\chi, t) = \sum_C \vartheta(\chi, C) t^{\deg C} .$$

We obtain using (44):

$$\begin{aligned} q^d t^d L(\chi, q^{-1}t^{-1}) &= \sum_C \vartheta(\chi, C)(qt)^{d-\deg C} \\ &= \vartheta(\chi, W\mathfrak{f}) \cdot \sum_C \vartheta\left(\overline{\chi}, \frac{W\mathfrak{f}}{C}\right) t^{d-\deg C} \\ &= \vartheta(\chi, W\mathfrak{f}) \cdot L(\overline{\chi}, t) . \end{aligned}$$

We have used that $C \mapsto W\mathfrak{f}/C$ is a permutation of the divisor classes of F, and that $d - \deg C = \deg W\mathfrak{f}/C$. For $C = 1$ we infer from (44) that $|\vartheta(\chi, W\mathfrak{f})|^2 = q^d$; this gives the functional equation (41).

[37] If A is not relatively prime to the conductor $\mathfrak{f}$ then $\chi(A) = 0$.

We see that the main step in the proof is indeed formula (44). Witt had chosen the notation ϑ on purpose, in order to emphasize the analogy to the theta functions which appear in the number field case in the proof of the functional equation. In fact, nowadays it is possible to give a unified proof for global fields, i.e., for number fields and function fields with finite field of constants.[38]

The cited text above is only part of Hasse's letter to Davenport. In the other part he sketches the way how Witt is going about to obtain the equation (44). Witt uses an arbitrary separating element $x \in F$ and constructs a so-called "normal basis" of $F|K(x)$ which is adapted to the situation at hand. We do not wish to go into the details here. Let us only mention that such construction had also been used by F. K. Schmidt [Sch31a] in order to prove the Riemann-Roch theorem for function fields – from which he then deduced the functional equation of the zeta function. Now Witt had formulated this construction in an abstract form in such a way that it applies as well in both situations, F. K. Schmidt's and Witt's. This abstract form had been named by Hasse as "Witt's Lemma" in his book "Zahlentheorie" [Has02]. There, Hasse used Witt's Lemma for the proof of the Riemann-Roch theorem.[39]

However the use of Witt's Lemma in this sitution is not quite satisfactory for the simple reason that it is not birationally invariant The choice of an arbitrary separating transcendental element x seems to be artificial. The later proof by H. L. Schmid and Teichmüller [ST43] works without referring to an artificial separating element. Therefore we will not explain here Witt's procedure in detail; instead we refer to section 5.6.

REMARK 1: It seems curious that Hasse when preparing his book "Zahlentheorie" (in the year 1938) still used Witt's Lemma in the proof of the Riemann-Roch theorem. We remember his writing to Davenport that he much prefers "*birationally invariant formulas and notions*". Certainly he would also prefer birationally invariant *methods* in the proofs. At the time when he completed his book such methods were available, mainly through the work of Chevalley (with his notion of "ideles" [Che36]) and of André Weil (with his new interpretation of differentials [Wei38b]). Hasse knew both of these papers and was very enthusiastic about this new local-global viewpoint. We have no explanation why he did not use it in his book. One possible reason may have been that he had to write the book under pressure of time (as it is evident from various letters of Hasse). The publisher (Springer) demanded

[38] See, e.g., Weil's book "Basic number theory" [Wei67].

[39] It has been pointed out by W.-D. Geyer [Gey81] that "Witt's Lemma" had already been used 1882 in Dedekind-Weber [DW82] and in various other situations in Mathematics.

harshly to get the manuscript since the delivering date which was agreed to in the original publication contract had passed long ago.

REMARK 2: In the year 1983, when Witt was 72 years, he reported to the Göttingen Academy:

> Tief beeindruckt haben mich 1932 die berühmten 3 Vorträge von Artin über Klassenkörpertheorie. Die anschließenden Ferien verbrachte ich in Hamburg, um dort die Klassenkörpertheorie für Zahlkörper intensiv zu studieren. In den folgenden Jahren war es mein Ziel, diese Klassenkörpertheorie auf Funktionenkörper zu übertragen.
>
> *I was deeply impressed by the 3 famous lectures of Artin in the year 1932. In the subsequent vacations I worked intensively in Hamburg in order to learn more about class field theory of number fields. In the following years I aimed at tranferring class field theory to function fields.*

The three lectures of Artin were given in Göttingen between February 29 and March 2, 1932. Artin, who resided in Hamburg, had been invited by Emmy Noether to talk about the new developments of class field theory. Witt was 21 years at that time. After these lectures, as he reported, he went to Artin in Hamburg to learn class field theory. Almost surely the also met Hecke there. His record during the following years is impressive. In his seminal paper [Wit34] (his Ph. D. thesis, suggested by Emmy Noether) he transferred the work of Käte Hey and Max Zorn to function fields, thereby proving the Local-Global Principle for algebras over function fields. This paper was submitted in 1933 already. One year later (when he was assistant to Hasse in Göttingen) he submitted his proof of the Existence Theorem of class field theory for function fields [Wit35], thereby introducing what today is called the theory of Kummer fields for abelian extensions, not necessarily cyclic. In the next year he settled the functional equation for L-series of function fields which we are discussing here. And after still another year he discovered the calculus of Witt vectors which are used to describe the arithmetic of cyclic extensions of p-power degree of function fields in characteristic p [Wit36]. This sequence of papers indeed was essential for the tranfer of class field theory to the function field case. And in each of these papers Witt developed new and striking ideas.

We see that Witt had been highly successful in carrying out his plan for class field theory of function fields. We also note again the influence of Artin's personality encouraging young mathematicians to produce high level results. Witt is not the only example for this.

5.5 Weissinger 1937

We have said above already that Witt did not publish his proof of the functional equation for L-series, since he did not wish to endanger the Ph.D. thesis of Weissinger in Hamburg. Thus while Weissinger worked on his thesis about the functional equation, there were already two unpublished proofs, namely Davenport's proof in the case of tame ramification and Witt's general proof. Weissinger's thesis appeared 1937 in the "Hamburger Abhandlungen" [Wei37]. This was short before Artin, his Ph.D. advisor, left Hamburg.[40] It seems that Weissinger did not know Davenport's or Witt's results since he did not cite them.

