# On the history of Artin's $L$-functions and conductors

## Seven letters from Artin to Hasse in the year 1930

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July 23, 2003

**Abstract**

In the year 1923 Emil Artin introduced his new $L$-functions belonging to Galois characters. But his theory was still incomplete in two respects. First, the theory depended on the validity of the General Reciprocity Law which Artin was unable at that time to prove in full generality. Secondly, in the explicit definition of $L$-functions the ramified primes could not be taken into account; hence that definition was of provisional character only whereas the final definition could be given in a rather indirect way only. In later years Artin filled both of these gaps: In 1927 he proved the General Reciprocity Law, and in 1930 he gave a complete definition of his $L$-functions, including the ramified and the infinite primes; at the same time he introduced his theory of conductors for Galois characters.

This development is well documented in the correspondence between Artin and Hasse of those years. In the present paper we discuss seven letters from Artin to Hasse, written in the year 1930, where he expounds his ideas about the final definition of the $L$-functions and about his conductors. We also discuss some letters from Emmy Noether to Hasse of the same time which are directly inspired by Artin's.

*This copy contains some minor corrections of the published version.

## Contents



# 1 Introduction

The legacy of Helmut Hasse contains 49 letters from Emil Artin, predominantly of mathematical content, written between 1923 and 1953. In this paper we discuss seven of those letters, written in 1930, which are concerned with Artin's theory of $L$-functions and conductors. Unfortunately, Hasse's replies to Artin's letters seem to be lost; we have to guess the contents of Hasse's replies by interpolating from Artin's comments in his subsequent letters.

The full text of all 49 letters are published by Günther Frei [30]. We have tried to write our paper in such a way that it can be read without knowledge of the full text. [1]

In the last chapter we will discuss some of Emmy Noether's letters to Hasse, written between 1930 and 1934. When Hasse had shown her Artin's letters about conductors for Galois characters, she responded enthusiastically and immediately started to bring her own ideas to bear on the subject, culminating in her famous theorem about normal integral bases. Thus it seems appropriate to view her letters in connection with Artin's.

The Noether letters to Hasse are preserved, like Artin's, in the University Library at Göttingen; they are not yet published in their full text. [2] This collection is one-sided in the same sense as Artin's: we know the letters from Emmy Noether to Hasse only, while most of the letters in the other direction are lost.

In the year 1930 when these letters were written, Emil Artin held a professorship at Hamburg University. He had come to Hamburg in 1922 as an assistant, and had been appointed full professor in 1926 at the age of 28. For biographical references on Artin see e.g., [90], [13].

Helmut Hasse in the summer of 1930 had just accepted an offer of professorship at Marburg University, as the successor of his academic teacher Kurt Hensel who had retired. Before that, Hasse had held a professorship at the University of Halle since 1925. Both Hasse and Artin were of the same age, born 1898. Biographical references: [20], [31]. – Hasse's voluminous legacy is kept at Göttingen University Library. All the letters from Artin to Hasse which are discussed here, are obtained from that source.

Emmy Noether, ten years older than Hasse and Artin, was *Privatdozent* and honorary professor at Göttingen University. Biographical references can be found in [28], [81]. – Noether's letters to Hasse can also be found in Hasse's legacy at Göttingen.

REMARK: The Hamburg Institute of History of Science keeps some papers, mostly handwritten material, of Artin. This legacy of Artin had been handed over by his son, Michael Artin, in the year 1999. A commented catalogue has been written by Peter Ullrich [106]. Among these papers there is a handwritten manuscript by Artin about $L$-functions with Galois characters. But that manuscript seems to be a draft for Artin's first paper 1923 on $L$-functions [4] only; I did not find there any information pointing towards Artin's later work on $L$-functions which we discuss here.

[1] Some of the letters contain also material which does not belong to the theory of Artin's $L$-functions and conductors; this will be discussed elsewhere.

[2] We are planning to do so in the near future.

# 2 Letter of 23 Aug 1930

## 2.1 The galley proofs

In the year 1930 Hasse had completed the manuscript of Part II of his great Class Field Report (*Klassenkörperbericht*) [43]. In the years before, on several occasions Artin had shown a vivid interest in the progress of this report. And Hasse seems to have informed him regularly about it. Now, with the manuscript completed, Hasse wished to inform him about its final version. Therefore he sent to Artin the galley-proofs of the report. This was a common practice in those times, without Xerox, to inform friends and colleagues about forthcoming publications.

In a letter of 23 August 1930 Artin acknowledges the receipt of these galley-proofs:

> *Vielen Dank für die Korrekturen, die ich mit großem Genuß gelesen habe. Wenn ich etwas daran auszusetzen habe, so ist es dies, dass ich viel zu gut dabei wegkomme und es aussieht, als ob ich viel mehr dabei geleistet hätte als es tatsächlich der Fall ist. Besonderes Vergnügen hat mir Ihre Theorie der Normenreste gemacht...*
>
> Many thanks for the galley-proofs which I have read with great pleasure. If I am to criticize it, then it is because I have been treated too well in it, and that it gives the impression that I had done much more in this direction than I did in reality. I was particularly delighted by your theory of norm residues...

When Artin mentions the theory of norm residues then this relates in part to their close cooperation in former years, starting in 1923, which had led to a joint publication of Artin and Hasse [12]. This subject of the Artin-Hasse correspondence will be discussed elsewhere. For our present purpose, which is directed towards Artin's $L$-functions, it is important that the reading of Hasse's galley-proofs stimulated Artin to think again about his $L$-functions. For, Artin continues:

> *...Natürlich hat mich Ihr Bericht wieder zum Nachdenken über diese Dinge gereizt, die ich jetzt so lange nicht angerührt habe. Ich will in Hamburg dann wieder ernstlich an die Dinge herangehen. Hier kann ich es nicht, da ich nicht Literatur zur Verfügung habe. Ich darf doch die Fahnen behalten ?...*
>
> ...Naturally, by reading your report I have been stimulated to think again about those things which I have not touched for such a long time. In Hamburg I intend to seriously approach these things again. I am not able to do this here, in this place, because I have no literature available. May I keep the galley-proof sheets ?...

Artin does not say what precisely he means by "those things". But from what follows in his subsequent letters it will become clear that indeed he has in mind his theory of $L$-functions $L(s, \chi)$ for Galois group characters $\chi$. After his return to Hamburg, he says, he intends to take up the work on $L$-functions again. [3]

Remark: In the first of the two citations from Artin's letter we observe that he tries to belittle his (Artin's) own contribution to class field theory. We do not know whether Artin really meant this or whether his remark is to be regarded as a polite gesture of modesty only. We tend to assume the latter. After all, Hasse's report was the first book containing Artin's Reciprocity Law embedded into a systematic account of class field theory. We may safely assume that Artin himself had been well aware of the high importance of his Reciprocity Law as a basic and fundamental fact within the framework of class field theory.

## 2.2 Artin's $L$-series and his Reciprocity Law

The introduction of Artin's $L$-functions dates back to the year 1923 when Artin's paper "*Über eine neue Art von L-Reihen*" (On a new kind of $L$-series) appeared in the *Hamburger Abhandlungen* [4]. In a letter to Hasse dated 9 July 1923, Artin had briefly informed him about these concepts and the results of his paper.

Part I of Hasse's Class Field Report [41] appeared in 1926, hence three years after Artin had presented his theory of new $L$-functions. Clearly Hasse had been well aware of Artin's results, but nevertheless he had not included Artin's $L$-functions into Part I. To understand this, we have to recall that by 1926, Artin's theory of $L$-functions was still incomplete.

In fact, Artin's 1923 paper contained an unproven theorem ("*Satz 2*") which today is known as *Artin's Reciprocity Law*. Although Artin had stated it as a theorem and not as a conjecture, in 1923 he was not yet able to prove it in full generality. He could verify it only for cyclic extensions of prime degree with the help of Tagaki's class field theory and then, somewhat more generally, for abelian extensions whose exponent is square free. This implied that Artin's theory of $L$-functions, as developed in his 1923 paper, remained incomplete as long as the validity of the Reciprocity Law was not yet fully established.

Perhaps this was one of the reasons why Hasse, when finishing his manuscript for Part I in 1925, decided not to include Artin's $L$-functions. But in any case, Hasse knew already in 1925 that Artin was well on his way towards a proof of his Reciprocity Law. For, in a letter to Hasse dated 10 Feb 1925 Artin had written:[4]

> *. . . Haben Sie die Arbeit von Tschebotareff in den Annalen Bd.95 gelesen? Ich konnte sie nicht verstehen und mich auch aus Zeit-*

[3] Artin's letter is dated "*Neuland, den 23. August 1930*". Neuland at that time was a small village south of Hamburg city, ducking under the dykes beyond the Elbe river.

[4] Remark (Jan 2007): While preparing the commented edition of the complete Artin-Hasse correspondence, I have read the letters again. It turned out by closer inspection that the letter in question cannot be written in February 1925, as Artin had dated it, since there are several details mentioned which happened later only. In particular, vol. 95 of the *Mathematische Annalen* which he mentions in his letter, had not yet appeared. It appears that Artin erroneously wrote 1925 when he meant 1926. For details we refer to our forthcoming book on the Artin-Hasse correspondence.

> *mangel noch nicht richtig dahinterklemmen. Wenn die richtig ist, hat man sicher die allgemeinen Abelschen Reziprozitätsgesetze in der Tasche...*
>
> ...Did you read Chebotarëv's paper in the *Annalen*, vol. 95? I could not understand it, and because of lack of time I was not able to dive deeper into it. If it turns out to be correct then, certainly, one has pocketed the general abelian reciprocity law...

It was in the summer of 1927 only that Artin finally succeeded to work out all details of the proof. [5]. This is documented in the letter to Hasse dated 17 July 1927:

> ...*Ich habe in diesem Semester eine zweistündige Vorlesung über Klassenkörper gehalten und dabei endlich das "allgemeine Reziprozitätsgesetz" bewiesen, in der Fassung, die ich ihm in der L-Reihenarbeit gegeben habe...*
>
> ...In this semester I have given a two hour course on class fields and on that occasion finally proved the "General Reciprocity Law" in the form which I had given it in my $L$-series paper...

As soon as Artin's proof had been published, the great importance of the Reciprocity Law in algebraic number theory was immediately and widely recognized. Takagi [103], in his review of Artin's paper, called it "one of the most beautiful recent results in algebraic number theory" [6]. Hasse [43] speaks of a "progress of the highest importance" (*"Fortschritt von der allergrößten Bedeutung"*). Although Hasse had already completed a large part of his manuscript for Part II of the Class Field Report, he decided to reorganize and rewrite his manuscript in order to include Artin's Reciprocity Law and put it into its proper perspective. [7]

Moreover, Hasse now decided to include the theory of Artin's $L$-functions as well. He considered it as one of the first important applications of Artin's Reciprocity Law. At that time, people dealing with class field theory were looking for its generalization from abelian to arbitrary Galois extensions. Both Artin and Hasse were convinced that $L$-functions with Galois characters would turn out to be an essential tool in future developments in this direction – although they were aware that, in addition, new concepts and ideas were still needed. Let us cite Hasse [43] (page 164) regarding this question:

[5] As announced in his 1925 letter to Hasse, Artin's final proof used ideas of Chebotarëv from his density paper [22]. Perhaps it is not widely known that Chebotarëv himself had independently worked on a proof at about the same time as Artin. In his memoirs Chebotarëv writes that in the summer of 1927 he studied class field theory, and he became convinced that it was possible to prove the Reciprocity Law by means of his (Chebotarëv's) device of taking composites with cyclotomic extensions. The outline of a proof began to dawn on him, although rather dimly. Then he discovered in the University Library in Odessa the new issue of the *Hamburger Abhandlungen* with Artin's proof. See the article on Chebotarëv by P. Stevenhagen and H.W. Lenstra [99] who pointed out: *"Chebotarëv was not far behind"*.

[6] The article [103] is written in Japanese. I am indebted to S. Iyanaga for providing me with a translation.

[7] The rewriting needed some time, and this explains to some extent why the publication of Part II was delayed so long; it appeared only four years after Part I.

> *...Es ist erstaunlich, wie weitgehend man so* [mit Hilfe der Artinschen $L$-Funktionen] *von der Theorie der Abelschen Zahlkörper aus die Theorie der beliebigen Galoisschen Zahlkörper beherrscht... Natürlich darf es nicht wunder nehmen, daß man auf diesem Wege nicht auch das letzte Ziel erreicht, nämlich die Aufstellung der Zerlegungsgesetze selbst in beliebigen Galoisschen Zahlkörpern...*
>
> ...It is remarkable how far the theory of arbitrary Galois number fields is thus [*by means of Artin's $L$-functions*] governed by the theory of abelian number fields...Of course it is not surprising that in this way one does not reach the final aim, namely the formulation of the decomposition law in arbitrary Galois number fields...

