Class Field Theory in Characteristic p, its Origin and Development

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Abstract

Today's notion of "global field" comprises number fields (algebraic, of finite degree) and function fields (algebraic, of dimension 1, finite base field). They have many similar arithmetic properties. The systematic study of these similarities seems to have been started by Dedekind (1857). A new impetus was given by the seminal thesis of E.Artin (1921, published in 1924). In this exposition I shall report on the development during the twenties and thirties of our century, with emphasis on the emergence of class field theory for function fields. The names of F.K.Schmidt, H. Hasse, E. Witt, C. Chevalley (among others) are closely connected with that development.

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1 Introduction

What today is called "class field theory" has deep roots in the history of mathematics, going back to Gauss, Kummer and Kronecker. The term "class field" was coined by Heinrich Weber in his book on elliptic functions and algebraic numbers [120] which appeared in 1891. It was Hilbert [68] who in 1898 proposed to establish class field theory as the theory of arbitrary abelian extensions of algebraic number fields. Although Hilbert himself discussed *unramified* abelian extensions only, i.e., what today is called the "Hilbert class field", it is evident from his introductory remarks that he clearly envisioned the possible generalization to the ramified case. [1] And Takagi, giving class field theory a new turn, succeeded in completing Hilbert's program to its full extent [110], [112]. His work was crowned by Artin's general reciprocity law [6] together with Furtwängler's proof of the principal ideal theorem [33].

Soon after Takagi's fundamental papers, there arose the question whether algebraic function fields with finite base field could be treated similarly, i.e., whether class field theory could be transferred to function fields. Today we know that this is the case.

In this article I shall outline the origin and the development of those ideas, and I shall follow up the main steps until finally class field theory for function fields was well established. The initial steps were done by F.K. Schmidt, Hasse and Witt; other mathematicians will be mentioned in due course. The time period covered will be from 1925 to about 1940. Thereafter class field theory for function fields ceased to be a separate topic; it became possible to deal with number fields and function fields simultaneously; the common name for both became *Global Field*. [2]

Class field theory for function fields was developed largely in analogy and parallel to class field theory for number fields. Hence, in order to understand what has happened in the function field case, it seems useful to give some comments to the development of class field theory in the number field case during the said time period. We shall do this briefly in the first preliminary section. [3]

2 Class field theory for number fields 1920-40

2.1 Zeittafel:

1920 Takagi's first great paper [111], establishing class field theory in its full generality according to Hilbert's program

1922 Takagi's second paper [112], on reciprocity laws in number fields

[1] This has been pointed out by Hasse [43]. In some contrast to this is the statement of K. Takagi in his memoirs that Hilbert seemed to be interested in the unramified case only and, hence, Takagi was "misled" by Hilbert into the wrong direction of study. See Kaplan's article [74] where several passages of Takagi's memoirs are translated from Japanese into French.

[2] In more recent times, however, the theory of function fields was revived under new aspects, among them also a new class field theory. See e.g., the book by D. Goss [35]. But this is outside the scope of this article.

[3] Information on the history of class field theory can also be obtained from [31], [32], [57], [72], [73], [74], [80].

1925 Hasse's report on Takagi's results at the Danzig meeting of the DMV (German Mathematical Society)

1926 Part I of Hasse's report "*Klassenkörperbericht*" [37]; the other two parts Ia and II appeared in 1927 and 1930 respectively, see [38] and [39]

1927 Publication of Artin's proof of the general reciprocity law [6], based on Chebotarev's ideas which were connected with his density theorem [11], [12]

1928 Furtwängler proves the principal ideal theorem (the proof appeared in print 1930 [33]; later simplifications by Magnus [79], Iyanaga [71] and Witt[131])

1929 Käte Hey's thesis: Class field theory on the basis of analytic number theory in non-commutative algebras [66]

1930 Hasse-F.K.Schmidt: Concept of local class field theory [40], [99] (later reorganized and simplified by Chevalley [15])

1931 Hasse determines the structure of the Brauer group over a local field [41], following ideas of E. Noether on crossed products;

1931 Herbrand: Essential simplification of computations pertaining to class field theory [64] [65]

1932 Hasse's Marburg lectures on class field theory [45]

1932 Brauer, Hasse, Noether: local-global principle for algebras [10]; connection with the product formula for the Hilbert symbol

1933 Hasse: Structure of Brauer group of number fields [44]

1933 Publication of Chevalley's fundamental thesis [15] which contributed greatly to the simplification and adequate organization of class field theory

1934 Deuring's book on algebras [23], based on E. Noether's lectures, containing a treatment of class field theory by means of algebras

1935 Chevalley and Nehrkorn present algebraic-arithmetic proofs for many of the main theorems of class field theory

1940 Chevalley's purely algebraic-arithmetic proof of Artin's reciprocity law in the framework of idèles [18], without using analytic functions

2.2 Comments

In 1920, the same year when Takagi's first main paper [110] had appeared, he attended the International Congress of Mathematicians in Strasbourg where he reported about his results [111]. However it seems that he did not receive any visible reaction to his report. [4] But two years later, 1922, after his second paper [112] had appeared and became available in Western libraries, it turned out that there was a number of young mathematicians who were keenly interested in Takagi's results and methods. Among them were Emil Artin and Helmut Hasse.

2.2.1 E. Artin

Artin in two papers [3], [4], obviously inspired by Takagi's, investigated ζ-functions and his new L-functions, and on this occasion he conjectured what is now called Artin's reciprocity law. Artin's proof appeared in 1927 [6] but already in 1925 he knew how to prove it, as we can infer from a letter dated February 10, 1925 and addressed to Hasse [29]: [5]

> *...Haben Sie die Arbeit von Tschebotareff in den Annalen Bd.95 gelesen? Ich konnte sie nicht verstehen und mich auch aus Zeitmangel noch nicht richtig dahinterklemmen. Wenn die richtig ist, hat man sicher die allgemeinen Abelschen Reziprozitätsgesetze in der Tasche...*
>
> ...Did you read Chebotarev's paper in the *Annalen*, vol. 95? I could not understand it, and because of lack of time I was not able to dive deeper into it. If it turns out to be correct then, certainly, one has pocketed the general abelian reciprocity law...

Artin's reciprocity law can be considered as the coronation of Takagi's class field theory. It was soon completed by Furtwängler who proved the principal ideal theorem [33] which had been conjectured by Hilbert [68]. The actual proof of this theorem had been obtained some time before its publication date (1930). In our *Zeittafel* we have dated it for 1928, because we have found a reference to its proof in a letter of Hasse to Mordell which is dated November 26, 1928: [6]

> *...Ich lege eine Arbeit von Artin (Hamburg) bei, die einen ganz grundlegenden Fortschritt in der Theorie der relativ-Abelschen Zahlkörper enthält. Vielleicht ist es nicht ohne Interesse für Sie, zu er-*

[4] This may have been due to the fact that until then, the development of class field theory took place mainly in Germany, and that German mathematicians were not admitted at the Strasbourg congress (probably on political grounds shortly after World War I). Thus Takagi did not meet the experts on class field theory at that congress.

[5] All letters which we cite in this article are contained among the Hasse papers which are deposited in the *Staats- und Universitätsbibliothek Göttingen* – except when it is explicitly stated otherwise.

[6] This letter is contained among the Mordell papers at the archive of St. John's College, Cambridge.

> *fahren, daß ganz kürzlich Furtwängler, auf dem Boden dieser Artinschen Arbeit, den Hauptidealsatz der Klassenkörpertheorie (vgl. meinen Bericht, S.45) vollständig bewiesen hat, durch Reduktion auf eine Frage der Theorie der endlichen Gruppen...*
>
> I enclose a paper by Artin (Hamburg) which contains a very important advance in the theory of relatively abelian fields. Perhaps it is not without interest for you to know that recently Furtwängler, based on this paper of Artin's, has completely proved the principal ideal theorem of class field theory (see my report, p.45), via reduction to a question of finite group theory.

The paper by Artin which Hasse was referring to, was Artin's proof of his reciprocity law [6]. And when Hasse mentioned his "report" then he referred to Part I of his "*Klassenkörperbericht*" which had appeared in 1926.

2.2.2 H. Hasse

A detailed historical analysis how Hasse became interested in class field theory is given by G. Frei in his article which appears in this same volume [32]. Already in 1923, in a letter dated April 21, 1923 and addressed to Hensel, Hasse explained the relevance of Takagi's new results and methods with respect to their project of studying the local norms for abelian extensions. At that time he was 24 years of age and held the position of *Privatdozent* at the University of Kiel. He had just completed the manuscript of a joint paper with Kurt Hensel, his former academic teacher at Marburg. That paper was to appear in the *Mathematische Annalen*; it gives the description of the local norm group for cyclic extensions of prime degree ℓ, under the assumption that the ℓ-th roots of unity are contained in the ground field [36]. Hasse now realized that Takagi's theory could be used to deal with the general case, without this assumption about roots of unity.

Moreover, Hasse leaves no doubt that he regards Takagi's papers as being of highest importance also for class field theory in general. He writes:

> *...Ich habe gerade die Ausarbeitung eines Kollegs über die Klassenkörpertheorie von Takagi vor, die ich mit unseren Methoden sehr schön einfach darstellen kann...*
>
> ...Just now I am writing the notes for a course about Takagi's class field theory, which I am able to present quite simply with our methods...

Clearly, when Hasse refers to "our methods" in this letter then he means the ℓ-adic methods as employed in their joint paper.

We all know that a good way of learning a mathematical subject is to give a course about that topic; the necessity of a clear and coherent presentation to the participants of the course will prompt the speaker to look for a better understanding of the subject. As evidenced by the *Vorlesungsverzeichnis* (list of lectures) of the University of Kiel, Hasse's course about Takagi's class field theory was given in the summer term 1923, and was supplemented in the winter term 1923/24 with a course on "Higher Reciprocity Laws". [7] Hasse's manuscript still exists and is available among Hasse's papers. It became the basis of Hasse's great class field theory report (*Klassenkörperbericht*) which appeared in three parts I, Ia, II. [37], [38] [39].

2.2.3 The class field report

As G. Frei states [30], it had been Hilbert who suggested to Hasse to write such a report, which then was conceived by Hasse as a follow-up of Hilbert's famous *Zahlbericht* [67]. Like Hilbert's report, Hasse's was commissioned by the DMV (German Mathematical Society), and it appeared in the *Jahresbericht* of the DMV; the last part as a supplement (*Ergänzungsband*). [8] The three parts were bound together as a single book which became known as the *Klassenkörperbericht*. [9]

Hasse delivered an excerpt from this report in a lecture at the annual DMV meeting 1925 at the town of Danzig.

The impact of Hasse's report, both the Danzig lecture and the printed report, can hardly be overestimated. Hasse was not content with merely presenting Takagi's results. He set out to give a comprehensive and systematic overview of all of class field theory known at that time; his treatment included quite a number of simplifications and additions – including proofs.

Actually, Part I of the *Klassenkörperbericht* does not yet contain proofs. It seems that for these, Hasse had originally planned an additional, separate publication in the *Mathematische Annalen*. For there is a letter dated Nov 1, 1926, from Hilbert (who was editor of the *Annalen* at that time) to Hasse, in which Hilbert said:

> *Sehr geehrter Herr Kollege, Ihr Anerbieten, mir für die Annalen ein Manuskript mit dem Titel: "Takagi's Theorie der relativ-Abelschen Zahlkörper, bearbeitet von Hasse" zur Verfügung zu stellen, nehme ich mit vielem Dank an – zugleich auch namens der Annalenleser u. der zahlenth. Wiss., der Sie damit einen wichtigen Dienst erweisen. Ich habe soeben einen Brief an Takagi aufgesetzt, darf aber von vorneherein seines Einverständnisses sicher sein...*
>
> Dear Colleague, with many thanks I shall accept your offer to let me have a manuscript for the *Annalen* with the title "Takagi's theory of relatively abelian number fields, presented by Hasse" – also in the

[7] I am indebted to W. Gaschütz for his help in obtaining the *Vorlesungsverzeichnis* of Kiel University for those years.

[8] It seems that Hasse's report was the last one which was commissioned by the DMV. Whereas in its earlier years, the DMV had tried to initiate a number of comprehensive reports in various mathematical disciplines, this usage came to an end in the 20's. Later, the role of the DMV reports was taken up by the publications in the series *Ergebnisse der Mathematik und ihrer Grenzgebiete* of Springer-Verlag, edited by the editorial board of *Zentralblatt für Mathematik*.

[9] There are some corrections [46] which, however, had not been included into that book.

> name of the readers of the *Annalen* and of the number theoretical science, to whom you will render an important service. Just now I have formulated a letter to Takagi but I am confident that he will agree...

Note that the date of this letter is late in 1926, hence *after* the appearance of Part I in the *Jahresbericht der DMV*. Hasse answered immediately, proposing to Hilbert several versions of his article. Thereupon Hilbert sent a second letter, dated Nov 5, 1926:

> *Ich bin gar nicht im Zweifel, daß wir Ihren ersten Vorschlag annehmen sollten und eine unbedingt vollständige Wiedergabe der Takagischen Theorie in Ihrer Ausführung und Korrektur in den Annalen bringen müssen; ich möchte Sie sogar bitten, nicht etwa auf Kosten der leichten Lesbarkeit und Verständlichkeit Textkürzungen vorzunehmen; es kann in diesem Fall auf einige Druckbogen mehr nicht ankommen. Ich möchte eine solche Darstellung wünschen, dass der Leser nicht noch andere Abhandlungen von Ihnen, Takagi oder anderen hinzuzuziehen braucht, sondern – wenn er etwa mit den Kenntnissen meines Berichts ausgestattet ist – Ihre Abhandlung verhältnismäßig leicht verstehen und auch die Grundgedanken sich ohne große Mühe aneignen kann. Ich bin ... überzeugt, dass das so entstehende Heft (bez. Doppelheft) den Annalen zur Zierde gereichen wird...*
>
> I have no doubts that we should accept your first proposal and have to publish in the *Annalen* a fully complete presentation of Takagi's theory, in your treatment and correction; in fact I would like to ask you not to shorten the paper on the expense of easy reading and understanding; in this case some more print sheets do not matter at all. I would prefer a presentation such that the reader does not have to consult other papers by yourself, by Takagi or by others but – if he is familiar with what is in my report – would be able to understand your article easily, and also become acquainted with the basic ideas without much trouble. I am confident that this fascicle (or double fascicle) will become a beautiful gem for the *Annalen*.

From these words we not only infer the high esteem in which Hilbert held the work of Hasse and his ability for presenting a good exposition. [10] We also see that Hasse was contemplating, at that time, to publish the full proofs for Takagi's theory in the *Mathematische Annalen*. instead, publishing his report (with proofs) in the *Jahresbericht der DMV*. [11]

[10] One of the biographers of Hasse says that "*...his books confirm Hasse's reputation as a writer who could be counted on to present the most difficult subjects in great clarity...*" [28]. We learn from Hilbert's letter that Hasse had that reputation already when he was young (and had not yet written any book at all).

[11] *Added in proof*: Recently we have found in the Hilbert legacy in Göttingen two letters

We have said above already that Hasse's report had a great impact on the further development of class field theory. As a consequence of this report, class field theory had become freely and easily accessible, as Hilbert had wished it to become, in a way which did not assume any further knowledge beyond what was generally known from Hilbert's *Zahlbericht*. Indeed, Hasse in his preface to [37] explicitly states that no essential prerequisites except chapters I-VII of Hilbert's *Zahlbericht* will be assumed. Alternatively, he said, the first six chapters of Hecke's book on algebraic numbers [60] (which had just appeared) would be sufficient. [12]

This triggered an enormous rise of interest in the subject, in particular in view of Artin's and Furtwängler's progress beyond Takagi, as mentioned earlier already. Both Artin's reciprocity law and Furtwängler's principal ideal theorem were included in Part II of Hasse's report.

It is remarkable, however, that *local class field theory* does not yet properly appear in Hasse's report. There is only a brief note in Part II §7 (which is concerned with the norm residue symbol) to the effect that the result derived there can be regarded as establishing the main theorems of local class field theory ("*Klassenkörpertheorie im Kleinen*"). In this connection Hasse cites his own paper [40] and the related one of F.K. Schmidt [99] which had just appeared in Crelle's Journal (1930). In those papers, local class field theory is derived from the global, contrary to what we are used to today. Hasse remarks, in the same context, that it would be highly desirable to have it the other way round, i.e., first to establish local class field theory and then, by some Local-Global-Principle, to switch to the global case. He informs the reader that, as a first step, F.K. Schmidt in a colloquium lecture at Halle [13] had developed local class

which throw new light upon Hilbert's plans to have Takagi's class field theory represented in the *Mathematische Annalen*.

One of those letters is from Takagi to Hilbert, dated already on Aug 20, 1926 in which he agrees to a proposal of Hilbert to reprint Takagi's articles [110], [112] in the *Annalen* (with some corrections, however). From this we infer that Hilbert originally had planned such reprint only.

The other letter is from Hasse to Hilbert, dated Nov 3, 1926, containing his proposals for several versions of an article on Takagi's class field theory, to be published in the *Annalen*. Hasse's letter is the one which we have mentioned in the text already but which we did not see it explicitly until very recently in the Hilbert legacy. The letter shows that, indeed, in the meantime Hilbert had changed his plan and now favored a new and systematic presentation of Takagi's theory, not just a reprint of Takagi's paper. But in this letter we can also read that Hasse, referring to Part I of his report which had already appeared in print, did not intend to prepare the proposed article for the *Annalen* himself. Instead, he informed Hilbert that he had already written for his personal use a complete ("*lückenlos*") manuscript on Takagi's class field theory, and that he was willing to hand over that manuscript to Bessel-Hagen who then should prepare it for publication. (Bessel-Hagen was Hilbert's assistant at that time.) We do not know the reason why that project was never realized. Perhaps Bessel-Hagen was not the most suited person who could complete such project as quickly as appeared necessary in view of the tremendously fast development of class field theory in the years after.

[12] However, in Part II, sections II and IV there are some arguments which belong to Hensel's theory of local fields – and these were mentioned neither in Hilbert's *Zahlbericht*, nor in Hecke's book, nor in Hasse's preface.

[13] In the spring of 1925, Hasse had moved from Kiel to the University of Halle where he had been offered a full professorship. From the correspondence between F.K. Schmidt and

field theory *ab ovo*, i.e., without the help of global class field theory. And he continues:

> *Von hier aus, durch Zusammenfassung der auf die einzelnen Primstellen bezüglichen Sätze der Klassenkörpertheorie im Kleinen zu den auf alle Primstellen gleichzeitig bezüglichen Sätze der Klassenkörpertheorie im Großen, verspreche ich mir eine erhebliche gedankliche und vielleicht auch sachliche Vereinfachung der Beweise der Klassenkörpertheorie im Großen, die ja in ihrem bisherigen Zustande wenig geeignet sind, das Studium dieser in ihren Resultaten so glatten Theorie verlockend erscheinen zu lassen.*
>
> Starting from here, combining the theorems of local class field theory referring to the individual primes, in order to obtain the theorems of global class field theory which refer to all primes simultaneously, I hope to get an essential simplification, conceptual and perhaps also factual, of the proofs of global class field theory; in their present state these proofs are not particularly inviting to study this theory which is so elegant in its results.

From this we see clearly *why* local class field theory was not included in Hasse's report: because it did not yet exist. It seems that during the process of writing those parts of his report Hasse became conscious of the fact that, indeed, what he was doing could be regarded as local class field theory. And immediately he developed the idea that class field theory could be better understood if it would first be developed locally, and then globally by somehow combining all the local theories.

2.2.4 Further development

It did not take long until these ideas could be realized. As we see from the *Zeittafel*, already in 1931 there appeared Hasse's paper where he determines the structure of the Brauer group over local number fields. Although in that paper class field theory is not explicitly mentioned, it is clear from the context (and it was certainly clear to Hasse) that the results obtained on local algebras can be translated to yield local class field theory. Explicitly this is carried out in the papers by Hasse [44] and Chevalley [14], [15]. [14]

Hasse we can infer that the colloquium lecture in question had been held in the first week of February, 1930.

[14] F.K. Schmidt's foundation of local class field theory "*ab ovo*" as announced in Hasse's report has never been published. In a letter to Hasse dated Dec 27, 1929 F.K. Schmidt asserts that he is able to handle *tame* abelian extensions – and he realizes that wild extensions will present more difficulties. In a second letter of January 21, 1930 he confirms that he intends to talk about this subject in the colloquium at Halle, and that meanwhile he has some more results (*"ich habe mir einiges weitere überlegt"*). This does not sound as if he had obtained the full solution. In all of the following correspondence – and there are many letters – he never returns to this problem. Perhaps F.K. Schmidt, in his colloquium lecture at Halle, was quite optimistic that he could solve the problems with wild extensions but later he found that the difficulties were larger than he had expected.

The global theory then follows through the local-global principle for algebras, proved jointly by Brauer, Hasse and Noether in 1932 [10]; see also Hasse's systematic treatment [44] one year later.

As Hasse says in the introduction to [44], it was a suggestion of Emmy Noether which had led him to introduce the theory of non-commutative algebras into commutative class field theory. Due to Emmy Noether, algebras can be represented as crossed products which are given by so-called factor sets; today we would call them 2-cocycles which represent cohomology classes of dimension 2. Hence [44] can be regarded as the first instance where cohomology was introduced and used in class field theory. In the course of time it was discovered that the formalism of general cohomology theory was well suited to serve the needs of class field theory, and that the reference to algebras was of secondary importance and could be dropped after all. But this came later, after the period (1925-1940) which we are discussing here.