In principle Weissinger's proof runs on the same lines as Witt's, but with an additional feature: He formulates and proves a certain duality theorem for divisor congruences in a function field, valid for an arbitrary field of constants. This plays a similar role for L-functions as does the Riemann-Roch theorem for the functional equation of the zeta function.

Weissinger's was the first pubslished paper which contained a proof of the functional equation for L-series in the function field case. Nevertheless the paper did not get much attention. The reason may have been that soon after, other proofs were published.

5.6 H. L. Schmid and Teichmüller 1941

Witt had presented his proof in the Göttingen seminar.[41] Consequently the details of Witt's proof became known among the specialists. There had been two young people[42] in the seminar who later published a joint paper with another proof of the functional equation, namely H. L. Schmid and Teichmüller. The paper appeared 1943 but was completed in 1941 already [ST43]. At that time the two authors lived in Berlin, H. L. Schmid as a staff member of the *Zentralblatt* and Teichmüller at Berlin University[43]. In their paper they cited

[40] In 1937 Artin had been ousted from Hamburg University because his wife Natasha was of Jewish descent. See [Wuß08]. As a consequence Artin emigrated to the USA. He returned to Hamburg in 1956.

[41] More precisely, this was the legendary *Arbeitsgemeinschaft* (workschop) organized by Witt. Hasse regularly attended the workshop and proposed the topics to be discussed. As mentioned in [SS92]: "*The workshop had developed quickly into a top-class research seminar*".

[42] H. L. Schmid was 28, Teichmüller 23, Witt 25 years of age.

[43] More precisely: Teichmüller was *Dozent* at Berlin University but was "on leave" since drafted to the army. In 1941 he worked on decoding problems, at a military unit which

Witt and also Weissinger.

Their proof starts more or less like Witt's proof but then Witt's Lemma is not used; instead it proceeds in birationally invariant manner, not depending on an auxiliary transcendental element. The authors use duality induced by the residues of differentials of the function field.

This idea resembles Weil's new concept of differential [Wei38b] in as much as they use differentials as linear mappings given by their residues. But they do not cite Weil's paper. Perhaps they had not realized Weil's new idea since this had been explained only in the last section of [Wei38b], as kind of side remark. In fact, Weil in his comments [Wei79] says about his paper:

> Du reste, ce travail finit par où il aurait dû commencer ...
>
> *By the way, that paper ends with what it should have started ...*

In any case, the idea of the authors, if not entirely, new, is remarkable since the duality was explicitly stated and used. Let us briefly report how they proceeded to obtain Witt's formula (44).

First some notations: In (44) we have denoted the given divisor class by C (following Hasse). Now let us change our notation and let C stand for a given divisor, representing his class, chosen to be relatively prime to the conductor $\mathfrak{f}$.[44] Then every integral divisor A of this class is of the form $A = C \cdot (x)$ where (x) is a principal divisor. We have

$$(x) = \frac{A}{C}$$

and see that x is contained in the module $\mathcal{L}(C)$ of multiples of $\frac{1}{C}$. Let us define $\chi(x) := \chi\big((x)\big)$, which means that χ is now a function on elements, not only on principal divisors. We put $\chi(0) = 0$. Then

$$\vartheta(\chi, C) = \frac{\chi(C)}{q-1} \sum_{x \in \mathcal{L}(C)} \chi(x). \tag{45}$$

The denominator $q-1$ appears since every principal divisor (x) is now counted $q-1$ times, namely as cx for each $c \in K^{\times}$.

By construction, every $x \in \mathcal{L}(C)$ is $\mathfrak{f}$-integral, i.e., P-integral für each prime divisor $P|\mathfrak{f}$. Let $\mathcal{O}_{\mathfrak{f}}$ denote the ring of $\mathfrak{f}$-integers and put[45]

$$R = \mathcal{O}_{\mathfrak{f}} / \, \mathfrak{f}\,, \tag{46}$$

was based at Berlin. For more on Teichmüller see [SS92].

[44] If there is no such divisor in the class, the relation (44) is trivial.

[45] See the remark on page 54.

the factor ring modulo $\mathfrak{f}$. Now, by definition of conductor the value $\chi(x)$ depends only on the residue class of x modulo $\mathfrak{f}$. Hence we project $\mathcal{L}(C)$ modulo $\mathfrak{f}$ into R, obtaining a K-module $M = M(\chi, C) \subset R$ and see that

$$\vartheta(\chi, C) = \frac{\chi(C)}{q-1} \sum_{x \in M} \chi(x) =: \frac{\chi(C)}{q-1} \theta(\chi, M). \tag{47}$$

where we have written $\theta(\chi, M)$ for the character sum along M. We see that this procedure has transported the problem into R and its submodules M.

So much the authors follow more or less Witt's idea. But now they describe and use the structure of R. By definition R is a finite commutative K-algebra, and χ is now a character of the multiplicative group $R^\times$ of its units, such that $\chi(c) = 1$ for $c \in K^\times$. Every non-unit $x \in R$ is a zero-divisor and we have put $\chi(x) = 0$ for such x.

But there is more to say. By definition $\mathfrak{f}$ is the precise conductor of χ. This means χ cannot be defined modulo any proper divisor $\mathfrak{f}_0$ of $\mathfrak{f}$, i.e., there exists an element $x \equiv 1 \bmod \mathfrak{f}_0$ such that $\chi(x) \neq 1$. This property translates within R as follows:

For any nonzero ideal $I \subsetneqq R$ *there exists* $z \in I$ *such that* $\chi(1+z) \neq 0, 1$.

We shall call such character χ a "proper" character of R. This corresponds to the classic terminology which says that the ray class character χ is a "proper" (*eigentlich*) character modulo its own conductor $\mathfrak{f}$.