This text shows Hasse's motivation for including Artin's $L$-functions in Part II of his Class Field Report. At the same time we see why Artin was so interested to put his theory onto a more adequate foundation.

As we know today, Hasse's and Artin's hope for obtaining a decomposition law for Galois number fields cannot be realized, at least not in the form which was expected at that time. The Langlands program for the generalization of Artin's Reciprocity Law to the general Galois case is of a different kind. But still, Artin's $L$-functions are part of the prerequisites of that program. [8]

# 3 Letter of 18 Sep 1930

About one month later Artin writes another letter to Hasse, in order to inform him about the outcome of his work which he had announced in his earlier letter. He does so in the form of critical comments to the relevant sections of Hasse's Class Field Report, Part II, whose text, as we know, Hasse had sent him in the form of galley-proof sheets. [9]

## 3.1 The complete definition of $L(s,\chi)$

The letter contains three sections. In section 1.) Artin starts as follows:

> *Mich störte in Ihrem Bericht der Satz auf Fahne 73, Zeile 10 von oben. Hier haben Sie die vollständige Definition von $L(s,\chi)$ ...*
>
> I did not like in your report the theorem on sheet 73, line 10 from above. Here you have the complete definition of $L(s,\chi)$...

[8] By the way, just recently the nonabelian reciprocity law for $p$-adic fields has been proved by M. Harris and R. Taylor, and a simplified version by G. Henniart [62]. See also the report [84]. The approach is global, and it is analogous to the derivation of local class field theory from global class field theory. For the latter see Hasse's original paper [45].

[9] As a side remark we mention that this letter was written in Berlin, not in Hamburg where Artin had announced he would be able to work with the literature. Artin writes that he had stayed in Hamburg only briefly but then went to Berlin where he found time to work. We do not know the reason why Artin had visited Berlin. In September, when he wrote this letter, there were university vacations and thus his presence in Hamburg was not required.

From the context it is apparent that Artin refers to Theorem IV on page 152 of the published version of Hasse's report. In this theorem Hasse presents, following the 1923 paper of Artin, the formula for $\log L(s,\chi)$ in its *temporary* definition, containing the contributions of the *unramified* primes only. The situation is as follows:

| | |
|---|---|
| $K\|k$ | is a Galois extension of number fields |
| $G$ | its Galois group, not necessarily abelian |
| $\chi$ | the character of a matrix representation of $G$, not necessarily irreducible |
| $\mathfrak{p}$ | ranges over the prime ideals of $k$ which are unramified in $K$ |
| $(\frac{K}{\mathfrak{p}})$ | $\in G$ denotes the $\mathfrak{p}$-adic Frobenius automorphism (determined by $\mathfrak{p}$ up to conjugation only) |

In this situation Hasse presents the following definition of Artin for his $L$-functions:

$$\log L(s,\chi) = \sum_{\mathfrak{p}} \sum_{m=1}^{\infty} \frac{\chi\Big((\frac{K}{\mathfrak{p}})^m\Big)}{mN(\mathfrak{p})^{ms}} \tag{1}$$

But this definition is to be regarded as temporary only, because the ramified primes are not yet taken into account. In his 1923 paper [4] Artin had given the final definition in a rather indirect way, by means of class field theory after the reduction to Hecke's $L$-series, and this method is reproduced in Hasse's Report.

But now Artin is able to give, right from the start, explicitly the contributions of the ramified primes which have to be added in order to obtain the final definition. If $\mathfrak{p}$ is ramified then the Frobenius automorphism $(\frac{K}{\mathfrak{p}})$ is not uniquely defined, but as a residue class modulo the inertia group only (up to conjugates). So is $(\frac{K}{\mathfrak{p}})^m$. In such case Artin now interprets $\chi((\frac{K}{\mathfrak{p}})^m)$ as the arithmetic mean of the values of $\chi$ on this residue class. The final definition of $L(s,\chi)$ is then given by the same formula (1), but now with $\mathfrak{p}$ ranging over *all* primes $\mathfrak{p}$ of $k$, regardless of whether $\mathfrak{p}$ is ramified in $K|k$ or not.

With this new definition, Artin says in his letter, all relations and theorems are now valid at once ("*von vornherein*") in the precise sense. In fact, this definition is the one which is used today, either in the additive form with the logarithm as given above, or in the corresponding multiplicative form

$$L(s,\chi) = \prod_{\mathfrak{p}} \frac{1}{\det(E - N(\mathfrak{p})^{-s}A_{\mathfrak{p}})} \tag{2}$$

which is also written in Artin's letter, although he does not elaborate on the properties of the matrix $A_{\mathfrak{p}}$ except that if $A_{\mathfrak{p}} \neq 0$ then its characteristic roots are roots of unity. The precise definition of $A_{\mathfrak{p}}$ is to be found in Artin's paper [7]. In fact, if the character $\chi$ belongs to the representation $\sigma \mapsto A_\sigma$ then $A_{\mathfrak{p}}$ is the arithmetic mean of the matrices $A_\sigma$ when $\sigma$ ranges through the residue class of $(\frac{K}{\mathfrak{p}})$ modulo the inertia group.

## 3.2 Functional equation

Section 2.) of Artin's letter starts as follows:

> *Mich störte noch viel mehr die Bemerkung auf Fahne 72, Absatz nach Satz XI ...*
>
> I disliked much more the remark on sheet 72, paragraph after Theorem XI ...

This remark can be found on page 159 of Hasse's report in its published version. [10] There, it is reported that the constants appearing in the functional equation of the $L$-series $L(s,\chi)$ are not given in explicit form in Artin's paper. But now, since Artin has succeeded to give a complete explicit definition of his $L$-functions, he is indeed able to give explicit formulas also for those constants, and he does so in this letter.

The functional equation relates the function $L(s,\chi)$ with the function $L(1-s,\overline{\chi})$ where $\overline{\chi}$ denotes the complex conjugate character. Artin writes the functional equation in the form

$$M(1-s,\overline{\chi}) = W(\chi)M(s,\chi) \tag{3}$$

where the function $M(s,\chi)$ is obtained from $L(s,\chi)$ by multiplication, firstly with explicitly given $\Gamma$-factors belonging to the infinite primes of $k$, and secondly with the function $c(\chi)^{s/2}$ where $c(\chi)$ is some constant reflecting the arithmetic structure of the field $k$ with respect to $\chi$. This constant is now given explicitly by Artin as follows:

$$c(\chi) = \frac{d^{\chi(1)} N_k(\mathfrak{f}(\chi, K|k))}{\pi^{n\chi(1)}} \tag{4}$$

where:

| | |
|---|---|
| $d$ | the discriminant of $k\|\mathbb{Q}$ |
| $N_k$ | the norm from $k\|\mathbb{Q}$ |
| $\mathfrak{f}(\chi, K\|k)$ | the conductor of $\chi$ which is explained in section 3.3 below |
| $n$ | the degree of $k\|\mathbb{Q}$ |
| $\chi(1)$ | the degree of the character $\chi$, i.e., the value of $\chi$ at the unit element of $G$ |

The only instance where Artin is not able to give an explicit description is the number $W(\chi)$ appearing in (3); here Artin is content with saying that this is some number with $|W(\chi)| = 1$.

Artin calls $W(\chi)$ the "Gaussian sum" belonging to $\chi$, and he puts the words "*Gauss'sche Summe*" into quotation marks. By this he indicates that in the abelian case, it was well known that the numbers $W(\chi)$ from the functional

[10] There seems to be some discrepancy in Artin's use of the sheet numbers of the galley-proofs. Although we do not know the galley proofs and their sheet numbers, we observe that the sheet number 73 which is mentioned in 1.), is greater than 72 mentioned in 2.) – but the corresponding page numbers 152 and 159 respectively of the printed version are ordered the other way: $152 < 159$. Perhaps Artin has erroneously mixed up those sheet numbers; we have also found other similar instances in his letters.

equation coincide essentially with suitably normalized Gaussian sums $\tau(\chi)$, up to a factor which is the square root of the conductor norm. Thus in the general Galois case those $W(\chi)$ are to be considered as "Galois Gaussian sums" – although there is not known a sum representation similar to the abelian case. Today the $W(\chi)$ are called "*Artin root numbers*".

It is curious that the problem of the local structure of these root numbers does not appear in the Artin-Hasse correspondence although certainly, both of them will have been aware of the importance of this problem. Many years later in 1954 Hasse published a paper [57] dedicated to the problem of finding a definition of local components $W_{\mathfrak{p}}(\chi)$. [11] The global root number $W(\chi)$ should then be the product of those local components:

$$W(\chi) = \prod_{\mathfrak{p}} W_{\mathfrak{p}}(\chi)\,.$$

The local definition should be independent of the (global) functional equation for Artin's $L$-functions. Hasse succeeded to state such a definition by using Brauer's theorem of induced characters, thus reducing the problem to the case of abelian Galois groups where the classical results on Gaussian sums are available. But then his definition depended on the representation of the given character as linear combinations of induced characters; Hasse at that time was not able to finally solve the problem of the structural invariance of his definition.

Much later development, leading to the notion of "extendable function", includes the work of Langlands, Deligne, Fröhlich, Tate and Koch; see the expositions in [37], [105], [72]. [12]

Remark: In sections 1.) and 2.) of his letter, Artin starts his comments with "*Mich störte in Ihrem Bericht...*" which we have translated as "I did not like in your report..." This sounds quite critical indeed. Later, near the end of the letter, Artin takes back this strong expression and says:

> *Sie sind doch nicht böse wegen der 'Einführung' bei 1.) und 2.). Sie sind natürlich nur Scherz.*
>
> Certainly you will not be angry because of the 'introduction' in 1.) and 2.). Obviously this is meant jokingly.

In fact, Artin does not criticize Hasse's exposition. Clearly, his aim is to contribute additional ideas and results which are to complete his own theory of

[11] An announcement had appeared earlier in 1952 [56]. – Hasse's paper [57] has 113 pages; this length can partly be explained by the fact that Hasse derives Artin's theory of $L$-functions and conductors *ab ovo*, thereby simplifying and systematizing the proofs, using several results which were not yet available in 1930 when Artin wrote his paper. In particular, Herbrand's ramification theory [63], [64] is used, as well as the theorem of R. Brauer about induced characters [17]. Because of this, Hasse's paper can be used as a good introduction not only to the structural problem of the $W(\chi)$ but also quite generally to the theory of Artin $L$-functions and Artin conductor.

[12] An interesting development in this direction is given by Boltje's theory of canonical group theoretic induction formulae [15] but the actual application to Artin root numbers is still missing.

$L$-functions. He did not like the fact that in the provisional definition he had to exclude the ramified primes, and now he is able to remedy this situation.

## 3.3 Artin conductors

The third section of Artin's letter is concerned with the definition of the "conductor" $\mathfrak{f}(\chi, K|k)$ and its properties. As above, $\chi$ denotes the character of a matrix representation of the Galois group $G$ of $K|k$. Artin presents his generalization of the discriminant-conductor formula for an arbitrary Galois field extension $K|k$, not necessarily abelian. He says that this interested him even more than the topics of the two foregoing sections.

For any prime ideal $\mathfrak{p}$ of $k$, the $\mathfrak{p}$-adic exponent of the conductor $\mathfrak{f}(\chi, K|k)$ is defined by Artin through the formula

$$f_{\mathfrak{p}}(\chi, K|k) = \frac{1}{e}\Big(e\chi(1) - \sum\chi(\tau) + p^{R_1}\chi(1) - \sum\chi(\tau_1) + p^{R_2}\chi(1) - \sum\chi(\tau_2) + \cdots\Big) \tag{5}$$

where:

- $e$ is the $\mathfrak{p}$-adic ramification degree of $K|k$, i.e., the order of the $\mathfrak{p}$-adic inertia group $T$,
- $\tau$ ranges over the elements of the inertia group $T$
- $p^{R_i}$ is the order of the $i$-th $\mathfrak{p}$-adic ramification group $V_i$ $(i \geq 1)$
- $\tau_i$ ranges over the elements of $V_i$.

