

Introduction

by Peter Roquette and Franz Lemmermeyer

This site contains the full text of all letters which had been exchanged between Emmy Noether and Helmut Hasse, as they are preserved in the *Handschriftenabteilung* of the Göttingen University Library. There are 82 such letters, dated from 1925 until Noether's sudden and tragic death in 1935.

The name of one of the correspondents, **Emmy Noether** (1882–1935), is known throughout the worldwide mathematical community. She has been said to be

- “*the creator of a new direction in algebra*”,
- “*the greatest woman mathematician who ever lived*”.¹

These are only some of the attributes which have meanwhile been bestowed upon her in so many articles and speeches. A number of scientific institutions and projects carry her name as an icon. There is a serious Noether literature trying to understand and evaluate her impact on the development of mathematics up to the present time. In the course of time her “*new direction in algebra*” has become standard not only in algebra but also in general mathematical thinking, namely to work with mathematical *structures*, these structures being based on *abstract axioms*. In a way our work can be viewed as part of this Noether literature, presenting a new and unique collection of *Noetheriana*. These Hasse - Noether letters are packed with mathematics, and they throw new light on how Emmy Noether arrived at her ideas and how she conveyed them to her peers. Through her open, unconventional and impulsive style, she allows us to have a glimpse not only into the working of her brain but also into her heart.

The other correspondent, **Helmut Hasse** (1898–1979), is known to the general mathematical public mainly through his *Local-Global Principle*. He has been said to be

- “*one of the most important mathematicians of the twentieth century*”,
- “*a man whose accomplishments spanned research, mathematical exposition, teaching and editorial work*”.²

¹Cited from Alexandrov's address in memory of Emmy Noether, in Moscow on September 5, 1935, reprinted in English translation in the Collected Papers of Emmy Noether.

²Cited from the Hasse-article by H. Edwards in the *Dictionary of Scientific Biography*, New York 1970-1990.

The main field of Hasse's research was Number Theory. Apart from the already mentioned Local-Global Principle (for quadratic forms and for algebras) he contributed substantial results to class field theory (explicit reciprocity laws, norm theorem, local class field theory, complex multiplication), the theory of algebraic function fields (in particular with finite base fields where he succeeded to prove the Riemann hypothesis for elliptic curves). He wrote several successful and seminal books.

From what we have said above, it may seem that these two mathematicians, Hasse and Noether, had somewhat different motivations and aims in their mathematical work. Whereas Hasse is remembered for his great *concrete results* in Number Theory, Emmy Noether's main claim to fame is not so much the theorems she proved but her *methods*. She herself has described her methods in one of her letters to Hasse (English translation): "*My methods are working and conceptual methods and therefore have spread everywhere*".³

But on the other hand, their letters show that there existed a mutual understanding on the basic intellectual foundations of mathematical work (if not to say of its "philosophy"). Both profited greatly from their contact. We see that not only did Hasse absorb Emmy's ideas on what was called "Modern Algebra" at that time, but conversely she became interested in the foundation of class field theory, to which she then contributed by proposing to use non-commutative arithmetic as a powerful tool. In the course of time there arose a close cooperation of the two, and also a friendship as is evidenced from the tone of their letters.

Accordingly this work is to be regarded as a contribution not only to the literature on Noether but also to the literature on Hasse.

Unfortunately, the Hasse - Noether correspondence is preserved on one side only, i.e., the letters from Noether to Hasse. The letters in the other direction, i.e., from Hasse to Noether, are probably lost – except for three letters of which Hasse had made a copy for himself. Thus the Hasse - Noether correspondence, as available today, consists of 79 letters from Noether to Hasse and only 3 letters from Hasse to Noether. For some time we have tried to locate the missing letters from Hasse to Noether. It is said that after her sudden death 1935 in Bryn Mawr, Noether's legacy had been put into a container and sent to her brother Fritz who was living in Tomsk in Siberia at that time. But apparently that container never arrived. And meanwhile we have found a letter from Fritz Noether to Hasse, dated October 2, 1935, from which we conclude that in all probability there are no more papers of his sister that have been preserved. We have included Fritz Noether's letter in our collection.