But R carries more structure, namely a non-degenerate bilinear form defined by differentials of F, as follows:

If $\omega \neq 0$ is a differential and P a prime divisor of F let $\mathrm{res}_P(\omega)$ denote its residuum at P. This is defined by first expanding ω into a power series with respect to a uniformizing variable u_P, then taking the coefficient of u_P^{-1} in this expansion (this yields an element of the P-adic residue field K_P) and finally taking the trace of that coefficient, with respect to the trace function $S_P : K_P \to K$. Hence $\mathrm{res}_P(\omega) \in K$. We put

$$\mathrm{res}_{\mathfrak{f}}(\omega) = \sum_{P|\mathfrak{f}} \mathrm{res}_P(\omega), \tag{48}$$

the sum over all prime divisors P occurring in the conductor $\mathfrak{f}$.

Now we fix a differential ω which at each place $P|\,\mathfrak{f}$ has a pole, of the same order as the multiplicity of P in $\mathfrak{f}$. If $x \equiv 0 \bmod \mathfrak{f}$ then the differential $x\omega$ has no pole at any prime $P|\,\mathfrak{f}$ and hence $\mathrm{res}_{\mathfrak{f}}(x\omega) = 0$. Therefore, $\mathrm{res}_{\mathfrak{f}}(x\omega)$

depends only on the class of x modulo $\mathfrak{f}$, which is to say that $\operatorname{res}_{\mathfrak{f}}$ induces in R a K-linear function

$$\varrho(x) := \operatorname{res}_{\mathfrak{f}}(x\omega)$$

with values in K. Since the pole order of ω at each $P|\,\mathfrak{f}$ coincides with the multiplicity of P in $\mathfrak{f}$, it is seen that the bilinear form $(x, y) \mapsto \varrho(xy)$ on R is *non-degenerate.*

Let $M^\perp$ be the orthogonal space to M, consisting of those $y \in R$ for which $\varrho(xy) = 0$ for all $x \in M$. Because of non-degeneracy we have

$$\dim M + \dim M^\perp = \dim R = f \tag{49}$$

and $(M^\perp)^\perp = M$. Recall that $M = M(\chi, C)$ has been defined as the projection of $\mathcal{L}(C)$ to R. It turns out that $M^\perp$ is the projection of $\mathcal{L}(\frac{W\mathfrak{f}}{C})$ where W denotes the divisor of the differential ω. Indeed, if $x \in \mathcal{L}(\mathcal{C})$ and $y \in \mathcal{L}(\frac{W\mathfrak{f}}{C})$ we have

$$(x) = \frac{A}{C}\,, \quad (y) = \frac{B\,C}{W\,\mathfrak{f}} \quad \text{hence} \quad (xy\omega) = \frac{AB}{\mathfrak{f}}\,,$$

where A, B are integral divisors. This means that all poles of the differential $xy\omega$ are occuring in $\mathfrak{f}$. It follows

$$\varrho(xy) = \operatorname{res}_{\mathfrak{f}}(xy\omega) = 0$$

since the sum of all residues of a differential in F vanishes. This shows that the projection of $\mathcal{L}(\frac{W\mathfrak{f}}{C})$ is contained in $M^\perp$. To verify that it actually coincides with $M^\perp$ one has to compute its dimension according to (49). Since M is the projection of $\mathcal{L}(C)$ modulo $\mathfrak{f}$ we have

$$\dim M = \dim C - \dim \frac{C}{\mathfrak{f}}.$$

Similarly, if for the moment we denote the projection of $\mathcal{L}(\frac{W\mathfrak{f}}{C})$ with M' we have

$$\dim M' = \dim \frac{W\,\mathfrak{f}}{C} - \dim \frac{W}{C}\,.$$

The relation

$$\dim M + \dim M' = \dim C - \dim \frac{W}{C} + \dim \frac{W\,\mathfrak{f}}{C} - \dim \frac{C}{\mathfrak{f}} = \deg \mathfrak{f} = f$$

is now verified in view of the Riemann-Roch theorem.

So we have the following situation: *R is a finite commutative K-algebra of dimension f with two additional structures:*

1. A proper character χ of $R^\times$ such that $\chi(c) = 1$ for $c \in K^\times$ (and χ is extended as a function on R by putting $\chi(x) = 0$ if x is a non-unit).
2. A K-linear map $\varrho : R \to K$ such that the bilinear form $\varrho(xy)$ is non-degenerate.

We are interested in a K-module $M \subset R$ and its orthogonal $M^\perp$, and we wish to compare the character sums along M and its dual $M^\perp$:

$$\theta(\chi, M) = \sum_{x \in M} \chi(x), \qquad \theta(\overline{\chi}, M^\perp) = \sum_{y \in M^\perp} \overline{\chi}(y)\,.$$

In this situation H. L. Schmid and Teichmüller have formulated their

Main Lemma:

$$q^{\dim M} \theta(\overline{\chi}, M^\perp) = \tau(\chi)\, \theta(\chi, M). \tag{50}$$

where $\tau(\chi)$ is defined as the generalized Gaussian sum:

$$\tau(\chi) = \sum_{x \in R} \chi(x) e(x) \quad \text{with} \quad e(x) = e^{\frac{2\pi i}{p} S\varrho(x)}\,. \tag{51}$$

Here, $S : K \to \mathbb{F}_p$ is the trace to the prime field. Moreover we have

$$|\tau(\chi)| = q^{f/2}\,. \tag{52}$$

In the proof of the Main Lemma the following property of $\tau(\chi)$ is needed:

$$\chi(x)\tau(\chi) = \sum_{y \in R} \overline{\chi}(y) e(xy)\,. \tag{53}$$

In the classical case this relation is well known; compare with Davenport's letter in section 3.1 (page 10). In our situation the proof is precisely the same provided $x \in R^\times$. For, in this case one can introduce $y' = xy$ as a new variable for the summation, then xy ranges over R if y ranges over R. But if $x \neq 0$ is a non-unit in R then this argument does not work. At this point one has to use that χ is a *proper* character: Since x is a zero divisor in R there esists $0 \neq u \in R$ such that $xu = 0$. The nonzero ideal Ru contains an element $z = au$ such that $\chi(1 + z) \neq 0, 1$. We have $x(1 + z) = x$ and therefore

$$\begin{aligned}\chi(1+z) \sum_{y \in R} \overline{\chi}(y) e(xy) &= \sum_{y \in R} \overline{\chi}((1+z)^{-1} y) e(xy) \\ &= \sum_{y \in R} \overline{\chi}(y) e(x(1+z)y) = \sum_{y \in R} \overline{\chi}(y) e(xy)\,.\end{aligned}$$

Since $\chi(1+z) \neq 0$ it follows that the right hand side in (53) vanishes. So does the left hand side since x is a zero divisor.

