Emmy Noether's contributions to the theory of group rings

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*This is a fragment of a manuscript written for a talk given at the Conference "*Around Group Rings*" at Jasper, Alberta, Feb. 18 - 21, 2002. It is planned to use it, together with our manuscript of March 17, 2001 on the Brauer-Hasse-Noether theorem, as a basis for a larger exposition on the history of the theory of algebras.

1 The beginning

1.1 The DMV-meeting in Danzig

The story which I will tell in this paper begins in the town of Danzig at the Baltic sea. There, in the year 1925, the German Mathematical Society (*DMV*) held its annual meeting from Sep 11 to 17. The conference program contained for the session of Tuesday Sep 15 afternoon the following entries:

Dienstag, den 15. September, nachmittags 4,00 Uhr
Vorsitz: *Hensel.*

1. H.Hasse, Halle a.S.: *Neuere Fortschritte in der Theorie der Klassenkörper.* (Referat, 60 Minuten)
2. Friedrich Karl Schmidt, Freiburg i.B.: *Zur Körpertheorie.* (20 Minuten.)
3. **E. Noether, Göttingen: *Gruppencharaktere und Idealtheorie.* (20 Minuten.)**
4. Karl Dörge, Köln: *Zum Hilbertschen Irreduzibilitätssatz.* (20 Minuten.)

For us, the third entry announcing a talk by Emmy Noether will be of interest. But before discussing this let us look briefly at the other entries.

The first talk is announced by Helmut Hasse and is labelled "*Referat*", which means "survey". This signifies that Hasse's talk was an invited lecture, and we see that its time allowance (60 minutes) was larger than the time allowance for the other talks (20 minutes). Hasse's talk stirred much interest at that time; it was the basis of Hasse's famous class field report [9] which appeared in three parts between 1926 and 1930, and which started an enormous development of class field theory and related problems. Through this talk, E. Noether seems to have got interested in class field theory. Some time later she started, in contact with Hasse and R. Brauer, the "abstract eludication of class field theory" [1]). At the time of the Danzig meeting, Hasse was 27, and he just had accepted his first professorship at the university of Halle.

[1]) cited from van der Waerden [28]

The second speaker, Friedrich Karl Schmidt (24), had just completed his doctoral thesis at Freiburg. We do not know precisely the contents of his talk but there is reason to believe that he talked about the subject of his thesis, a foundation of the arithmetic in function fields over finite base fields. See our discussion in [23]. The important fact in Danzig was that he met Hasse, that he was impressed by the results and problems of class field theory which Hasse talked about, and that in consequence he started his investigations on class field theory for function fields. In his paper [24] he introduced the zeta-function of a curve and its connection to the Riemann-Roch theorem. This paper became a classic and led to a quick development of what today would be called arithmetic geometry for curves over finite fields – including Hasse's proof of the Riemann hypothesis for elliptic curves, followed by A. Weil's proof for arbitrary curves. See [22].

The fourth talk was by Dörge who has published several papers on Hilbert's irreducibility theorem, simplifying and strenghtening the known results. It seems that through his talk, Hasse became interested in this subject; some time later he suggested to one of his doctoral students, W. Franz, to investigate the validity of Hilbert's irreducibility theorem over arbitrary abstract fields [8]. Again, this started a long and interesting development, which has today become part of what is called Field Arithmetic for which we refer to [11].

E. Noether's talk, which we will discuss here, also contained seminal ideas which triggered the development of the arithmetic of algebras, in particular group rings. We see that this afternoon session at the Danzig meeting 1925 carried high potential for future developments, in a concentration rarely found at such meetings.

The chairman of the meeting was Kurt Hensel, at that time the grand old man (64), respected as the discoverer and explorer of p-adic numbers.

1.2 Emmy Noether

The abstract of E. Noether's talk is contained in her Collected Papers [21]. Its first sentence reads:

> *Die Frobeniussche Theorie der Gruppencharaktere – also der Darstellung endlicher Gruppen – wird aufgefasst als Idealtheorie eines vollständig reduziblen Ringes, des Gruppenringes.*
>
> Frobenius' theory of group characters – that is, of representation of finite groups – is interpreted as ideal theory of a completely reducible ring, the group ring.

Here, "group ring" is to be understood over a field whose characteristic is not a divisor of the group order. In particular the field may be the field of complex numbers, as it is the in Frobenius' theory.