Because of the lost Hasse letters, many parts of Noether's text which refer to his letters seem incomprehensible on first sight. However we have been able, by using other sources, to clear up most of the doubtful passages. Accordingly we have supplemented Noether's letters by detailed comments. In these comments we not only provide explanations in the technical sense but we also try to describe, as to our present knowledge, the mathematical environment of

³The German text can be found in the last paragraph of the letter of November 12, 1931 and reads: "*Meine Methoden sind Arbeits- und Auffassungsmethoden, und daher anonym überall eingedrungen.*" We are not sure whether our translation of "*Arbeits- und Auffassungsmethoden*" faithfully reflects Noether's intentions but we have found no alternative.

Noether and that of Hasse, the mathematical tendencies of the time, and what was going on parallel to Hasse–Noether, as far as it is relevant to the text of the letters. Our comments appear immediately after the transcription of the respective letter. They are given in the language of Hasse and Noether, i.e., in German. We have abstained from translating the Hasse–Noether letters. In our opinion, the impulsive and unmistakable distinctive style of Emmy Noether can best be appreciated in her original language. Nevertheless, if there is some demand then we may offer English translations some time in the future.

As already said, all of the Hasse–Noether letters (and postcards, respectively) are packed with mathematics. The correspondence partners did not only inform each other about their final results, but they also freely exchanged mathematical ideas and speculations, even when these could not yet be formulated in a precise manner, or when a convincing reason was still lacking. Noether called them “fantasies”; we would perhaps say “visions”. Some of these visions have become reality and today belong to the basics of mathematics. Others have turned out not to be sustainable and had to be revised or abandoned. The Hasse–Noether correspondence is a rich source for those who are interested in the rise and the development of mathematical notions and ideas.

Not all the results and projects of Hasse or of Noether are touched in their correspondence. Their letters are restricted to those topics which were close to their common interest. These include:

- Axiomatic algebra,
- Class field theory ,
- Algebras and their arithmetics,
- Function fields.

In the 1920s and 1930s, those areas witnessed a particularly strong development whose effects can still be felt today. Emmy Noether and Helmut Hasse belonged to the outstanding protagonists of that development. From this viewpoint, the Hasse–Noether correspondence appears as a first rate historical document.

The exchange of letters between Hasse and Noether started in January 1925. At that time Hasse was 26 and held a position of *Privatdozent* at the University of Kiel. He had studied with Kurt Hensel in Marburg, and now strongly promoted the use of Hensel’s p -adic numbers in Number Theory. While in Kiel, Hasse had started a lively correspondence with Emil Artin in Hamburg, on explicit reciprocity laws and other topics of algebraic number theory; this continued until 1934. (The correspondence file of Artin–Hasse will be published some time in the future.) In 1924 Hasse had delivered a lecture course on class field theory in Kiel, of which there were lecture notes written by Reinhold Baer. Hasse’s aim was to simplify and streamline the foundations of class field theory. In September 1925 he would deliver, at the meeting of the German Mathematical Society in Danzig, his famous survey on class field theory which finally led to his 3-volume class field theory report. In April 1925 Hasse would be appointed full professor at the University of Halle. It appears that he became the youngest professor at German universities at the time.

Emmy Noether had come to Göttingen in 1915, and in 1925 she held a position as *Privatdozent* with the official title of *Außerordentlicher Professor*