And then the conductor itself is defined as

$$\mathfrak{f}(\chi, K|k) = \prod_{\mathfrak{p}} \mathfrak{p}^{f_{\mathfrak{p}}(\chi, K|k)}.$$

Artin comments this as follows:

> *Sie werden jetzt an Heckesche Größencharaktere denken und an die Beziehungen zur Relativdiskriminante. Natürlich geht das, es ist aber nicht erforderlich, da das angegebene $\mathfrak{f}$ von vorneherein die gewünschten Eigenschaften hat.*
>
> Perhaps you think of Hecke's *Größencharacters* and their relations to the relative discriminant. Of course this is possible but it is not necessary since the $\mathfrak{f}$ as given above has from the start the desired properties.

In other words: Artin has found a definition of the conductor which allows him to prove the conductor-discriminant formula directly, without resorting to the theory of Hecke's *Größencharacters*. The "desired properties" mentioned above are stated by Artin as follows:

a) $\mathfrak{f}(\chi, K|k)$ is an integral ideal in $k$.

b) Functorial behavior with respect to inflation, and

c) with respect to induction of characters.

d) Generalization of the conductor-discriminant formula.

e) If $K|k$ is abelian then $\mathfrak{f}(\chi, K|k)$ is the conductor of class field theory.

f) $\mathfrak{f}(\chi, K|k)$ is divisible precisely by the ramified primes – provided $K|k$ is the smallest Galois extension in which $\mathfrak{f}(\chi, K|k)$ is definable ("*erklärbar*").

g) If $k = \mathbb{Q}$ then $\mathfrak{f}(\chi, K|k) = 1$ if and only if $\chi$ is the principal character.

h) For a given ideal $\mathfrak{f}$ of $k$, there are only finitely many Galois extensions of $K|k$ of bounded degree which have $\mathfrak{f}$ as their conductor – provided again one counts the smallest extensions in which the conductor is definable.

i) Algebraically conjugate characters have the same conductor.

Artin says that a) is the deepest theorem of all, and is the one which raises the theory above trivialities. It is easy to see that the exponent $f_{\mathfrak{p}}(\chi, K|k) \geq 0$; the difficulty lies in the proof that it is an integer. And Artin adds: "*Das ist mir gelungen*", which means: "I succeeded" – indicating that it needed some nontrivial work for him to find the proof, and that now he feels satisfaction about his success. Artin does not sketch the proof here. But he does so in a later letter (see section 5 below).

On first sight, formula (5) looks somewhat complicated, and there arises the question of how Artin arrived at this formula. Artin himself does not say much about this except in the next letter (of 23 Sep 1930) where he says that he had to guess much of it. ("*In der Führersache und der Funktionalgleichung musste ich auch alles erraten.*") Today it has become standard to write the right hand side of (5) as an inner product, taken over the $\mathfrak{p}$-adic inertia group $T = V_0$, of the given character $\chi$ against the so-called "Artin character" $\alpha$. The latter gives (up to a minus sign) for every $\tau \neq 1$ in $T$ the smallest index $i$ such that $\tau$ is not contained in the $i$-th ramification group, i.e.,

$$\alpha(\tau) = -i \qquad \text{if } \tau \in V_{i-1} \setminus V_i$$

and $\alpha(1)$ is defined in such a way that

$$\sum_{\tau \in T} \alpha(\tau) = 0\,.$$

See Serre's exposition in [95]. By this definition, the Artin character is a "virtual" character, i.e., a linear combination of the irreducible characters. Artin's statement that the numbers (5) are positive integers is equivalent to saying that $\alpha$ is the character function of some matrix representation of the inertia group. But in Artin's letters and in his paper [8] we do not find any mention of this interpretation of (5) as an inner product.

Artin's discriminant-conductor formula reads:

$$\mathfrak{d}_{K|k} = \prod_{\chi} \mathfrak{f}(\chi, K|k)^{g_\chi}\,. \tag{6}$$

Here, $\mathfrak{d}_{K|k}$ denotes the discriminant [13] of $K|k$ and $\chi$ ranges over the *irreducible* characters of the Galois group $G$ of $K|k$. The conductor $\mathfrak{f}(\chi, K|k)$ is defined via (5), and $g_\chi$ is the multiplicity of $\chi$ in the regular permutation representation of $G$, i.e., $g_\chi = \chi(1)$.

More generally, Artin considers a subextension $\Omega|k$ of $K|k$; let $H \subset G$ denote the corresponding subgroup, i.e., the Galois group of $K|\Omega$. Then the discriminant $\mathfrak{d}_{\Omega|k}$ splits similarly as on the right hand side of (6), but with different exponents $g_\chi$, namely: $g_\chi$ is now the multiplicity of $\chi$ in the permutation character of $G$ defined by its subgroup $H$. [14]

In a postscript to his letter, Artin says that his results seem to support strongly his conjecture that his $L(s, \chi)$ are integral (holomorphic) functions – provided $\chi$ does not contain the principal character. But until today this conjecture of Artin has not yet been verified in general; it remains as "one of the great challenges in Number Theory" [76]. Richard Brauer proved in 1947, by means of his celebrated Theorem on Induced Characters, that Artin's $L(\chi, K|k)$ are *meromorphic*; see [17] , also [86] and [18].

# 4 Letter of 23 Sep 1930

## 4.1 Two papers

It seems that Hasse, in his reply to Artin's foregoing letter, had spontaneously offered him to publish his results, or some part of it, in *Crelle's Journal* [15]. For in the present letter, dated five days after the foregoing one, Artin explains that he intends to divide his investigations into two papers and to submit only one of them to Crelle's Journal, while he will put the other into the *Hamburger Abhandlungen.* [16]

His first paper, Artin says, will contain his results about discriminants and conductors (see section 3.3); this he believes will fit well with Crelle's Journal since Hasse's paper on conductors for ray class characters had appeared there. Obviously he means Hasse's paper [46] which had appeared earlier the same year. The second paper will then deal with the applications to his new results on $L$-functions (sections 3.1 and 3.2); this would fit with the *Hamburger Abhandlungen* since there Artin's first paper on $L$-functions [4] had appeared already.

Artin justifies the division into two parts because, he says, the first paper on conductors will be of general importance, independent of the application to $L$-functions. From further development of conductor theory he expects many

[13] In the letter and in the published version [7] Artin denotes the discriminant with capital $\mathfrak{D}$ while $\mathfrak{d}$ stands for the different. We choose the notation as it is usual today (and was used by Hasse).

[14] For the definition of these $g$'s, Artin in his letter refers to page 94 of his former $L$-series paper [4]. But that definition is to be found on page 96 (not 94).

[15] Hasse was one of the editors of *Crelle's Journal* since 1929; see [85].

[16] Artin was one of the editors of the *Hamburger Abhandlungen* since 1926.

more results. For instance, he has a small hope for the solution of the Class Field Tower Problem.

Furtwängler's class field tower problem had been mentioned in Part I of Hasse's class field report [41] (which had appeared in 1926):

*Does there exist a number field whose tower of successive absolute class fields is infinite?*

There, on page 46 of his report, Hasse had presented Artin's idea that some suitable sharpening of the Minkowski bound for the discriminant would solve the problem in the sense that every class field tower terminates. In view of this we understand Artin's hope that further development of conductor theory may perhaps prove to be useful, through the discriminant-conductor formula, to obtain the desired better estimate for the discriminant.

But now Artin is cautious in his letter: he says that his hope is "small". This caution is in some contrast to Artin's statement three years earlier in his letter of 19 Aug 1927 to Hasse: There, Artin had written:

> *Was ich zum Klassenkörperturm meine ? ... Nach wie vor glaube ich, dass der beste Beweisansatz die Verschärfung der Minkowskischen Abschätzung ist. Wie man an diese herankommen kann, ist eine Frage für sich.*
>
> My opinion on the class field tower problem ? ... I still believe that the best approach will be the strengthening of Minkowski's estimate. It is a question of its own how to do this.

But he had added that this problem does not seem to be of a pure group theoretic nature because he believes that one may construct an infinite tower of groups $G_i$ such that $G_{i-1}$ is the $(i-1)$-th commutator factor group of $G_i$. It is not only group theory but the arithmetic nature of the ground field, he says, which will give the final clue.

Perhaps Artin's caution in 1930 was due to the fact that in 1929, less than one year earlier, the existence of class field towers of arbitrarily large length had been proved by Arnold Scholz [91]. Today we know through the work of Golod and Shafarevich [39] that infinite class field towers do exist; see also [96], [87]. Hence Artin's caution was justified. On the other hand, his belief as expressed in his former letter is also verified, namely, that special arithmetic properties of the ground field (for instance, its ramification structure) are responsible for the existence or non-existence of an infinite class field tower. For information about the present state of knowledge we refer to the article of René Schoof [97] and the literature cited there.

Artin's letter contains already the final titles of his two announced papers, namely:

1. *Gruppentheoretische Struktur der Diskriminante algebraischer Zahlkörper*
2. *Zur Theorie der L-Reihen mit allgemeinen Gruppencharakteren*

As to the first title, Artin was careful to avoid mention of $L$-functions because, he says, he wishes the paper to be read also by those people who are not too much interested in analytic functions ("*. . . die beim Lesen des Titels 'L-Reihe' die Arbeit mit Grauen beiseite legen würden.*"). This remark reflects the situation at that particular time, the twenties and thirties, namely a sharp division of number theoretic research with respect to the methods: *analytic* versus *algebraic* number theory. Artin's own work could not be classified into one of those divisions: although his publications mainly were concerned with the algebraic side, he was quite knowledgeable also in analytic number theory and ready to use it if adequate. And, after all, he had invented his new analytic $L$-functions. But he seemed to be aware that there were a number of people working in algebraic number theory who would not easily read a paper in analytic number theory.

Paper no.1 appeared in the first issue of Crelle's Journal in 1931, and no.2 even earlier, in the last issue of *Hamburger Abhandlungen* in 1930. We note the extremely short time between submission of papers and publication: Artin's first paper was submitted to Hasse on 7 November 1930, and the second paper is signed "October 1930".

# 5 Undated Letter

This letter has been found among Hasse's papers between the one of 23 September and that of 7 November 1930. Although undated, we may assume that it has been written some time in early October 1930.

## 5.1 Hasse's congruences

In his foregoing letter Artin had asked Hasse: *Did you know the formula* (5) *for* $\mathfrak{f}(\chi, K|k)$ *in the abelian case*? If not then, he said, that formula may also be regarded as a small contribution to class field theory.

It seems that in the meantime Hasse had replied to this question. For, Artin begins this letter as follows:

> *Vielen Dank für Ihre ausserordentlich interessanten Mitteilungen. Sie sind da wirklich einen grossen Schritt weiter gekommen . . .*
>
> Many thanks for your extremely interesting communications. Really, you have made a big step forward . . .

We do not know precisely the content of Hasse's letter; so we have to rely on what we can infer from the context. Artin writes:

> *Sie fragen nach den Kongruenzen für die* $v_i$*. Ich glaube nicht, daß diese Kongruenzen auch im allgemeinen Galois'schen Fall stimmen. . . .*
>
> You are asking about the congruences for the $v_i$. I do not believe that these congruences will hold in the general Galois case too.

What kind of congruences had Hasse mentioned ? We have searched for those congruences and found them in Hasse's paper [46] which is entitled "*Führer,* *Diskriminante und Verzweigungskörper relativ-abelscher Zahlkörper*". This is Hasse's paper on conductors for abelian extensions which Artin was referring to in his foregoing letter already.

In that paper Hasse had proved for abelian extensions $K|k$ of number fields the discriminant-conductor formula (6) – but with the conductors of class field theory and not with Artin's conductors. Thus in Hasse's paper, $\chi$ denotes an irreducible character of the ray class group in $k$ belonging to $K|k$ as a class field, and $\mathfrak{f}(\chi, K|k)$ is the smallest module of definition (*Erklärungsmodul*) of $\chi$. [17] Class field theory shows that those ray class characters correspond to irreducible characters of the abelian Galois group $G$ of $K|k$.