Proof of the Main Lemma: In view of (53) we have:

$$\sum_{x\in M}\sum_{y\in R}\overline{\chi}(y)e(xy) = \tau(\chi)\sum_{x\in M}\chi(x) = \tau(\chi)\,\theta(\chi,M)\,. \tag{54}$$

On the other hand:

$$\sum_{y\in R}\overline{\chi}(y)\sum_{x\in M}e(xy) \;= q^{\dim M}\sum_{y\in M^{\perp}}\overline{\chi}(y) = q^{\dim M}\theta(\overline{\chi},M^{\perp})\,, \tag{55}$$

since $\sum_{x\in M}e(xy) = 0$ if $y \notin M^{\perp}$ and $= q^{\dim M}$ otherwise. (The map $x \mapsto e(xy)$ is a character of the additive group of M, and it is nontrivial or trivial according to $y \notin M^{\perp}$ or $y \in M^{\perp}$.)

Multiplying (50) with $\tau(\overline{\chi})$ and applying the relation twice we obtain

$$\tau(\chi)\tau(\overline{\chi})\,\theta(\chi,M) = q^{f}\,\theta(\chi,M)$$

which gives $|\tau(\chi)|^2 = q^f$ – provided $\theta(\chi,M) \neq 0$. This can be achieved by taking, e.g., $M = K$ since $\theta(\chi,K) = q-1$.

$\square$.

Applying the Main Lemma to $M = M(\chi,C)$ and $M^{\perp} = M(\overline{\chi},\frac{W\mathfrak{f}}{C})$, and remembering (47) we obtain Witt's formula (44), hence the functional equation.

Looking at formulas (54), (55) we are reminded of the calculations which Davenport sent to Hasse in his letter of January 1932 (see page 10). There, Davenport finished with the comment: "*Quite trivial!*" Here we are tempted to give the same comment. Indeed, those two lines (54), (55) make up the essential part of this proof of the functional equation; on the preceding pages we had just given a description of the situation. Perhaps Davenport's first proof of the functional equation in October 1934 had run along similar lines when Hasse praised it as a "*precious gem*" (see section 5.3). But in the published version [Dav39] the computations are quite involved and not really lucid. It may be that in October 1934 Davenport had sent Hasse a preliminary version for a special case only, perhaps for the case of the Davenport-Hasse fields where $f = \deg\mathfrak{f} = 3$ and the character χ is of order $\not\equiv 0 \bmod p$. In that case the algebra R reduces to a direct sum of three fields isomorphic to K and Davenport's calculations could have been simplified so that they look similar to (54), (55). But Davenport's proof from that time is not preserved, and so we can only speculate.

We have no record about Hasse's reaction to the proof of H. L. Schmid and Teichmüller. We do not know whether he even had realized the paper [ST43]. We note that the paper appeared not in Crelle's Journal of which Hasse was the managing editor, but in the *Hamburger Abhandlungen* which were edited by the Mathematics professors in Hamburg, and Witt was one of them at that time. So we may assume that Witt was informed about the paper but again, we have no record about his reaction to it. It is true that H. L. Schmid and Teichmüller followed Witt's *Ansatz* to some degree, and even the calculations (54), (55) appear in some form in Witt's proof.[46] But the main idea of the two authors was to put into evidence that the functional equation was closely connected with duality, given by differentials, of the conductor algebra R. This was not evident in Witt's proof.

Remark: The reader will have observed that the above proof breaks down when the conductor $\mathfrak{f} = 1$, i.e., when the cyclic extension to which the character belongs, is unramified. For, in this case the definition of R as given in (46) (page 49) would give $R = 0$ which does not make sense in this situation. H. L. Schmid and Teichmüller were well aware of this fact and provided another proof for the case $\mathfrak{f} = 1$ which is short and nice (and follows Witt's elegant proof in [Wit34] of the Riemann-Roch theorem). But this does not quite fit into their scheme. In this sense the main purpose of their paper was not completely accomplished.

Searching for the cause of this failure we note that the authors were bound to use duality in *finite dimensional* vector spaces, according to the state of knowledge at the time. If they would have at their disposal also the generalization of this to compact and even *locally compact* spaces, they could have argued in the ring of valuation vectors R and its group of units J, called ideles. The valuation vectors are the "adeles". [47]

In the meantime it is common knowledge that instead of ray class characters one should use continuous characters of the idele group which are trivial on the principal ideles. And differentials should be looked at as continuous

[46] Except that Witt did not take for $e(x)$ the exponential but defined a somewhat artificial looking function for this purpose, guided by the analogy to the classical situation in number fields.

[47] Chevalley had introduced the notion of "idele" in [Che36], and had called it in French "élement idéal". Hasse in his review of Chevalley's paper had proposed the name "Idel" which he had created from the German "Ideal" by deleting the letter "a". This became translated into French as "idèle" with an accent, and then was translated into English as "idele". The notion of "valuation vector" is found in Artin's Lecture Notes [Art51], but seems to have been introduced earlier by Chevalley in his book [Che51] under the name "repartition". We do not know who had coined the word "adele" suggesting "additive idele". Perhaps it was A. Weil.

homomorphisms of the additive group of adeles which vanish on the principal adeles. In this setup instead of finite sums there appear integrals, and duality is given by Fourier transforms, as well as the so-called Poisson summation formula. It seems to me of interest that the nucleus of this development is contained in the papers of Witt, H. L. Schmid and Teichmüller in the 1930s and 1940s.