Today we are used to Noether's viewpoint that matrix representations of a group are given by ideals or modules of the group ring. But it was a revolutionary viewpoint in the early twenties. Emmy Noether not only introduced and propagated it but she developed the theory to a high degree in its simplicity and beauty. The first time she had come into the open with these ideas had been 1924 in Göttingen when she delivered a lecture course on the close relationship between representations, modules and ideals. For this lecture course the general description of van der Waerden [28] about Noether's lectures applied:

> *Es wurden fast nie fertige Theorien vorgetragen, sonderns meistens solche, die erst im Werden begriffen waren. Jede ihrer Vorlesungen war ein Programm.*
>
> Completed theories were almost never presented, but usually those that were still in the making. Each of her lecture series was programmatic.

Now in Danzig she reported about this program. The last sentence of the abstract reads:

> *Eine ausführliche Darstellung soll in den Mathematischen Annalen erscheinen.*
>
> A detailed account is to appear in the *Mathematische Annalen.*

But it took four more years until her paper [20] finally appeared in 1929 (and not in *Mathematische Annalen* but in *Mathematische Zeitschrift*). This delay reflects the style of how she was working: When she saw the general lines clearly in her head, the details were gradually developing in private conversations or in the lecture room, or by writing letters. She was a great talker and a great letter-writer. [2]) Her talk and her letters were spontaneous. It could happen in one of her lectures that she abruptly interrupted a proof and started it anew, using a new idea which had just occurred to her. [3]) As to her letter-writing: in many of her letters and postcards we find words or sentences crossed out, and the replacement text scribbled between the lines or on any free white spot on the paper which seemed sufficiently large to her.

[2]) The letters of E. Noether to Hasse, as well as her letters to R. Brauer, are preserved in the archives of Göttingen and Bryn Mawr, respectively.

[3]) According to oral communication by Heinrich Grell. – Grell had been one of the "Noether boys", as were called the students in her mathematical circle.

But always we see her great vision in her letters and lectures.

It fits into this picture of E. Noether that her great paper [20] on representation theory and group rings of 1929 was essentially prepared not by herself but by van der Waerden, based on her Göttingen lecture course in the winter semester 1927/1928. He put down her spoken word into writing. Not without some corrections though, for in a footnote of the paper she expresses her thanks to him for his critical remarks.

Prior to publication in 1929, Noether presented a brief summary of the paper at the International Congress of Mathematicians in Bologna 1928. At the time of the Bologna talk, the paper was already completed and submitted.[4]) The title of her talk was:

> *Hyperkomplexe Größen und Darstellungstheorie in arithmetischer Auffassung.*
>
> Algebras and representation theory in an arithmetical setting.

As to the terminology, let us note that Noether always used "*Hyperkomplexes System*" instead of "algebra" which is used today.[5]) The "hyperkomplex" terminology was widely used in the 19th century. The term "algebra" was common with english writing authors, e.g. with Wedderburn, Dickson, Albert. But by and by the terminology "algebra" became popular also for european writers, even in the circle around Noether. For instance, Deuring uses already "*algebra*" in his book [7]. But in any case, whether "algebra" or "hypercomplex system" was used, it was always understood over a base field. Today we use "algebra" also over a commutative ring. Thus, if k is a commutative ring then a k-algebra A is defined to be a (possibly non-commutative) ring, which at the same time is a right k-module such that

$$ab \cdot \kappa = a \cdot b\kappa = a\kappa \cdot b \qquad \text{for} \;\; a, b \in A, \kappa \in k\,.$$

As to the expression "arithmetical" setting which Noether used in the title of her Bologna talk, today we would say that it is an "ideal theoretic" or, better still, "module theoretic" setting. The concept of ideals and modules originated from algebraic number theory (Dedekind), and for some time the term "arithmetic" was used for theories or proofs which used ideals and modules.[6]) The title of her final paper [20] avoids mention of this "arithmetical

[4]) The manuscript had been received by the editors of *Mathematische Zeitschrift* on Aug 12, 1928.

[5]) If Noether speaks of hypercomplex "*Größen*" then this includes all entities which are connected with hypercomplex systems: elements, ideals, modules, automorphisms etc.

[6]) But this terminology was not uniformly used. A number of authors reserved "arithmetic" for topics which directly belong to theory of numbers. In the older literature, "arithmetic" was sometimes used to describe what we would call "algebraic", in contrast to "geometric" or "analytic".

setting".