(=associate professor; this title did not carry a salary; she never was promoted in Göttingen). In 1925 she could already look back on a successful mathematical career, in the sense that she had become a respected and highly valued member of the mathematical community. Originally she had worked under the influence of Hilbert and Klein, and her papers on differential invariants have become classic. In 1921 her seminal paper on ideal theory had appeared, where she had introduced and studied what today are called “Noetherian rings” on an axiomatic basis; more precisely: she had shown that in a commutative ring the various decomposition laws of ideals into primary ideals (which were well known for polynomial rings) can be obtained solely under the assumption that every ideal admits a finite basis. With this paper she started her own “*completely original mathematical path*” (Alexandrov) into abstract algebra. At the meeting of the German Mathematical Society 1924 in Innsbruck she had presented the axiomatic basis for the factorization of ideals into prime ideals (known for rings of integers in an algebraic number field). Today those rings are called “Dedekind rings”. Hasse was in attendance at that lecture and it left a great impression on him. In September 1925 she would deliver, at the meeting of the German Mathematical Society, her talk in which she outlined her ideas how to do representation theory, namely in the framework of the theory of abstract algebras. (This was at the same meeting and even in the same section in which Hasse delivered his survey on class field theory.)

In the first period of their correspondence, from 1925–1927, the topics of their letters do not seem to be particularly remarkable. They discussed Hasse’s attempt to obtain an axiomatic description of unique factorization domains, there was some discussion of a possible axiomatic foundation of class field theory, and there were Noether’s comments on how to present Galois theory in a text book or in a lecture course.

In this first period the tone of Noether’s letters sounds somewhat like from a teacher to her student: criticising and praising, passing good and not so good marks, encouraging and teaching. Noether, 16 years older than Hasse, was always ready to support young talents. We can observe that during this period, Hasse became quite interested in abstract axiomatic algebra as promoted by Emmy Noether.

This “teacher-student relationship” changed after 1927 when Noether had asked Hasse about the existence of certain cyclic number fields. Within a few days Hasse was able to provide her with the construction of those fields. These were to be minimal splitting fields of high degree for the quaternion algebra. In his construction Hasse used (among other tools) the Local-Global Principle for quadratic forms which he had discovered in his thesis. Emmy Noether was quite satisfied with Hasse’s result. In a joint note with Richard Brauer on algebras and their splitting fields, she referred to Hasse’s construction; Hasse’s own note appeared immediately thereafter. These two notes together contain the first instance where the Local-Global Principle for some algebra was established, namely for the quaternion algebra.

Subsequently Hasse became interested in the theory of algebras and their arithmetic, and on the other side Emmy Noether looked more closely into class field theory. In the following years there arose a close cooperation between Noether, Hasse and R. Brauer. As a result Hasse developed the arithmetic of

algebras by introducing and studying algebras over local fields; this turned out to be a powerful tool also for the foundation of local class field theory – shortly after he had discovered local class field theory through his (global) theory of the norm residue symbol. At the end of 1930, Hasse ventured to formulate his conjectures about simple algebras over number fields, including their Local-Global Principle and the fact that they are all cyclic. Noether first did not believe in those conjectures but soon became convinced, and both started their quest for the proofs. This finally culminated in the famous Brauer-Hasse-Noether theorem, obtained at the end of 1931. This was an exciting year, and the excitement is mirrored in the Hasse–Noether correspondence.

Parallel to this we see the making of Noether’s famous paper on integral bases for unramified extensions, we learn of her motivation, her aim and her ideas around it.

In this second period 1927–1931, Noether more and more addresses Hasse as a colleague and partner on equal terms.

In the third period, 1932–1935, we see from the letters the development of a heartfelt friendship. This starts with Hasse’s letter to Noether on the occasion of her 50th birthday on March 23, 1932. In the University of Göttingen her birthday was not officially noticed, but Hasse had made sure that the algebraists in and outside of Göttingen were informed. As for himself, he dedicated to her his paper on the structure of the Brauer groups of number fields, including a proof of Artin’s reciprocity law of class field theory by means of algebras. In Noether’s reply, and in all her subsequent letters we see her heartfelt affection towards her younger colleague Hasse. Mathematically, the later letters contain the attempts to generalize class field theory from abelian to arbitrary Galois extensions. Although these attempts did not reach their goal, it is of interest that the ideas of Noether, as well as the ideas of Artin and Hasse, went in the direction of what today is called cohomology theory.