In the academic year 1932/33, when Hasse was already in Marburg, he had the opportunity to deliver again a course on class field theory, as he had done nine years ago in Kiel. But now the methods employed were quite different from those in earlier times, reflecting the state of the art at the time (but without explicit use of algebras). There were notes taken from these lectures, which were widely circulated and for a long time constituted a valuable source for many mathematicians who wanted to become acquainted with class field theory without cohomology. [15]

In these lectures Hasse still had to use complex analysis, namely in order to compute the norm residue index for cyclic extensions. More precisely, analytic properties of certain L-series were used in the proof of one of the two fundamental inequalities for the norm residue index. According to the trend of that time, the use of analytical tools in order to prove theorems of class field theory was not considered to be quite adequate. Since the main theorems of class field theory had become statements about algebraic structures, e.g., the reciprocity law as an isomorphism statement, it was desired to have a proof which would open more insight into the structures involved. The analytic methods of that time did not do this. Based on Hasse's methods, Chevalley and Nehrkorn [16] were able to go a long way towards this goal. Finally, the seminal paper by Chevalley [18] in which the proofs were given in the setting of idèles, marked a cornerstone in the development of class field theory. Today it is generally accepted that the framework of idèles is most appropriate for questions concerned with class field theory. [16]

[15] In 1933, an English translation of the Marburg lectures was planned. It seems that Mordell was interested in such a translation, probably because class field theory had been used in Hasse's first proof (1933) of the Riemann hypothesis for elliptic curves, and therefore Mordell wanted class field theory to become better known in England. In a letter to Mordell dated Nov 1, 1933, Hasse suggested that on the occasion of such a translation certain improvements should be carried out, the most important one being the inclusion of the theory of the norm residue symbol and the power residue symbol, which Hasse had covered in the lecture but which were not included in the notes. (This letter is found in the archive of St. John's College, Cambridge.) However, the translation plan had to be given up in 1934.

[16] A description and assessment of Chevalley's work on class field theory is given by

Later in the sixties, the interest in Hasse's Marburg lectures rose again, and therefore the old lecture notes were printed and published in book form.

The foregoing comments refer to class field theory for number fields. They are meant to provide a background for the following discussion of the development of class field theory for function fields. That story begins in 1925 at the Danzig meeting of the DMV.

3 Arithmetic foundation

3.1 The conference program

As mentioned in the foregoing section already, in the year 1925 the DMV (German Mathematical Society) held its annual meeting at the town of Danzig. The meeting lasted from the 11th to 17th of September. In the program we find the following entry for the session on Tuesday, September 15 afternoon: [26]

> Dienstag, den 15. September, nachmittags 4,00 Uhr
> Vorsitz: *Hensel.*
>
> 1. H.Hasse, Halle a.S.: *Neuere Fortschritte in der Theorie der Klassenkörper.* (Referat, 60 Minuten)
> 2. Friedrich Karl Schmidt, Freiburg i.B.: *Zur Körpertheorie.* (20 Minuten.)
> 3. E. Noether, Göttingen: *Gruppencharaktere und Idealtheorie.* (20 Minuten.)
> 4. Karl Dörge, Köln: *Zum Hilbertschen Irreduzibilitätssatz.* (20 Minuten.)

The first entry represents Hasse's talk which we have discussed above already. At the time of the Danzig meeting Hasse was affiliated with the University of Halle, where he had just accepted a full professorship. Hasse's talk is labelled *Referat* (report) which means that it was an invited lecture. The time allocated for it was 60 minutes, more than for the following talks. [17] Immediately after Hasse's report we see the announcement of a talk by F.K. Schmidt. [18]

S. Iyanaga in [73]. By the way, Iyanaga reports that the terminology of "idèle" is due to a suggestion of Hasse. Chevalley originally used two words: "éléments idéal".

[17] In the final report about the meeting [26] it is said that the session started already at 3:25 p.m. instead of 4 p.m. as originally planned. It is conceivable that Hasse had asked for more time for his report which, after all, was a formidable task since it was to cover the whole of Takagi's class field theory.

[18] In Germany the name "Schmidt" is quite common. There are several known mathematicians with this name. In order to identify them it is common to use their first names, or first name initials. We shall follow this habit here too; this is the reason why we always use the initials when mentioning F.K. Schmidt, whereas with other mathematicians the initials are not used in general.

F.K. Schmidt was 24 at the time of the Danzig meeting [19], hence three years younger than Hasse. He had just completed his *Doktorexamen* (Ph.D.) at the University of Freiburg. His formal advisor had been Alfred Loewy but in fact he had been guided in his work by Wolfgang Krull who at the time was assistent to Loewy in Freiburg. [20] It appears that the Danzig meeting was the first mathematical congress which the young F.K. Schmidt attended.

The title of his talk "On field theory" is not very informative. In the *Jahresbericht der DMV* [26] we find an abstract which says that arbitrary algebraic function fields F of one variable will be considered, over a base field K which is absolutely algebraic of prime characteristic p. Given a transcendental element $x \in F$, it is announced that the speaker will present the ideal theory, the theory of units and the theory of the discriminant for the ring R_x of x-integral elements in F. The abstract ends with the words:

> *Eine erweiterte Fassung des Vortrags erscheint in diesem Jahresbericht.*
>
> An extended version of the talk will appear in this journal.

However, this "extended version" never appeared, neither in the *Jahresbericht der DMV* nor elsewhere. Hence, in order to find out more about the content of F.K. Schmidt's talk we have to consult his thesis, for it seems likely that he talked about the results which he had recently obtained there.

3.2 F.K. Schmidt's thesis

3.2.1 Arithmetic in subrings of function fields

Again, this thesis has never been published. But the University of Freiburg still keeps the original and I could obtain a copy of it. [21] The thesis is written in clear, legible handwriting and contains essentially the following results. (The notation as well as the terminology is ours, not F.K. Schmidt's.) As already introduced above, R_x denotes the ring of elements in F which are integral over $K[x]$. The base field K is assumed to be finite. [22]

- R_x is a Dedekind ring. [23]
- The discriminant of R_x over $K[x]$ contains precisely those primes of $K[x]$ which are ramified in R_x.

[19] More precisely: Five days after the meeting he had his 24th birthday.

[20] Biographical information about Loewy may be found in the article by Volker Remmert [88]; about Krull in the obituary by H. Schöneborn [103] – or in the "Collected Works" of Krull which are in preparation and are to appear in the de Gruyter Verlag.

[21] I am indebted to Volker Remmert for his help in this matter.

[22] Some of the following results remain true and accordingly are proved under the more general assumption that K is absolutely algebraic of prime characteristic p.

[23] F.K. Schmidt does not use this term which is common today. He speaks of "*Multiplikationsring*" (multiplication ring). This name should indicate that the non-zero fractional ideals form a group with respect to ordinary ideal multiplication.

– The ideal class group of R_x is finite.

– The unit group $R_x^\times$ is finitely generated, and the number of generators modulo torsion is one less than the number of infinite primes of F with respect to x.

– In $K[x]$ there holds an n-th power reciprocity law under the assumption that the n-th roots of unity are contained in K (in analogy to Kummer's reciprocity law in the n-th cyclotomic number field if n is prime).

For the moment, let us disregard the last item which we shall discuss later. The other items belong today to the basic prerequisites for every student who wishes to study algebraic function fields. In the mid-twenties, however, it seems that these things were not general knowledge, at least there was no standard reference. Hence it was a good problem for a young Ph.D. student to develop this theory *ab ovo*, i.e. from scratch. F.K. Schmidt solved the problem by standard methods which were well known and used by that time, referring to the analogy with Dedekind's foundation of the theory of algebraic numbers. [24] The title of F.K. Schmidt's thesis reads:

> *Allgemeine Körper im Gebiet der höheren Kongruenzen*
> (Arbitrary fields in the domain of higher congruences)

This is a rather queer title, and the notion of "*Gebiet der höheren Kongruenzen*" does not appear in the proper text of the thesis. But after reading the introduction it is clear *why* this title had been chosen. Namely, the author wished to refer to Artin's thesis [5] (which had been completed 1921 but appeared 1924 only), and which carried the title:

> *Quadratische Körper im Gebiet der höheren Kongruenzen*
> (Quadratic fields in the domain of higher congruences)

Artin had considered quadratic extensions F of the rational function field $K(x)$ (with $K = \mathbb{F}_p$). [25] Through the choice of the title F.K. Schmidt wished to signalize that he is generalizing Artin's work by considering not only quadratic but *arbitrary* field extensions F of $K(x)$ of finite degree (with K algebraic over $\mathbb{F}_p$). Artin in turn had chosen his title in order to refer to Dedekind's classical paper of 1857 whose title read:

> *Abriss einer Theorie der höheren Congruenzen in Bezug auf einen reellen Primzahl-Modulus*
>
> (Outline of a theory of higher congruences with respect to a real prime number module)

[24] He could not refer to E. Noether's axiomatic characterization of Dedekind rings because her paper [86] appeared in 1927 only.

[25] Hence Artin's thesis covered hyperelliptic function fields.

There Dedekind discusses the number theory of the polynomial ring $\mathbb{F}_p[x]$ in analogy to the ordinary ring of integers $\mathbb{Z}$.

So we see that F.K. Schmidt's thesis had been written with the aim of establishing the fundamental facts of the arithmetic in function fields over finite base fields – in analogy to the arithmetic of algebraic number fields, and in generalization of Artin's thesis, in reference to ideas going back to Dedekind.

This gives us an explanation why F.K. Schmidt's thesis has never been published. For, the same results had appeared about the same time in another paper [104] by the author Sengenhorst. As F.K. Schmidt explains in [98], at the time of completing his thesis he did not know about Sengenhorst's paper which already contained his results. Likewise, he did not know about the work of Rauter, a Ph.D. student of Hasse, who also at the same time (and also without knowledge of Sengenhorst) came to the same conclusions [87]. It seems that in those days the need for a solid foundation of the arithmetic of function fields was felt widespread, so that there were three dissertations, almost at the same time, dealing with the same subject. [26]

Both Sengenhorst and Rauter became gymnasium teachers, the first one in Berlin and the other in the town of Tilsit in East Prussia. They did not remain active in mathematical research. But F.K. Schmidt did; he realized that the results in his thesis could only be the beginning, and that the next aim should be to establish Takagi's class field theory in the function field case. And he started to work in that direction.

3.2.2 The n-th power reciprocity law

Actually, in F.K. Schmidt's thesis there is one chapter which already has some bearing on class field theory, namely the chapter on the n-th power reciprocity law in the rational function field $K(x)$. (We had mentioned this above already.)

Let n be an integer not divisible by the characteristic p of K. Suppose that K contains the n-th roots of unity, i.e., that n divides $q-1$ where q is the order of K. Then, for any two elements $a, b \in K[x]$ which are relatively prime, the n-th power residue symbol $\left(\frac{a}{b}\right)_n$ can be defined in complete analogy to the number field case (i.e., when a, b are integers in a number field containing the n-th roots of unity). Suppose that a, b, when considered as polynomials in x, are monic of degree r and s respectively. Then the power reciprocity law according to F.K. Schmidt reads as follows:

$$\left(\frac{a}{b}\right)_n = (-1)^{rs\frac{q-1}{n}} \left(\frac{b}{a}\right)_n . \tag{1}$$

In particular, if n is odd then the inversion factor is trivial and we obtain

$$\left(\frac{a}{b}\right)_n = \left(\frac{b}{a}\right)_n .$$

[26] Rauter, in addition, dealt also with the Hilbert ramification theory for Galois extensions of function fields.

The case $n = 2$ had been treated by Artin in his thesis [5]. But already Dedekind in 1857 [21] had written down the quadratic reciprocity law in $K[x]$ (for $K = \mathbb{Z}/p$) with the comment: "*Der Beweis kann ganz analog dem fünften Gaußschen Beweis für den Satz von Legendre geführt werden...*" (The proof can be done in complete analogy to the fifth proof of Gauss for Legendre's theorem...) – thereby Dedekind assumed that the reader is familiar with the various proofs of Gauss and their numbering.

F.K. Schmidt pointed out in his thesis that the proof of his n-th power reciprocity law in $K[x]$ is elementary, in contrast to Kummer's proof in the number field case over the field of n-th roots of unity. And one year later in [98] he presented a formula which made this law a triviality. Namely, if $a = \prod_{1 \leq i \leq r}(x - \alpha_i)$ is the decomposition into linear factors of the polynomial a, and similarly $b = \prod_{1 \leq j \leq s}(x - \beta_j)$ then he observed that

$$\left(\frac{a}{b}\right)_n = \prod_{\substack{1 \leq i \leq r \\ 1 \leq j \leq s}} (\alpha_i - \beta_j)^{\frac{q-1}{n}} \tag{2}$$

which is putting (1) into evidence.

Later in 1934, Hasse [52] said that (1) is a "well known reciprocity formula" (*eine bekannte Reziprozitätsformel*). Although he referred to F.K. Schmidt [98], Hasse did not specify whether he considered this formula to be known *because* of F.K. Schmidt's paper, or it had been "well known" before already. In any case, Hasse in his paper showed that this reciprocity formula finds its interpretation within the theory of cyclic class fields over a rational function field $F = K(x)$ as ground field. One year later in 1935, Hasse's student H.L. Schmid then generalized this to an arbitrary function field F as ground field [93]. See section 7.1.

3.3 Further remarks

As we have pointed out, it seems that during the Danzig meeting F.K. Schmidt became aware that most of his results in his thesis had been obtained elsewhere already. On the other hand, he did realize that there was interesting and important work ahead in the form of a project to transfer class field theory from number fields to function fields. His result about the n-th power reciprocity law in rational function fields could be regarded as a beginning in this direction, however small.

He seems to have been stimulated by Hasse's Danzig lecture which, as we have seen, had been delivered just before his own talk, and he surely had attended Hasse's. Perhaps F.K. Schmidt had not known Takagi's class field theory before, and he became interested in it through Hasse's lecture. An indication for this is the fact that the notion of "class field" does not appear in his 1925 thesis – but in his 1926 paper [98] already he refers to Takagi's theory of class fields as his main aim in the case of function fields. Another indication of Hasse's influence is the fact that starting in the spring of 1926, F.K. Schmidt regularly wrote to Hasse and informed him about his progress. Hasse seemed to have not only stimulated F.K. Schmidt's further work but he was continuously interested in its progress. [27]

The proofs of Takagi's main statements on class field theory depended, at that time, heavily on analytic methods; more precisely: on the properties of the Dedekind ζ-function and the L-functions of the base field. Therefore, in order to transfer Takagi's theory to the function field case, as a first step one would have to transfer the relevant theory of ζ-functions and L-functions. Accordingly, F.K. Schmidt started to develop just such a theory, which became his first major and widely known paper.

3.4 Summary

- *F.K. Schmidt in his thesis (Freiburg 1925) proved the basic facts about the arithmetic in function fields with finite base fields. Thereby he generalized the arithmetic part of Artin's thesis (Hamburg 1921) where hyperelliptic function fields only were considered. But F.K. Schmidt's thesis was never published because the same results had been obtained independently by other authors, about the same time.*

- *F.K. Schmidt's thesis contained one section which had some bearing on class field theory; it contained the n-th power reciprocity law for polynomials in the ring $K[x]$ if K contains the n-th roots of unity. This generalized Dedekind's reprocity law (1857) for the case $n = 2$. Artin in his thesis (1921) had also given a proof for $n = 2$. It turned out that F.K. Schmidt's proof was very simple and almost trivial; nevertheless ten years later it was recognized, after suitable generalization, as an important ingredient of general class field theory.*

- *At the DMV meeting in Danzig (1925) F.K. Schmidt met Hasse and attended his great lecture which reported about Takagi's class field theory. Stimulated by this experience he decided to direct his further work towards establishing class field theory for function fields.*

4 Analytic foundation

4.1 F.K. Schmidt's letters to Hasse 1926

The first letter from F.K. Schmidt to Hasse which is preserved in the collection of the Hasse papers at Göttingen, is dated May 6, 1926. F.K. Schmidt wrote:

> *. . . Was die Grenzformel für die ζ-Funktion in Körpern von der Charakteristik p angeht, so ist mir in meiner Dissertation die Übertragung auf den ersten Anhieb nicht gelungen. . .*

[27] Unfortunately, only one side of their correspondence is preserved, namely the letters from F.K. Schmidt to Hasse; they are to be found among the Hasse papers in the Göttingen library. The letters from Hasse to F.K. Schmidt seem to be lost.

>... Concerning the limit formula for the ζ-function in fields of characteristic p, on the first try I did not succeed in my dissertation to transfer it...

By "limit formula" he means a formula for the residue of $\zeta(s)$ at the point $s = 1$. And "transfer" means the transfer from the number field case to the function field case. It seems that Hasse had asked him whether he had obtained the result already, which as we see was not the case. But soon, on August 8, 1926 F.K. Schmidt could announce success:

> *Es ist mir bei erneuter Betrachtung ziemlich bald möglich gewesen, die bekannten Dedekindschen Resultate in vollem Umfang auf Körper der Charakteristik p auszudehnen...*
>
> After taking up the subject anew, I fairly soon succeeded in transferring completely Dedekind's well known results to fields of characteristic p...

And then he continues to report to Hasse about his definition of the zeta function and the limit formula. Given a function field $F|K$ with finite base field K, the zeta function $\zeta(s)$ in his definition depends on the choice of a transcendental $x \in F$, and it refers in the well known manner to the prime ideals of the ring R_x of x-integers in F. The limit formula for this function, according to F.K. Schmidt, reads as follows:

$$\lim_{s\to 1}(s-1)\zeta(s) = \frac{(q-1)^{n-1}\,\mathcal{R}}{\sqrt{|\mathcal{D}|}\,\log q}\cdot h \qquad (3)$$

where $q = |K|$ is the order of the base field, $n = [F : K(x)]$ the field degree, $\mathcal{D}$ is the discriminant of R_x over $K[x]$ with absolute norm $|\mathcal{D}| = q^{\deg \mathcal{D}}$, and $\mathcal{R}$ is the "regulator" which F.K. Schmidt had some difficulty to define but finally succeeded, replacing the logarithms (which appear in the number field case) by the valuation degrees of the units at the infinite primes (the poles of x). The essential term on the right hand side is h, the number of ideal classes of R_x. The above formula holds only in the case when all infinite primes are of degree 1, which is the analogue to totally real fields in the number field case. In his letter, F.K. Schmidt had restricted himself to this "totally real" case for reasons of brevity only. At the end of the letter F.K. Schmidt writes:

> *Auf Veranlassung von Herrn Prof. Haupt soll demnächst in den Erlanger Berichten eine vorläufige Mitteilung meiner Resultate und Methoden erscheinen; da der Druck dort sehr schnell geht...*
>
> On the suggestion of Prof. Haupt, a preliminary announcement about my results and methods is to appear shortly in the Erlangen reports; since there the printing will be very fast...

The reference to Erlangen shows that F.K. Schmidt had changed his place of activity from Freiburg to Erlangen, where he had accepted a position of assistant to Professor Otto Haupt. The latter, although his primary interests were in real analysis and geometry, was also keenly interested in the modern developments of algebra and number theory. Haupt kept contact with Emmy Noether who whenever she visited her home town Erlangen, was heartily welcomed in the Haupt residence. [28]

From the remarks in F.K. Schmidt's letter we infer that Haupt was impressed by F.K. Schmidt's work and therefore wished to secure priority for him in publication, in particular in view of F.K. Schmidt's earlier experiences with his thesis. The *Erlanger Berichte* could quickly publish but otherwise this journal was not so well known, devoted not only to mathematics but also to science at large, and not available in many universities' mathematics libraries.

In his above cited letter F.K. Schmidt did not mention class field theory but in his next letter to Hasse, dated December 6, 1926 he does. Obviously replying to a question of Hasse, he writes that he did not plan a general axiomatic foundation of class field theory but he believes this could be done – similar to E. Noether's axiomatic characterization of rings which admit classical ideal theory. [29] However, he continues, there may arise difficulties concerning the existence theorem of class field theory in the case when the class number is divisible by the characteristic p. Then he offers to send a brief summary of his results on class field theory in characteristic p – but he does not mention any details in the letter; for those we are dependent on F.K. Schmidt's publications.

4.2 The preliminary announcement

F.K. Schmidt's *Vorläufige Mitteilung* [98] is signed by the author with the date "August 1926", soon after his above mentioned letter in which he announced this paper to Hasse. It appeared in November that year with the title:

> *Zur Zahlentheorie in Körpern der Charakteristik* p.
> *(Vorläufige Mitteilung.)*
>
> On number theory in fields of characteristic p.
> (Preliminary announcement.)

In the introduction he refers to his thesis and acknowledges that both Sengenhorst and Rauter had obtained identical results. But now, he says, he is going to start with the transfer of the *analytic theory*. In the quadratic case (i.e., quadratic extensions of rational function fields) the analytic theory had been

[28] Inspired by the discussions with Emmy Noether, Otto Haupt wrote a textbook on the then "modern" algebra [58], which appeared in 1929 and was the first such textbook, before van der Waerden's appeared. Haupt's book covered more material than van der Waerden's; the fact that the latter became more widely known than the former seems to be due to the style of writing.

[29] He is referring to the paper [86] of Emmy Noether of which he seems to know the content already, and he also assumes that Hasse knows it although the paper had not yet appeared in print (it appeared in 1927). Note that both Hasse and F.K. Schmidt had met E. Noether one year earlier in Danzig, as is evident from the program excerpt which we gave in section 3.1. – An axiomatic treatment of class field theory was given much later, in the early fifties, in the seminal lecture notes by Artin-Tate [9].

covered in the second part of Artin's thesis. Now he (F.K. Schmidt) would generalize also the second part of Artin's thesis to the case of arbitrary function fields with finite base field (which had not been done neither by Sengenhorst nor by Rauter). And he continues:

> *Die hier angeführten Ergebnisse eröffnen u.a. die Möglichkeit, die Takagische Theorie der Klassenkörper und der höheren Reziprozitätsgesetze [auf Funktionenkörper] zu übertragen, worauf ich demnächst einzugehen gedenke.*
>
> The results given here open up, e.g., the possibility of transferring Takagi's class field theory and higher reciprocity laws [to the case of function fields]; I intend to discuss this soon.

Thus F.K. Schmidt announced publicly that he was aiming at class field theory in characteristic p. From his correspondence with Hasse as discussed above we may infer that in December 1926 he was already in the possession of the main class field theorems, at least in a first and maybe incomplete version. We shall return to this in section 5. Here we wish to discuss what seems to be the most important part of the preliminary announcement:

Namely, at the end of the paper we find a *Zusatz bei der Korrektur* (Added in proof), dated "October 1926". There F.K. Schmidt says that further considerations have led him to change his viewpoint, as follows.