In the following we will use the terminology which is standard today, even if it differs from Noether's. In particular, we use "algebra" also over commutative rings, not necessarily fields.

The avowed aim of Noether's paper [20] was the development of a new theory of group representations via the group ring. Under this aspect, it seems odd on first sight that in this long paper only the very last section (§26) deals explicitly with group rings, and this only on $1\frac{1}{2}$ pages. The other 50 pages are concerned with concepts from abtract algebra for which, as Noether explains, the group ring and its ideals are an example.

We are reminded of what van der Waerden has said about Noether's maxim:

> *Alle Beziehungen zwischen Zahlen, Funktionen und Operationen werden erst dann durchsichtig, verallgemeinerungsfähig und wirklich fruchtbar, wenn sie von ihren besonderen Objekten losgelöst und auf allgemeine begriffliche Zusammenhänge zurückgeführt sind . . . Sie konnte keinen Satz, keinen Beweis in ihren Geist aufnehmen und verarbeiten, ehe er nicht abstrakt gefaßt und dadurch für ihr Geistesauge durchsichtig gemacht war.*
>
> All relations between numbers, functions and operations become clear, generalizable and truly fruitful only when they are separated from their particular objects and reduced to general concepts . . . She could not accept and assimilate a theorem or proof before it had been abstracted and thus made clear in her mind's eye.

Indeed, Noether's paper [20] in all its beauty and glory is an excellent specimen for this kind of thinking. Although she was heading for group algebras over a field, the first 50 pages are devoted to abstract algebraic theorems.

2 Algebras and representations

2.1 Groups with operators

Let us now discuss her paper [20] in some more detail. The great length of this paper is partly explained by the fact that Noether had to start from scratch. She could not expect the reader to have a background of the fundamental algebraic concepts which are familiar to everyone today. Van der

Waerden's famous textbook "Moderne Algebra" [27] had not yet appeared.[7]) Hence she included a preliminary chapter (16 pages) developing the algebraic prerequisites which were necessary to follow her arguments.

This Chapter I discusses groups with operators – a notion which Noether attributes to Krull [16].[8]) An abelian group with operator is nothing else than a module-

The basic functorial properties of groups with operators are developed:

- isomorphisms, homomorphisms and factor groups
- theorem of Jordan-Hölder
- direct product decompositions
- completely reducible groups

This is covered in a form which essentially has become textbook standard today.

A group is completely reducible if it is the direct product of finitely many simple groups. Noether proves the uniqueness (up to isomorphisms) of the direct components, using the Jordan-Hölder theorem.[9])

2.2 Wedderburn theorems

The next Chapter II is devoted to the Wedderburn structure theorems for algebras. Noether considers algebras A over a commutative ring k. Such an algebra A is called *completely reducible* if it is the direct sum of finitely many simple right ideals. An equivalent condition is that A satifies the minimum condition and contains no nilpotent ideal. The following structure theorems are proved and turn out to be central in the paper:

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[7]) In one of the footnotes in [20] she mentions that van der Waerden is writing a book which is scheduled to appear in Springer's "*Grundlehren der Mathematischen Wissenschaften*". In this book, she says, some further simplifications (by further abstraction) of her theory will be given. The book appeared 1930/31. Van der Waerden says in [29] that two chapters of his book, those on algebras and representations, are almost entirely due to Emmy Noether (except one §).

[8]) As a young student W. Krull had studied two years 1920/1921 in Göttingen. These years under the guidance of Emmy Noether had a formative influence on all of his later mathematical work. – Krull received his doctors degree 1921 in Freiburg with Loewy. In Freiburg he met F. K. Schmidt who was some years younger, and so the latter became acquainted early, through Krull, with Noether's ideas about modern algebra.

[9]) Noether does not discuss the more general uniqueness theorem, valid for arbitrary groups with a finite composition series, concerning direct decomposition into directly indecomposable groups. For that she refers to Krull [16] in the commutative case, and the Russian mathematician O. Schmidt [25] in general. Today this uniqueness theorem is named Krull-Schmidt. Sometimes the name of Remak is added since Remak had proved the theorem for finite groups in his Berlin dissertation (1911).