By the way, in the introduction of [Sch41b] it is said that:

> 1941 brachte der zweitgenannte Verfasser den Wittschen Beweisansatz mit dem Begriff der dualen Moduln in Verbindung und führte dadurch die Funktionalgleichung auf eine rein algebraisch formulierbare Identität über Charaktere zurück. Für diese Identität (Haupthilfssatz) gab der erstgenannte Verfasser einen einfachen Beweis.
>
> *In the year 1941 the second author* [Teichmüller] *construed a connection between Witt's Ansatz and the notion of dual module, and with this he reduced the functional equation to an identity between characters which can be formulated by purely algebraic means. For this identity (Main Lemma) the first author* [H. L. Schmid] *provided a simple proof .*

Thus it was Teichmüller who had the essential idea to use duality. Teichmüller died 1943 as a soldier in the war.

To be sure, the definition of the notions of "idele" and "adele" had been already defined (although under different names) when Teichmüller and H. L. Schmid wrote their paper; see footnote 47. But it appears that the importance of these notions had not yet been appreciated in full by the mathematical community. Weil in his paper [Wei38b] discussed adeles in the last section only, somewhat as a side remark. But later he said in his comments in [Wei79] about this paper:

> . . . ce travail finit par où il aurait dû commencer . . .
>
> *. . . that paper terminates where it should start . . .*

Thus in retrospective the author sees the main merit of his paper in the last section which opens new aspects of the theory but realizes that in those times this had not yet be seen.[48]

[48]We use this occasion to point out that Weil's paper [Wei38b] is written in German language and appeared in Crelle's Journal. The paper is the result of a correspondence between Weil and Hasse in 1937. In seems worthwhile to note that in 1938, five years after the Nazis had come to power in Germany, Hasse accepted a paper whose author was of Jewish descent.

5.7 Weil 1939

In 1939 Weil announced to Hasse that he had obtained a proof of the functional equation for L-series. In a letter of January 20, 1939 he writes:

> Ich habe seit kurzer Zeit meine Untersuchungen über p-Gruppen unterbrochen, um früher gefasste Gedanken über die Analogie zwischen Zahlen- und Funktionenkörpern fortzusetzen. Ein erster Erfolg war, dass ich die Funktionalgleichung für die mit beliebigen Charakteren gebildeten Zetafunktionen in Funktionenkörpern mit endlichem Konstantenkörper bewies ...
>
> *Recently I have disrupted my investigations on p-groups in order to give more thoughts to the analogy between number fields and function fields. A first success was that I proved the functional equation for zeta functions belonging to an arbitrary character, in function fields with finite field of constants.*[49]

It appears that Weil was not aware of the two preceding proofs by Witt and by Weissinger. (The proof by Davenport had not yet appeared, and the proof by H. L. Schmid-Teichmüller had not yet been written.)

Hasse replied on February 4 as follows:

> Der Beweis der Funktionalgleichung für beliebige Charaktere ist allerdings schon bekannt, wenn auch nicht veröffentlicht. Witt hat ihn durch eine sehr elegante und formale Analogisierung der Thetafunktionen geführt ... Die Verallgemeinerung des Riemann-Rochschen Satzes auf Strahlklassen hat auch in etwas anderer Weise ein Schüler von Artin gegeben in einem der letzten Bände der Hamburger Nachrichten. Ich glaube der Name dieses Schülers ist Weiss. Schliesslich hat auch Davenport auf mehr rechnerische Art das Bestehen der Funktionalgleichung bewiesen. Leider ist auch diese Rechnung nicht veröffentlicht ...
>
> *But the proof of the functional equation for arbitrary characters is already known, if not published. Witt had done this by a very elegant and formal method in analogy to to the theta functions ... The generalization of the Riemann-Roch theorem for ray classes has also been given, in a somewhat different form, by a student of Artin in one of the recent volumes of the Hamburger Abhandlungen.*[50] *I believe the name of that*

[49]When Weil writes "zeta functions belonging to an arbitrary character" he means L-functions in our terminology as defined above.

[50]Hasse writes "Hamburger Nachrichten" but he means the journal "Hamburger Abhandlungen".

student is Weiss.[51] *Finally Davenport has verified the functional equation by more computational means. Unfortunately these computations are also not published ...*

Upon this Weil thanked for the information and replied, that after checking Weissinger's paper he had found that his own procedure coincides exactly with Weissinger's. Moreover he asked Hasse to inform him about Witt's proof. It seems that he had forgotten that in 1936 Hasse had already sent him an outline of Witt's proof, for on February 24, 1939 he wrote again:

> Mit grosser Beschämung habe ich gefunden, dass Sie mir schon in einem Brief vom 12. VI. 1936 die wesentlichen Züge des Wittschen Beweises für die Funktionalgleichung der L–Reihen in Funktionenkörpern mitgeteilt hatten. Sie sollen sich also bitte nicht die Mühe geben, mir darüber wiederum zu schreiben, obwohl ich Sie in meinem vorigen Brief überflüssigerweise damit belästigen wollte.
>
> *With great embarrassment I have found out that you have already informed me, in a letter of June 12, 1936, about the essential features of Witt's proof of the functional equation of the L-series in function fields. So please do not take the trouble to write again about it, although I had asked you for it in my last letter quite unneccesarily.*

In fact, Hasse had already written to him three years ago, in a letter dated July 12, 1936 :

> Ihrer Bitte, Ihnen von zahlentheoretischen Neuigkeiten zu berichten, entspreche ich sehr gerne. Zunächst wird es Sie sicher interessieren, zu erfahren, dass Herr Witt die Funktionalgleichung der L–Funktionen in Kongruenzfunktionenkörpern nunmehr bewiesen hat, und zwar durch eine sehr hübsche Analogisierung des klassischen Beweises mit den Thetafunktionen. Ich lege Ihnen eine kurze Skizze seines Beweises bei. Die Einzelheiten habe ich nicht aufgeschrieben, weil sie sich für jeden Kenner des klassischen Beweises und der Methodik der arithmetischen Theorie der algebraischen Funktionen wohl von selbst ergeben ...
>
> *It is my pleasure to respond to your request for information about number theoretical news. First, it will surely be of interest to you that Mr. Witt has succeeded to prove the functional equation of the L-functions in congruence function fields.*[52] *He used a very nice idea of creating an ana-*

[51] The correct name was Weissinger.