Up to now, when transferring arithmetic or analytic notions to the case of a function field $F|K$, the theory had been developed with respect to a given transcendental element $x \in F$. The ring R_x of x-integral functions had been regarded as the analogue of the ring of algebraic integers in a number field, and all the notions and theorems for function fields had referred to the structure of R_x and its prime ideals. The same point of view had been taken also by the other authors, i.e., Artin, Sengenhorst and Rauter. But from the new viewpoint, F.K. Schmidt says, no transcendental element in F is distinguished. Today we would say that his new viewpoint was "birationally invariant" but F.K. Schmidt did not use this expression. Instead, he refers to the classical theory of complex algebraic functions, where the "birationally invariant" point of view means that one works with an abstract Riemann surface, independent of its representation as a covering of the complex plane. Let us cite F.K. Schmidt himself [98]:

> *. . . Wir nehmen also jetzt den Standpunkt ein, der in der Theorie der algebraischen Funktionen zuerst bei Dedekind und Weber zu finden ist. Diese beiden Autoren haben bekanntlich für die von ihnen behandelten Körper algebraischer Funktionen eine arithmetische Definition des Punktbegriffes gegeben, der von jeder Bezugnahme auf eine Variable frei ist. . .*
>
> . . . Thus we now take the same viewpoint which in the theory of algebraic functions had been taken the first time by Dedekind and Weber. As is well known, those two authors had given, for the fields of algebraic functions as considered by them, an arithmetic

> definition of the notion of point, which is free from any reference to a variable...

He is referring to the classical paper by Dedekind and Weber [22] on the algebraic theory of function fields over the base field $\mathbb{C}$ (the complex number field). [30] The "arithmetic definition" he alludes to, is today's usual definition: [31] a point is given by a "place" of the function field or, equivalently, by a valuation which is trivial on the base field. Here again, we can verify the enormous conceptual influence which the paper by Dedekind and Weber has exerted in the course of time.

The remarkable fact is not so much that F.K. Schmidt had adopted the viewpoint of Dedekind-Weber which to us looks quite natural, but that it was not adopted earlier, neither by himself in his thesis nor by any of the other authors: Artin, Sengenhorst and Rauter. An explanation for this may be that the theory of algebraic function fields with finite base field had first been developed in close analogy to the theory of algebraic number fields. In the latter case the ring of all algebraic integers in the field is a natural and distinguished object of study. In the first attempts to transfer number theory to function fields, one was looking for an analogue of this ring and found it in the ring R_x of all x-integral elements in the field, with respect to a given transcendental x.

However in number theory it became more and more evident that the various "infinite primes" (as we call them today), which belong to the archimedean valuations of the number field, play an important role and should be treated, as far as possible, on the same footing as the "finite primes", which belong to the non-archimedean valuations and hence to the prime ideals of the ring of integers. These ideas were adopted via the analogy between number theory and the theory of complex algebraic functions on a compact Riemann surface – an analogy which had been pointed out on many occasions. We only mention Hilbert in his famous Paris lecture [70] in the year 1900; see also the report [85] by E. Noether on this subject, published 1919.

In particular during the development of *class field theory* for number fields the need to consider those "infinite primes" was strongly felt. For in the definition of a "class group" in the sense of Weber one has to consider modules which consist of finite as well as of those infinite primes. See, e.g., Part Ia of Hasse's *Klassenkörperbericht* [38].

We may imagine that F.K. Schmidt, during his attempts to transfer class field theory, observed that for a function field $F|K$ one has to consider a similar situation: given a transcendental element $x \in F$ its poles should be treated on the same footing as the finite places for x. And then he recalled that this viewpoint had been adopted much earlier by Dedekind-Weber in the case of

[30] That paper was submitted to Crelle's Journal in 1880 but appeared in 1882 only – due to a decision of Kronecker, the editor of Crelle's Journal at the time, that he did not wish to publish the Dedekind-Weber paper before he himself had completed his own paper [76] on a related topic.

[31] See, e.g., Stichtenoth's introduction to the theory of function fields [109]. The first systematic treatment in a textbook on the basis of F.K. Schmidt's viewpoint was given by Hasse [55] in his "*Zahlentheorie*" which had been completed in 1938 but appeared in 1949 only.

complex algebraic functions. In this way it now became possible to appeal directly to the analogy with the fields of complex analytic functions on a compact Riemann surface – without the detour over the number field case. This then led F.K. Schmidt to the birationally invariant viewpoint, as announced in his "Note added in proof".

In that note he briefly outlined the basic definitions and results (but without proofs for which he referred to the forthcoming final version). Given a function field F over a finite field K with q elements, his new definition of the zeta function is as follows:

$$\zeta(s) = \prod_{\mathfrak{p}} \frac{1}{1 - |\mathfrak{p}|^{-s}} = \sum_{\mathfrak{a} \geq 0} |\mathfrak{a}|^{-s} \tag{4}$$

where the product is taken over *all* places (primes) $\mathfrak{p}$ of F ("points" in F.K. Schmidt's terminology), regardless of whether $\mathfrak{p}$ is a pole of any given transcendental or not. On the right hand side, $\mathfrak{a}$ ranges over the positive divisors of the function field and $|\mathfrak{a}| = q^{\deg \mathfrak{a}}$ denotes its absolute norm. Since every $\mathfrak{a}$ is composed uniquely from the places $\mathfrak{p}$ it follows that the Euler product equals the Dirichlet series.

Indeed, this definition of $\zeta(s)$ is birationally invariant with respect to the function field $F|K$. It is the analogue not to the classical zeta function of a number field, but to the modified zeta function which, besides of the Euler factors belonging to the finite primes, contains factors corresponding to the archimedean primes.

4.3 Riemann-Roch theorem and zeta function

4.3.1 The final version

The final version with the title *Analytische Zahlentheorie in Körpern der Charakteristik p* (Analytic number theory in fields of characteristic p) appeared in 1931 only, in the *Mathematische Zeitschrift* [100]. The manuscript was received by the editors on April 30, 1929. But it was essentially finished already in the summer of 1927 because F.K. Schmidt had used it as the first part in his *Habilitationsschrift* (thesis for his second academic degree). He did his *Habilitation* at the University of Erlangen during the summer semester of 1927. The *Habilitationsschrift* carried the title: [32]

> *Abelsche Körper im Gebiet der höheren Kongruenzen.*
> (Abelian fields in the domain of higher congruences.)

and it consisted of two parts:

I. *Analytische Zahlentheorie in Körpern der Charakteristik p.*
   (Analytic number theory in fields of characteristic p.)

[32] I am indebted to W. Schmidt (Erlangen) for providing me with a copy of F.K. Schmidt's *Habilitationsschrift*.

II. *Die Theorie der Klassenkörper über einem Körper algebraischer Funktionen in einer Unbestimmten und mit endlichem Koeffizientenbereich.* (Class field theory over a field of algebraic functions in one variable and with finite coefficient domain.)

Part II appeared in 1931 in the *Erlanger Nachrichten* [101]; see section 5. And Part I is identical with the paper in the *Mathematische Zeitschrift* which we are discussing now.

The main object of the paper is to develop the analytic properties of the zeta funtion $\zeta(s)$ of a function field $F|K$ with finite base field – in a birationally invariant manner as sketched in the "Note added in proof" of the preliminary announcement. For this purpose, the results of his thesis seemed to F.K. Schmidt not well suited as a framework because they depend on the choice of a transcendental element $x \in F$ and hence are not birationally invariant. Accordingly F.K. Schmidt developed the entire arithmetic theory of function fields anew, in a birationally invariant setting. He had discovered, firstly, that the classical theorem of Riemann-Roch [33] can be transferred to function fields with finite base field, and secondly that this Riemann-Roch theorem is intimately connected with the main analytical properties of his new zeta function. Accordingly he divided the paper into two parts: In the first part he developed the theory of divisors, and in the second part the theory of the zeta function.

4.3.2 Theory of divisors

In the first part F.K. Schmidt relies heavily on the analogy with the theory of complex algebraic functions; for the latter he refers to the paper by Dedekind and Weber already mentioned above, and also to the book by Hensel and Landsberg [62] from the year 1902.

If we compare those sources with F.K. Schmidt's paper then we discover that in the latter almost the same methods and arguments are used as in the former; we are tempted to say that F.K. Schmidt just copies his classical sources. But we should not underestimate the conceptual difficulties which F.K. Schmidt had to overcome. Today we could just say that the arguments used by Dedekind-Weber are applicable *mutatis mutandis* in the cases discussed by F.K. Schmidt, i.e., for finite base fields and, more generally, for arbitrary perfect base fields. But such general statement is accepted today only because now *it is well known how to modify* the arguments of Dedekind-Weber for the cases at hand – thanks to F.K. Schmidt. Before something is accepted to be "well known" it has to be done first.

What seems to be trivial or easy to us was by no means trivial to F.K. Schmidt at the time. Let us discuss the various steps which were to be taken in

[33] F.K. Schmidt always writes "Roche" in his paper, instead of "Roch". This could possibly lead to confusion because the (German) mathematician Roch is not identical with the (French) mathematician(s) Roche. In a postcard to Hasse dated January 4, 1934 he apologizes for his mistake of constantly appending an *"e"* to the name of Roch. And he adds somewhat jokingly: *Leider wird diese Konstante "e" neuerdings, wohl im Anschluß an mich, auch von anderer Seite geschrieben. Also wieder einmal der bekannte "Fluch der bösen Tat".*

the transfer process from Dedekind-Weber:

1. Dedekind-Weber [22] already had mentioned that their whole theory remains valid if the base field $\mathbb{C}$ is replaced by, e.g., the field of all algebraic numbers. Today we read this remark as to say that their theory is valid over an arbitrary algebraically closed field of characteristic zero. This is evident (to us) by just looking at the Dedekind-Weber paper which is of purely algebraic nature.

2. A closer look convinces us (today) that the paper remains valid in characteristic $p > 0$ *provided* the choice of transcendental elements x will be restricted to *separating* elements whenever necessary, e.g., when computing formal derivatives. And we know today that separating elements do exist if the base field K is perfect. F.K. Schmidt was the first to prove this. [34] Thereafter he is able to define the genus g of the function field $F|K$ in the same way as Dedekind-Weber:

$$g = \frac{w_x}{2} - n_x + 1 \tag{5}$$

where $x \in F$ is a separating variable, $n_x = [F : K(x)]$, and w_x is the so-called "ramification number" of F over $K(x)$ which he defines to be the degree of the Dedekind different. Note that this definition covers the case of wild ramification which can appear in characteristic p. In contrast, Hensel-Landsberg [62] used a definition which looks simpler but is applicable in case of tame ramification only. It seems that F.K. Schmidt was aware of this situation and hence took care to choose the correct definition.

Although the definition (5) is not birationally invariant *per se*, F.K. Schmidt shows that the result of the expression on the right hand side of (5) does not depend on x. Hence g is indeed well defined as a birational invariant of the field. [35]

3. A certain difficulty arises for F.K. Schmidt because the base field K is not assumed to be algebraically closed. On several occasions he has to enlarge the base field K in order to be able to follow the lead of Dedekind-Weber. Then

[34] Without, however, using the terminology "separable" or "separating". He still uses the terminology "of the first kind" (*von erster Art*) as introduced by Steinitz [107]. The term "separable" which is common today was introduced by van der Waerden in his textbook [117] whose first edition appeared in 1930.

[35] E. Witt, who had attended F.K. Schmidt's lectures on function fields in the winter semester 1933/34 at Göttingen, presented in [128] what he calls a simplification of this invariance proof. He explicitly refers to §4 of F.K. Schmidt's paper and proposes to replace that section by his (Witt's) proof. The "simplification" of Witt consists essentially of proving, in the algebraic setting including characteristic p, the well known explicit formula for the divisor of a differential, whereas F.K. Schmidt works with derivations only, not with differentials. – Independently of Witt and at the same time, Hasse [53] gave the same proof in the framework of his general theory of differentials. – F.K. Schmidt himself, in his later paper [102] which appeared in 1936, proved the Riemann-Roch theorem for arbitrary function fields whose base field need not be perfect and, hence, there may not exist separating elements. In this general situation he *defined* the genus as the constant which appears in the Riemann part of the Riemann-Roch theorem; this is a truly birationally invariant definition and is generally used today. Perhaps it is not without interest to add that this idea for the invariant definition of the genus arose directly from the correspondence of F.K. Schmidt with Hasse. In a letter to Hasse dated May 22, 1934 F.K. Schmidt outlined already the plan for his paper [102].

he has to show that those base changes do not disturb the arithmetic of the function field and are admissible for the respective problem. In doing this he relies heavily on the fact that K is perfect and, hence, algebraic extensions of the base field are separable. Today we would say that a function field over a perfect base field is *conservative*. But this notion did not exist at the time; in fact, F.K. Schmidt just proves it and uses the consequences. [36]

4. After those preparations F.K. Schmidt is now ready for the proof of the Riemann-Roch theorem. Let C be a divisor class of $F|K$ and $C' = W - C$ its dual class [37], where W denotes the differential class of $F|K$. Then the Riemann-Roch theorem says that

$$\dim C = \deg C - g + 1 + \dim(C') \tag{6}$$

or, in symmetric form:

$$\dim C - \frac{1}{2}\deg C = \dim C' - \frac{1}{2}\deg C' . \tag{7}$$

But F.K. Schmidt does not present the proof explicitly. He assumes the reader to be familiar with the book of Hensel-Landsberg and is content with saying that the Riemann-Roch theorem is important and can now be proved quite as in Hensel-Landsberg ("*... der wichtige Satz, der sich nunmehr ganz wie bei Hensel-Landsberg (S.301–304) beweisen läßt...*").

In fact, this is correct: When checking the cited pages of Hensel-Landsberg the reader will find that all the notions and facts which are used in the proof there, had been transferred by F.K. Schmidt to the more general case of a perfect base field. The main fact is the construction of so-called "normal bases" which permit the explicit determination of the dimension of a divisor.

To repeat: Concerning the Riemann-Roch theorem, the transition from the base field $\mathbb{C}$ (Dedekind-Weber) to an arbitrary perfect base field and hence to finite base fields (F.K. Schmidt) is familiar to us and easily performed – but only thanks to F.K. Schmidt who did it first in his 1931 paper. His procedure was adequate, elegant and paved the way for further development. With this paper, F.K. Schmidt opened the general arithmetic theory of algebraic function fields.

4.3.3 Theory of the zeta function

But most important of all is his discovery that the Riemann-Roch theorem, in case the base field K is finite, is intimately connected with the properties of the zeta function $\zeta(s)$. In fact, in the second part of the paper he gives as almost immediate consequences of the Riemann-Roch theorem the following fundamental results.

[36] F.K. Schmidt's proof of the Riemann-Roch theorem in [100] holds for any conservative function field, even if the base field is not perfect.

[37] "*Ergänzungsklasse*" in the terminology of F.K. Schmidt.

Let $q = |K|$ denote the order of the base field. The zeta function $\zeta(s)$ of F is defined by the expansions (4) which converge if the real part of the complex variable s is $\Re(s) > 1$.

(i) *$\zeta(s)$ is a rational function of the variable $t = q^{-s}$. In particular it follows that $\zeta(s)$ is analytically extendable to the whole complex plane as a periodic function with period $\frac{2\pi i}{\log q}$. It admits essentially only two poles, of order 1, at the points $s = 1$ and $s = 0$ (and at those points which differ from these by an integral multiple of the period).*

(ii) *The residue of $\zeta(s)$ at $s = 1$ is*

$$\lim_{s\to 1}(s-1)\zeta(s) = \frac{q^{g-1}}{(q-1)\log q}\cdot h \tag{8}$$

where h is the class number of the field F, i.e., the number of the divisor classes of degree 0, and where g is the genus of F.

(iii) *$\zeta(s)$ satisfies the functional equation*

$$\zeta(1-s) = q^{(g-1)(2s-1)}\zeta(s) \tag{9}$$

or, in symmetric form:

$$q^{(g-1)(s-\frac{1}{2})}\zeta(s) = q^{(g-1)(-s+\frac{1}{2})}\zeta(1-s) \tag{10}$$

Let us add some comments:

Ad (i) In the context of the proof of (i) F.K. Schmidt discovered the important fact that every function field with finite base field admits a divisor of degree 1. This is today known as "F.K. Schmidt's theorem". It is remarkable that the proof of this algebraic statement was discovered and proved by analytic means. Later Witt provided an algebraic proof; we shall discuss this in section 8.1.

Ad (ii) The limit formula (8) looks somewhat different from the original limit formula (3). The reason is that now F.K. Schmidt uses a different zeta function, i.e., the birationally invariant one. In a separate section of his paper he explains the relation of the new zeta function $\zeta(s)$ with the former zeta function which now should be denoted by $\zeta_x(s)$ since it refers to a given transcendental element $x \in F$. [38] This is easily explained: $\zeta_x(s)$ is obtained from $\zeta(s)$ by multiplying with the finite product $\prod_{\mathfrak{q}}(1-|\mathfrak{q}|^{-s})$ where $\mathfrak{q}$ ranges over the poles of x. Hence the earlier limit formula (3) can be deduced from (8).

Ad (iii) In fact, the Riemann-Roch theorem is *equivalent* to the functional equation of $\zeta(s)$. To put this into evidence we may perhaps borrow from an idea of Witt which consists of rewriting the definition (4) of $\zeta(s)$ in such a way that the functional equation becomes obvious. [39]

[38] Note that, as said above already, our notation differs from F.K. Schmidt's.

[39] This idea has been recorded by Hasse in his survey [54]. – Witt used this idea in the paper [126] where he proved the functional equation for the zeta function of a simple algebra. We shall discuss that paper in section 6.2.

We use the variable $u = q^{\frac{1}{2}-s}$; then the transformation $s \mapsto 1-s$ of the functional equation appears as $u \mapsto u^{-1}$. Witt introduces the formal relation

$$\sum_{-\infty<n<\infty} u^n = 0 \tag{11}$$

which is interpreted in the following way: breaking up this relation at any index n into two partial sums, and adding the two rational functions in u which arise that way, this will always yield zero.

Now in the expansion on the right hand side of (4) we combine those divisors $\mathfrak{a}$ which belong to the same divisor class C; they have the same degree $\deg C$ and their number is $\frac{q^{\dim C}-1}{q-1}$. A straightforward manipulation, adding a suitable multiple of (11) to the right hand side of (4) gives the following expansion:

$$q^{(g-1)(s-\frac{1}{2})}\zeta(s) = \frac{1}{q-1}\sum_{C} q^{\dim C - \frac{1}{2}\deg C} u^{\deg C - \frac{1}{2}\deg W} \tag{12}$$

where C ranges over *all* divisor classes of the function field $F|K$, including those of negative degree. (Note that $\dim C = 0$ if $\deg C < 0$).) The Riemann-Roch theorem in its symmetric form (7) now says that the coefficient $q^{\dim C - \frac{1}{2}\deg C}$ remains invariant under the substitution $C \mapsto C'$. On the other hand, this substitution transforms $u^{\deg C - \frac{1}{2}\deg W}$ into its inverse. In other words: the substitution $C \mapsto C'$ (which is just a permutation of all divisor classes and hence leaves the right hand side of (12) invariant) is equivalent to $u \mapsto u^{-1}$. This is the functional equation in its symmetric form (10).

4.3.4 General comments

From what has been said above it is clear that F.K. Schmidt had conceived this paper with the explicit aim to transfer those tools of analytic number theory which are necessary to develop class field theory in the function field case. But the paper has exerted its influence much further than class field theory. The paper constitutes the first systematic presentation of the theory of algebraic function fields over arbitary base fields (or at least over perfect ones). It has served several generations of mathematicians as a basis for further research; in this sense it became a classic.

In this connection I would like to point out that F.K. Schmidt's paper appeared just in time in order to serve as a basis for Hasse's investigation of the Riemann hypothesis for function fields. As I have mentioned in another article [90], Hasse had been introduced by Davenport to the problem of diophantine congruences. Hasse first met Davenport in the summer of 1931; at that time Hasse was already familiar with F.K. Schmidt's paper. Therefore he was able to realize at once that Davenport's problem was equivalent to the Riemann hypothesis for F.K. Schmidt's zeta functions.

In a later publication [51] Hasse presented the theory of F.K. Schmidt's zeta function in a form which he wished to use in further references. There he added some facts which were not explicitly mentioned in F.K. Schmidt's paper but which had been communicated to him by F.K. Schmidt in writing. One of those facts is the following representation of $\zeta(s)$ in terms of the variable $t = q^{-s}$:

$$\zeta(s) = \frac{L(t)}{(1-t)(1-qt)} \tag{13}$$

where $L(t)$ is a polynomial with integer coefficients. Of course this is an immediate consequence of F.K. Schmidt's theorem (i) above, the poles $t = 1$ and $t = q^{-1}$ corresponding to $s = 0$ and $s = 1$ respectively. The numerator polynomial $L(t)$ is known to play an important role in connection with the Riemann hypothesis for $\zeta(s)$. The degree of $L(t)$ is $2g$, and this is given correctly by Hasse [51]. In F.K. Schmidt's paper [100] the formula (13) does not appear but if we would follow up F.K. Schmidt's arguments in [100] then we would obtain the degree $2g - 1$. The reason is that F.K. Schmidt's formulas in [100] contain an "annoying misprint" as Hasse calls it (*durchweg ein störender Druckfehler*). [40] It seems that F.K. Schmidt had observed this "misprint" and informed Hasse about it, and at the same time he pointed out to Hasse the formula (13) with

$$L(t) = 1 + (N_1 - (q+1))t + \cdots + q^g t^{2g} \tag{14}$$

which gives the correct degree of $L(t)$. (Here, N_1 is the number of places of degree 1 in the given function field). By the way, a similar "misprint" occurs in the follow-up paper of F.K. Schmidt about class field theory [101] which we are going to discuss in section 5.