This discriminant-conductor formula of class field theory was not new in the abelian case. A first proof had been given using the functional equation of Hecke's $L$-series with *Größencharacters* [61]; Hasse had presented this in his report [41] already. [18] But now, Hasse says in [46], his new proof is "arithmetical" in the sense that its methods use only concepts and arguments which belong to the *arithmetic* of class field theory. In the course of proof Hasse showed that for each prime $\mathfrak{p}$ of $k$, after localizing, the exponent $f_{\mathfrak{p}}$ of the $\mathfrak{p}$-part of the class field conductor $\mathfrak{f}$ of $K|k$ is given by the the following formula:

$$f_{\mathfrak{p}} = 1 + \frac{v_1}{e_0} + \frac{v_2 - v_1}{e_0 p^{r_1}} + \cdots + \frac{v_n - v_{n-1}}{e_0 p^{r_1 + r_2 + \cdots + r_{n-1}}} \tag{7}$$

Here, $e_0$ is that part of the ramification degree $e$ of $\mathfrak{p}$ which is relatively prime to $p$. The $v_i$, $r_i$ are defined as follows: the group sequence

$$V_{v_1} > V_{v_2} > \cdots V_{v_n} > 1$$

is the sequence of *different* $\mathfrak{p}$-adic ramification groups starting with $V_1$, such that

$$V_1 = \cdots = V_{v_1} > V_{v_1+1} = \cdots = V_{v_2} > V_{v_2+1} = \cdots > V_{v_n+1} = 1\,.$$

And

$$p^{r_i} = (V_{v_i} : V_{v_{i+1}}) = p^{R_{v_i} - R_{v_{i+1}}} \tag{8}$$

[17] In Hasse's terminology the formula (6) is called "*Führerdiskriminantenproduktformel*". This strikes as a noteworthy example of the possibility in the German language to form long nouns (33 letters in this case) as compounds of shorter ones.

[18] See [41] p.38. Hasse does not say whether that proof had been found by himself or whether he reported on a proof given by someone else. Since he does not give any reference we have reason to suppose that, indeed, the proof as given by Hasse is due to himself. In Hasse's legacy we have found a complete manuscript, handwritten with proofs, about Hecke's $L$-functions with *Größencharacters*. The manuscript is not dated, but from the context it seems that it had been written in the early twenties. It includes the proof of the discriminant-conductor formula by means of Hecke's *Größencharacters*. – Sugawara in his 1926 paper [100] says: "*Herr H. Hasse hat aus der Hecke'schen Funktionalgleichung der L-Funktion den folgenden Satz bewiesen...*" (Mr. H. Hasse has proved the following theorem from Hecke's functional equation for $L$-functions...) – and the statement of the discriminant-conductor formula follows. Thus Sugawara attributes this proof to Hasse, and we may also suppose that Takagi did, because Takagi had presented Sugawara's paper to the Academy and so had probably read it. – See also footnote 28.

where the $R_j$ are from Artin's notation in (5). [19]

Hasse's proof of (7) in [46] was based on a detailed study of norm residues which he had just recently developed and on that occasion had discovered local class field theory [45]. [20]

In the course of proof of formula (7) Hasse had shown in [46] the following congruences for the ramification levels $v_i$:

$$\begin{aligned} v_1 &\equiv 0 \mod e_0 \\ v_i &\equiv v_{i-1} \mod e_0 p^{r_1+\cdots+r_{i-1}} \qquad (i > 1)\,. \end{aligned} \tag{9}$$

Very likely these are the congruences which Hasse did communicate to Artin. Moreover, Hasse may have pointed out that these congruences, if valid also in the non-abelian case, would yield the integrality of Artin's conductor exponent (5). We do not know Hasse's argument for this but it could have been as follows:

Suppose first that $\chi$ is *nontrivial and irreducible* on the inertia group $V_0$. Let $i$ be some integer such that $V_i$ is not contained in the kernel of the representation belonging to $\chi$. Since $V_i$ is normal in $V_0$ it follows [21] that the restriction of $\chi$ to $V_i$ does not contain the trivial character and hence $\sum \chi(\tau_i) = 0$ if $\tau_i$ ranges over $V_i$. Consider the largest integer $i$ such that $V_i$ is not contained in the kernel of the representation of $\chi$; we have $i = v_m$ with certain $m \leq n$. For $j > v_m$ we have $\sum \chi(\tau_j) = \chi(1)p^{R_j}$. We see that Artin's formula (5) yields

$$f_{\mathfrak{p}}(\chi, K|k) = \frac{1}{e}\left(e\chi(1) + p^{R_1}\chi(1) + p^{R_2}\chi(1) + \cdots + p^{R_{v_m}}\chi(1)\right)$$

Using the notations introduced in (7) this can be rewritten as [22]

$$f_{\mathfrak{p}}(\chi, K|k) = \chi(1)\left(1 + \frac{v_1}{e_0} + \frac{v_2 - v_1}{e_0 p^{r_1}} + \cdots + \frac{v_m - v_{m-1}}{e_0 p^{r_1+r_2+\cdots+r_{m-1}}}\right) \tag{10}$$

We see: The validity of the congruences (9) would imply that the right hand side of (10) and hence $f_{\mathfrak{p}}(\chi, K|k)$ is an integer. If $\chi$ is not irreducible on $V_0$ then we may decompose $\chi|_{V_0}$ into irreducible characters and for each nontrivial constituent use the same argument to conclude again the integrality of $f_{\mathfrak{p}}(\chi, K|k)$.

We also see that Hasse's formula (7) may be considered as a special case of Artin's formula in the form (10); in the latter we have to take $\chi$ to be a *faithful* character of $G$ (and so $G$ has to be cyclic). Hence Hasse might have replied "yes" to Artin's question: yes, he indeed knew Artin's formula at least in the cyclic case – but only after manipulating that formula as indicated above.

[19] Hasse's enumeration of the ramification groups is different from that used by Artin. The latter denotes the full ramification group by $V_1$ whereas Hasse writes $V_0$ for it. We follow Artin's which seems to be standard today. This means that $V_i$ consists of all $\sigma \in T$ for which $v(\pi^{\sigma-1} - 1) \geq i$ where $v$ is the normalized $\mathfrak{p}$-adic valuation of $K$ and $\pi \in K$ is a $\mathfrak{p}$-adic prime element.

[20] That paper had appeared in the same volume of Crelle's Journal as had [46].

[21] using the so-called Frobenius reciprocity

[22] Formula (10) also appears in the paper [69] by Ikeda; there it is called a "generalization of the Hasse-Arf theorem".

Perhaps Artin did not accept this, for in the published version of his paper [7] Artin kept the remark that his conductor formula (5) constitutes a new addition to class field theory.

Anyhow, whether Hasse's argument went this way or not, it seems clear from the context that he had informed Artin about the congruences (9) in the abelian case, and he had asked Artin about the possible validity of (9) in the general Galois case. It is remarkable that Artin immediately answered no, he did *not* believe the congruences (9) to hold in general, although from the context we may infer that he had no immediate counter-example.

Today we know that Artin had the right intuition in this instance. For, the congruences (9) are essentially equivalent to what today is called the "Theorem of Hasse-Arf". [23] Fesenko [29] has shown 1995 that in some sense, the Hasse-Arf property characterizes abelian extensions. More precisely: Let $K|k$ be a local, totally ramified Galois extension with perfect residue fields. If $K|k$ is not abelian then there exists a finite abelian, totally ramified extension $E|k$ such that the theorem of Hasse-Arf does not hold for the composite extension $KE|k$.

## 5.2 On the theorem of Hasse-Arf

Let us insert a few words concerning the theorem of Hasse-Arf. In our days, this theorem is usually presented using the function

$$\varphi(u) = \int_0^u \frac{dt}{(V_0 : V_t)} \tag{11}$$

where $V_t$ is defined to be $V_i$ when $i$ is the smallest integer $\geq t$. This function $\varphi$ is continuous and piecewise linear. Today's version of the Hasse-Arf theorem is stated as follows:

*If $G$ is abelian then the values $\varphi(u)$ are integers at those points $u$ at which the derivative $\varphi'$ is not continuous, i.e., if $V_u \neq V_{u+1}$.*

Of course, since the problem is local, it is sufficient to require the inertia group $V_0$ to be abelian.

Now, the points of discontinuity of $\varphi'$ are precisely the $v_m$ which had been defined by Hasse as indicated above. Hence the folloing numbers are integers:

$$\begin{aligned} \varphi(v_m) &= 1 + \sum_{1\leq j\leq v_m} \frac{1}{(V_0 : V_j)} = 1 + \frac{v_1}{(V_0 : V_{v_1})} + \sum_{2\leq i\leq m} \frac{v_i - v_{i-1}}{(V_0 : V_{v_i})} \\ &= 1 + \frac{v_1}{e_0} + \sum_{1\leq i\leq m} \frac{v_i - v_{i-1}}{e_0 p^{r_1+\cdots r_{i-1}}} \end{aligned}$$

This puts into evidence that the theorem of Hasse-Arf in today's version is equivalent to the validity of the congruences (9) (for all $i$).

For the application to Artin's conductor problem, it would have been sufficient to know the congruences (9) in the case of *cyclic* field extensions which are

[23] More precisely, this is Hasse's part of the Hasse-Arf theorem. We shall mention below what Arf, in later years, has added to this.

purely ramified of $p$-power degree. In that case, a proof of (9) had been given in 1926 by M. Sugawara [100], some years earlier than Hasse's paper. Hasse [46] cites Sugawara's paper and points out that Sugawara's proof was based on the discriminant-conductor formula (6) for ray class characters which, at that time, had been proved by analytic means only, i.e., via $L$-series with Hecke's *Größencharacters*; we had mentioned this in section 5.1 already. Three years later in 1929 another proof was presented by S. Iyanaga [70]. It seems that by 1930, neither Hasse nor Artin knew of Iyanaga's paper because it is not mentioned in their correspondence, nor cited in [46] or [7].

However, later in 1934 Hasse published another paper [53] where he presented a new and more systematic proof of (9) – and on that occasion he discussed in detail the history of the problem, giving the proper references including Iyanaga's. [24] That other paper was written after Chevalley and Herbrand had greatly simplified class field theory [25]; in particular local class field theory could be developed on its own, independent of global class field theory. With all those new methods and notions at hand, Hasse gave a new treatment of (9) and hence of the Hasse-Arf theorem. In this paper he also introduced the piecewise linear function (11) which sometimes is called Hasse function; see e.g., Tamagawa in [104].

What is remarkable in our present context is that the first part of Hasse's new paper [53] concerns general Galois extensions, not necessarily abelian. This first part contains a detailed and exhaustive, yet beautiful study of local norm residues in the Galois case. It seems to us that this was an attempt of Hasse to carry the arguments from the abelian to the Galois case as far as possible, perhaps with the hope (which was not fulfilled) of finally obtaining some general congruences which would solve Artin's conductor problem in the Galois case. In this sense the first part of [53] could be regarded as being directly influenced by Artin who always had proposed to extend the methods of class field theory

[24] Apart from the papers by Sugawara and by Iyanaga, Hasse also cites the paper [108] by Ph. Vassiliou which appeared 1933 in Crelle's Journal (received 22 March 1932). Vassiliou's paper gives a simplification of Hasse's proof [46] of the discriminant-conductor formula for ray class characters. It was soon superseded by Hasse's further simplification [53]. – Vassiliou (1904–1983) was a Greek mathematician. After his Ph.D. 1929 in Athens he obtained, on the recommendation of Carathéodory, a Rockefeller grant to study in Hamburg in the years 1930–1932; certainly he studied not only with Hecke but also with Artin. In 1937 he became Professor at the Polytechnicum in Athens, and later was member of the Greek Academy of Sciences. (I am indebted to J. Antoniadis for this information who also pointed out that Vassiliou was the first Greek mathematician who worked in modern algebraic number theory.) – In the introduction of [108] Vassiliou expresses his thanks to Hasse for help in the composition of the paper. Although we could not find letters of correspondence between Hasse and Vassiliou (except those which are concerned with the submission of Vassiliou's manuscript [108] to Crelle's Journal) it seems that both have kept scientific contact. On the invitation of Vassiliou, Hasse visited Athens at least twice: once in 1957 and another time in 1967. In 1957 Hasse's Athen lectures concerned certain questions on conductors; those lectures have appeared in the publication series of the Greek Academy of Sciences [58].

[25] In a letter to Hasse of 16 June 1931 Artin writes: "*Begeistert bin ich über die neuen ungeheuren Vereinfachungen der Klassenkörpertheorie, die von Herbrand und Chevalley stammen*" – "I am quite enthusiastic about the immense simplifications of class field theory due to Chevalley and Herbrand".

to Galois extensions, as far as possible.