[52] In the terminology of the time, a congruence function field is a function field (of one variable) over a finite field of constants.

logue to the classiccal proof with theta functions. I enclose a short sketch of his proof. I did not mention the details since they will be evident to any one who is familiar with the classical proof and the methods used in the arithmetic theory of algebraic functions ... [53]

We have not found the "short sketch" which Hasse had provided for Weil in 1936. But now in 1939, Hasse sent Weil an extended exposition of Witt's proof. Although Weil had written that this would not be necessary any more, nevertheless Hasse did send it. He wrote on March 7, 1939:

> Als Ihr Brief vom 24. 2. eintraf, hatte ich bereits angefangen, eine Ausarbeitung des Wittschen Beweises der Funktionalgleichung zu machen. Ich möchte Ihnen nun doch die jetzt fertig gewordene Ausarbeitung vorlegen, allerdings mit der Bitte um demnächstige Rücksendung. Es war mir sehr erwünscht, dass Ihre Anfrage endlich einen Anlass gab, meine damaligen sehr kursorischen Aufzeichnungen in geschlossener Form auszuarbeiten.
>
> *When your letter of February 24 arrived I had already started to prepare an exposition of Witt's proof of the functional equation. Still I would like to put this exposition, which is now completed, into your hands, but asking for return sometime soon. Having been motivated by your request, I took the opportunity to prepare this detailed exposition whereas my former notes were quite sketchy only.*

This detailed exposition of Witt's proof which Hasse had sent to Weil is preserved since the exposition had indeed been returned as requested. (Remember that in those times there was no Xerox and no photo copy, and mathematical expositions used to be written by hand. It was not unusual that the adressee was asked to return a handwritten manuscript which he had received, since it was assumed that he would have taken notes if he wanted to.)

[53] In 1936 there were a number of letters exchanged between Weil and Hasse. This started with the work of Elisabeth Lutz, a student of Weil, on elliptic curves over p-adic fields. Hasse was quite interested in this result and offered publication in Crelle's Journal. Weil considered this as a "*sign of continued cooperation*" and Lutz' paper appeared in 1936 [Lut36]. In a letter of July 8, 1936 Weil asked Hasse to inform him about news from number theory. And Hasse did so in the above cited letter of July 12. That letter was quite long with more than 8 pages. Hasse reported not only about Witt's proof of the functional equation, but also about his own proof of the characterization of the p^n-primary elements in p-adic fields. However the main part of Hasse's letter was devoted to the work of Deuring who had just started to develop an algebraic theory of correspondences. That part of Hasse's letter will be discussed in detail in Part 5.

Thus although Witt had never published his proof, there are two expositions of the proof preserved, both by Hasse. One in Davenport's *Nachlass* at Trinity College in Cambridge, England, and the other in Hasse's *Nachlass* in Göttingen. Finally, even Witt's original manuscript was found in the *Nachlass* of Witt himself. It is included by Ina Kersten in Witt's "Collected Papers"[Wit98], with comments by R. Schulze-Pillot.

5.8 *Summary*

The relevant L-functions for the Davenport-Hasse function fields are of degree 1. This implies that the Riemann hypothesis is an imediate consequence of the functional equation of these L-functions. But in 1934, at the time of publication of the Davenport-Hasse paper, a proof of the functional equation was not yet known. Although the Riemann hypothesis for the Davenport-Hasse fields had finally been established without the functional equation, there was a general feeling that there exists a close connection between the Riemann hypothesis and the functional equation. The letters which were exchanged between Hasse and Davenport in the years 1934-36 show much activity towards finding a general proof of the functional equation, not only for the Davenport-Hasse fields. At the end of 1934 Davenport appears to have succeeded, at least for tame characters.

However Davenport did not immediately publish his proof; the paper appeared in 1939 only. The reason for this delay may have been that Witt had given a general proof, not restricted to tame characters. This was in March 1936; at that time Witt was assistant in Göttingen with Hasse. The idea of Witt's proof was to exploit the analogy between function fields and number fields. But Witt's proof was not published either. Witt had abstained from publication in order not to hamper the Ph.D. thesis of Weissinger (a student of Artin in Hamburg). Weissinger's paper appeared 1937. In the year 1941 still another proof was found by H. L. Schmid and Teichmüller, both former students with Hasse in Göttingen. That proof exploited the duality in the function field given by the residues of differentials. Moreover, in the year 1939 A. Weil had informed Hasse that he had found a proof, but it turned out that this was the same as Weissinger's.

Although Witt's proof was never published, there are two expositions of his proof preserved, both by Hasse. One of these had been sent to Davenport in 1936, and it is contained in the Nachlass of Davenport at Trinity College in Cambridge. The other one Hasse had sent to Weil in 1939 with the request for returning the manuscript; it is now contained in Hasse's Nachlass *in Göttingen. Witt's original handwritten notes have been found in Witt's Nachlass and they were included in the Collected Papers of Witt.*

6 More comments

6.1 Exponential sums

In section 3.1 we have cited a passage in a letter of Davenport to Hasse in January 1932. In another passage of the same letter we read:

> *I have extended the $p^{2/3}$ for Kloosterman sums to $\sum_x \chi(x)\, e(ax + b/x)$ for any χ, hence to $\sum_x e(ax^n + bx^{-n})$.*

Let us explain:

A *Kloosterman sum* is of the form

$$\sum_{\substack{x \bmod p \\ x \not\equiv 0}} e(ax + bx^{-1}) \quad \text{where} \quad e(x) = e^{\frac{2\pi i x}{p}} \,.$$

The problem was to estimate it in its order of magnitude for $p \to \infty$. At the time of Davenport's letter he was preparing a paper where, among other results, he proved that such a Kloosterman sum is $O(p^{2/3})$, in generalization of former results which yield only larger exponents (the best exponent to be expected was 1/2). And similarly for the sum

$$\sum_{\substack{x \bmod p \\ x \not\equiv 0}} e(ax^n + bx^{-n})$$

as he announced in his letter. Later in the year Hasse accepted Davenport's paper for Crelle's Journal where it appeared in 1933 [Dav33].