F.K. Schmidt's discovery that analytic properties of its zeta function are equivalent to the Riemann-Roch theorem of a function field, inspired several authors to look for an analogue of the Riemann-Roch theorem in a number field. One such analogue can be found in Tate's thesis [113].

4.4 Summary

- *As a first step towards transferring class field theory, F.K. Schmidt transferred the necessary tools from analytic number theory. Thereby he generalized the analytic part of Artin's thesis and developed the theory of the zeta function of an arbitrary function field. Already in August 1926 he was able to report to Hasse the residue formula for the zeta function at $s = 1$. A preliminary announcement about his results was published in November 1926. But in a note "*Added in Proof*" F.K. Schmidt changed his point of view and introduced his new, birationally invariant definition of the zeta function.*

- *F.K. Schmidt's final version of his paper, dealing with the birationally invariant zeta function, appeared in 1931 only, but the manuscript had been*

[40] This misprint occurs in the statement of the Riemann part of the Riemann-Roch theorem. It says that for a divisor class C we have $\dim C = \deg C - g + 1$, *provided* $\deg C > 2g - 2$. But F.K. Schmidt says in his paper that the condition $\deg C \geq 2g - 2$ is already sufficient which is obviously not true.

finished already in 1927 when he had submitted it to the Faculty in Erlangen for his "Habilitationsschrift".

- *In its first part he developed the birationally invariant theory of divisors of function fields up to the Riemann-Roch theorem. His theory was modeled after the Dedekind-Weber paper (1880) on the classical theory of algebraic functions, and after the book of Hensel-Landsberg (1902). F.K. Schmidt was able to transfer the methods of those classical sources to the case of arbitrary perfect base fields.*
- *In its second part F.K. Schmidt developed the birationally invariant theory of the zeta function $\zeta(s)$ of a function field. He had discovered that the main properties of the zeta function were closely connected with, and in fact immediate consequences of the Riemann-Roch theorem. This includes the rationality of the zeta function, the determination of the poles, their order and residues, and also the functional equation.*
- *Although the main aim of F.K. Schmidt was directed towards the establishment of class field theory in characteristic p, this paper obtained importance in a much wider context. It constitutes the beginning of a systematic theory of algebraic function fields from the algebraic-arithmetic point of view. In addition, his theory of the zeta function proved to become the proper background for Hasse's investigations on the Riemann hypothesis in characteristic p.*

5 Class field theory: the first step

5.1 General comments

In his next paper [101] F.K. Schmidt started to deal with class field theory proper. As with the foregoing paper on analytic number theory in characteristic p, this paper [101] appeared in 1931 but it had been completed in the summer of 1927 already, when F.K. Schmidt had used it as Part II of his *Habilitationsschrift* in Erlangen.

The title announces "*Class field theory in the case of function fields with finite base fields*". But a closer examination of the content of the paper shows that there are serious shortcomings and that this paper does not contain a full account of class field theory as announced in the title. The paper can be viewed only as a first approach to class field theory. It seems that F.K. Schmidt was well aware of this and had conceived the paper as a kind of a preliminary announcement, similarly as [98] which was published as a preliminary announcement of [100]. An indication for this is the fact that [101] appeared in the same not widely known journal as did [98], i.e., in the "*Erlanger Berichte*". Moreover, the presentation of the material is not as clear and final as it is in F.K. Schmidt's paper [100] on analytic number theory in function fields. While the latter has become a "classic" (we had mentioned this above already) this attribute cannot be given to the paper under discussion now.

The first serious shortcoming is stated already in the introduction of [101]. There the author says:

> *Die vorliegende Darstellung beschränkt sich zunächst auf den Fall derjenigen Abelschen Erweiterungen, deren Grad zur Charakteristik prim ist. Diejenigen Abelschen Erweiterungen, deren Grad durch die Charakteristik teilbar ist, erfordern noch einige weitere Betrachtungen und sollen an anderer Stelle behandelt werden.*
>
> The present account is restricted to the case of abelian extensions whose degree is relatively prime to the characteristic. Those abelian extensions whose degree is divisible by the characterictic do require some further considerations and will be dealt with elsewhere.

Abelian extensions whose degree are divisible by the characteristic cannot be generated by radicals, not even after adjoining the proper roots of unity. Thus F.K. Schmidt excludes precisely those cases which cannot be dealt with by the classical methods employed by Takagi. Those new cases would require a new idea which is adapted to characteristic p particularly and cannot be obtained by transfer from characteristic 0.

Such an idea appeared in the same year 1927, namely in the paper by Artin and Schreier [8]. There it was shown that cyclic extensions of degree p in characteristic p are generated by the roots of (today) so-called *Artin-Schreier equations*: $y^p - y = a$. Since F.K. Schmidt does not mention this result of Artin-Schreier we have to assume that he did not yet know about it; perhaps he had something different in mind when he mentioned "some further considerations" which he would deal with elsewhere. In section 4.1 we said that F.K. Schmidt may have had fixed his main ideas about class field theory for function fields already in December 1926, when he wrote his letter to Hasse. Already in that letter he mentioned that the case of p dividing the class number (which is the field degree in case of the Hilbert class fields) may produce difficulties. We conclude that F.K. Schmidt was well aware of the new kind of problems which those cases present when compared to classical class field theory, but that he was unable to solve those problems in the absence of Artin-Schreier theory which had not yet been published when he conceived his paper on class field theory.

In any case, he never came back to this and it was Hasse in [52] who introduced Artin-Schreier theory into class field theory for function fields. See section 6.1.

5.2 The main theorems of class field theory 1927

Now let us see what F.K. Schmidt *did* prove in [101].

We have said in the introduction already that class field theory in characteristic p was developed parallel and in analogy to the characteristic 0 case. This can be well observed here, for the main theorems in characteristic p as formulated and proved in this paper, do reflect precisely the state of Takagi's class field theory in characteristic 0 in the year 1927. The source for F.K. Schmidt was Hasse's class field report [37] about which he had heard Hasse lecture at the Danzig meeting; Hasse later had sent him an offprint of the published version.

Let $F|K$ be a function field with finite base field K. Given a positive divisor $\mathfrak{m}$, F.K. Schmidt defines the *ray modulo* $\mathfrak{m}$ in the usual way: it consists of those principal divisors which can be generated by elements α with $\alpha \equiv 1 \mod \mathfrak{m}$. The full ray class group $\mathcal{C}_\mathfrak{m}$ is defined to be the factor group of the group of all divisors relatively prime to $\mathfrak{m}$, by the ray modulo $\mathfrak{m}$. Class field theory deals with subgroups $H_\mathfrak{m} \subset \mathcal{C}_\mathfrak{m}$. Unlike in the number field case, however, $\mathcal{C}_\mathfrak{m}$ is not finite in the function field case. Therefore, in the context of class field theory one has to add the additional requirement that the index $h_\mathfrak{m} = (\mathcal{C}_\mathfrak{m} : H_\mathfrak{m})$ is finite. F.K. Schmidt observes that this is satisfied if and only if $H_\mathfrak{m}$ contains at least one ray class of positive degree.

For ray class groups belonging to different modules the following equivalence relation is introduced: $H_\mathfrak{m} \sim H_{\mathfrak{m}'}$ if and only if there exists $\mathfrak{m}'' \geq \mathfrak{m}, \mathfrak{m}'$ such that $H_\mathfrak{m}$ and $H_{\mathfrak{m}'}$ have the same inverse image under the natural projections $\mathcal{C}_{\mathfrak{m}''} \to \mathcal{C}_\mathfrak{m}$ and $\mathcal{C}_{\mathfrak{m}''} \to \mathcal{C}_{\mathfrak{m}'}$ respectively. If this is the case then, following Hasse [38] the groups $H_\mathfrak{m}$ and $H_{\mathfrak{m}'}$ are said to be "equal". In this way every $H_\mathfrak{m} \subset \mathcal{C}_\mathfrak{m}$ defines an equivalence class of "equal" ray class groups; this then is regarded as some kind of abstract ray class group H which at $\mathfrak{m}$ admits $H_\mathfrak{m}$ as its "realization". $\mathfrak{m}$ is called a "module of definition" (*Erklärungsmodul*) for H. The smallest module of definition for H is called the "conductor" (*Führer*) of H, to be denoted by the letter $\mathfrak{f}$.

In modern terms, H indeed can be viewed as a group in the proper sense, namely as an open subgroup of the inverse limit

$$\mathcal{C}_* = \lim_{\leftarrow} \mathcal{C}_\mathfrak{m}$$

which may be called the "universal ray class group" of F. The equivalence class corresponding to an open subgroup $H \subset \mathcal{C}_*$ consists of all $H_\mathfrak{m}$ which have H as their inverse image under the natural map $\mathcal{C}_* \to \mathcal{C}_\mathfrak{m}$. Thus it does not matter whether we talk about equivalence classes of "equal" ray class groups, or of open subgroups of $\mathcal{C}_*$.

But the notion of inverse limit of algebraic structures was not yet well established at the time when F.K. Schmidt wrote his paper. It was Chevalley who, at a later stage, realized $\mathcal{C}_*$ as the *idèle class group* of F and so simplified the conceptual framework of class field theory considerably [18]. Viewed from today, it does not matter whether we use the language of inverse limit of ray class groups, or avoid this and talk about equivalence classes of ray class groups; these are but two ways of describing the same object. F.K. Schmidt still used the old definition of Hasse [37] referring to equivalence classes of ray class groups.

Now let $E|F$ be a Galois extension of degree n, say. Consider a module $\mathfrak{m}$ in F and its ray class group $\mathcal{C}_\mathfrak{m}$. The norm map $N\colon E \to F$ yields a map of ray class groups whose image $N_\mathfrak{m} \subset \mathcal{C}_\mathfrak{m}$ is of finite index, say $h_\mathfrak{m}$. Following Takagi, $E|F$ is called a *class field defined modulo* $\mathfrak{m}$ if $h_\mathfrak{m} = n$.

This being said, the main theorems of class field theory as announced by

F.K. Schmidt can be stated as follows:

I. Existence- and uniqueness theorem: Given any module $\mathfrak{m}$ in F and any subgroup $H_{\mathfrak{m}} \subset \mathcal{C}_{\mathfrak{m}}$ of finite index, there exists one and only one class field $E|F$ defined over $\mathfrak{m}$ which admits $H_{\mathfrak{m}}$ as its norm group, i.e., $H_{\mathfrak{m}} = N_{\mathfrak{m}}$. If $H_{\mathfrak{m}'}$ is "equal" to $H_{\mathfrak{m}}$ in the sense as explained above then its class field coincides with E, and conversely.

II. Isomorphism theorem: $E|F$ is abelian and its Galois group is isomorphic to the norm factor group $\mathcal{C}_{\mathfrak{m}}/H_{\mathfrak{m}}$.

III. Discriminant-conductor theorem: The discriminant of $E|F$ contains precisely those places which are contained in the conductor $\mathfrak{f}$ of H.

IV. Decomposition theorem: If $\mathfrak{p}$ is a prime of F not contained in $\mathfrak{m}$ and if f denotes the order of $\mathfrak{p}$ modulo $H_{\mathfrak{m}}$ then $\mathfrak{p}$ splits in E into different primes of relative degree f.

V. Inversion theorem: Every abelian field extension $E|F$ is a class field in the above sense.

In fact, these are the main theorems of class field theory which had been stated essentially in this form in Hasse's report [37]. We see that F.K. Schmidt is closely following Hasse's presentation indeed. He is going to prove those theorems under the additional hypotheses that the group index (in theorem I) and the field degree (in theorem V) are not divisible by the characteristic p.

Looking at the above list we observe the second serious shortcoming of this paper, namely that *Artin's Reciprocity Law is completely missing*. Artin had published his proof (in the case of number fields) in 1927 already, and according to Hasse it constituted a "progress of greatest importance" (*einen Fortschritt von der allergrößten Bedeutung*) [39]. F.K. Schmidt certainly must have heard of this by the time when his paper was sent to print (1930). So why didn't he attempt to prove Artin's reciprocity law in the function field case? Why didn't he even mention the reciprocity law? Again, we have only one explanation, namely that he had completed his paper in 1927 already (for his *Habilitationsschrift*) and at that time he did not yet know about Artin's result. Later, he did not change the text at all.

We shall see in section 6.1.1 that Hasse took up the problem and proved the reciprocity law in the function field case.

5.3 The L-series of F.K. Schmidt

Having stated the above 5 theorems F.K. Schmidt says:

> *Der Beweis dieser Sätze vollzieht sich in entsprechenden Schritten wie in der Takagischen Theorie.*
>
> The proof of these theorems proceeds in analoguous steps as in Takagi's theory.

Accordingly he follows Hasse's report and starts with the proof of the so-called "first inequality" of class field theory. This inequality refers to the following situation: $E|F$ is a Galois extension of finite degree n and $\mathfrak{m}$ is a positive divisor in F. Let $h_{\mathfrak{m}} = (\mathcal{C}_{\mathfrak{m}} : N_{\mathfrak{m}})$ denote the corresponding norm index. Then the first inequality says that

$$h_{\mathfrak{m}} \leq n \,. \tag{15}$$

It is the proof of this inequality where F.K. Schmidt had to use analytic methods; the situation is just like in the number field case. More precisely, he had to use:

1. the theory of the zeta function; in particular the fact that $\zeta(s)$ has a pole of order 1 at $s = 1$;

2. the theory of L-series; in particular the fact that for any non-principal character χ of $\mathcal{C}_{\mathfrak{m}}$ of finite order, its L-series $L(s,\chi)$ assumes a finite value at $s = 1$.

F.K. Schmidt had dealt with item 1. in his former paper [100] which we have discussed above in section 4.3.3. In order to cover item 2., F.K. Schmidt introduces the L-series in the usual way:

$$L(s,\chi) = \prod_{\mathfrak{p}}{}' \frac{1}{1-\chi(\mathfrak{p})|\mathfrak{p}|^{-s}} = \sum_{\mathfrak{a}\geq 0}{}' \chi(\mathfrak{a})|\mathfrak{a}|^{-s} \tag{16}$$

where the dashes $'$ at the product sign and at the sum sign indicate that only those places $\mathfrak{p}$ and divisors $\mathfrak{a}$ are to be considered which are relatively prime to the given modulus $\mathfrak{m}$.

Using the Riemann-Roch theorem F.K. Schmidt is able to show that for every non-principal character χ the series on the right hand side of (16) terminates, i.e. that $L(s,\chi)$ is a polynomial in the variable $t = q^{-s}$. Therefore, of course, $L(1,\chi)$ is finite. [41] Thus the analytic theory in function fields turns out to be much simpler than in the number field case – thanks to the Riemann-Roch theorem.

Once having obtained item 2. above, F.K. Schmidt does not bother to present the proof of the first inequality (15) but he is content with saying:

> *Die L-Reihen des Funktionenkörpers verhalten sich bei Annäherung an $s = 1$ genau ebenso wie die L-Reihen eines endlichen algebraischen Zahlkörpers. Man kann daher die bekannten zahlentheoretischen Schlußweisen auf die L-Reihen des Funktionenkörpers übertragen und gewinnt so nach dem Vorbild von Hasse die Ungleichung. . .*
>
> Approaching $s = 1$, the L-series of the function field show the same behavior as the L-series of a finite algebraic number field. Therefore, it is possible to transfer the known number theoretic arguments to the L-series of the function field, and one obtains in this way, following Hasse, the inequality. . .

[41] F.K. Schmidt doesn't say anything about $L(1,\chi) \neq 0$.

He is referring to Part Ia of Hasse's report [38] (the part where the proofs are presented). In other words: F.K. Schmidt assumes the reader to be familiar with Hasse's report including the proofs, and his arguments here are given only to the extent that the reader can do the transfer by himself: from characteristic 0 to characteristic p.

The paper does not contain any further systematic study of L-series in function fields. In particular the functional equation of the L-series is not discussed. This has been proved later by Witt; see section 7.4.

5.4 Further remarks

Takagi's original proof of the main theorems of class field theory for number fields is not straightforward. The structure of proof is a complicated net of back-and-forth arguments which finally yield the desired theorems but otherwise are quite unsatisfactory, in as much as they do not yield sufficient insight into the structure of the mathematical objects to be studied. [42]

This initiated the search for simplification and re-organisation of class field theory. That process was quite under way in 1931 when F.K. Schmidt's paper [101] appeared. But we do not see any sign of that development reflected in this paper. The paper refers to Hasse's report Parts I and Ia only, and it closely follows the lines of arguments as given there.

In accordance with this, after having proved the first inequality F.K. Schmidt now switches to the proof of the inversion theorem for cyclic extensions $E|F$ of prime degree n (where $n \not\equiv 0 \bmod p$). To this end he is performing a lengthy computation of group indices, including the so-called "*Hauptgeschlechtssatz*" which (from today's viewpoint) asserts the vanishing of a certain 1-cohomology group of ray classes. F.K. Schmidt puts into evidence that those computations (which were later much simplified by Herbrand [65]) can be carried out quite in the same manner as in the number field case. There are even certain simplifications due to the simple unit structure (every unit is a constant) and to the simple structure of cyclotomic fields (every cyclotomic extension is a base field extension and hence unramified).

This being done, the rest of the paper is more or less hand waving. F.K. Schmidt seems to be in haste and therefore leaves all the rest to the reader, with the following comment:

> *Dabei wird man ganz von selbst auf einige leichte Abweichungen von den zahlentheoretischen Schlußweisen geführt, die aber durch das oben Gesagte bereits so nahe gelegt sind, daß es sich erübrigt, näher auf sie einzugehen.*
>
> One will be led automatically to some minor differences to the number theoretical arguments; but it does not seem necessary to discuss them in detail since they are suggested sufficiently by what has been said above already.

[42] See the remarks by Hasse which we have cited in section 2.2.3 from part II of his class field report.

This does not sound very convincing. In particular the transfer of the existence theorem of class field theory, which uses several delicate index computations, would remain doubtful unless it is presented explicitly – even if one restricts the discussion to the case where the characteristic p does not divide the relevant group index n. In fact, some years later in 1935 Witt, when presenting a simple proof of the existence theorem, did not say that F.K. Schmidt had already proved it for $n \not\equiv 0 \bmod p$, but that F.K. Schmidt "had already discussed the possibility of transferring the proof" (*hat die Möglichkeit einer Übertragung schon erörtert*). See section 7.2 below.

This paper is the last one by F.K. Schmidt about class field theory in function fields. In the late twenties and thirties he had a number of other important papers on algebraic function fields and also on other topics, e.g., from the theory of local fields, some of them in cooperation with or inspired by Hasse. [43] I am planning in a separate publication to cover in more detail the results of his cooperation with Hasse. But since we are concerned with class field theory in function fields we have now to turn to other authors who completed the work initiated by F.K. Schmidt.

5.5 Summary

- *As a follow-up to his paper on analytic number theory in characteristic p, F.K. Schmidt published a second paper announcing class field theory in characteristic p. As with the former paper, the manuscript for this one too was finished already in the summer of 1927.*

- *Notwithstanding its title the paper does not give a comprehensive presentation of all of class field theory. The following items are missing in the function field case:*
  1. Abelian extensions of degree divisible by the characteristic p,
  2. Artin's Reciprocity Law,
  3. Functional equation of the L-series.
  Thus the paper can be regarded as a first approach only to class field theory.

- *The arguments of the paper follow closely the presentation of class field theory in Parts I and Ia of Hasse's class field report. The proofs are given only partially, and the reader is assumed to be able to transfer himself,* mutatis mutandis*, the proofs given by Hasse in his class field report.*

6 The reciprocity law

6.1 Hasse's paper on cyclic function fields

After F.K. Schmidt's paper [101], the next one which contains a contribution to class field theory for function fields was Hasse's [52], published in 1934 with

[43] For a list of publications of F.K. Schmidt see [77].

the title: *Theorie der relativ-zyklischen algebraischen Funktionenkörper, insbesondere bei endlichem Konstantenkörper* (Theory of relatively cyclic algebraic function fields, in particular with finite base field).

This paper is a product of Hasse's cooperation with Davenport who, as said in section 4.3.4 already, had introduced him to the problem of diophantine congruences which is equivalent to the Riemann hypothesis for the zeta function of function fields. In January 1933 Hasse had succeeded in proving the Riemann hypothesis in the case of elliptic function fields [47]. [44] As a step towards the general case, i.e., function fields of arbitrary genus, Davenport and Hasse investigated fields of the form $K(x, y)$ with defining relation:

$$y^n = 1 - x^m \qquad \text{or :} \qquad y^p - y = x^m \tag{17}$$

where p is the characteristic and m, n are integers not divisible by p (the base field K is assumed to be finite and containing the m-th and the n-th roots of unity, respectively). In fact, in these cases Davenport and Hasse succeeded in proving the Riemann hypothesis since the roots of the zeta function can be interpreted by means of certain Gaussian sums and related expressions whose absolute value is known [20].

Now, from (17) we see that the field $E = K(x, y)$ can be regarded as a cyclic extension of $F = K(x)$ (and also of $F' = K(y)$). Therefore Davenport and Hasse wished to write a preparatory paper for reference purposes, containing the necessary general facts from the theory of cyclic extensions of function fields and their corresponding L-series.

The paper [52] under consideration was written for this purpose; it appeared in the same volume of Crelle's Journal as the Hasse-Davenport paper [20]. However, as it is often the case in Hasse's papers, he not only presented the facts which were necessary for the intended application but in addition he developed a comprehensive and systematic study of the objects under consideration, in this case the cyclic extensions $E|F$ of function fields.

Thus Hasse's paper [52] was not written primarily with class field theory in mind. Class field theory is only one aspect of the theory of cyclic extensions of function fields, and Hasse deals with it only in passing. From the 18 pages of the paper, only 2 are concerned with class field theory proper (pages 45–46). Nevertheless these pages constitute an important step in the development of class field theory for function fields. For, Hasse proves the analogue of *Artin's Reciprocity Law* in the case of function fields.

In its proof Hasse uses quite new ideas when compared to the former papers by F.K. Schmidt or by Artin. This reflects the state of the art in class field theory as of 1934: recently Hasse had introduced the *theory of algebras* into class field theory of number fields, following an idea of Emmy Noether [41], [44]. Now he uses algebras also in the case of function fields.