The friendship between Hasse and Emmy Noether stood its test in the hard times of summer 1933 when Hasse tried everything in his powers to keep Noether in Göttingen. It is well known that all this was in vain and she had to emigrate. But the friendly contact between Hasse and Noether continued, as we see in her letters from Bryn Mawr. As always, these letters were packed with mathematics. She never complained about her situation but we observe a sad undertone in her letters, reflecting Noether’s wish to return to her beloved Göttingen – the place which she had become used to consider as her home, where she had started her mathematical career and where she had risen to the height of her power.

We have included in our collection 14 additional letters which, after the sudden death of Emmy Noether in April 1935, Hasse had exchanged with Hermann Weyl, Richard Brauer and Fritz Noether.

The transcription of the Noether letters turned out to be quite difficult, not only because Emmy used the old German handwriting of the 19th century, but also because of her really original style and her impulsive way of expressing herself. Most of the time she uses mathematical symbols without explanation, under the assumption (which was probably justified) that her correspondence partner will be able to find out what is going on. Formulas and diagrams are condensed to tiny spaces on postcards with much additional text, which often

is hardly decipherable. Except for some very few isolated spots we were finally able to read everything. Sometimes we wondered how Hasse has managed to deal with her writing.

For the convenience of the reader we have included a list of names, a list of mathematical terms, and a list of literature. Moreover, there is an appendix with short biographic sketches for all persons who are mentioned in this work. Whenever possible we have added references to more detailed biographical sources.

As to our two protagonists, Helmut Hasse and Emmy Noether, we have refrained from including biographies since, after all, the relevant data can be readily found in the literature. (See, e.g., our bibliography.) Instead, as an introduction to their world of ideas we have included two articles written by contemporaries. These are, first, van der Waerden's obituary for Emmy Noether and, second, Leopoldt's address on the occasion of Hasse's 50th anniversary of his doctorate. The two authors belong to the leading mathematicians of their time. What is more, each of them, in some period of his mathematical life, had been a disciple of his protagonist: Van der Waerden of Emmy Noether and Leopoldt of Hasse. So both were acquainted at first hand with the mathematical viewpoints and philosophy of their mentor. They also had close personal contacts to her/him. Their articles, although written in different times and under different circumstances, show not only competence but also a warm attachment which makes them the more valuable as contemporary witnesses for the influence of Noether's and of Hasse's personalities. Thus these articles themselves have become important historical documents.

REMARK: In 1970, on the occasion of the 35th year after Emmy Noether's death, there appeared the short and empathetic biography of Emmy Noether by Auguste Dick. By looking through the pages of that biography one finds a number of citations from Noether's letters to Hasse. We conclude that Auguste Dick had access to those letters. But in 1970 Hasse was still alive; at that time the Hasse legacy was not yet in the Göttingen library. So it seems that Dick had established personal contact to Hasse and that he had shown her the letters. In fact, in the meantime we have found a letter from Auguste Dick to Hasse, dated September 7, 1967 where we read (English translation):

“Respected Herr Professor, your mailing has surprised me and at the same time made me very happy. I do not know how to thank you. Already while preliminarily browsing through the 85 documents I have found much material which is quite important to me . . . May I ask you to let me have those valuable documents until the beginning of November? . . .”

We see that Hasse had sent her the whole collection of his correspondence with Emmy Noether. (And he had sent the originals, for at that time the Xerox copy method was not as familiar as it is today.) Thus all the letters which are published in this volume have already been in the hands of Auguste Dick. But apparently the possibility of publishing *all* the letters was *not* discussed between Hasse and Dick, when they met in November 1967. She only asked for permission to cite those pieces of the text which we now can read in her book. And Hasse granted this. Thus the Hasse–Noether letters had to wait 35 more

years for their complete publication.

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NOTE: All the letters from which we are citing in this work are contained in the Hasse legacy at the *Handschriftenabteilung* of the University Library in Göttingen (Cod. Ms. H. Hasse) – except when we explicitly refer to another source. The cited letters from Hasse to Davenport are contained in the Davenport legacy at Trinity College, Cambridge, England. The cited letters from Hasse to R. Brauer are contained in the Brauer legacy which is now at the *Handschriftenabteilung* in Göttingen too.

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