Quite generally, Mordell and also Davenport had studied so-called *exponential sums.* They are of the form

$$\sigma(f) = \sum_{x \bmod p} e\big(f(x)\big)$$

where $f(x)$ is a rational function modulo p. (The sum ranges over those x modulo p which are not poles of $f(x)$.) In various cases they were able to give estimates of the form $\sigma(f) = O(p^\gamma)$ with some $\gamma < 1$.

Here again Hasse, in his discussion with his friend Davenport, criticized the unsystematic, purely computational methods and looked for a more structural approach. This he found while investigating Artin-Schreier extensions of function fields. We have already mentioned in section 3.3 (on page 15) that Hasse has studied function fields with the defining equation $y^p - y = f(x)$ and had obtained new results, in particular he computed the genus of such function field. In this connection he also mentioned L-functions, and he said:

> *In particular the Kloosterman sums belong to L-series with 2 zeros.*

This shows that Hasse had found the Kloosterman sums in connection with L-functions. The same was the case for arbitrary exponential sums. Let us cite his paper [Has34a]:

> ...Hieraus geht hervor, dass das Problem der Abschätzung der Summen $\sigma(f)$ endgültig gelöst ist, wenn die Riemannsche Vermutung für die Zetafunktion bewiesen ist. Dieser Zusammenhang war bisher ...für die Exponentialsummen nicht bekannt, zu denen z.Bsp. auch die Kloostermanschen Summen gehören.
>
> *...We see from this that the problem of estimating the sums $\sigma(f)$ will be definitely solved, if the Riemann hypothesis for the zeta function is proved. Until now this connection has not been known for the exponential sums, including the Kloosterman sums.*

From this we can deduce that Hasse had realized for arbitrary exponential sums[54] the Riemann hypothesis as the source of the estimate $\sigma(f) = O(\sqrt{p})$ which Davenport and Mordell were looking for – although at that time the Riemann hypothesis was not yet generally proved.

Many years later, in 1948, André Weil published a paper entitled "On some exponential sums" [Wei48c]. The paper starts as follows:

> *It seems to have been known for some time that there is a connection between various types of exponential sums, occurring in number theory, and the so-called Riemann hypothesis in function fields. However, as I was unable to find in the literature a precise statement for this relationship, I shall indicate it here,and derive from it precise estimates for such sums, including the Kloosterman sums.*

[54] Well, for those exponential sums for which the estimate is to be expected: $f(x)$ should not be of the form $c + g(x)^p - g(x)$ with $c \in K$ and $g(x) \in K(x)$. This means that the field extension of $K(x)$ generated by $y^p - y = f(x)$ should be of genus $g > 0$. See, e.g., [Roq98].

The author cites several papers were the said relationship is indicated, e.g., Rademacher's report [Rad42] and Davenport's Crelle paper [Dav33]. But he does not cite Hasse's [Has34a] where, as we have seen, Hasse had already mentioned exactly that relationship.

6.2 Cyclic extensions of p-power degree

In section 5.1 we have reported on Weil's question and have cited Hasse's answer, namely that the Davenport-Hasse fields are those for which all the relevant L-functions are of degree 1. (See page 36). But Hasse had added the following remark which we did not yet mention:

> Zu diesen [Funktionenkörpern] kommen noch solche hinzu, wo der Grad eine höhere Potenz der Charakteristik ist. Ihre Theorie ergibt sich aus den Ergebnissen von Witt über zyklische Körper dieser Art.
>
> *There have to be added those* [function fields] *where the degree is a higher power of p. Their theory is a consequence of the results of Witt about cyclic fields of this kind.*

This sounds as if Hasse would claim that the same or a similar proof works also in a situation more general than in the diagram (12) on page 13. Namely, the field extension $K(y)|K(z)$ of degree p may be replaced by a cyclic extension of arbitrary p-power degree where only one prime P_∞ is ramified. In particular this would imply that the conductor of any non-trivial ray character of that extension would be of degree ≤ 3.

But this is not the case. There is a paper by H. L. Schmid of 1936 where he develops the arithmetic theory of cyclic p-extensions, using Witt vectors [Sch36b]. Among other results, Schmid had computed the conductor degree for any nontrivial subfield. It turned out that the conductor degree is $> p$ if the subfield is of degree $> p$ – at least if the ground field is the rational function field $K(z)$, hence admits no unramified extension. Accordingly the proof in the Davenport-Hasse paper, where these conductors are of degree ≤ 3, does not work in the same way for p-power degree $> p$, regardless which cyclic extension $K(y)|K(z)$ in the diagram (12) is chosen.

In the year 1936 Schmid had been an assistant to Hasse at Göttingen and his above mentioned paper had been written on the suggestion of Hasse.[55]

[55] Later in 1941, when Schmid was already in Berlin, he published a second paper with more detailed computations [Sch41a]. But he indicated that these compuations were of interest only as long as the Riemann hypothesis was not yet proved for these fields.

Hence Hasse knew Schmid's results. Therefore Hasse's above mentioned remark in the letter to A. Weil of 1939 has to be interpreted differently.