The general reciprocity law as conceived by Artin is concerned with abelian extensions. But it suffices to prove it for cyclic extensions only. For, as Hasse remarks, the general abelian case is reduced "immediately in a well known

[44] For more details on that story see [90].

manner" to the cyclic case (*ohne weiteres in geläufiger Weise*). This is the justification for Hasse to include class field theory and the reciprocity law in a paper which is devoted to the study of cyclic function fields. In the following discussion we shall formulate the reciprocity law for arbitrary abelian extensions. Later, while discussing Hasse's proof we shall point out where and how Hasse uses the assumption that the extension is cyclic. In section 6.1.4 we shall discuss what Hasse may have had in mind when he mentioned, without reference, the "immediate and well known" reduction to the cyclic case.

6.1.1 The reciprocity law: Theorems A, B and C.

Let $E|F$ be an abelian extension of function fields, with Galois group G. For each unramified prime $\mathfrak{p}$ of F, let $\left(\frac{E|F}{\mathfrak{p}}\right) \in G$ denote its Frobenius automorphism. [45] The map $\mathfrak{p} \mapsto \left(\frac{E|F}{\mathfrak{p}}\right)$ extends uniquely to a homomorphism $\mathfrak{a} \mapsto \left(\frac{E|F}{\mathfrak{a}}\right)$ of the group of unramified divisors (i.e., those divisors $\mathfrak{a}$ which are composed of unramified primes only) into G. This is the "Artin homomorphism". Artin's reciprocity law is concerned with this homomorphism and can be formulated as follows. [46]

Theorem A. *The kernel of the Artin homomorphism contains the ray modulo $\mathfrak{m}$ where $\mathfrak{m}$ denotes a sufficiently large positive divisor which contains all primes which are ramified in E. The smallest $\mathfrak{m}$ with this property is the conductor $\mathfrak{m} = \mathfrak{f}$ of the extension $E|F$. (Definitions see below.)*

Theorem B. *Regarded as a homomorphism from the ray class group, the Artin homomorphism $\mathcal{C}_{\mathfrak{m}} \to G$ is surjective, and hence yields an isomorphism of the factor group $\mathcal{C}_{\mathfrak{m}}/H_{\mathfrak{m}}$ onto G, where $H_{\mathfrak{m}}$ denotes the kernel of the Artin homomorphism. This kernel is called the Artin group of $E|F$ modulo $\mathfrak{m}$. If $\mathfrak{m}'$ is another module with the same properties and $H_{\mathfrak{m}'}$ its Artin group then $H_{\mathfrak{m}}$ and $H_{\mathfrak{m}'}$ are "equal" in the sense as explained above in section 5.2.*

Theorem C. *The Artin group $H_{\mathfrak{m}}$ equals the norm group $N_{\mathfrak{m}}$ which consists of those ray classes modulo $\mathfrak{m}$ which are norms from E, i.e., $H_{\mathfrak{m}} = N_{\mathfrak{m}}$. The norm group is called the Takagi group of $E|F$ (modulo $\mathfrak{m}$).* [47]

As to the definition of a "ray" modulo $\mathfrak{m}$ and the corresponding full ray class group $\mathcal{C}_{\mathfrak{m}}$ we refer to section 5.2.

The notion of "conductor" (*Führer*) also had been defined in section 5.2 but there the definition is of group theoretic nature: it refers to a ray class group of finite index. It is only *after* establishing the main theorems of class field

[45] Hasse [52] calls it "Artin automorphism".

[46] The division into three parts follows roughly Hasse's presentation but the notation **A, B, C** for the following theorems is ours, introduced for later reference.

[47] The terminology of "Artin group" and "Takagi group" has been introduced by Chevalley. It is not used in Hasse's paper [52]. (But in his Marburg Lecture Notes [45] Hasse uses the terminology "Artin classes" for the residue classes modulo the Artin group.)

theory that the conductor, as defined in section 5.2, can be associated to an abelian extension. In the present context, however, the conductor is defined *a priori*, without recourse to class field theory, for any finite abelian extension $E|F$, namely by local norm conditions. For each prime $\mathfrak{p}$ of F let $E_{\mathfrak{p}}|F_{\mathfrak{p}}$ denote the corresponding local extension. Then the $\mathfrak{p}$-component $\mathfrak{f}_{\mathfrak{p}}$ of $\mathfrak{f}$ is defined to be minimal such that every $\alpha \in F_{\mathfrak{p}}$ with $\alpha \equiv 1 \mod \mathfrak{f}_{\mathfrak{p}}$ is a norm from $E_{\mathfrak{p}}$. [48]

As to the existence of this conductor, Hasse refers to his paper [50] which had just appeared in the Science Journal of Tokyo University. There, Hasse discusses local *number fields* only. But the paper [50] is based on the theory of local division algebras [41] which immediately can be transferred to the function field case; therefore, Hasse says, all the results of [50] hold also in the function field case. [49]

Examples: If $\mathfrak{p}$ is not ramified in E then $\mathfrak{f}_{\mathfrak{p}}$ is trivial. If $\mathfrak{p}$ is ramified and $[E : F]$ is not divisible by the characteristic p then $\mathfrak{f}_{\mathfrak{p}} = \mathfrak{p}$ (there is tame ramification only). If $[E : F] = p$ then $E = F(y)$ with $y^p - y = a \in F$; supposing that the pole order of a at $\mathfrak{p}$ is $m \not\equiv 0 \mod p$, the multiplicity of $\mathfrak{p}$ in $\mathfrak{f}$ is $m + 1$. For the proof of these examples Hasse again refers to [50] but mentions that one could easily obtain them directly.

The above reciprocity law contains all the main theorems on class field theory which had been listed by F.K. Schmidt (see section 5.2) except the existence theorem (Theorem I in 5.2).

6.1.2 Hasse's proof of Theorem A, in the cyclic case, by means of algebras

Let A be a simple algebra over F. [50] For each prime $\mathfrak{p}$ of F consider the $\mathfrak{p}$-adic completion $\mathsf{A}_{\mathfrak{p}}$ over $F_{\mathfrak{p}}$. Hasse refers to his former paper [44], published one year earlier in the *Mathematische Annalen*, where he had defined what today is called the *Hasse invariant* of $\mathsf{A}_{\mathfrak{p}}$ which is a rational number modulo 1 . [51] In that former paper Hasse had worked with local *number fields* but, as said earlier, the local theory can be transferred without problems to the case of local function fields (which are power series fields over finite base fields). In particular, the $\mathfrak{p}$-adic Hasse invariant $\left(\frac{\mathsf{A}}{\mathfrak{p}}\right)$ is defined also in the function field case, as a rational

[48] After establishing the main facts of class field theory it turns out that both notions of conductor become equivalent: the conductor of an abelian extension coincides with the group theoretical conductor of its Artin group.

[49] Witt has pointed out in [129] that the existence of the conductor is equivalent to the fact that the norm map is *open* in the topology of local fields, and that this is an easy consequence of Hensel's lemma.

[50] It is tacitly assumed that A is finite dimensional and that F is its center.

[51] That paper, dedicated to Emmy Noether, is the one where Hasse succeeded to prove the Artin reciprocity law in the number field case by means of the theory of algebras. The starting point for this was the fact that the local Hasse invariant $(\frac{\mathsf{A}}{\mathfrak{p}})$ could be defined, following Chevalley [14], by purely local considerations whereas formerly, as in [42], Hasse could give the definition by means of the global class field product formula only. – As Auguste Dick [27] reports, Emmy Noether was extremely glad ("*ganz besonders erfreut*") about Hasse's results in this paper which confirmed her belief that non-commutative arithmetic can be profitably used to study commutative number fields.

number modulo 1.

The essential step in proving Theorem **A** is the proof of the sum formula

$$\sum_{\mathfrak{p}} \left(\frac{\mathsf{A}}{\mathfrak{p}}\right) \equiv 0 \mod 1 \tag{18}$$

where $\mathfrak{p}$ ranges over all primes of F. For this, the proof in the number field case cannot be transferred directly to the function field case, because of the different behavior of cyclotomic fields in these two cases. Now in the function field case, Hasse proves (18) with the help of what today is known as *Tsen's theorem*.

In his Göttingen thesis [115], [116] Ch.C. Tsen had proved, shortly before Hasse's paper, that there are no nontrivial simple algebras over a function field with algebraically closed base field. Tsen had studied with Emmy Noether as his advisor; in the preface of his thesis he mentions that he had also received valuable help from Artin. [52]

Because of Tsen's theorem, for each algebra A over F there exists a finite base field extension $L|K$ such that A splits over FL. This is quite analogous to the fact, known from number theory, that every algebra admits a cyclotomic splitting field. But in the function field case the situation is much simpler because the splitting field FL is unramified over F. Moreover, FL is cyclic over F and its Galois group admits the Frobenius automorphism π of $L|K$ as its generator.

In this situation A is similar to a crossed product algebra: $\mathsf{A} \sim (\beta, FL, \pi)$, defined as being generated by FL and an element u with the defining relations

$$u^m = \beta\,, \qquad u^{-1}xu = x^{\pi} \qquad (x \in FL)$$

where $m = [L : K]$ and β is an element of F which is determined modulo norms from FL only. Now since $FL|F$ is unramified the local Hasse invariants of such a crossed product are easily read off from their definition, namely $\left(\frac{\mathsf{A}}{\mathfrak{p}}\right) \equiv \frac{\deg(\mathfrak{p})v_{\mathfrak{p}}(\beta)}{m} \mod 1$. (Here, $v_{\mathfrak{p}}(\beta)$ is the order of β at $\mathfrak{p}$.) Hence the sum formula (18) is a consequence of the formula

$$\sum_{\mathfrak{p}} \deg(\mathfrak{p})v_{\mathfrak{p}}(\beta) = 0 \tag{19}$$

which expresses the fact that every $\beta \neq 0$ admits as many poles as there are zeros. Having established the sum relation (18), the proof of Theorem **A** above

[52] I am indebted to Falko Lorenz for pointing out to me that also F.K. Schmidt is mentioned in Tsen's thesis, namely as his referee (*Referent*). This reflects the state of affairs at the Göttingen mathematical scene in 1933/34. Emmy Noether had been dismissed from her university position in early 1933, due to the antisemitic policy of the National-Socialist regime in Germany since 1933. Also, many other mathematicians had left Göttingen; see the report by Schappacher and M. Kneser [92]. F.K. Schmidt had been called to Göttingen in the fall of 1933 as a temporary replacement of H. Weyl. In this position he took care of a number of students who had been advised by E. Noether, and in particular of Tsen. – F.K. Schmidt remained in Göttingen for one year; after that he received a position as a full professor at the University of Jena. For more biographical information about F.K. Schmidt see [77].

is straightforward, once one has accepted Hasse's use of the theory of algebras in arithmetic.

At this point Hasse uses the assumption that $E|F$ is cyclic. Accordingly let σ be a fixed generator of the Galois group G. Let $n = [E : F]$. For each $0 \neq \alpha \in F$ consider the cyclic crossed product algebra $\mathsf{A} = (\alpha, E, \sigma)$. Writing the $\mathfrak{p}$-adic Hasse invariant in the form $\left(\frac{\mathsf{A}}{\mathfrak{p}}\right) \equiv \frac{r_\mathfrak{p}}{n} \mod 1$ with $r_\mathfrak{p} \in \mathbb{Z}$, Hasse defines the local norm symbol as follows: [53]

$$\left(\frac{\alpha, E|F}{\mathfrak{p}}\right) = \sigma^{-r_\mathfrak{p}}\,. \tag{20}$$

We have $\left(\frac{\alpha,E|F}{\mathfrak{p}}\right) = 1$ if and only if α is a norm from $E_\mathfrak{p}|F_\mathfrak{p}$.

By means of the definition (20) the sum formula (18) is translated into the product formula

$$\prod_{\mathfrak{p}} \left(\frac{\alpha, E|F}{\mathfrak{p}}\right) = 1\,. \tag{21}$$

Now if $\alpha \equiv 1 \mod \mathfrak{f}$ then for every ramified $\mathfrak{p}$ we have by definition that α is a local norm from $E_\mathfrak{p}$, hence the corresponding algebra A splits at $\mathfrak{p}$ and therefore $\left(\frac{\alpha,E|F}{\mathfrak{p}}\right) = 1$. On the other hand, for unramified $\mathfrak{p}$ it follows from the definition that $\left(\frac{\alpha,E|F}{\mathfrak{p}}\right)$ is a power of the Frobenius automorphism, namely $\left(\frac{\alpha,E|F}{\mathfrak{p}}\right) = \left(\frac{E|F}{\mathfrak{p}}\right)^{-v_\mathfrak{p}(\alpha)}$ and hence the product formula (21) yields for the principal divisor $\mathfrak{a} = (\alpha)$:

$$\left(\frac{E|F}{\mathfrak{a}}\right)^{-1} = \prod_{\text{unramified } \mathfrak{p}} \left(\frac{E|F}{\mathfrak{p}}\right)^{-v_\mathfrak{p}(\alpha)} = \prod_{\text{all } \mathfrak{p}} \left(\frac{\alpha, E|F}{\mathfrak{p}}\right) = 1$$

which shows that, indeed, $\mathfrak{a} = (\alpha)$ is in the kernel of the Artin homomorphism.

6.1.3 The proof of Theorems B and C

For the proof of Theorem **B** Hasse uses the fact, proved by F.K. Schmidt, that the zeta function of every function field has a pole of order 1 at the point $s = 1$. He argues as follows: Let G' be the image of the Artin homomorphism and E' the subfield of E corresponding to G' by Galois theory; put $n' = [E' : F]$. Then every prime $\mathfrak{p}$ of F which is not contained in $\mathfrak{m}$ splits completely in E', i.e., it has precisely n' extensions in E'. It follows from the product representation of the zeta function of E' that $\zeta_{E'}(s)$ is the n'-th power of $\zeta_F(s)$ – except perhaps for finitely many Euler factors belonging to the primes of $\mathfrak{m}$. In any case, the zeta function of E' has a pole of order n' at $s = 1$. Thus $n' = 1$ and $E' = F$.

[53] The minus sign in front of the exponent $r_\mathfrak{p}$ on the right hand side is for normalizing purposes only and is not important for the following argument.

We see that for Theorem **B**, Hasse used the following lemma which he proved with the help of F.K. Schmidt's zeta function:

LEMMA 1: *Let $E|F$ be an abelian field extension such that almost every prime* [54] $\mathfrak{p}$ *of F splits completely in E. Then $E = F$.*

For the proof of Theorem **C** Hasse uses the "first inequality" $h_{\mathfrak{m}} \leq n$ of (15) which had been proved by F.K. Schmidt by means of L-series. Here, $h_{\mathfrak{m}}$ is the index of the Takagi group. Hasse remarks that this part of F.K. Schmidt's paper [101] in which he proved the first inequality, is generally valid and does not depend on the assumption, otherwise imposed in [101], that the field degree $n \not\equiv 0 \mod p$.

According to Theorems **A** and **B**, the field degree n equals the index of the Artin group. Since the Takagi group is contained in the Artin group it follows that $h_{\mathfrak{m}} = n$ and both groups coincide.

6.1.4 Further remarks

We have discussed Hasse's proof in such detail in order to put into evidence that his idea of using algebras in class field theory did contribute essentially to simplify and systematize the proofs. There are two comments of Hasse on his proof which perhaps need some further attention. The first of these comments, found on page 46, we have mentioned above already:

> *Der obige Klassenkörperhauptsatz überträgt sich ohne weiteres in geläufiger Weise auf beliebige separable abelsche Erweiterungskörper.*
>
> The above main theorem of class field theory can be extended immediately and in a well known manner to the case of arbitrary separable abelian extensions fields.

By "main theorem" Hasse means the union of what we have called Theorems **A**, **B** and **C**. His proof, as presented above, covers only cyclic extensions. In order to obtain class field theory in its full extent one has to reduce the general abelian case to the cyclic case. Hasse does not give any reference, nor does he explain what he means by an "immediate" and "well known" method to carry out this reduction. A closer look may perhaps reveal what he had in mind. Let us explain the situation:

Our above presentation puts into evidence that the proofs of Theorems **B** and **C** are generally valid, and it is only Hasse's proof of Theorem **A** where the cyclic property of the extension $E|F$ is used.

Let $E|F$ be an arbitrary abelian extension with group G. Consider the cyclic subextensions $E_i|F$ of $E|F$, with conductors $\mathfrak{f}_{E_i}$. Let the divisor $\mathfrak{a}$ of F be unramified in E and $(\frac{E|F}{\mathfrak{a}}) \in G$ its image under the Artin map. When restricted to E_i this gives $(\frac{E_i|F}{\mathfrak{a}})$. Hence, if $\mathfrak{a}$ is contained in the ray modulo $\mathfrak{f}_{E_i}$ then, by the cyclic case of Theorem **A**, the restriction of $(\frac{E|F}{\mathfrak{a}})$ to E_i is trivial;

[54] This means every prime but finitely many.

if this is true for all i then $(\frac{E|F}{\mathfrak{a}}) = 1$ in G. Now by the very definition of the conductor, we have

$$\mathfrak{f}_E \geq \mathfrak{m} = \max_i \mathfrak{f}_{E_i} \tag{22}$$

That is, the ray modulo the conductor $\mathfrak{f}_E$ is contained in the intersection of the rays modulo the conductors $\mathfrak{f}_{E_i}$ of its cyclic subextensions. It follows that the Artin homomorphism vanishes on the ray modulo $\mathfrak{f}_E$.

Thus indeed, this "immediate" argument shows that Theorem **A** holds for any abelian extension $E|F$ – except for the fact that the conductor $\mathfrak{f}_E$ is the *smallest* divisor with the property as stated in the theorem. For the proof of this additional contention one has to use that equality holds in (22):

$$\mathfrak{f}_E = \max_i \mathfrak{f}_{E_i} \, . \tag{23}$$

Since conductors are defined locally, this is a purely local affair, concerning the local abelian extensions $E_\mathfrak{p}|F_\mathfrak{p}$ for each prime $\mathfrak{p}$ of F. It is clear that this formula is a consequence of *local class field theory*.

Already in 1930 Hasse and F.K. Schmidt had developed local class field theory [40], [99] in the number field case. But in their first approach the local theory was based on the global theory, because it was not possible, at that time, to define the norm residue symbol on purely local terms. It was Chevalley who in his famous thesis [15] had developed the local class field theory directly, without using global arguments.

But, considering the year of publication (1934), where Chevalley's thesis had just appeared, is it conceivable that Hasse would refer to it by naming it "immediate and well known", without mentioning explicitly what he has in mind? And without giving any particular reference?

In looking for further evidence we discover that two pages earlier, on page 44, Hasse gives a reference to another of his papers [50], on the norm residue theory of Galois fields with applications to conductor and discriminant of abelian fields. (We have mentioned this earlier already.) That paper appeared right after Chevalley's thesis in the same Japanese journal. [55] It contains a detailed study of the norm map when compared with the higher ramification groups. Although it is concerned with local *number fields*, it is clear from the context and mentioned explicitly by Hasse that the local theory can be transferred directly to the function field case. [56] For us it is of interest that this paper [50] contains an explicit formula for the conductor of an abelian extension, from which (23) can be deduced. In proving that formula, Hasse had to use certain facts from local class field theory, and he said about it:

[55] See also Hasse's Comptes Rendus Notes [48],[49] where he announced the results of [50].

[56] Hasse's paper [50] became widely known because it contains the Hasse part of the "Theorem of Hasse-Arf" on the ramification numbers of local abelian extensions. The Hasse part is concerned with local fields whose residue field is finite. Hasse conjectured that the same result would hold for arbitrary perfect residue fields and he gave this problem to his student Cahit Arf who solved it in his thesis [2].

> *Was den Beweis ... angeht, so bildet der Spezialfall, wo die Erweiterung zyklisch ist, den einen Hauptpunkt... Für diesen zyklischen Spezialfall hat Herbrand einen sehr eleganten Beweis gegeben. (Eine Darstellung dieses Beweises siehe in der Thèse von C. Chevalley, die dieser Arbeit unmittelbar vorangeht...) Den Übergang zum allgemein abelschen Fall kann man* entweder *unter voller Ausnutzung der Hauptsätze der Klassenkörpertheorie im Grossen durch Entwicklung der Theorie des Normenrestsymbols vollziehen – das ist aber methodisch unschön –* oder *aber auch direkt durch methodisch in die Klassenkörpertheorie im Kleinen gehörende Betrachtungen ausführen. (Eine Ausführung dieses Beweises siehe ebenfalls in der Chevalleyschen Thèse.) – Die Ausführungen meiner vorliegenden Arbeit ergänzen den Herbrandschen Beweis für den zyklischen Fall und den Chevalleyschen Übergang zum allgemein-abelschen Fall eben in der Weise, daß sie die genaue Bestimmung des $\mathfrak{p}$-Führers liefern...*
>
> Concerning the proof..., the main point is the special case where the extension is cyclic... For this cyclic case a very elegant proof has been given by Herbrand. (For an exposition of this proof see Chevalley's thesis immediately preceding this paper...) The transition to the general abelian case can be given *either* with full use of the main theorems of global class field theory by developing the norm residue symbol – but that is not desirable from a methodical point of view – *or else* directly, using arguments which methodically belong to local class field theory. (For an exposition of this see Chevalley's thesis again.) – The discussion in my present paper supplement Herbrand's proof in the cyclic case and Chevalley's transition to the general abelian case in such a way that they yield the exact determination of the $\mathfrak{p}$-conductor.

Thus here in [50] we find what we have missed in [52], namely an explanation of how Hasse envisages the transition from the cyclic to the abelian case, i.e. the proof of (23). From today's viewpoint, since local class field theory is well known nowadays, the proof of (23) is indeed "immediate and well known", but it seems doubtful whether this could be said in 1934 already. In 1934, the reader of [52] would perhaps have preferred a more detailed explanation of what Hasse had in mind.

In any case, as said earlier already, Hasse says explicitly that the results of [50] which were stated and proved there for local *number fields*, remain valid for local *function fields*.