Hasse's paper [53] appeared 1934 in the same volume of Japanese Journal of Mathematics as did the great paper by Chevalley (his thesis) on class field theory. Announcements of [53] had been published 1933 as notes in the Comptes Rendus [51], [52]. It was Chevalley who had asked Hasse for these Comptes Rendus notes. We infer this from a letter of Iyanaga to Hasse dated 25 Feb 1933, from Paris. In that letter, Iyanaga expresses his thanks to Hasse for sending him a letter, the manuscript of the paper [53] and an additional postcard; he gave extensive comments and said:

> *Herr Chevalley wollte schon Ihren ersten Brief publizieren; jetzt sagt er aber: in diesen Umständen wolle er Sie lieber um eine Note für die Comptes Rendus bitten.*
>
> Mr. Chevalley originally wanted to publish your first letter; but now he says: in these circumstances he wants to ask you for a note in the Comptes Rendus.

We do not know the content of Hasse's letter and postcard to Iyanaga and hence do not know what kind of "circumstances" Iyanaga was referring to. In any case, Hasse's Comptes Rendus notes were published as an announcement of the paper [53]. [26]

Iyanaga stayed three years in Europe, from 1931 to 1934, part of this time in Hamburg (with Artin) and partly in Paris (with Chevalley) before he returned to Japan on Oct 11, 1934. In those years (and also later) he kept close scientific contact to Hasse, as is evident from their correspondence which is preserved in Hasse's legacy in Göttingen. Through this contact Hasse seems to have learned about Iyanaga's 1929 paper [70] on the discriminant-conductor formula for ray class characters. In his letter to Hasse cited above, Iyanaga said that he could recognize several details which he (Iyanaga) also had thought about but which he could prove through tedious recursion only, and with the use of genus theory. He concludes:

> *Alles in allem bewundere ich Ihren Beweis. Ich glaube, er ist schliesslich ein definitiver Beweis und fühle mich gleichzeitig etwas erleichtert.*
>
> All together I admire your proof. I believe this proof is definite, and I feel somewhat relieved.

From the last statement we infer that Iyanaga had not been wholly satisfied with his own proof and still had in mind to find a better proof but had not been successful.

But Iyanaga did not only think about the abelian case where, by class field theory, the characters of the Galois group correspond to the ray class characters. In fact – and this is of special interest in our present context – in his letter he explicitly mentions Artin's conductors for Galois characters. After discussing Herbrand's genus formula (which we do not cite here) he writes:

[26] Since Hasse's note turned out to be too long for the Comptes Rendus it was divided into two shorter notes. The proof reading for those notes was done by Iyanaga.

> *Wenn Sie sie* [die Herbrandsche Aussage] *irgendwie ohne Benutzung der abelschen Ergebnisse bewiesen hätten, so würde auch vielleicht die Ganzheit des Exponenten des Artinschen Führers für allgemein galoissche Körper neuartig bewiesen werden, denke ich.*
>
> If you would have proved it [*Herbrand's statement*] somehow without using the abelian results, then I believe that perhaps also the integrality of the exponent of Artin's conductor for general Galois fields would be proved in a new way.

This letter, written $2\frac{1}{2}$ years after Artin's 1930 letters to Hasse, shows again not only the influence of Artin who was looking for a direct proof not using arguments referring to class field theory, but it is also witness for the importance which people attached to Artin's problem. We can understand this as part of the quest for non-abelian class field theory.

## 5.3 Artin's proof

Returning to Artin's letter: Not knowing yet the details of Hasse's proof of (9) in [46], Artin explains *his* idea of showing that his conductor exponent (5) is an integer, i.e., that the numerator in (5) is divisible by $e = e_0 p^{R_1}$. Divisibility by $e_0$, Artin says, he could get from a simple lemma (which, however, had first given him some difficulties). In his next letter [27] he admits that this proof was quite similar to that of Speiser [98]; it seems that Hasse had given him the reference to Speiser since Artin says that he had not known Speiser's beautiful paper before.

But divisibility by $p^{R_1}$, the order of the ramification group, is more subtle. In the abelian case, Artin says, this is equivalent to Hasse's congruences (9). For a long time he had tried to prove that divisibility directly, but he did not succeed, not even in the abelian case which now had been taken care of by Hasse.

Then Artin informs Hasse about his "detour" ("*Umweg*") which he had to take in order to arrive at his result. This "detour" consists of three steps:

1. Reduction to the ramification group, which is a $p$-group.
2. Reduction to the abelian case using the theorem of Blichfeldt that every character of a $p$-group is monomial.
3. In the abelian case using Hasse's congruences to show that Artin's conductors are integral. Alternatively, Artin says, he could use the discriminant-conductor formula of class field theory from which Hasse's congruences could be retrieved. Moreover, the discriminant-conductor formula is essential in order to identify his (Artin's) conductors with the class field conductors.

As to the proof of the discriminant-conductor formula of class field theory, Artin says, one may take either Hasse's proof or Hecke's ("*nach Belieben Ihren Beweis*

[27] of 7 Nov 1930; see section 6.

*oder den Heckeschen*"). Obviously, when he talks about "Hasse's proof" then he means the proof given by Hasse in [46] using his congruences (9). It is not quite clear what Artin means when he talks about "Hecke's" proof. We have not found in Hecke's papers anything resembling a proof or even a statement of the discriminant-conductor formula (except perhaps in the quadratic case where, however, there is only one non-trivial character). But we recall that Hasse, as we said earlier already, had sketched a proof of the discriminant-conductor formula in Part I of his Class Field Theory Report [41], using the theory of $L$-functions with Hecke's *Größencharaktere*. Very likely Artin has in mind that proof; since Hasse's argument is a straightforward consequence of Hecke's functional equation for $L$-series, Artin may have called it "Hecke's proof". [28]

Today it is possible to somewhat shorten Artin's "detour" by using Richard Brauer's theorem on induced characters [17], [86] so that step 1, the reduction to $p$-groups is not necessary. But otherwise we would consider Artin's proof not so difficult, and quite adequate.

But not Artin. He is looking for a "simpler" proof, not using this "detour" and, what seems to be even more important to him, not depending on class field theory. He suspects Hasse's proof of (9) in the abelian case (which he does not yet know) to contain some new idea which he could perhaps transfer to the non-abelian case. Therefore he is asking:

> *Darf ich meine Bitte um ein Separatum nochmals erneuern und Sie ersuchen es nach Hamburg zu schicken ?*
>
> May I again renew my request for a reprint, and ask you to send it to Hamburg ?

Certainly, he means reprints of Hasse's paper [46] which contained Hasse's proof of (9).

## 5.4 The Frobenius-Schur theorem

In the rest of the letter Artin informs Hasse about several additional results which he had obtained in the meantime and which will appear in his papers. Among them is the statement that the number of irreducible rational characters of a finite group $G$ equals the number of "*Abteilungen*" of $G$ in the sense of Frobenius [34], i.e., the number of classes of conjugate cyclic subgroups of $G$. Artin says that this statement was not known to him, and he believes it would be new. But it was not new; the result is due to Frobenius and Schur (1906). Somebody, perhaps Hasse, seems to have told him because in Artin's published paper [7] we find a footnote pointing to a paper by Frobenius and Schur of 1906 where, he said, the same theorem had been proved. Frobenius and Schur had two joint papers in 1906, both in the same volume of "*Berliner Berichte*" [35],

[28] It is conceivable, however, that Hecke had orally communicated this idea to Artin and/or Hasse and therefore Artin spoke of "Hecke's proof". But we have no evidence for this. See also footnote 18.

[36]. From these two, the first one contains the theorem in question but Artin cited the second. It seems that he had inserted the reference in haste, without proper checking.

Today, the proof of the Frobenius-Schur-Artin theorem is usually presented following Artin, using his theorem about induced characters from cyclic subgroups in [8]. See e.g., Huppert's book on characters [68].

# 6 Letter of 7 Nov 1930

About 6 weeks after his previous letter, Artin sends his manuscript for the paper [8] as a submission to Crelle's Journal. In the published version the paper carries the date of receipt as of 9 Nov 1930. The letter to be discussed now is the accompanying letter for this manuscript.

## 6.1 Artin's dream and Arf's theorem

It seems that in the meantime, Hasse had sent to Artin a reprint of his conductor paper [46] which Artin had asked for in his foregoing letter. But Artin, contrary to his hope, could not use it to simplify his proof of the integrality of the conductor exponents $f_{\mathfrak{p}}(\chi, K|k)$. He writes:

> *Sie ersehen daraus* [aus dem Manuskript]*, dass mir die gewünschte Vereinfachung nicht geglückt ist…Ich hegte die Hoffnung, dass Sie Ihre Kongruenzen nicht aus der Klassenkörpertheorie beziehen, sondern direkt, etwa wie Speiser beweisen…Sie ersehen jetzt auch, weshalb ich nicht an eine allgemeine Gültigkeit der Kongruenzen glaube. Wie aber der entsprechende Satz allgemein lautet, davon habe ich keine Ahnung.*
>
> You will see from it [*from the manuscript*] that I did not succeed to obtain the simplification which I had in mind…I was hoping that you would get your congruences not from class field theory but would proceed directly, e.g., in the same spirit as Speiser…You will also see why I do not believe in the general validity of those congruences. But I have no idea how to formulate the corresponding theorem in the general case.

Here, the word "general" (*allgemein*) refers to arbitrary local fields, including those which do not admit ordinary class field theory, i.e., whose residue field is not finite.

Several years later, in 1938, Artin's dream to have a proof without class field theory came true. Cahit Arf, a student from Turkey who had graduated at the École Normale Supérieure in Paris, went 1937 to Göttingen for his doctorate under the supervision of Hasse. It seems that Hasse had not forgotten about Artin's dream and so he proposed to Arf (who was 27) to work on this problem. After one year, in 1938, Arf succeeded to prove the congruences (9), in the abelian case, for an arbitrary complete local field with perfect residue field of characteristic $p > 0$, irrespective of whether the field admits class field theory or not. Since then the theorem is called after Hasse-Arf. [29] It implies that Artin's conductor exponent $f_{\mathfrak{p}}$ from (5) is a positive integer, for an arbitrary Galois extension $K|k$ of local fields, irrespective whether the base field $k$ admits class field theory or not.

Arf's proof rests, again, on Hasse's detailed study of the local norm map, in particular its behavior with respect to the filtration of the local unit group by its natural congruence subgroups. [30] In addition Arf has given, in case when $\chi$ is irreducible on the inertia group $V_0$, a characterization of the conductor exponent (5) within the arithmetic theory of local non-commutative algebras. The proof is by no means trivial and is far from the simplicity of Speiser's proof which Artin mentioned in his letter as a model.

Much later in 1961, another proof of the Hasse-Arf theorem was given by J.-P. Serre [94]. Serre's idea is, first to reduce the theorem to the case where the residue class field is algebraically closed, and secondly, in that case develop some substitute of class field theory which turns out to yield the desired result. Later in 1969 still another proof was given by Shankar Sen, a Ph.D. student of Tate; that proof is, however, restricted to purely ramified cyclic extensions of $p$-power degree. Sen shows that in this case

$$v_{m+1} \equiv v_m \bmod p^m \tag{12}$$

(in our notation introduced above). In fact, these are the same congruences as had been proved by Sugawara [100] already in 1926. The beauty of Sen's paper is the simplicity of the argument which is quite elementary and straighforward (and which covers also more general situations in characteristic $p$).

As we have pointed out above already, the congruences (12) in the case of cyclic totally ramified extensions of $p$-power degree, are sufficient for the solution of Artin's problem as to the integrality of his conductor (5). However, it is still necessary today to use induced group characters in order to reduce the general Galois case to the cyclic case where the Hasse-Arf theorem is available. It seems that this is not quite what Artin had in mind. As of today, a really satisfying proof of the integrality of Artin's conductor in the Galois case has not been found. The aim is to exhibit explicitly a matrix representation of the inertia group whose character coincides with the Artin character, and from which the main properties of the Artin character can be read off easily.

Serre [93] has shown that in general the Artin representation cannot be realized over $\mathbb{Q}$, but it can be realized over the Hensel $\ell$-adic field $\mathbb{Q}_\ell$ for every prime number $\ell \neq p$. [31]

---

[29] In later years, Arf became one of the leading figures of the mathematical community in Turkey. Many of the present day mathematicians in Turkey were deeply influenced by his devotion to mathematics and to science in general. He kept a life-long warm friendship with his academic teacher Hasse. As to Arf's collected papers, see [3].

[30] A lucid presentation can be found in Serre's *Corps Locaux* [95].

[31] In the function field case, Laumon [75] has succeeded to obtain a cohomological construction of a representation module for the Artin representation.