Perhaps Hasse wanted to say the following: If in the diagram (12) the extension $K(y)|K(z)$ is replaced by a suitable cyclic extension of degree p^ν, then he (Hasse) is still able to give an explicit description of the corresponding ray class characters and the coefficients of their L-functions. And this in such a way that a suitable estimate of those coefficients would imply the Riemann hypothesis.

In fact, we have found a manuscript by Hasse where he expounds just this idea. We found it among the Davenport papers in Trinity College. The title of this manuscript is

"L-Reihen zyklischer p-Körper."

"L-series of cyclic p-fields."

Hasse seems to have written this manuscript for Davenport. For, although the language is German Hasse did not use in his handwriting the German script which he commonly did when writing German, but here he used Latin script so that Davenport would be able to read it. The manuscript is not dated but we believe it is written shortly after the Witt vectors had been discovered, i.e. around 1937. For, Hasse starts with an explanation about the construction of cyclic p-extensions by Witt vectors, from which we conclude that he did not assume Davenport to know the details already.

Hasse's interest in these questions is also documented by an unpublished manuscript which we have found in his *Nachlass* in Göttingen [Has52]. This manuscript is dated many years later, namely December, 1952. Here we see Hasse's renewed interest in this problem. The title is:

Verallg. von Davenport-Hasse:
Gaußsche Summen in algebraischen Funktionenkörpern.
(Zu den Arbeiten von H. L. Schmid, Berl. Akad. Abh. 1941
und H. L. Schmid-Teichmüller, Hamb. Abh. *15* (1943).)

Generalization of Davenport-Hasse:
Gaussian sums in algebraic function fields.
(*Comments to the papers of H. L. Schmid* [Sch41a]
and H. L. Schmid-Teichmüller [ST43].)

The main result in this manuscript is an explicit description of the characters of the extension $F|K(z)$ in the following diagram:

$$\begin{tikzcd} & F = K(x,\underline{y}) \arrow[dl, dash] \arrow[dr, dash] & \\ K(x) \arrow[dr, dash, "x^m = z"'] & & K(\underline{y}) \arrow[dl, dash, "\underline{y}^p - \underline{y} = \underline{z}"] \\ & K(z) & \end{tikzcd} \qquad (56)$$

This is similar to the diagram (12) on page 13 but now $\underline{y} = (y_0, y_1, \ldots, y_{\nu-1})$ denotes a Witt vector of length ν and $\underline{z} = (z, 0, \ldots, 0)$. Thus $K(\underline{y})$ is cyclic over $K(z)$ of degree p^ν with P_∞ as the only ramified prime.

In this manuscript Hasse gives an explicit formula for the L-functions of the extension $F|K(z)$. Specifically, the Gaussian sums which appear in the functional equation of the L-functions are computed.

But in this way Hasse could not prove the Riemann hypothesis for these fields. After all, if we realize that this manuscript is dated 1952 we remember that a full proof of the Riemann hypothesis had already been published by A. Weil [Wei48a]. So why did Hasse care to compute in such detail these Gaussian sums with Witt vectors?

We do not know. Perhaps Hasse wished to have explicit formulas at hand for his investigation of the so-called Hasse-Weil zeta function for the function fields of Fermat type over number fields?

We observe that just in the year 1952 Weil's paper "*On Jacobi sums as Grössencharacters*" had appeared, where Weil investigated the function fields F of Fermat type over a number field K and their zeta functions [Wei52]. This can be regarded as a 2-dimensional problem, in as much as two kinds of arithmetics come into play: first the arithmetic of the number field K and secondly the arithmetic of the function field $\mathsf{F}|\mathsf{K}$.

From the correspondence Hasse-Weil it is apparent that Hasse in the late 1930s had already taken into account the question whether there exist zeta functions of algebraic function fields over algebraic number fields. The idea was to define the zeta function of such a function field $\mathsf{F}|\mathsf{K}$ as the product of the zeta functions of the reduced function fields modulo the primes $\mathfrak{p}$ of the base field K. (Excepting perhaps those finitely many $\mathfrak{p}$ at which F has no good reduction.)

Hasse had already proposed to two of his students to follow up this idea.[56] But nothing had come of it, not the least due to the political happenings in the subsequent years. Now in 1952 Weil had taken up this question. It is apparent that Hasse was highly interested in Weil's work. Weil had studied the function fields $\mathsf{F}|\mathsf{K}$ of Fermat type, and these are precisely those which after reduction modulo $\mathfrak{p}$ yield Davenport-Hasse fields, however without being restricted to the case $[K(y) : K(z)] = p$ as in diagram (12). Therefore it becomes necessary to consider also the case of the digram (56) where $K(y)|K(z)$ is cyclic with degree an arbitrary p-power.

Thus perhaps we may interpret the above-mentioned unpublished Hasse manuscript [Has52] as a preparatory work for his study of the *Größencharacter* in question [Has55].

6.3 *Summary*

The so-called Kloosterman sums and, more generally, the exponential sums had appeared in various number theoretical investigations, and it was conjectured that they are of order of magnitude $O(\sqrt{p})$ for $p \to \infty$. Hasse was well aware of the fact that the Riemann hypothesis for the Davenport-Hasse fields implies this estimate. This is apparent from the correspondence between Davenport and Hasse. Moreover, Hasse had mentioned this in his 1934 paper on cyclic extensions of function fields. But this seems not to have become common knowledge among number theorists. Later in 1948, after having proved the Riemann hypothesis, André Weil published an account of these matters. –

In order to extend his results on the Davenport-Hasse fields, Hasse investigated cyclic extensions of arbitrary p-power degree, of their ray class characters and L-functions. He did this in the framework of Witt vectors. He sent his manuscript to Davenport, but it was never published. Later in 1952 he wrote another manuscript where Gaussian sums for Witt vectors were computed, in connection with the functional equation for L-functions. We suspect that Hasse intended to use it for the computation of the conductor of the Hecke "Größencharacter" appearing in the zeta function of the function fields of Fermat type over algebraic number fields.

[56] These were Hanna von Caemmerer (later Hanna Neumann) and Pierre Humbert.

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