By the way, in 1933 Hasse had already accepted a paper by Chevalley for Crelle's Journal [14], where the latter also presented his method how to prove (23). It is quite apparent that his method is of cohomological nature, as the "crossed products" (*verschränkte Produkte*) of E. Noether are used to compare the norm groups of different cyclic extensions. –

Now we quote the second comment of Hasse on page 44 in [52], on his proof of Artin's reciprocity law:

> *Damit ist eine dem heutigen Stande angepaßte Begründung der von F.K. Schmidt entwickelten Klassenkörpertheorie gegeben und insbesondere die dortige Beschränkung auf zur Charakteristik p prime Grade beseitigt.*
>
> Herewith we have given a presentation, adapted to the present state of knowledge, of the class field theory which had first been developed by F.K. Schmidt; in particular we have eliminated the restriction to those degrees which are relatively prime to the characteristic. [57]

Hasse's wording that his presentation corresponds to the "present state" of knowledge may reflect that he did not consider it as final; he leaves it open that further simplifications are to come in due time. Had he envisaged already the penetration of cohomology into class field theory?

If we review the above proof of Theorem **A** we see that simple algebras are used mostly in a formal way: as crossed products which, in the cyclic case, reflect the norm class structure for the splitting field. [58] Nevertheless it seems unlikely that Hasse was contemplating to substitute, as regards class field theory, the theory of algebras by a more formal calculus of cohomological nature. In fact, he has always propagated Emmy Noether's dictum: *Use non-commutative arithmetic to get results in the commutative case!* And later in the forties and fifties, when cohomology indeed had found its place in class field theory due to the works of Hochschild, Nakayama, Artin and Tate [9], [114], then Hasse did never take up those ideas in his own work – although of course he himself had started all this by using crossed products in class field theory. It seems that he did not wish to part from Emmy Noether's idea about the role of non-commutative algebras in commutative number theory. [59]

More likely, Hasse may have had in mind to free class field theory from *analytic* methods and to find proofs which are based solely on algebra and arithmetic. Such tendency was spreading in the thirties, with the intention to gain more insight into the structures connected with class field theory.

Hasse's proof of Theorem **A** is certainly of algebraic-arithmetic nature. But Theorems **B** and **C** still rested on analytic arguments. We shall see in section 8 how it became possible to replace these analytic arguments by algebraic ones.

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[57] Hasse's citation list includes all 4 papers by F.K. Schmidt which we have discussed above: [97], [98], [100], [101]. He gives full credit to F.K. Schmidt for having developed the general theory of function fields, in particular with finite base fields. (Side remark: Erroneously Hasse cites [97] as F.K. Schmidt's thesis in *Erlangen* but as we have mentioned above, this thesis was written at the university of *Freiburg*. This seems quite curious since Hasse had sent the proof sheets of his paper to F.K. Schmidt and asked for his comments. It seems that F.K. Schmidt himself did not discover this error.)

[58] It is only in the local case that Hasse had to regard algebras not only through the crossed product formalism but in fact with their arithmetic structure: In order to prove that a local division algebra admits an unramified splitting field Hasse extended the canonical valuation of the center to the division algebra and studied its arithmetic properties [41].

[59] See the last words of Hasse in his paper on the history of class field theory [57].

6.2 Witt: Riemann-Roch theorem and zeta function for algebras

In the year 1934, at about the same time when Hasse's paper [52] was published, there appeared a paper by Witt [126] which also contained important contributions to class field theory for function fields. This was Witt's Göttingen thesis of 1933. The aim of Witt's thesis was to transfer the theory of Käte Hey to the function field case.

In the year 1929 Käte Hey had completed her thesis [66] in Hamburg, with E. Artin as her advisor. She had considered simple algebras over a number field and developed analytic number theory in this setting; in particular the zeta function was defined and investigated in the non-commutative case, in analogy to the Dedekind zeta function of a number field. Hey's thesis has never been published [60] but it was well known at that time in the context of algebraic and analytic number theory. It contained also a new analytic foundation of the main theorems in class field theory; according to Deuring [23] "*die stärkste Zusammenfassung der analytischen Hilfsmittel zur Erreichung des Zieles*" (the strongest concentration of analytic tools in order to reach the goal).

Now, after F.K. Schmidt had succeeded in transferring the theory of the Dedekind zeta function to characteristic p there arose the question whether Hey's theory of the zeta function for division algebras could be transferred too. E. Noether had posed this question to Witt and he answered it in his thesis.

E. Witt had studied one year in Freiburg (since 1929) and then 3 years in Göttingen. As he himself recalls [133]:

> *Tief beeindruckt haben mich 1932 die berühmten 3 Vorträge von Artin über Klassenkörpertheorie. Die anschliessenden Ferien verbrachte ich in Hamburg, um dort die Klassenkörpertheorie intensiv zu studieren. In den folgenden Jahren war es mein Ziel, diese Klassenkörpertheorie auf Funktionenkörper zu übertragen.*
>
> In the year 1932 I was deeply impressed by the famous three lectures of Artin on class field theory. [61] In the next academic vacations I went to Hamburg for an intensive study of class field theory for number fields. In the following years it was my aim to transfer class field theory to function fields.

Witt was 21 when he decided to complete what F.K. Schmidt had started. From the above we see that Witt in his work was much influenced by E. Artin. Other people who influenced Witt were Emmy Noether, his thesis advisor, and H. Hasse whose assistant in Göttingen he became in 1934. [62]

[60] The thesis contained some errors which, however, could be corrected. See e.g., Zorn [135], Deuring [23] chap.VII, §8.

[61] There were notes taken by Olga Taussky from Artin's lectures. A copy is preserved in the library of the Mathematics Institute in Göttingen [7]. I am indebted to F. Lemmermeyer for pointing out to me that an English translation of these lecture notes has been included as an appendix to H. Cohn's "Classical Invitation to Algebraic Numbers and Class Fields" [19].

[62] More biographical information about Witt can be obtained from [75].

The title of Witt's thesis is: "*Riemann-Rochscher Satz und ζ-Funktion im Hyperkomplexen*" (Riemann-Roch theorem and ζ-function in the hypercomplex domain). Witt cites F.K. Schmidt's already classical paper [100]. In fact, Witt's proof of the Riemann-Roch theorem in the non-commutative case copies F.K. Schmidt's proof very closely; he says that F.K. Schmidt's proof served him as a model ("*nach dem Vorbild von F.K. Schmidt*"). After establishing the proper notions of "divisor" etc. of a simple algebra, Witt showed that the Riemann-Roch theorem can be formulated and proved precisely as in the commutative case, with the exception that the genus of the algebra may be negative. In fact, the treatment by Witt puts into evidence that the Riemann-Roch theorem essentially belongs to linear algebra, hence the non-commutativity of the multiplication does not disturb the general picture.

Thus this paper continues the historical line which had been started 1880 by Dedekind-Weber [22], which was followed 1902 by Hensel-Landsberg [62] and had been taken up 1927 by F.K. Schmidt [100]. Witt seems to have been fully aware of this background; he says that his construction of a normal basis follows the usual path ("*in der üblichen Weise*"). But in contrast to F.K. Schmidt [100], Witt presented fully the algebraic proof of the Riemann-Roch theorem. (Recall that F.K. Schmidt had been content to wave his hands and just said that the Riemann-Roch theorem can be proved in quite the same way as in Hensel-Landsberg; see section 4.3.2.)

Similarly as in the case of fields, in the case of division algebras the Riemann-Roch theorem leads to a birationally invariant zeta function; this is the function field analogue to Hey's zeta function. Comparison of Witt's new zeta function of the division algebra with F.K. Schmidt's zeta function of its center field leads to the following conclusion (as in Hey's thesis for number fields):

Every non-trivial division algebra (or, more generally, simple algebra) over a function field F admits at least two places where the algebra does not split. In other words: *If a simple algebra splits locally for all but possibly one place then it splits globally.*

Based on this local-global principle Witt presents an alternative proof of the sum formula (18) which Hasse had used for the proof of Artin's reciprocity law in function fields.

Thus in this paper, Witt's result concerning class field theory for function fields overlaps widely with Hasse's [52]. But the methods are different: whereas Hasse's proof of the sum formula (18), based on Tsen's theorem, is essentially of algebraic nature, Witt's proof of (18) is based very much on analytic properties of the zeta function of algebras. It constitutes, to use Deuring's words once more, "*the strongest concentration of analytic tools in order to reach the goal*". But the trend in the further development was more towards the algebraic direction. Witt's analytic proof of the reciprocity law is not widely known today, and his paper [126] is known mainly for the Riemann-Roch theorem for algebras, i.e., as a contribution to non-commutative algebraic geometry. [63]

By the way, in this paper Witt also gives a complete description of the Brauer

[63] See e.g., the Remarks by Günter Tamme in [134], page 60.

group of algebras over function fields. The result is of the same type as Hasse had found a year ago in the number field case [44].

6.3 Summary

- *In 1934 Hasse published a paper on cyclic extensions of function fields. His original motivation came from his joint work with Davenport on the Riemann hypothesis for certain function fields of higher genus. But Hasse's paper contained also important contributions to class field theory for function fields. Its main achievement regarding class field theory was Hasse's proof of Artin's general reciprocity law in the function field case.*

- *The methods used in that proof belong to the arithmetic theory of algebras and their splitting behavior; these methods had recently been successfully used by Hasse in the number field case (responding to a question of Emmy Noether) and were now transferred to the function field case.*

- *Hasse's proof in the function field case relied heavily on the theorem which Tsen had just obtained in his Göttingen thesis (with Emmy Noether as his main advisor).*

- *In 1934, parallel to Hasse's paper, there appeared Witt's Göttingen thesis (again with Emmy Noether as thesis advisor). This paper was concerned with the transfer of Käte Hey's theory to the function field case; i.e., developing the theory of zeta functions for simple algebras over function fields. To this end Witt proved the Riemann-Roch theorem for simple algebras over function fields, in generalization of F.K. Schmidt's work. Witt's theory of zeta functions for division algebras leads to a local-global principle for algebras over function fields and, consequently, to a new proof of the Artin reciprocity law for function fields. Hence, Witt's results overlap with those of Hasse but the methods used are different.*

- *Witt's paper was conceived as the first of a series in which Witt planned to complete class field theory for function fields, which had been started by F.K. Schmidt.*

7 The final steps

7.1 H.L. Schmid: Explicit reciprocity formulas

In the case when the ground field is a rational function field, $F = K(x)$, Hasse in his 1934 paper [52] provided a second proof of Artin's reciprocity law, not depending on Tsen's theorem and being of "elementary" nature in the sense that only elementary manipulations of polynomials and rational functions are used. In doing this he distinguished two different cases, depending on the degree $n = [E : F]$, namely $n \not\equiv 0 \mod p$ and $n = p$. (Recall that p denotes the characteristic.) In the case $n \not\equiv 0 \mod p$ Hasse observed that his arguments are essentially identical to those which lead to the power reciprocity law (1) in F.K. Schmidt's thesis which we have discussed in section 3.2.2.

But the arguments in the case $n = p$, where Artin-Schreier theory had to be used, were new. Hasse's computations in this case involved logarithmic derivatives of rational functions. He pointed out that these are the precise analogues of Kummer's logarithmic derivatives which appear in the explicit reciprocity formulas in the number theory case. [64]

There arose the question whether those computations could be generalized to arbitrary function fields F, not necessarily rational. Hasse had put this question to his student H.L. Schmid. [65]

H.L. Schmid solved Hasse's question in his 1934 Marburg thesis which appeared in print one year later [93]. The paper has the title *Über das Reziprozitätsgesetz in relativ-zyklischen Funktionenkörpern mit endlichem Konstantenkörper* (On the reciprocity law in relatively cyclic function fields with finite fields of constants). It is conceived as a follow-up to Hasse's paper [52], with the aim of supplementing it by giving explicit formulas for the local norm symbols.

Let $E|F$ be cyclic of degree n. Following Hasse, H.L. Schmid deals separately with the two cases $n \not\equiv 0 \mod p$ and $n = p$.

The most interesting is the case $n = p$. Then $E|F$ admits an Artin-Schreier generation

$$E = F(y)\,, \qquad y^p - y = \beta$$

with $\beta \in F$. For any $\alpha \in F^\times$ and any prime $\mathfrak{p}$ of F consider the local norm symbol $\left(\frac{\alpha, E|F}{\mathfrak{p}}\right)$ as defined above in (20). For computational purposes it is convenient to replace this symbol, which is an element in the Galois group G, by another symbol which is an element in the prime field $\mathbb{Z}/p$. Namely, if the automorphism $\left(\frac{\alpha, E|F}{\mathfrak{p}}\right)$ is applied to y then the result is $y + c$ with c in the prime field $\mathbb{Z}/p$. This c is then denoted by $\left\{\frac{\alpha,\beta}{\mathfrak{p}}\right\}$; in other words, the defining relation for the new symbol is

$$y^{\left(\frac{\alpha, E|F}{\mathfrak{p}}\right)} = y + \left\{\frac{\alpha,\beta}{\mathfrak{p}}\right\}.$$

This symbol is multiplicative in the first variable α and additive in the second variable β. [66]

Now, H.L. Schmid gives the following explicit formula for the computation of this symbol:

$$\left\{\frac{\alpha,\beta}{\mathfrak{p}}\right\} = \mathfrak{S}\,\mathrm{Sp}_{\mathfrak{p}}\mathrm{res}_{\mathfrak{p}}\left(\beta\,\frac{d\alpha}{\alpha}\right). \tag{24}$$

[64] Hasse had discussed and generalized Kummer's formulas in Part II of his class field report [39].

[65] Not to be confused with F.K. Schmidt. For biographical information about H.L. Schmid see the obituary [56], written by Hasse in 1958.

[66] In the literature there is no unique notation for this symbol. Here we use H.L. Schmid's notation. Note that this symbol is asymmetric; there is no formula for exchanging the arguments α and β. See e.g., the notation used by Witt in [129].

Here, $\text{res}_{\mathfrak{p}}(\ldots)$ denotes the residue at $\mathfrak{p}$ of the differential in question; this is an element in the residue field $K_{\mathfrak{p}}$. [67] And $\text{Sp}_{\mathfrak{p}}: K_{\mathfrak{p}} \to K$ is the trace function ("*Spur*") to K, whereas $\mathfrak{S}: K \to \mathbb{Z}/p$ is the absolute trace from K to its prime field.

The formula (24) contains the logarithmic differential $\frac{d\alpha}{\alpha}$ which again puts into evidence the analogy to Kummer's formulas in number theory – this time for an arbitrary function field F instead of the rational field as in Hasse's paper. The importance of the formula lies in the following:

Firstly, in view of the theorem of the residues in function fields:

$$\sum_{\mathfrak{p}} \text{Sp}_{\mathfrak{p}} \text{res}_{\mathfrak{p}} \left(\beta \, \frac{d\alpha}{\alpha} \right) = 0 \tag{25}$$

it follows immediately from (24) that

$$\sum_{\mathfrak{p}} \left\{ \frac{\alpha, \beta}{\mathfrak{p}} \right\} = 0 \tag{26}$$

which is equivalent to the sum formula (18) for the algebra $\mathsf{A} = (\alpha, E, \sigma)$, i.e., for all algebras A over F which admit a cyclic splitting field $E|F$ of degree p. This proof of (18) does not need the theorem of Tsen. In this way it is possible to prove Artin's reciprocity law for cyclic extensions of degree p without Tsen's theorem, using the theorem of the residues instead.

Secondly, the formula (24) immediately gives the multiplicity of $\mathfrak{p}$ in the conductor of $E|F$: it is $m+1$ if $m \not\equiv 0 \mod p$ is the pole order of β at $\mathfrak{p}$. (Note that by definition, α is a norm from $E_{\mathfrak{p}}$ if and only if $\left\{ \frac{\alpha,\beta}{\mathfrak{p}} \right\} = 0$.) Whereas Hasse [52] had to rely on the theory of higher ramification groups and its connection to conductors, as developed in [50], the formula (24) shows this result immediately.

Thirdly, the formula (24) gives rise to a formalism about p-algebras over arbitrary fields of characteristic p; this has later been observed and used, e.g., by Witt [129] (see section 7.2).

Let us add some remarks concerning the case $n \not\equiv 0 \mod p$. It is assumed that the n-th roots of unity are in F. Then $E|F$ is a cyclic Kummer extension:

$$E = F(y)\,, \qquad y^n = \beta$$

with $\beta \in F$. Again for any prime $\mathfrak{p}$ of F, H.L. Schmid is concerned with the local norm symbol $\left(\frac{\alpha, E|F}{\mathfrak{p}} \right)$ which is an element of the Galois group G. The corresponding numerical symbol $\left(\frac{\alpha,\beta}{\mathfrak{p}} \right)_n$ is now to be defined multiplicatively, in the form:

$$y^{\left(\frac{\alpha, E|F}{\mathfrak{p}} \right)} = y \cdot \left(\frac{\alpha,\, \beta}{\mathfrak{p}} \right)_n .$$

[67] H.L. Schmid defines $\text{Sp}_{\mathfrak{p}} \text{res}_{\mathfrak{p}}(\cdots)$ to be the residue of the differential.

This time the symbol $\left(\frac{\alpha,\beta}{\mathfrak{p}}\right)_n$ is an n-th root of unity; it is multiplicative in both variables $\alpha\,,\beta$. [68] Now H.L. Schmid arrives at the following explicit formula. For simplicity let us write $a = v_{\mathfrak{p}}(\alpha)$ and $b = v_{\mathfrak{p}}(\beta)$. Then

$$\left(\frac{\alpha,\beta}{\mathfrak{p}}\right)_n = N_{\mathfrak{p}}\left((-1)^{ab}\frac{\alpha^b}{\beta^a}(\mathfrak{p})\right)^{\frac{q-1}{n}} \tag{27}$$

where $N_{\mathfrak{p}}$ is the norm function from the residue field to K. (For any function $f \in F$ we denote by $f(\mathfrak{p})$ its image in the residue field.)

H.L. Schmid points out that the formula (27) is the multiplicative analogue of (24). But, he says, while (24) leads to a new proof of Hasse's sum formula (18) (via the theorem of the residues) and hence to Artin's reciprocity law in the case $n = p$, the formula (27) does not so in the case $n \not\equiv 0 \bmod p$.

Hasse, when reporting about H.L. Schmid's work in 1958, also says that a multiplicative analogue of the residue theorem has not been found in this connection [56].

It seems that both H.L. Schmid and Hasse had overlooked the relation of the n-th norm symbol $\left(\frac{\alpha,\beta}{\mathfrak{p}}\right)_n$ to the universal symbol in arbitrary conservative function fields, over any base field. That universal symbol $\left(\frac{\alpha,\beta}{\mathfrak{p}}\right)$ is defined by the same formula (27) but without the exponent $\frac{q-1}{n}$ on the right hand side. It is well known that for this universal symbol the product formula

$$\prod_{\mathfrak{p}}\left(\frac{\alpha,\beta}{\mathfrak{p}}\right) = 1$$

holds. See, e.g., the treatment of those symbols in Serre's book on algebraic groups and class fields [105], or in [89]. This product formula, when raised to the $\frac{q-1}{n}$-th power, yields the product formula for $\left(\frac{\alpha,\beta}{\mathfrak{p}}\right)_n$ and hence formula (21), thus it gives a new proof of Artin's reciprocity law, independent of Tsen's theorem, also in the case $n \not\equiv 0 \bmod p$ – at least if the n-th roots of unity are contained in K.

7.2 The existence theorem

As we have said above already, Artin's general reciprocity law does not cover the existence theorem of class field theory as formulated in statement **I** in section 5.2. Witt takes up the challenge in his second paper [129] of his series on class field theory, which appeared 1935 in Crelle's Journal with the title "*Der Existenzsatz für abelsche Funktionenkörper*" (The existence theorem for abelian function fields).

The existence theorem is a major ingredient in general class field theory. In the case of number fields, the existence theorem had been part of the results of

[68] In the literature this symbol is called the *Hilbert symbol*.

Takagi [110]. The proof had been included in Hasse's class field report but was later much simplified by Herbrand and in Chevalley's thesis [15].

For function fields, F.K. Schmidt [101] had claimed to have a proof in the case when the index is not divisible by p. As I have said already in section 5.4 his claim was not too convincingly substantiated since F.K. Schmidt did not go into the details of proof which would involve delicate index computations. And for subgroups of index divisible by p, before Witt's paper there had been no hint of how to approach this problem.

Witt's paper constitutes a major advance in the development of class field theory for function fields. It is a masterpiece not only because of its results but also because of its concise and precise style which became the characteristic of Witt's papers. Witt's reputation as a first rate and very original mathematician was fully established with this paper.

The existence theorem can now be formulated as follows:

Theorem D. *Given a module* $\mathfrak{m}$ *in a function field* F *and a subgroup* $H_{\mathfrak{m}}$ *of finite index in the ray class group* $\mathcal{C}_{\mathfrak{m}}$*, there exists a unique abelian extension* $E|F$ *such that* (i) *every prime* $\mathfrak{p}$ *of* F *which does not appear in* $\mathfrak{m}$ *is unramified in* E*;* (ii) $H_{\mathfrak{m}}$ *is the kernel of the Artin homomorphism from* $\mathcal{C}_{\mathfrak{m}}$ *to the Galois group* G *of* $E|F$*.*

Consequently the factor group $\mathcal{C}_{\mathfrak{m}}/H_{\mathfrak{m}}$ is isomorphic to G in view of part **B** of Artin's reciprocity law (see section 6.1.1). Moreover, $H_{\mathfrak{m}} = N_{\mathfrak{m}}$ in view of part **C**.

Let n denote the exponent of the factor group $\mathcal{C}_{\mathfrak{m}}/H_{\mathfrak{m}}$. Witt discusses separately the two cases $n \not\equiv 0 \bmod p$ and $n = p$; the general case is then treated by induction.