## 6.2 Artin's introduction

Let us return to Artin's letter of 7 Nov 1930 which accompanies the manuscript of [8] which he submits to Crelle's Journal. About his introduction to this paper Artin writes to Hasse:

> *Hoffentlich finden Sie nicht, dass ich in der Einleitung mit Zukunftsaussichten zu phantastisch gewesen bin, aber ich bin der Meinung, dass man mit seinen Vermutungen nicht immer hinter dem Berge bleiben soll.*
>
> I hope you do not find that in the introduction I have been too fantastic with respect to future aspects, but I believe that one should not always hold back one's conjectures.

Let us review that introduction as to be found in [8].

**(i)** Artin says that his conductors in the Galois case will be of importance in the future development of Number Theory.

*Today we would fully agree to this.*

**(ii)** Artin expresses his hope that the study of his conductors would clear up the mystery of the decomposition laws in non-abelian Galois groups.

*We have already said earlier* (in section 2.2) *that this hope for a non-abelian decomposition law has not been fulfilled.*

**(iii)** Artin points out that all formal properties of his conductors can be proved in an elementary manner, except the fact that they are integral ideals in the base field (which is to say that the local exponents (5) are rational integers). Here he had to use class field theory but he does not believe that class field theory is really necessary for this.

*As explained in the foregoing section this has indeed been verified by Cahit Arf, followed by Serre and Sen. On the other hand, a canonical realization of the Artin representation has not yet been given in the number field case.*

**(iv)** Artin observes that the discriminant-conductor formula (6) for subfields suggests that the discriminant could be written as a group determinant. Perhaps, he says, the discriminant may indeed be written this way and this could be proved directly. If so then, he says, all the theorems of this paper could be read off directly from this. Again, Artin expresses his hope that the unknown decomposition laws could be extracted from this representation of the discriminant as group determinant.

*As to the decomposition laws, see our comments to* (ii). *The question regarding group determinants will be discussed below in section* 9, *in connection with Emmy Noether's letters.*

At the end of his letter Artin is asking Hasse which issue of Crelle's Journal his paper will eventually appear in. He would like to have the correct volume number available for reference, to use in his other paper on $L$-functions [7], the galley proofs of which are expected to arrive in a few days already. Well, Hasse was going to put Artin's paper into the next volume of Crelle's Journal which was to appear early in 1931, and certainly he had responded to Artin's request and given him the volume number. But somehow the digits got mixed up: While Artin's conductor paper [8] appeared in volume 164 of Crelle's Journal, the reference to this in [7] reads vol. 146.

# 7 Letter of 11 Nov 1930

This letter is written 4 days after Artin's foregoing letter where Artin had submitted his manuscript to Hasse. In the meantime Hasse had refereed the paper, and Artin thanks Hasse for his comments.

The letter contains three paragraphs.

In the first paragraph, Artin refers to Hasse's comments on Artin's manuscript. He mentions the "congruences in question" ("*die fraglichen Kongruenzen*"), and clearly he means Hasse's congruences (9) which were the subject of their discussion in the foregoing letters. It seems that Hasse had asked him to cite his (Hasse's) paper [46] where those congruences are proved. Now Artin points out that only the first of the congruences (9) are under discussion, namely those modulo $e_0$. The other congruences are not explicitly dealt with in Artin's paper. Artin says that originally he had planned a reference to Hasse but then he forgot about it. He promises to insert a corresponding remark while reading the galley-proofs. Indeed, in the published paper [8] we find a footnote (numbered 6) referring to Hasse [46].

Artin also asks which problem he should be referring to ("... *auf welches Problem ich bezug nehmen soll*"). It seems that Hasse had asked him to mention the problem list at the end of Part II of Hasse's class field report, where (among others) several problems on Artin $L$-functions are listed. One of those problems called for explicit formulas for the contributions of the ramified primes to Artin's $L$-functions: this is precisely what Artin solves in his paper. In the published version of the paper, Artin cites this problem list of Hasse in footnote 2.

The second paragraph of the letter sounds somewhat mysterious at first glance. Artin seems to comment on some questions which Hasse had raised in his last letter. But what were those questions? We shall discuss this in section 9.2 in connection with Emmy Noether's letters.

In the third and last paragraph of his letter, Artin announces that he had sent to Hasse the galley-proofs of his "other paper", meaning his second paper on $L$-functions which is to appear in the *Hamburger Abhandlungen*. He is sending the galley-proofs not for proof reading; instead he is asking Hasse for his comments, in particular he wishes to have references for the handling of the infinite primes in an arithmetical framework. Apart from Hasse's Class Field Report, Artin says, he does not know any other reference for dealing with infinite primes.

# 8 Letter of 27 Nov 1930

Artin starts with apologizing for not having been able to send the proof sheets earlier. Obviously, these are the proof sheets of Artin's paper [8] on the group theoretic structure of the discriminant, to appear in Crelle's Journal.

Artin says that he followed all of Hasse's proposals for correction – except one. We do not know what kind of proposals Hasse did forward – except the one which Artin did not heed, namely that the same letter $k$ should not be used in two different meanings in one paper. In fact, in §1 of [8] Artin uses the letter $k$ as an index (e.g., $\psi_k$) whereas in §2 $k$ is introduced as the base field of a Galois extension $K|k$. But Artin believes this could not lead to misunderstandings on the part of the reader.

Artin also mentions "the other galley-proofs" (*die anderen Korrekturen*). By this he means the galley proofs of his other paper [7], i.e., the paper on $L$-functions which was to appear in the *Hamburger Abhandlungen*. Recall that in the foregoing letter Artin had announced he would send those galley-proofs to Hasse. And he had asked for Hasse's comments, in particular for references concerning the handling of the infinite primes of a number field. It seems that, indeed, Hasse did send him his comments. For, in a footnote (numbered 6) in [7] we find references, not only to Hasse's class field report (Part Ia) which was the only source Artin had been aware of, but also to Hasse's *Habilitationsschrift* [40] and, in the case of the rational number field, to Hensel's book on $p$-adic numbers [59]. Very likely those references had been provided by Hasse.

Also, it seems that Hasse had criticized Artin's introduction of the notion of a valuation in [7]. For, in section 3 of that paper a "valuation" of a number field $K$ is defined as "*a mapping of $K$ into an ordinary number field*". Hasse may have pointed out to Artin that the notion of "valuation" has long been fixed in a more general way, according to Kürschák [73] and Ostrowski [82], including both the archimedean and the non-archimedean case. And he may have proposed that Artin should use, accordingly, the "proper" notion of valuation ("*den richtigen Bewertungsbegriff*"). But in his letter Artin says:

> *... den "richtigen Bewertungsbegriff" habe ich doch nicht eingeführt. Ich hätte sonst den ganzen Paragraphen neu schreiben müssen und das wäre immerhin kostspielig geworden. Ich habe aber die richtige Definition in einer Fußnote gebracht.*
>
> ...... I have not introduced the "proper notion of valuation" after all. Otherwise I would have to rewrite the whole section, and that would have been quite expensive. But I have stated the proper definition in a footnote.

This turned out to become footnote 8 in the published version of [7].

It may have been that Artin, by restricting the notion of valuation to archimedean ones, had wished to address the analytic number theorists, perhaps the people around Hecke. They would not care too much about abstract definitions of algebraic notions since their main interest would be in the analytic behaviour of these new functions.

But clearly, Artin himself was well aware of the usefulness of abstract notions in mathematics at the right place. This of course also applies to the notion of non-archimedean valuation. [32] In fact, two years later in 1932 he published a paper on the determination of all valuations (in the Kürschàk-Hensel sense) of a number field. The problem had been solved by Ostrowski early in 1918 already [82] but now Artin gives a much simplified proof. We may imagine that Artin had found this simplification during a number theory course, after recalling Hasse's letter which said that one should use the "proper" definition of valuation. This paper appeared in volume 167 of Crelle's Journal which was dedicated to Kurt Hensel on the occasion of his 70-th birthday [10].

Hasse was so delighted about Artin's simple proof of Ostrowski's theorem that he used it in his textbook "*Zahlentheorie*" [55] with the comment: "*Dieser schöne Beweis geht auf Artin zurück.*" (This beautiful proof is due to Artin.) Today Artin's proof has become standard in every elementary introductional course on valuations.

It is curious that this valuation paper of Artin is never mentioned in the Artin-Hasse correspondence although, as we have pointed out, it can be viewed as an indirect consequence of it, namely, of Hasse's criticism to the use of the word "valuation" in Artin's paper [7].

Artin expresses his gratefulness to Hasse for his very exact proof-reading:

> *Ich danke Ihnen noch einmal für die Mühe, die Sie sich mit den Korrekturen gemacht haben. Sie sind wirklich ein ungeheuer fleissiger Mensch. Ich glaube, daß Sie der einzige sein werden, der die Arbeit überhaupt liest.*
>
> Thanks again for the trouble you have had with reading the galley-proofs. Indeed you are an extremely diligent person. I believe you will be the only person who will read the paper after all.

The last sentence is, of course, a wilful understatement; we believe that Artin was well aware of the importance of his paper and that, consequently, he expected that many people would be reading it.

In a final paragraph of his letter Artin thanks Hasse for sending him proof sheets of still another paper, i.e., on "hypercomplex arithmetic". This refers to Hasse's Annalen paper on $\wp$-adic skew fields [47] which appeared in 1931. This cannot be regarded as properly belonging to the theory of $L$-functions and conductors, and we shall discuss it elsewhere. [33]

---

[32] A set of mimeographed Lecture Notes from Artin's lecture on class field theory in Hamburg 1931/1932, is preserved. The first sentence of those notes reads: "*Es sei $k$ ein endlicher Zahlkörper, $\mathfrak{p}$ ein Primideal entsprechend einer nichtarchimedischen Bewertung $\varphi$; $\overline{k}$ der perfekte Abschluß bezüglich dieser Bewertung.*" (Let $k$ be a finite number field, $\mathfrak{p}$ a prime ideal corresponding to a non-archimedean valuation $\varphi$; $\overline{k}$ the completion with respect to this valuation.) We conclude that in this course Artin supposed the audience to be familiar with the notion of non-archimedean valuation.

[33] But as a side remark we may mention here already that Artin expresses his belief that every finite skew field over a number field will be cyclic – this is a fundamental theorem which Hasse, jointly with Emmy Noether and Richard Brauer, proves two years later [19]. Artin's comment is in contrast to that of Emmy Noether who, when Hasse confronted her with his ideas and asked her advice, at first did not believe it. In fact, she wrote to Hasse on 19 December 1930 that she had a counter example – only to admit a week later that there was

# 9 Emmy Noether

Hasse kept close scientific and personal contacts also with Emmy Noether; their friendly correspondence started in 1925 and persisted until shortly before her untimely death in 1935. It is planned to edit the Noether-Hasse correspondence separately but let us discuss here already those letters of Emmy Noether which are connected with Artin's theory of conductors and discriminants.

Prior to 1930, Emmy Noether had published her well known paper on the different and the discriminant of orders in number fields or function fields [77] which appeared 1927 in Crelle's Journal. One year later she had finished a second paper on the different; this however was not published during her lifetime; she intended to rewrite certain parts of it before publication. That second manuscript was published posthumously 1950 in Crelle's Journal [80]. [34]

From this it is evident that Emmy Noether was keenly interested in the structure of the different, the discriminant and ramification. No wonder that she was fascinated by Artin's theory of conductors which leads to a group theoretic decomposition of the discriminant. She knew Artin's theory already before its actual publication, namely from Hasse.

## 9.1 Emmy Noether's letter – 10 Oct 1930

We have found a letter from Emmy Noether to Hasse, dated 10 October 1930, which begins as follows:

> *Lieber Herr Hasse! Schönen Dank für Artin! Die Sachen sind wirklich wunderschön!*
>
> Dear Mr. Hasse! Thanks for Artin! Those things are really beautiful!

We deduce from these words that Hasse had sent her a letter of Artin and she was really excited about its content. Hasse had sent her the original, asking for return. From the context as well as from the date of Noether's letter, it is clear that she refers to Artin's letter of 18 Sep 1930, in particular its third part about Artin's conductors and the discriminant-conductor formula (6). Perhaps Hasse had sent her Artin's undated letter too (section 5) which contains more details about Artin's proofs, but we have no direct evidence for this. Emmy Noether continues:

> *Mich reizen besonders die darin steckenden formalen Grundlagen; einiges Hyperkomplexe – einstweilen noch ganz unabhängig – habe ich mir überlegt, so das folgende ...*
>
> I am particularly interested in the formal foundations on which it is based; I have given some thoughts to it – quite independent for

an error, and from then on joined Hasse in the search for a proof. See also [1] where the contribution of Albert to this result is explained.