It suffices to prove the existence theorem for the smallest ray class group in $\mathcal{C}_{\mathfrak{m}}$ whose factor group is of exponent n, i.e., for the group $\mathcal{C}_{\mathfrak{m}}^n$. For, if there exists a class field $E|F$ for this group, then the subgroups $H_{\mathfrak{m}}$ between $\mathcal{C}_{\mathfrak{m}}^n$ and $\mathcal{C}_{\mathfrak{m}}$ correspond, via the isomorphism of Artin's reciprocity law and Galois theory, $1-1$ to the intermediate fields between E and F; it is immediate that for each such subgroup (ii) holds with respect to the corresponding field.

Witt cites Hasse's class field results in [52] and says that, by Hasse, every abelian field extension $E|F$ is a class field for *some* ray class group. So he is going, for given $\mathfrak{m}$ and n, to construct a certain field extension $E|F$ by the usual algebraic procedures, namely Kummer extension in case $n \not\equiv 0 \bmod p$ (after adjoining the n-th roots of unity) and Artin-Schreier extension in case $n = p$; then he verifies that because of his careful construction of $E|F$ its Artin group is precisely $\mathcal{C}_{\mathfrak{m}}^n$. This requires some rather straightforward index computations.

In case $n \not\equiv 0 \bmod p$ there will be tame ramification only and $\mathfrak{m}$ can be assumed to be "square free", which means that every prime occurring in $\mathfrak{m}$ has multiplicity 1 in $\mathfrak{m}$. Witt says that his proof in this case is just a copy of "Herbrand's proof". For this he cites Hasse's Marburg lecture notes [45] where Herbrand's computations are presented. (As it is to be expected, the computations in the function field case require some modifications.) It seems strange that Witt does not cite Chevalley's thesis [15]. It is also strange that

Witt does not cite the paper by Chevalley and Nehrkorn [16] which appeared at about the same time as Witt's. In that paper the existence theorem is discussed (in the number field case) from the point of view of arithmetic-algebraic proofs (see section 8). Neither do Chevalley-Nehrkorn cite Witt, and hence it seems that none of the two parties knew about the work of the other party before it was too late to insert a reference. [69]

As a side result, Witt develops the theory of arbitrary abelian Kummer extensions (not necessarily cyclic) of a given exponent $n \not\equiv 0 \bmod p$. This is the form which today is usually given in algebra textbooks. [70]

The case $n = p$ was new and Witt could not rely on analogues in number fields; Herbrand and Chevalley did not cover this case. Witt relied, however, on H.L. Schmid's paper [93] and the formula (24); it permits to estimate in advance the conductor of an abelian extension of exponent p. Actually, Witt generalized H.L. Schmid's formula in the following way: For $\alpha \neq 0$ and β in F Witt defines the algebra $(\alpha, \beta\,]$ over F given by generators u, y with the defining relations:

$$u^p = \alpha\,, \qquad y^p - y = \beta\,, \qquad u^{-1}yu = y + 1\,. \tag{28}$$

Let $\mathfrak{p}$ be a prime of the function field F and $F_\mathfrak{p}$ its completion. Over $F_\mathfrak{p}$ one can perform a similar construction; the corresponding algebra is denoted by $\left(\frac{\alpha,\beta}{\mathfrak{p}}\right]$. (Witt's notation is $(\alpha, \beta]$ since he considers $\mathfrak{p}$ as being fixed.) Now Witt states and uses the formula

$$\left(\frac{\alpha,\,\beta}{\mathfrak{p}}\right] \sim \left(\frac{t,\,\mathrm{res}_\mathfrak{p}\,\beta\frac{d\alpha}{\alpha}}{\mathfrak{p}}\right] \tag{29}$$

where $\sim$ denotes the equivalence of algebras and t is a uniformizing variable at $\mathfrak{p}$. In this formula, like in H.L. Schmid's formula (24), there appears a logarithmic derivative. Witt mentions H.L. Schmid's paper [71] and refers to the proof there, although the formula (24) is not quite the same as Witt's (29): for, (24) concerns Hasse invariants of algebras, whereas (29) holds for the algebras themselves. Accordingly Witt's formula is more general: it holds over arbitrary perfect base fields while H.L. Schmid's formula (24) makes sense only if the base field is finite. But his formula, Witt says, is proved by the same methods as H.L. Schmid's. It is straightforward to extract H.L. Schmid's formula from Witt's.

Again as a side result, Witt develops the theory of abelian extensions of exponent p, not necessarily cyclic, thereby generalizing Artin-Schreier [8].

[69] In November 1934, Chevalley informed Hasse about the results of his paper with Nehrkorn. At the same time Chevalley announced that he was working on a proof of the existence theorem in characteristic p. Hasse replied that just recently Witt had submitted to him a paper containing the proof of the existence theorem, and he added a sketch of Witt's proof. He also informed Witt about Chevalley's letter. It appears that Chevalley's ideas were quite similar to Witt's on this matter. Thus Witt and Chevalley were informed about the work of the other through Hasse.

[70] See e.g., Lorenz [78].

[71] This had not yet appeared when Witt wrote his manuscript; so he referred to [93] by saying: "*erscheint demnächst in der Math. Zeitschr.*" (will appear soon in the Mathematische Zeitschrift).

It turns out that the computations in case $n = p$ are easier than those in case $n \not\equiv 0 \bmod p$. Let us cite Witt:

> *Ein Analogon der Theorie der Kummerschen Körper erhalten wir im Falle $n = p$, indem wir die Produkte additiv schreiben. Es ist bemerkenswert, daß der Existenzbeweis im vorliegenden Falle viel einfacher geführt werden kann. Durch direktes Schließen mit Hilfe des Riemann-Rochschen Satzes werden lange Indexrechnungen vermieden.*
>
> In case $n = p$ we obtain an analogue to the theory of Kummer fields by writing all products in an additive manner. It is remarkable that in this case the existence proof can be given in a much easier way. By direct recourse to the Riemann-Roch theorem one can avoid long index computations.

Added in Proof: In the meantime we have found among the papers of Hasse's legacy the correspondence with Claude Chevalley from 1931 to 1939. We plan to prepare a separate manuscript commenting on this correspondence. But it seems necessary to point out, here already, that Chevalley had a proof of the existence theorem at about the same time as Witt had. There is a letter of Chevalley to Hasse, dated 5 Nov 1934, in which he writes:

> *Cher Monsieur et Maître, Je viens de lire avec grand interêt vos deux derniers mémoires du Journ. de Crelle, relatif à la théorie des corps de caractéristique p. Je m'occupe actuellement de demontrer le théorème d'existence des corps de classes pour ces corps, dans le cas le plus difficile des extensions de degré p. J'espère pouvoir vous communiquer prochainement la démonstration...*
>
> Dear Sir and Master, I have read with great interest your two last papers in Crelle's Journal, concerning the theory of fields in characterique p. At present, I am working to prove the existence theorem of class field theory for those fields, in the most difficult case of extensions of degree p. I hope to be able to communicate to you the proof shortly...

In the 1934 volume of Crelle's Journal (vol. 172), Hasse had 5 papers but from the context of Chevalley's letter it is clear that he refers foremost to Hasse's paper [52] on cyclic extensions of function fields (see section 6.1). [72] Hasse replied on 16 Nov 1934, and wrote:

> *... Gerade als Ihr Brief ankam, gab mir Herr Witt eine sehr schöne Arbeit über das Existenztheorem des Klassenkörpers bei algebraischen Funktionenkörpern, in der insbesondere auch der Fall des Grades p (Charakteristik) sehr einfach behandelt wird. Dies scheint sich ja mit Ihren neuen Untersuchungen zu berühren...*

[72]The second article which Chevalley meant is Hasse's paper on differentials in function fields; Chevalley commented on this in the rest of his letter.

> ... Just as your letter arrived, Mr. Witt handed me a very beautiful paper on the existence theorem of class fields for algebraic function fields, in which in particular the case of degree p (characteristic) is discussed very simply. It seems that this overlaps with your new investigations...

Already one day later, on 17 Nov 1934, Chevalley replied to Hasse and sent him a preliminary exposé of his proof of the existence theorem. He explained his method, where he used the Riemann-Roch theorem for extensions with many ramified primes, and he is curious whether Witt uses the same method. Well, as we have seen, Witt indeed had used similar arguments. So we conclude that Chevalley and Witt had found very similar proofs at about the same time. Witt was a little earlier because he had already presented to Hasse a finished paper (which appeared in the next volume of Crelle's Journal: vol. 173) whereas Chevalley could only present a sketch of proof at this time.

7.3 Cyclic field extensions of degree p^n

With Artin's reciprocity law and the existence theorem, the foundation of general class field theory was now achieved in the function field case. But we have still to mention two other items which concern class field theory for function fields: Explicit reciprocity formula for cyclic extensions of p-power degree, and the functional equation for F.K. Schmidt's L-series. The relevant papers for these were published by H.L. Schmid and Witt who, both being assistants to Hasse in Göttingen, seem to have worked closely together.

In 1937, as a result of the legendary Göttinger *Arbeitsgemeinschaft* (workshop) headed by E. Witt, there appeared his great paper [132] where he introduced what is now known as Witt vectors. The construction of Witt vectors "*is of fundamental importance for modern algebra and some of the most recent developments in arithmetical algebraic geometry*" (G. Harder in [134], page 165).

It seems not to be widely known that Witt vectors were discovered in connection with a problem belonging to class field theory in function fields. The problem was to generalize H.L. Schmid's explicit formula for the norm symbol (24) (section 7.1) to cyclic extensions of p-power degree, not just of degree p. This was not necessary for the proof of Artin's reciprocity law or of the existence theorem since for those purposes one had an argument using induction with respect to the degree. But the problem was of importance in order to fully transfer class field theory, including the explicit reciprocity formulas, to the function field case.

Witt had transformed H.L. Schmid's formula (24) into (29) which concerned algebras of rank p; now the problem was to arrive at similar formulas for cyclic algebras of p-power rank.

In order to attack this problem one first had to generalize the Artin-Schreier generation of cyclic fields of degree p to cyclic fields of p-power. This had been done in 1936 by H.L. Schmid [94]. He had found that an earlier solution of the problem, given by Albert [1], was not suited for the intended arithmetical application. Instead, he had discovered that a cyclic field extension $E|F$ of degree p^n in characteristic p can be generated in the form

$$E = F(y) = F(y_0, \ldots, y_{n-1})$$

where the Witt vector $y = (y_0, y_1, \ldots, y_{n-1})$ of length n has components in E and satisfies an equation of the form

$$y^p - y = \beta \tag{30}$$

with a vector $\beta = (\beta_0, \beta_1, \ldots, \beta_{n-1})$ over F. In formula (30) one has to interpret y^p as the vector with the components y_i^p (as it is usual with Witt vectors), and the minus sign is to be interpreted in the sense of the additive group of Witt vectors.

H.L. Schmid, however, had not yet the formalism of Witt vectors at his disposal. Recall that every Witt vector $y = (y_0, y_1, \ldots)$ is also given by its "ghost components" (*Nebenkomponenten*) $y = (y^{(0)}, y^{(1)}, \ldots)$; the algebraic operations are given componentwise in the ghost components which yield polynomially defined operations for the main components y_i. [73]

Those polynomials are quite complicated to work with explicitly. It was a high accomplishment that H.L. Schmid was able to get through with the very complicated polynomial computations, proving associativity, distributivity etc. for those operations and, moreover, using this to study the arithmetic notions like Artin-symbol, norm symbol etc. in this situation.

H.L. Schmid had reported about his results in Witt's workshop. It was again a high accomplishment, this time by Witt, to see through this jungle of polynomial identities and to find out that it could be reduced to simple operations on the ghost components. This then was the birth of the Witt vector calculus, soon to be amended by Teichmüller's multiplication and so providing a solid foundation for the structure theory of complete unramified local fields.

Here we are interested in that part of Witt's paper [132] which concerns the arithmetic of function fields.

Given an element $\alpha \neq 0$ in F and a Witt vector $\beta = (\beta_0, \ldots, \beta_{n-1})$ over F of length n, Witt considers the algebra $(\alpha \,|\, \beta\,]$ defined by generators $u, y_0, \ldots, y_{n-1}$ where the y_i are commuting with each other and the following defining relations hold, with $y = (y_0, \ldots, y_{n-1})$ considered as a Witt vector:

$$u^{p^n} = \alpha\,, \qquad y^p - y = \beta\,, \qquad uyu^{-1} = y + \mathbf{1}\,.$$

Here, $\mathbf{1} = (1, 0, \ldots, 0)$ denotes the unit element of the ring of Witt vectors, and uyu^{-1} means $(uy_0u^{-1}, \ldots, uy_{n-1}u^{-1})$. These relations define a simple algebra with center F; it has the cyclic splitting field $E = F(y)$, and this is of precise

[73] The connection between ghost components and main components is defined in characteristic 0. Hence if we talk about ghost components of vectors over a field of characteristic p then we tacitly assume that the given field has been represented as the reduction mod p of some integral domain in characteristic 0, and the ghost components belong to foreimages of those vectors.

degree p^n if β_0 is not of the form $b^p - b$ with $b \in F$. The symbol $(\alpha|\,\beta]$, when considered as an element in the Brauer group over F, is multiplicative in the first variable α and additive in the second Witt vector variable β.

Note that the algebra $(\alpha\,|\,\beta\,]$, defined with Witt vectors, is a direct generalization of $(\alpha, \beta\,]$ which is defined by field elements (28). There arises the question whether for these new algebras the formula (29) can be generalized, and what this generalization looks like. Witt gives the following solution.

If $\mathfrak{p}$ is a prime of F then one can consider the same algebra over the corresponding $\mathfrak{p}$-adic completion $F_\mathfrak{p}$. This is to be denoted by $\left(\frac{\alpha\,|\,\beta}{\mathfrak{p}}\right]$. [74]

Consider the $\mathfrak{p}$-adic completion $F_\mathfrak{p}$ as power series field over the $\mathfrak{p}$-adic residue field. Consider the ghost components $\beta^{(i)}$ as power series and let the operation $\mathrm{res}_\mathfrak{p}\, \beta\, \frac{d\alpha}{\alpha}$ be defined ghost-componentwise. There results a Witt vector which Witt calls the "residue vector" and denotes by $(\alpha, \beta)_\mathfrak{p}$. The components of this residue vector are contained in the $\mathfrak{p}$-adic residue field. Now the analogue of (29) is as follows:

$$\left(\frac{\alpha\,|\,\beta}{\mathfrak{p}}\right] \sim \left(\frac{t,\,(\alpha,\,\beta)_\mathfrak{p}}{\mathfrak{p}}\right]$$

where, again, t is a uniformizing variable at $\mathfrak{p}$. For the computation of the Hasse invariant of this algebra similar formulas are available, which generalize H.L. Schmid's formua (24).

Without proof Witt mentions the relation

$$\sum_\mathfrak{p} S_\mathfrak{p}(\alpha, \beta)_\mathfrak{p} = 0$$

which is a generalization of the residue theorem, now for residues of Witt vectors. ($S_\mathfrak{p}$ denotes the trace from the $\mathfrak{p}$-adic residue field $K_\mathfrak{p}$ to K, extended to Witt vectors.)

In the same volume of Crelle's Journal as Witt's paper [132] there appeared another paper of H.L. Schmid on the arithmetic of cyclic fields of p-power degree [95]. There, building on the now established theory of Witt vectors he continues his investigation of [94]. Given a cyclic extension $E = F(y)$ of degree p^n with Witt vector generation $y^p - y = \beta$, H.L. Schmid establishes formulas for the conductor, the discriminant and the genus of E in terms of β. These formulas are very useful in various arithmetic and geometric applications. [75]

7.4 The functional equation for the L-series

In the preface to his paper [129] on the existence theorem, Witt mentioned the proof of the functional equation of the L-series for function fields as a further desideratum. According to his own testimony [133] he had completed the proof

[74] Witt writes $(\alpha\,|\,\beta\,]$ and regards $\mathfrak{p}$ as fixed.

[75] H.L. Schmid's paper [95] and Witt's [132] were two papers out of seven which all arose in the Göttingen workshop and which all appeared in a single fascicle of Crelle's Journal, together with a paper by Hasse.

one year later in 1936. But he abstained from publication because he was asked by Artin to do so; Artin had a doctoral student who was working on the same subject. [76]

Let χ be a non-trivial ray class character in F with conductor $\mathfrak{f}$. Thus $\chi(\mathfrak{a})$ is defined for divisors which are relatively prime to $\mathfrak{f}$, and $\chi(\mathfrak{a}) = 1$ if $\mathfrak{a} = (\alpha)$ with $\alpha \equiv 1 \bmod \mathfrak{f}$. For divisors $\mathfrak{a}$ which are not relatively prime to $\mathfrak{f}$ we may put $\chi(\mathfrak{a}) = 0$. Then F.K. Schmidt's L-series is

$$L(s,\chi) = \prod_{\mathfrak{p}} \frac{1}{1-\chi(\mathfrak{p})|\mathfrak{p}|^{-s}} = \sum_{\mathfrak{a}\geq 0} \chi(\mathfrak{a})|\mathfrak{a}|^{-s} .$$

The product ranges over *all* primes $\mathfrak{p}$ of F and the sum over *all* positive divisors; hence the dashes at $\prod$ and $\sum$ which appear in formula (16) are not necessary here.

The functional equation establishes a relation between $L(s,\chi)$ and $L(1-s,\overline{\chi})$ where $\overline{\chi}$ denotes the conjugate complex character; $\overline{\chi}$ has the same conductor $\mathfrak{f}$ as does χ. Let $d = \deg \mathfrak{f}$; then the functional equation can be written in the following form:

$$q^{(2g-2+d)s/2} L(s,\chi) = \varepsilon(\chi) \cdot q^{(2g-2+d)(1-s)/2} L(1-s,\overline{\chi}) \tag{31}$$

with $|\varepsilon(\chi)| = 1$.

Although not publishing this result, Witt presented his proof in the Göttingen seminar, and so in the course of time it became known in wider circles. We know about the proof from various sources, namely:

1. In a letter from Hasse to Davenport dated April 30, 1936, Hasse gave a three page outline of Witt's proof. [77]

2. In a letter from Hasse to A. Weil dated July 12, 1936, Hasse informed Weil about Witt's proof (among other number theoretic news) and included a sketch of it. [78]

[76] This was J. Weissinger; his proof of the functional equation appeared in 1938 [124]. – Later, Weissinger went to applied mathematics.

[77] This letter is contained among the Davenport papers at the archive of Trinity College, Cambridge. – Somewhat later Davenport himself gave another proof which, as Hasse said in a letter to Weil (Feb 4, 1939), proceeds in a more computational way ("*auf mehr rechnerische Art*").

[78] Weil seemed to have forgotten about it, for on Jan 20, 1939 he informed Hasse that he had a proof of the functional equation. In his reply Hasse mentioned Witt's proof again and sent Weil a new, more detailed exposition; but he also mentioned Weissinger's and Davenport's proof. Upon this Weil wrote to Hasse that he had checked Weissinger's proof which had already appeared in [124], and he found that his (Weil's) proof was essentially the same as Weissinger's. He also apologized to Hasse that he had forgotten Hasse's former information about Witt's proof in 1936. –

As a side remark it may be mentioned that in this letter Hasse informs Weil also about other news, one of them being Deuring's algebraic theory of correspondences of algebraic function fields (published later 1937 and 1941 in [24],[25]). Hasse explains to Weil that this theory will open the way to the proof of the Riemann hypothesis for function fields of arbitrary genus: one would have to prove that the algebraic analogue of the hermitian form from the period matrix

3. In the year 1943 the *Hamburger Abhandlungen* accepted a paper by H.L. Schmid and O. Teichmüller for volume 15, which contained essentially a presentation of Witt's proof as seen by those authors [96]. [79]

4. The recently published Collected Papers of E. Witt [134] contain a note, written by Schulze-Pillot, where Witt's proof is sketched after Witt's own handwritten notes (which are not dated, however).

It seems remarkable that Witt had conceived his proof as an analogue to the classical proof by Hecke [59] who worked with theta functions and the theta transformation formulas. Although in the function field case the analogues to these are purely algebraic identities and hence of quite another type, Witt named those algebraic lemmas in the same way as their classical counterparts – in order to stress the analogy between both. This analogy is not so transparent, however, in the presentation given by H.L. Schmid and Teichmüller [96].

Any known proof so far is based on a generalization of the Riemann-Roch theorem, much the same way as the functional equation of the ordinary zeta function by F.K. Schmidt is based on the ordinary Riemann Roch theorem. Perhaps it is not without interest to cite from the first paragraph of Hasse's letter to Davenport 1936 where he gave an outline of Witt's proof. (This letter is handwritten in English.)

> *The main source for Riemann-Roch's theorem and generalizations to character classes is, according to Witt, the following theorem:*
>
> *Let k be an arbitrary field and K the field of all power series $\sum_{\nu=\nu_0}^{\infty} a_\nu t^\nu$ with a_ν in k and t an indeterminate; furthermore:*
>
> R_1 *the ring of all polynomials in $\frac{1}{t}$ over k,*
> R_2 *the ring of all integral power series $\sum_{\nu=0}^{\infty} a_\nu t^\nu$ over k,*
>
> *both subrings of K.*
>
> *Let M be a matrix with determinant $\neq 0$, consisting of elements in K. Then there are a matrix A_1 over R_1 and a matrix A_2 over R_2, both with determinant a unit (element $\neq 0$ in k) such that*
>
> $$A_1 M A_2 = \begin{pmatrix} t^{\nu_1} & & 0 \\ & \ddots & \\ 0 & & t^{\nu_n} \end{pmatrix}$$

We observe that this theorem embodies the classical method of so-called "normal bases" which had been used by Dedekind-Weber and Hensel-Landsberg in

is positive definite. In his reply (dated July 17, 1936) Weil appreciated Deuring's promising idea ("*. . . es ist sehr schön, dass durch die Idee von Deuring nunmehr die Lösung dieses Problems in Aussicht gestellt wird.*"). And he adds a reference to Severi's "Trattato" [106]. It seems remarkable that Deuring's idea turned out to be precisely the same which A.Weil several years later used when he indeed arrived at the proof of the Riemann hypothesis [122].