[34] Emmy Noether had handed a copy to Heinrich Grell (one of her students in Göttingen) who preserved it.

the time being – from the hypercomplex point of view, namely the following ...

She uses the old-fashioned terminology "*hypercomplex systems*", instead of Dickson's "*algebras*" which has become standard today. In those days, algebras (or hypercomplex systems for that matter) were always to be understood with respect to a base field. On many occasions Emmy Noether had propagated her dictum that non-commutative algebras and their arithmetic can be profitably used for a better understanding of the arithmetic of commutative fields, e.g., number fields. And she is doing this here again, trying to approach Artin's discriminant-conductor formula (6) within the framework of non-commutative algebras.

Emmy Noether writes that she is looking for decomposition formulas for the *different* $\mathfrak{D}_{K|k}$ of a Galois extension $K|k$ which, after applying the norm operator, will yield Artin's formula (6) for the *discriminant* $\mathfrak{d}_{K|k}$ – or at least there will show up a connection with Artin's formula. And she immediately goes about to set the stage for this:

She considers the split crossed product algebra of a Galois extension $K|k$ with its Galois group $G$. This is defined as the $k$-algebra $A$ which is generated by $K$ and by the group ring $k[G]$ with the defining relations

$$z\sigma = \sigma z^{\sigma} \qquad (z \in K,\ \sigma \in G) \tag{13}$$

where $z^{\sigma}$ denotes the result of the action of $\sigma$ (regarded as automorphism of $K$) on $z$. This algebra $A$ can be represented as a full matrix algebra over $k$. Explicitly, Noether points out that:

> *Jede Basis von $K|k$ – zusammen mit der Einheit der identischen Darstellung des Gruppenrings – liefert eine Zerlegung in einseitig einfache, etwa Rechtsideale; die entsprechende Linkszerlegung wird dann durch die komplementäre Basis von $K|k$ erzeugt ...*
>
> Every basis of $K|k$ – together with the unit of the identity representation of the group ring – defines a decomposition of $A$ into one-sided simple, say, right ideals; the corresponding decomposition into left ideals is then given by the dual basis of $K|k$ ...

She obviously assumes that the details of this would be known to Hasse – which they probably were since Hasse had thoroughly studied Noether's theory of crossed products. [35] We have found those details explicitly presented in Noether's 1934 paper [79] namely as follows:

[35] Emmy Noether had developed the theory of crossed products in her lectures "*Algebra der hyperkomplexen Größen*", delivered in Göttingen in the winter semester 1929/1930. Deuring had taken notes in the lecture, and these had been circulated among the interested mathematicians. Hasse published, authorized by Emmy Noether, the theory of crossed product algebras as part of his "American" paper [48] which was completed in 1931 and appeared 1932 in the Transactions of the American Mathematical Society.

Let $a_i$ $(1 \le i \le n)$ be a $k$-basis of $K$ and $\widehat{a}_j$ $(1 \le j \le n)$ its dual basis, defined by the relations

$$\mathrm{tr}(\widehat{a}_i a_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

where $\mathrm{tr}(\cdots)$ denotes the trace from $K$ to $k$. Put

$$E = \sum_{\sigma \in G} \sigma \in k[G] \tag{14}$$

this is what Noether calls the "unit of the identity representation of the group ring". It is immediately verified from the defining relations (13) that the elements

$$e_{ij} = \widehat{a}_i E a_j$$

form a system of $n^2$ matrix units in $A$, which is to say that

$$e_{ij} e_{k\ell} = \begin{cases} e_{i\ell} & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} .$$

This puts into evidence that $A$ is a full matrix algebra over $k$. The corresponding direct sum decompositions of $A$ into minimal right resp. left ideals, which Noether mentions, is then given by

$$A = \oplus \sum_{1 \le i \le n} \widehat{a}_i E A = \oplus \sum_{1 \le j \le n} A E a_j \, .$$

In this situation, she writes in her letter, one should restrict the investigation to *integral ideals*, i.e., in case of a local number field $k$ the coefficients should be restricted to $\mathfrak{o}_k$, the valuation ring of $k$. She observes that the right- and left decompositions then belong to complementary ideals and from this she expects information about the decomposition of the different. Recall that the inverse of the different ideal of $K|k$ is the complementary ideal of the valuation ring of $K$. – But Emmy Noether is well aware that these ideas are quite vague:

> *Aber das ist Zukunftsphantasie! Jedenfalls schönen Dank für die Überlassung des Briefs.*
>
> But these are fantasies for the future! In any case, many thanks for showing me the letter.

We shall see below that and how Noether will make these ideas more precise.

She closes her letter by apologizing for the delay in returning Artin's letter to Hasse. The reason which she gave was that Deuring was out of town and she wished to show Artin's letter to him. Max Deuring was one of the promising students who had gathered around Emmy Noether at that time. [36]

[36] When Emmy Noether was asked to write a survey on the theory of algebras in the newly founded Springer series "*Ergebnisse der Mathematik*" then she declined but proposed her

## 9.2 Artin's "*Zukunftsmusik*"

Let us return to Artin's letter of 11 Nov 1930 which we have discussed in section 7. There, we had omitted from our discussion the second paragraph of that letter. That paragraph, as we believe, can be understood only vis-à-vis Noether's letter to Hasse. Note that Noether's letter is dated 10 Oct 1930 while Artin's is dated 11 Nov 1930, about one month later. Hence it is well conceivable that by 11 November, Artin knew about the content of Noether's letter. Maybe Hasse had sent him Noether's letter for information, in the same spirit as he had sent Artin's to Noether in the first place. But perhaps Emmy Noether herself had written a similar letter to Artin as she had written to Hasse; unfortunately we are not able to check this because Artin's as well as Noether's papers are lost.

Artin starts the second paragraph with the words: "*Ein bischen Zukunftsmusik.*" (Some music for the future.) This sounds quite similar to Emmy Noether's "*Zukunftsphantasien*" (fantasies for the future) which we just mentioned in the last section. It is conceivable that Artin purposely used a similar word as Emmy Noether had used in her letter, in order to relate his words to hers. [37] We are almost convinced about this if we observe the remarkable coincidence between his ideas and Noether's. For, Artin writes:

> *Ich denke mir, dass man jetzt den Ring untersuchen muss, der durch Erweiterung des Körpers mit seiner eigenen Gruppe entsteht, also den nichtkommutativen Ring den schon Dickson eingeführt hat.*
>
> I think that now one has to investigate the ring which arises after extending the field with its own group, i.e., the non-commutative ring which has already been defined by Dickson.

In fact, this ring is precisely the split crossed product whose investigation had been proposed by Emmy Noether ! Hence these comments by Artin, which otherwise sound somewhat cryptical, can be understood as agreeing with Noether's proposal to investigate the arithmetic of the split crosssed product in more detail.

Artin is convinced that the matter (i.e., his discriminant formula (6)) has something to do with the matrix representations of the ring he mentions and that, consequently, a systematic investigation of its non-commutative arithmetic structure is needed – again he is backing Noether's opinion on this point. It seems that Hasse, in his foregoing letter to Artin (which we do not know) had mentioned some recent literature on non-commutative arithmetic which may

student Deuring as the author of that book. Deuring's "*Algebren*" [27] appeared in 1934 and contained all the relevant results on the algebraic and arithmetic theory of algebras known at the time. It has served many generations of mathematicians as an introduction to the arithmetic theory of algebras; even today it is still an important source. Most of the book is based on the results of Noether, Artin and Hasse but the excellent exposition is certainly due to Deuring himself.

[37] In German language, it is not uncommon to use the word "*Zukunftsmusik*" to express some optimism for the future, maybe a little more optimism than the subject deserves.

be useful in this investigation. But Artin replied he does not know yet this literature.

Artin says (like Noether had done) that at present he has no time to follow up these ideas. But he is hoping that "*wir in spätestens einem Jahre darüber Bescheid wissen werden*" (after at most one year we shall know about it). But perhaps, he adds, he is too optimistic on this.

Well, his caution was justified because even today it cannot be said that the situation is cleared up completely. On the other hand, one year later Noether wrote her paper on integral bases which throws light on those questions at least in the tamely ramified case, as we shall see in the next section.

## 9.3 Noether's Hensel note – 1932

On 29 December 1931 Kurt Hensel celebrated his 70-th birthday. On this occasion Hasse planned a dedication volume in Crelle's Journal, and he had asked Artin as well as Emmy Noether, among others, to contribute to this volume. Artin, as we have mentioned already in section 8, sent his paper on the foundation of valuation theory [10]. [38]. And Emmy Noether sent her paper [78] on local normal integral bases which later became well known and often cited. [39]

In her letters to Hasse, Emmy Noether always called this paper [78] the "*Hensel note*". She submitted the manuscript to Hasse on 24 August 1931, hence 10 months after her first letter to Hasse on Artin's conductor theory. It seems that the manuscript was written in haste because in a number of later letters to Hasse she gives additions and corrections to her Hensel note. In a postcard of 23 October 1931 she is apologizing for her many corrections and gives as a reason that this time she had published more quickly than she was used to. This may refer to the deadline of 1 September 1931 which had been set for the dedication volume. [40]

The main result of Emmy Noether's Hensel note is the existence of local normal integral bases when the local degree is not divisible by the characteristic $p$ of the residue field. Perhaps it is not generally known that this note and its famous result had been directly influenced by Artin's paper [8] on discriminants and conductors, more precisely: by Artin's letter of 23 Sep 1930 to Hasse where Artin outlines his results. In this paper she is setting the tune for her "*Zukunfts-phantasien*", at least in the tame case. Her aim is to produce, in the framework of Galois modules, a decomposition of the local field discriminant by means of

[38] Artin had a second paper [9] in the Hensel volume, namely the one where he presented his simple proof of Herbrand's theorem on the Galois module structure of the unit group.

[39] She too had a second paper [19] in the Hensel volume, namely the one where, jointly with Hasse and Richard Brauer, the first proof of the Local-Global Principle for algebras over number fields was given. By the way, also Hasse had planned to submit a second paper to the Hensel volume (besides the one mentioned above with R. Brauer and Emmy Noether). This was the paper on the structure of discrete valued complete fields, written jointly with F.K. Schmidt. However that paper was not ready for publication in time, due to repeated wishes of F.K. Schmidt to rewrite and reorganize the paper. See [89].

[40] Moreover, Emmy Noether says, the manuscript had been written on the beach. We can identify this beach because we know from other sources that in August 1931, Emmy Noether stayed in Rantum. This is a small village on the North Sea island of Sylt, near Westerland.

group characters and then to identify it with Artin's decomposition (6). The situation is as follows:

| | |
|---|---|
| $K\vert k$ | is a Galois extension of local fields [41] |
| $p > 0$ | the characteristic of the residue field |
| $n$ | $= [K : k]$ its degree |
| $G$ | the Galois group of $K\vert k$, not necessarily abelian |
| $\mathfrak{o}\,, \mathfrak{O}$ | the valuation rings of $k\,, K$ |

$\mathfrak{O}$ is an $\mathfrak{o}[G]$-module, where $\mathfrak{o}[G]$ denotes the group ring of $G$ with coefficients in $\mathfrak{o}$. The aim is to show that if $n$ is not divisible by $p$, then $\mathfrak{O}$ is *free* as an $\mathfrak{o}[G]$-module, i.e., that there exists an element $z \in \mathfrak{O}$ whose conjugates $z^\sigma$ (for $\sigma \in G$) form an $\mathfrak{o}$-basis of $\mathfrak{O}$. To show this, Noether introduces the group ring $k[G]$ and observes that $\mathfrak{o}[G]$ is a *maximal order* of $k[G]$ – as a consequence of the hypothesis that the order $n$ of $G$ is not divisible by $p$. Consequently, citing Hasse's 1931 paper [47] on local arithmetic of algebras, Noether concludes that $\mathfrak{o}[G]$ is *principal*, i.e., that every fractional right or left $\mathfrak{o}[G]$-ideal is principal. Now, since $K|k$ admits a normal basis we may identify $K$, as a $k[G]$-module, with $k[G]$ itself. Under this identification $\mathfrak{O}$ appears as a fractional $\mathfrak{o}[G]$-ideal, say right ideal. Being principal it follows that it is generated by one element $z$, hence $z$ and its conjugates $z^\sigma$ generate $\mathfrak{O}$ over $\mathfrak{o}$.

This being settled, Noether proceeds to study the discriminant. Because of the module isomorphism $\mathfrak{O} \approx \mathfrak{o}[G]$ she observes that the discriminant of $K|k$ is the square of what she calls *group determinant of* $G$. More precisely, it is the square of the determinant

$$D = \det(z^{\sigma\tau^{-1}})_{\sigma,\tau\in G}$$

where $z \in \mathfrak{O}$ denotes an element which generates an integral normal basis. By general representation theory, this group determinant is decomposed into factors corresponding to the irreducible characters $\chi$ of $G$. Accordingly, the discriminant itself splits into factors

$$\mathfrak{d}_{K|k} = \prod_{\chi} \mathfrak{c}(\chi, K|k)^{g_\chi}$$

with certain ideals $\mathfrak{c}(\chi, K|k)$. The exponents $g_\chi$ are the same as the exponents in Artin's formula (6). Emmy Noether conjectured that the $\mathfrak{c}(\chi, K|k)$ coincide with Artin's conductors $\mathfrak{f}(\chi, K|k)$ from (6). She could not prove it but she said that its verification would need more ideal theoretic investigation; it would not follow directly from the formal properties of group determinants as Artin had believed in the introduction of his paper [8]. In fact, in a footnote Noether produces a counter example, due to Max Deuring, to Artin's statement.

Noether's conjecture had been verified many years later only, by A. Fröhlich. See his book [38] which appeared in 1983.

Remark: Noether's *Hensel note* contains some discrepancy: The title of the paper announces that it would concern *tame* field extensions ("*Körper ohne*

[41] Noether considers also global fields but her main result refers to the local case.

*höhere Verzweigung*"). But the theorems are formulated and proved for those field extensions only whose local degree is not divisible by the residue characteristic $p$. These two notions are not equivalent; in general there do exist local field extensions (containing unramified subfields) which are tame but whose degree is divisible by $p$. Today we know that Noether's theorem on local integral normal bases does indeed hold for arbitrary tame extensions but Noether's proof does not give this result.

Perhaps this discrepancy in Noether's paper can be explained by the fact that she started from Artin's conductor problem. For this purpose it is indeed sufficient to consider purely ramified local extensions only – hence in the background she always thought about purely ramified extensions although she did not say so in the paper. We should also remember that the paper had to be written in haste (on the Sylt beach) in order to meet the deadline for the Hensel dedication volume. –

If we compare Noether's *Hensel note* with the program she outlined in her letter to Hasse of 10 Oct 1930, then we see that her ideas in the letter had not fully come into the play in the note. It is true that "hypercomplex" notions and ideas were used in the Hensel note, namely Hasse's theorem that in the local case, regular ideals of maximal orders are free. But we do not find the split crossed product algebra appearing in the Hensel note. Consequently Emmy Noether obtained a group theoretic decomposition of the *discriminant* only, whereas in her letter to Hasse she had envisaged a similar decomposition of the *different*.

Emmy Noether was quite aware of this fact. In a letter to Hasse dated 4 October 1931, when she sent another addendum to her Hensel note (namely Deuring's counterexample, mentioned above already), she continued:

> *Im allgemeinen Fall, wo keine Gruppendeterminante existieren kann . . . zeigt sich das Zusammenspiel von formal Hyperkomplexem und Zahlentheoretischem noch deutlicher; die Galoismoduln müssen mit Trägheits- und Verzweigungskörper und -gruppen verknüpft werden. Über erste Ansätze bin ich aber noch nicht hinaus.*
>
> In the general case where group determinants cannot exist . . . the mutual connection between formally hypercomplex and arithmetic ideas is seen more clearly; the Galois modules have to be combined with inertia- and ramification fields and -groups. But I have not yet gone further than first ideas about this.

Today this is quite clear to us: In discussing Galois modules one indeed has to bring the inertia- and ramification groups into the picture. It seems interesting that this idea, as a general principle, had first been explicitly stated by Emmy Noether in her letter to Hasse 1931. But of course, this principle is implicitly contained already in Artin's theory of conductors although he does not speak of Galois representation modules but rather of Galois characters.

## 9.4 Noether's Herbrand note – 1934

An opportunity to make her ideas more explicit arose when Emmy Noether was asked to send a contribution to a publication dedicated to the memory of the late Jacques Herbrand.

Jacques Herbrand was a young French mathematician who in 1930/31 came to Göttingen as a Rockefeller fellow to study with Emmy Noether. He also visited Hamburg (Artin), Marburg (Hasse) and Berlin (v. Neumann). Wherever he went he was accepted by the mathematical community as a bright and talented young colleague from which important work was to be expected in the future. His warm and friendly manner won him the hearts of those whom he met. As an example, we can cite a letter of Emmy Noether to Hasse dated 8 February 1931. She had just returned from Halle where she had delivered a colloquium lecture. She wrote:

> *Mein Rockefeller-Stipendiat... Dr.J. Herbrand... kam nach Halle* [aus Berlin]*, und er hat am meisten von allen von meinen Sachen verstanden. Er hat bis jetzt außer Logik nur Zahlentheorie gearbeitet, die er aus Ihrem "Bericht" und Ihrer Normenresttheorie gelernt hat... Wir hatten in Halle alle einen ausgezeichneten Eindruck von ihm.*
>
> My Rockefeller awardee... Dr.J. Herbrand... came to Halle [*from Berlin*], and he understood more of my things than anybody else. Until now he has worked, apart from logic [42], in number theory only, which he had learned from your "report" and your theory of norm residues... We in Halle all got an excellent impression of him. [43]

But in the summer of 1931 Herbrand, during a mountain tour in the French mountains, had a fatal accident. The date of his death is documented in a letter written by André Weil to Hasse on 4 August 1931:

> *...Ich muss Ihnen leider eine betrübliche Nachricht mitteilen, die des Todes Jacques Herbrands, der vor wenigen Tagen bei einer Bergbesteigung im Dauphiné tödlich verunglückt ist...*
>
> I am sorry to have to tell you a sad message, of the death of Jacques Herbrand, who only a few days ago had an accident at a moutain tour in the Dauphiné...

Weil went on to inform Hasse that Herbrand's unfinished manuscripts will be published by Claude Chevalley. [44] Weil said that Chevalley would have written

[42] In fact, Herbrand had given essential contributions to formal logic and philosophy; see e.g., [67].

[43] It is reported in [16] that Herbrand, probably on a second visit to Halle, had given a colloquium lecture there on his papers about algebraic number fields of infinite degree [65], [66]. With this he had won the admiration in particular of Heinrich Brandt who was Professor in Halle at that time (in succession of Hasse who had left Halle for Marburg the year before).

[44] Professor Catherine Chevalley, the daughter of Claude Chevalley, has written to me that "*...he* [*Herbrand*] *was maybe my father's dearest friend...*". This explains in part why Chevalley was actively engaged in the posthumous publication of Herbrand's papers.

himself but he did not feel up to writing a letter in German language.

The message about the tragic death of Herbrand came as a shock to the mathematicans who had known him. Emmy Noether wrote to Hasse on 24 August 1931 (from Rantum, Sylt):

> *Mir geht der Tod von Herbrand nicht aus dem Sinn... Sein Vater hat mir heute Genaueres geschrieben...*
>
> The death of Herbrand occupies my thoughts... Today I received a message from his father containing more details...

When in December 1931 there arrived a letter from Chevalley asking whether she would send a contribution to a volume published in the memory of Herbrand, then she readily consented. This "Herbrand note" [79] of Emmy Noether appeared in 1934. It gives an explicit and exhausting description of the arithmetic of maximal orders in a split crossed product; this is in fact the scenery which she had envisioned in her letter to Hasse on 10 October 1930.

More precisely: Consider the situation as explained in section 9.1, i.e., $K|k$ is a Galois extension of local fields [45], and let $A$ be the split crossed product of $K|k$ with its Galois group $G$. We use the notations introduced in section 9.1; in particular $E$ denotes the "unit of the identity representation" as defined by (14). Recall that $\mathfrak{o}$, $\mathfrak{O}$ denote the valuation rings of $k$, $K$. Besides of this, we have also to consider maximal orders of $A$ which we denote by $\mathcal{O}$.

Noether shows that the maximal orders of $A$ correspond to the $\mathfrak{o}$-modules $\mathfrak{a} \subset K$ of rank $n$. If $\mathcal{O}_\mathfrak{a}$ belongs to $\mathfrak{a}$ then

$$\mathcal{O}_\mathfrak{a} = \widehat{\mathfrak{a}} E \mathfrak{a}\,, \qquad\qquad K \cap \mathcal{O}_\mathfrak{a} = \mathit{Order}(\mathfrak{a})$$

where $\widehat{\mathfrak{a}}$ denotes the complementary module of $\mathfrak{a}$, and $\mathit{Order}(\mathfrak{a})$ denotes the order of $\mathfrak{a}$, consisting of all elements $x \in K$ with $x\mathfrak{a} \subset \mathfrak{a}$.

Moreover, Noether is able to describe explicitly the right ideals and the left ideals of the maximal orders $\mathcal{O}_\mathfrak{a}$, as well as their multiplication in the sense of Brandt's grupoid. [46]

As we have seen, this "Herbrand note" too had been directly influenced by Artin's theory of Galois conductors. Emmy Noether had conceived the note as a (hypercomplex) foundation for a proof of Artin's discriminant-conductor formula. But it must be said that, to our knowledge, her idea has not been followed up, neither by Noether herself nor by someone else. The Herbrand note did not contribute to the further development of Artin's theory of conductors, as Noether obviously had envisaged. It is not easy to determine the reason for

[45] Noether also considered global fields but we shall restrict our discussion to the (essential) case of local fields.

[46] The multiplicative theory of maximal orders and their ideals in the sense of Brandt's grupoid had been developed in "modern" terms by Artin [6]; see also Hasse [47] and Deuring's book [27]. (We have used the expression "modern" in the same sense as van der Waerden has used in the title of his book "Modern Algebra" which appeared 1930.) In this form Brandt's theory became widely known as the analogue of the theory of Dedekind rings in the non-commutative case.

this, whether this became a dead end because of the insufficiency of the notions and the structures of Noether's construction, or whether the paper was simply forgotten by the following generations of mathematicians. [47]

But for its own sake, as a contribution to the arithmetic of algebras, this short paper deserves to be dug up again; besides of its exhaustive and complete results it is beautifully written and should rank among the pearls of, in Emmy Noether's words, hypercomplex arithmetics.

Finally, we would like to remark that among the papers dedicated to the memory of Herbrand we find, besides Noether's, a paper by Hasse [54] and one by Chevalley [26], both on the same topic as Noether's. Noether describes the relation of her note to Hasse's in a footnote as follows:

> *Diese expliziten Darstellungen* [der Maximalordnungen und ihrer Ideale] *waren mir lange bekannt; den Anstoß zu den anschließenden Überlegungen gab die Mitteilung des zitierten Satzes von Hasse.*
>
> Those explicit representations [*of maximal orders and their ideals*] were known to me since a long time; the occasion for the following discussions was given by Hasse who informed me about his theorem as cited.

"Hasse's theorem" as cited by Noether is a special case of Noether's, namely the representation of those maximal orders which contain the valuation ring $\mathfrak{O}$ of $K$; this means that the above module $\mathfrak{a}$ should be an $\mathfrak{O}$-ideal. [48]

Thus, when Hasse told her about this result then Noether replied that she knew this all along and more. But Hasse's methods are different from Noether's and so his note is not completely superseded by hers.

It seems that Chevalley's paper [26] too was inspired by Emmy Noether; we deduce this from a letter of Chevalley to Hasse about questions on ideals in hypercomplex systems, dated 31 December 1931. There he refers to Emmy Noether and to Artin. [49] However in his paper [26] he does not cite nor mention the name of Emmy Noether. This may be justified since his point of departure as well as his methods are quite different from Noether's. In particular he is able to deal with arbitrary crossed products, not necessarily split. It may be because of this that Noether finally added a section to her Herbrand note, indicating how she could handle with her methods also the case of non-split crossed products.

[47] The paper is not mentioned in the Introduction by Jacobson to Noether's Collected Papers [81]. It is not cited in Fröhlich's book [38] on Galois module structures, nor in Reiner's book [83] on maximal orders.

[48] Remark: Hasse's paper contains a reference to a communication by Artin. We found that in Artin's letter of 16 June 1931, which we will discuss elsewhere.

[49] It is curious that Artin did not contribute a paper to the memory of Herbrand. It seems certain that he had been asked for it because Herbrand had met him in Hamburg. Perhaps Artin did not have a manuscript suitable for publication at that time. We observe that after 1932 Artin did not have any publication at all until 1940; in 1937 he emigrated to the U.S.A.

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