[79] Due to war time difficulties of publication, volume 15 was completed in 1947 only.

proving the Riemann-Roch theorem, taken up by F.K. Schmidt (see section 4.3.2), and also used by Witt himself in his proof of the Riemann-Roch theorem in non-commutative algebras (see section 6.2). This same theorem appears here again in connection with Witt's proof of the functional equation. And this time it is formulated as a separate result, independent of the intended application.

Hasse was convinced that this theorem is an important result which should be included as a separate lemma (or theorem) in his textbook "*Zahlentheorie*" which he had finished in 1938. And the name "Witt's lemma" seemed to him justified because Witt had fully seen its importance, had used it in various different situations, and now had formulated it as a separate statement. Hasse was fully aware of the role of this lemma in the historical development connected to the Riemann-Roch theorem. In his letter to Weil on March 7, 1939 he says:

> *Den Inhalt von §1 habe ich übrigens in meinem Buch verarbeitet, indem ich von diesem Hilfssatz aus – der ja im Wesentlichen der Satz von der Existenz der Normalbasis ist – direkt zum Riemann-Rochschen Satz vorstosse, also ohne Einführung des Wittschen Formalismus mit der Theta-Funktion.*
>
> By the way, I have worked the content of §1 into my book, in such a way that starting from this lemma – which is essentially the theorem of the existence of a normal basis – I advance directly to the Riemann-Roch theorem, hence without introduction of Witt's formalism with the theta function. [80]

Geyer [34], p.125 has pointed out, however, that this "Witt's lemma" had been formulated already in various other situations in the course of history.

7.5 Summary

- *A student of Hasse, H.L. Schmid, gave in his thesis (1934) an amendment to Hasse's paper on cyclic extensions of function fields (see section 6.1). H.L. Schmid's main achievement was an explicit formula, involving logarithmic differentials, for the p-th norm symbol when p is the characteristic. As a side result, this yields another proof of Artin's reciprocity law, without referring to Tsen's theorem, for cyclic extensions of degree p, namely with the help of the residue theorem in function fields. H.L. Schmid found also an explicit formula for the n-th norm symbol in case* $n \not\equiv 0 \bmod p$*. But he failed to see that this too could have been used to give another proof of Artin's reciprocity law, in case* $n \not\equiv 0 \bmod p$*, namely with the help of the product formula for the universal symbol.*

[80] Hasse's "*Zahlentheorie*" appeared in 1949 only, with another publisher as originally planned. 11 years earlier (more precisely: in a letter dated Nov 28, 1938) the original publisher had rejected the book because it had become larger than originally planned. When Weil heard of this situation he wrote that he was highly indignant ("*aufs höchste empört*") about this situation and he offered to try to have Hasse's book published in France. But Hasse did not consider this possibility. Later, on June 8, 1939, the publisher accepted a recommendation by C.L. Siegel and, reversing his former decision, agreed to publish Hasse's book. But due to the outbreak of the second world war this could not be realized.

- *In 1935 Witt published a proof of the class field existence theorem for function fields. In the case of exponent $n \not\equiv 0 \bmod p$ his proof supersedes that of F.K. Schmidt, and he follows the methods of index computations as given earlier by Herbrand and Chevalley for number fields. In the case of exponent p he relies on H.L. Schmid's formulas for the norm symbol in order to have an estimate for the conductor of an abelian extension of exponent p. His methods are original and quite new. – This paper was the second in Witt's planned series devoted to class field theory for function fields. It provided the last missing stone for the building of general class field theory in function fields.*

- *In 1936 there appeared Witt's great paper where he introduces what is now called Witt vectors. The discovery of Witt vectors was intimately connected with problems from class field theory for function fields, namely the search for explicit formulas for the norm symbol in function fields. H.L. Schmid had done this in case of degree p, and now this became possible also in the cyclic case of p-power degree, thanks to the calculus of Witt vectors.*

- *In 1936 Witt arrived at a proof of the functional equation for F.K. Schmidt's L-series with ray class characters for function fields. The proof presents, in the function field case, the algebraic analogues to the analytic tools which Hecke had used in the number field case. Witt never published his proof; it is preserved as handwritten note only, by him and by other mathematicians who had heard him lecture on this. The main ingredient is still another variant of the Riemann-Roch theorem, whose proof is based on the classical method of "normal bases".* [81] *There is also a paper by H.L. Schmid and Teichmüller in which they present Witt's proof as they saw it.*

8 Algebraization

Let us recall the main steps in the foundation of class field theory for function fields which we have discussed above.

1. F.K. Schmidt's theory of the zeta function and the L-functions: 1927–31 (sections 4 and 5)
2. Hasse's proof of the Artin reciprocity law: 1934 (section 6)
3. Witt's existence theorem for function fields: 1935 (section 7)

In the thirties there arose the question whether the use of analysis was really necessary for the foundation of class field theory. Would it be possible to prove the Artin reciprocity law and the existence theorem without F.K. Schmidt's analytic theory?

[81] It seems not to be widely known in the mathematical public that Witt, in proving this Riemann-Roch theorem, preceded Rosenlicht [91] by 16 years.

Looking more closely into the matter we see that only little "analysis" was involved. For, F.K. Schmidt had proved that his zeta functions and L-series were rational functions and polynomials, respectively, in the variable $t = q^{-s}$. Hence what was considered an "analytic" argument turns out, from this point of view, to be "algebraic" after all. But this is algebra over the field of complex numbers, not over the given function field. Hence the search for algebraic proofs in the function field case did not so much care about the relation between algebra and analysis: it was the search for intrinsic, structural proofs which yield more insight into the relevant structures of function fields.

But the terminology was not quite clear. Some authors spoke of "algebraic" proofs, and some of "algebraic-arithmetic" proofs. It is not quite clear what should have been the difference between both. Some authors used "arithmetic" in the sense that the proof works for function fields over *finite* base fields: these are global function fields which resemble the global number fields most. But other authors used "arithmetic" also for function fields over arbitrary base fields; in such case the *methods* used were valuation theoretic or ideal theoretic and, in this sense the methods came from the study of ordinary arithmetic of algebraic numbers.

Let us here use the terminology of "algebraic proof" for a proof which avoids the use of zeta functions and L-functions and work with structures inside the function field only. In this sense, it turned out that the algebraization of class field theory for function fields was indeed possible. In the following we shall report on the work in this direction, and the results.

8.1 F.K. Schmidt's theorem

The first instance where the use of analysis had been found to be unnecessary was F.K. Schmidt's theorem. As explained in section 4.3.3, this theorem asserts that every function field $F|K$ with finite base field admits a divisor of degree 1. F.K. Schmidt was quite aware of the curious fact that his original proof used analytic arguments. In a letter to Hasse dated January 21, 1933 he comments on this: [82]

> *Bekannte Tatsache im Fall der algebraischen Funktionen mit bel. komplexen Zahlkoeffizienten! Aber hier, wo der Konstantenkörper ein Galoisfeld ist, keineswegs trivial, ja bisher nicht einmal rein algebraisch, sondern nur mit transzendenten Methoden beweisbar.*
>
> Known fact in case of algebraic functions with arbitrary complex numbers as coefficients! But here, where the field of constants is a Galois field, it is by no means trivial, up to now it is not even provable purely algebraically, but with transcendental methods only.

But some months later, on August 7, 1933, he reported to Hasse on a postcard:

[82]Strictly speaking, in this letter F.K. Schmidt did not directly refer to his theorem but to the following result which is a consequence of his theorem: *Every function field of genus zero is rational.*

> *Witt schrieb mir vorige Tage, er könne nun bei algebr. Fkt. mit einem Galoisfeld als Konstantenkörper rein arithmetisch einen Divisor von der Ordnung* 1 *nachweisen. Sein Beweis sei allerdings länger als mein analytischer. Leider teilte er mir aber seinen Beweis nicht mit.*
>
> Witt wrote me some days ago that he was able, in algebraic function fields with a Galois field as its field of constants, to construct a divisor of degree 1. However his proof was longer than my analytic proof. Unfortunately he did not convey his proof to me.

Witt included his proof in the first section of his paper [127] which appeared in 1934. Its title is "*Über ein Gegenbeispiel zum Normensatz*" (On a counter example to the norm theorem). This title does not give any hint that the paper also contains a new algebraic proof of F.K. Schmidt's theorem; this is perhaps the reason why Witt's proof did not become widely known at the time and was rediscovered several times. It seems that Witt included that proof because of the similarity of methods used in both cases: for F.K. Schmidt's theorem and for the discussion of the norm theorem for function fields. In the introduction to this paper Witt writes:

> *Für den Satz* "In einem Funktionenkörper über einem Galoisfeld gibt es Divisoren jeder Ordnung*" hat F.K. Schmidt einen sehr kurzen und eleganten analytischen Beweis gegeben. Vom algebraischen Standpunkt ist es wohl nicht unnütz, wenn bei dieser Gelegenheit ein gruppentheoretischer Beweis mitgeteilt wird.*
>
> The theorem "*In a function field over a Galois field there exist divisors of every degree*" has been given by F.K. Schmidt with a very short and elegant analytical proof. From the algebraic point of view it may not be superfluous if on this occasion we present a group theoretical proof.

The "group theoretical" proof which Witt mentions is of cohomological nature. Of course, Witt does not explicitly use the modern notions and notations of algebraic cohomology; they did not yet exist at the time. But in fact, Witt's computations can be interpreted as determining the Galois cohomology of the divisor group and related groups, with respect to the Galois group G of a finite base field extension $FL|L$. Here, L is chosen as the field whose degree $[L:K]$ equals the smallest positive divisor degree of $F|K$. This choice implies that every prime $\mathfrak{p}$ of $F|K$ splits completely in the extension FL. Hence the divisor group, as a G-module, is cohomologically trivial and from this Witt deduces that $[L:K]=1$.

In his computations Witt uses a technique very similar to what today is known as "Herbrand's lemma" in cohomology; note that G is cyclic and hence Herbrand's lemma is applicable. With today's cohomological formalism it is possible to rewrite Witt's algebraic proof such that it does not appear longer than F.K. Schmidt's. In fact, with only minor changes [83] Witt's proof yields the following more general

LEMMA 2. *If $E|F$ is a cyclic extension such that every prime $\mathfrak{p}$ of F splits completely in E then $E = F$.*

This is almost the statement of the Lemma 1 (see section 6.1.3) which Hasse had used in the proof of Theorem **B**. [84] The differences are that, firstly, here we deal with cyclic extensions whereas Lemma 1 refers to arbitrary abelian extensions. But this is not essential: if Lemma 1 holds for cyclic extensions (or only for cyclic extensions of prime degree) then trivially it holds for arbitrary abelian extensions. The second difference seems to be more essential: whereas in Lemma 1 it is assumed that *almost all* primes are completely split, in Lemma 2 this is required for *all* primes.

Now it has been shown by Chevalley and Nehrkorn [16] how to reduce Lemma 1 to Lemma 2. They show (in case $E|F$ is cyclic of prime degree) that if almost all primes $\mathfrak{p}$ of F split completely in E then there exists a field F' containing F, linearly disjoint to $E|F$, such that indeed *all* primes $\mathfrak{p}'$ of F' split completely in the composite field EF'; hence (using Lemma 2) $EF' = F'$ and so $E = F$.

Chevalley and Nehrkorn, however, discuss only number fields; their construction uses radicals, i.e., Kummer theory, and this is not always applicable in the function field case, not if the field degree equals the characteristic p. It has been observed by Moriya [82], [83] that the Chevalley-Nehrkorn construction works also in the case of degree p if Kummer theory is replaced by Artin-Schreier theory.

Accordingly, Lemma 1 can be reduced to Lemma 2, also in the function field case. Now we have said above already that Lemma 2 had been proved algebraically by Witt; more precisely, it could have been proved with the same cohomological arguments as are used in Witt's proof [127] of F.K. Schmidt's theorem. In 1937 Moriya [82] published a proof along similar lines as Witt's proof. He does not seem to have known Witt's paper because he says:

> *Ich vermeide es, diese Tatsache* [*daß es einen Divisor* 1*-ten Grades gibt*] *zu benutzen, weil man, soweit ich weiß, zum Beweis die Kongruenzzetafunktion zu Hilfe nehmen muß.*
>
> I avoid to use this fact [that there exists a divisor of degree 1] because, as far as I know, for its proof it is necessary to use the congruence zeta function.

[83] The changes are as follows: Witt uses the fact that the multiplicative group $L^\times$ of any finite extension L of K is cohomologically trivial, with respect to the action of the Galois group. Now with respect to any cyclic group action the cohomology of $L^\times$ may not be trivial, but since $L^\times$ is finite both cohomology groups $H^0(L^\times)$ and $H^1(L^\times)$ have the same order. That is what is actually needed. (H^0 is to be understood in the sense of Tate's modified cohomology.)

[84] We use the notations as introduced earlier: Theorems **A**,**B** and **C** are stated in section 6.1.1; they concern Artin's reciprocity law. Theorem **D** is the existence theorem in 7.2.

Moriya's paper carries the title: "*Rein arithmetisch-algebraischer Aufbau der Klassenkörpertheorie über algebraischen Funktionenkörpern einer Unbestimmten mit endlichem Konstantenkörper*" (Purely arithmetic-algebraic foundation of the class field theory for algebraic function fields in one indeterminate with finite field of constants). Thus his aim is precisely to eliminate the use of analytic arguments from class field theory for function fields. In the course of this he discusses Hasse's Theorem **B** and, as we have said above, reduces it to Lemma 2 and then presents a proof of Lemma 2. [85]

We have seen: Hasse's 1934 proof of Theorem **A** [52] was of algebraic nature. His proof of Theorem **B** was not, but Moriya [83] gave an algebraic proof in 1938. The methods are of cohomological nature and very similar to those which Witt used 1934 in his proof of F.K. Schmidt's theorem [127].

8.2 The new face of class field theory

Witt says in the introduction to his 1935 paper [129] on the existence theorem:

> *Die Voranstellung des Artinschen Reziprozitätsgesetzes hat eine große Wandlung mit sich gebracht. Die frühere Klassenkörpertheorie ist heute einer Theorie der abelschen Körper gewichen. Die früher an die Spitze gestellte Takagische Definition des Klassenkörpers hat heute eine andere Bedeutung. Sie dient nur noch zur Gewinnung eines handlichen Kriteriums für abelsche Körper. Ein solches Kriterium wird nämlich für den vollständigen Beweis des Existenzsatzes benötigt.*
>
> Putting Artin's Reciprocity Law first has brought great changes. Today the former class field theory has given way to a theory of abelian fields. Takagi's definition of a class field, which formerly had been the starting point, is today regarded from a different perspective. It is considered as a convenient criterion for abelian fields only. For,

[85] In the Hasse legacy at Göttingen we found several letters from Moriya to Hasse. One of them, dated 26 May 1937, informs Hasse that he, Moriya, is working on the algebraization of class field theory in characteristic p, in extension of work of F.K. Schmidt, of Hasse and of Chevalley–Nehrkorn. And he announces to Hasse a preprint of his paper [82]. Moreover, Moriya told Hasse that he was able to do class field theory for function fields with arbitrary absolutely algebraic field of constants of prime characteristic – with the same methods as were used in the class field theory over algebraic number fields of infinite degree. (Obviously he is referring to Chevalley's paper [17].) Moriya's results were then accepted by Hasse for publication in Crelle's Journal [84]. In another letter, dated 14 July 1967, Moriya says "*Ich erinnere mich immer an mein Leben in Marburg und ich bin sehr dankbar, weil ich dort bei Ihnen viel studieren durfte.*" (I will always remember my stay in Marburg and I am very grateful that I had the opportunity to study with you.) – We have mentioned this because these letters seem to indicate that Moriya's work on class field theory for function fields had been stimulated and supported by Hasse. Moreover, we infer from these letters that Moriya stayed in Germany during the Marburg years of Hasse, i.e., before 1934. This may explain why Moriya had not been informed, at the time he wrote his paper [82], about Witt's proof [127] of F.K. Schmidt's theorem which, as mentioned above, appeared in 1934 only. As pointed out already the title of Witt's paper does not indicate that it includes a proof of F.K. Schmidt's theorem.

such a criterion is necessary for the complete proof of the existence theorem.

What does this mean? What kind of changes did Witt have in mind?

Witt distinguishes between "former class field theory" and "theory of abelian fields". Takagi's definition of a class field, he says, is not fundamental in the body of the "theory of abelian fields" but of secondary importance only.

This seems to indicate that Witt proposes to include into the body of his "theory of abelian fields" the union of Theorems **A**, **B** and **D** only while Theorem **C**, referring to the Takagi groups, is separated and not regarded any more to be fundamental.

Note that Theorems **A** and **B** are those which we just have listed as having been proved algebraically. Is Witt's proof of the existence theorem **D** also algebraic ?

Witt himself does not discuss this question. But since he is separating Theorem **C** (which was not yet proved algebraically) from the body of the other theorems, he seems to have been aware of the problem. In his proof of the existence theorem he says that he accepts the full result of Hasse's paper which is partly based on analytic properties of L-series. But if one looks more closely into his proofs then it turns out that in fact, from Hasse he uses only Theorems **A** and **B** in order to prove **D**. Consequently, Witt's proof yields an algebraic foundation of what he calls "theory of abelian fields" in the function field case.

As to Theorem **C**, the case was discussed carefully by Chevalley-Nehrkorn in their 1935 paper [16] (which Witt seemed not to know). They presented a method showing how to reinterpret Witt's proof in a purely algebraic manner, such that at the same time it also yields **C**. This is a nice idea and is worthwhile to be discussed a little bit further. To be sure, Chevalley and Nehrkorn did not discuss function fields; they were concerned with class field theory in number fields (and apparently did not know Witt's paper). But the same idea applies to the function field case, and this was explicitly pointed out by Moriya [83].

Let $E|F$ be an abelian extension of degree n, and $H_{\mathfrak{m}}$ its Artin group for a suitable module $\mathfrak{m}$ (e.g., we can take for $\mathfrak{m}$ the conductor of $E|F$). By Theorems **A** and **B** we see that the index $(\mathcal{C}_{\mathfrak{m}} : H_{\mathfrak{m}}) = n$. The Takagi group, or norm group $N_{\mathfrak{m}}$ is a subgroup of $H_{\mathfrak{m}}$, as follows immediately from the definitions. Hence for the index $h_{\mathfrak{m}} = (\mathcal{C}_{\mathfrak{m}} : N_{\mathfrak{m}})$ of the Takagi group it follows

$$h_{\mathfrak{m}} \geq n\,. \tag{32}$$

This is the "*second*" inequality of classical class field theory. Usually, in classical class field theory one proves first the "*first*" inequality $h_{\mathfrak{m}} \leq n$, namely by analytic means. The "second" inequality then shows $h_{\mathfrak{m}} = n$ and hence $H_{\mathfrak{m}} = N_{\mathfrak{m}}$.

Thus in the new algebraic setting, the second inequality is proved before the first inequality! This has led several authors to rename those inequalities: what formerly was the first was now named second, and vice versa. As could be expected, this produced a certain amount of uncertainty. Anyone reading the literature of the time should be aware that the terminology in this respect is not uniform.

Using the inequality (32), be it called "first" or "second", Chevalley and Nehrkorn proved the following result which we state as a lemma.

LEMMA 3. *Let $F \subset E' \subset E$ be a tower of abelian fields over F, of degree n', n over F respectively. Let $\mathfrak{m}$ contain the conductor of $E|F$. If $h_{\mathfrak{m}} = n$ then also $h'_{\mathfrak{m}} = n'$. In other words: If $E|F$ is a class field in Takagi's sense then $E'|F$ is so too.*

This being said, Witt's existence proof can now be interpreted as being purely algebraic, also yielding **C**, as follows: We use the same notations as in our discussion of Witt's proof in section 7.2. Given $\mathfrak{m}$ and n, Witt's construction yields an abelian field $E|F$ whose Artin group is precisely $\mathcal{C}_{\mathfrak{m}}^n$ and coincides with its Takagi group. Then for all intermediate groups $H_{\mathfrak{m}}$ between $\mathcal{C}_{\mathfrak{m}}^n$ and $\mathcal{C}_{\mathfrak{m}}$ there exists, by Artin's reciprocity isomorphism and Galois theory, an intermediate field between F and E whose Artin group is $H_{\mathfrak{m}}$; using Lemma 3 we conclude that $H_{\mathfrak{m}}$ coincides with the Takagi group of that field.

A slight difficulty arises when $n \not\equiv 0 \bmod p$ and the n-th roots of unity are not contained in F. Then Witt, in order to construct the field E, has first to adjoin the n-th roots of unity and then perform his construction via Kummer theory. One has to be sure that the constructed field is abelian over the original field F. For this Witt uses Takagi's theorem **C** as a "convenient criterion for abelian fields", as he had announced in the introduction (see above). He sketches a new proof of this criterion [86] and informs us that it is based on an idea of Iyanaga.

8.3 Summary

- *In the thirties we observe a tendency to eliminate analytic arguments from the foundations of class field theory, in particular from class field theory for function fields. The motivation was to arrive at a better understanding of the underlying structures of class field theory.*

- *The first theorem which was freed from analytical proofs was F.K. Schmidt's theorem on the existence of a divisor of degree* 1*. Witt discovered an algebraic proof in 1933. He included his proof in his paper on a counterexample of the norm theorem, but it seems that it did not become widely known at the time.*

- *Chevalley and Nehrkorn 1935 supplied useful ideas for the algebraization of the proofs in class field theory. They discussed number fields only but Moriya in 1937 showed that their results could be transferred to the function field case.*

- *Consequently it became possible to give algebraic proofs of all main theorems of class field theory in the function field case. The main ingredients are* (i) *Hasse's algebraic proof of the sum relation for the local invariants of a*

[86] The criterion had been stated and proved already by Hasse in §5 of Part II of his class field report [39].

simple algebra, and (ii) *Witt's proof of the existence theorem. The latter rests on an idea of Herbrand in the case when the index is not divisible by the characteristic, and otherwise on H.L. Schmid's explicit formulas for the norm residue symbol.*

- *An exposition of the algebraic foundation of class field theory in function fields was given by Moriya in 1937.*

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