Wedderburn Structure Theorems. (i) Any completely reducible algebra A admits a unique direct sum decomposition into finitely many simple two-sided ideals:

$$A = A_1 \oplus A_2 \oplus \cdots \oplus A_s\,.$$

Each A_i is a completely reducible simple algebra.

(ii) *Any completely reducible simple algebra A is isomorphic to the ring of $n \times n$ matrices over a division algebra D. The integer n is uniquely determined as the length of a composition series (for right ideals). And D is determined (up to isomorphisms) as the field of endomorphisms of a simple right ideal in A.*

Because of (i), the direct decomposition into simple algebras, A is also called "semisimple" instead of "completely reducible", but Noether in [20] does not use this terminology.

The essential statement is part (ii) which Noether labeled as "*Hauptsatz*" (main theorem). Its original proof is due to Wedderburn [30] 1908, for algebras over a field. Artin in [4] says:

> This extraordinary result has excited the fantasy of every algebraist and it does so in our day.

Another proof was contained in Dickson's book "Algebras and their Arithmetics" [6]. Dickson's book had appeared in 1923 and we know that Noether had read it. She had put Dickson's theorem and proof into her abstract setting – quite according to van der Waerden's statements cited above. And the above theorem was the result. It is true that the ideas of her proof are rooted in Wedderburn's original, but it is her setting into the abstract theory of rings and modules which turned out to be fruitful for future developments. Subsequently there have been given a number of simplifications and generalizations but any of those proofs bear the marks of Noether's ideas, up to Tate's proof arrangement presented by Artin in [4] which is of utmost elegance.

Today we consider it somewhat strange that Noether restricted her investigation to semi-simple algebras, thus excluding group rings over fields whose characteristic divides the group order. With not much more effort she could have treated the case of arbitrary algebras (containing a unit element) which satisfy the minimum and the maximum condition for right ideals. [10]) We

[10]) The fact that for algebras, the minimum condition implies the maximum condition, was discovered in 1938 only, by C. Hopkins [13], [14] and about the same time by J .Levitzki [19]. – Levitzki had been a student of Emmy Noether. In his 1929 doctoral thesis [18] he had presented an extension of Noether's paper [20]. Namely, Frobenius' theory of induced characters, which relate characters of a subgroup to those of the whole group, is put into the framework of algebras.

observe in other situations that she always strives to cover the most general result which may be achieved by her methods. Why didn't she do it here?

An explanation may be that her starting point and main concern was the Frobenius theory of representations which referred to the field of complex numbers as coefficients, where complete reducibility holds for group rings. She was well aware that in the general case complete reducibility may not hold. In that case, a representation space is not necessarily a direct sum of simple ones, but it can still be decomposed into simple ones by a Jordan-Hölder series. This lead Noether to concentrate on the simple, i.e., irreducible representations. Such simple representation of A can be regarded as a representation space of the radical factor algebra A/R. Noether proves that any algebra A satisfying the minimal and the maximal condition for right ideals contains a largest nilpotent ideal R which she calls the radical, and that A/R is completely reducible. In this way she reduces representation theory of algebras to the semi-simple case.

The structure of group algebras over fields of characteristic $p > 0$ is much harder to describe if p divides the group order. The investigation of these group algebras was started by R. Brauer [5].

Noether is fully aware of the possible generalization of the structure theorem to non-semisimple algebras with minimum- and maximum condition. She cites Artin's paper [2] of 1927 which does precisely this. According to Artin, the necessary changes in the Structure Theorem are:

- ad (i): The components A_i are not necessarily simple but what Artin calls "primary" (*primär*), which means that its radical factor algebra is simple.
- ad (ii): D is not necessarily a division algebra but it is what Artin calls "completely primary" (vollständig primär), which means that its radical factor algebra is a division algebra.

Artin does not cite Noether (except generally for her ideas about chain conditions for ideals) but it seems safe to assume that Artin knew about Noether preparing her paper on representation theory because they were in close mathematical contact. We note the dates: Artin's paper had been submitted to the "*Hamburger Abhandlungen*" in January 1927; Noether's lecture whose notes were afterwords edited as her paper, was delivered in the winter semester 1927/28. Thus Artin could not have seen Noether's completed paper before writing his own. But it is intersting to note that Artin, like Noether, cites Dickson's book [6]. That book had initiated quite a remarkable development in Europe.

We would like to mention that Jacobson [15] in 1945 presented a further generalization of Wedderburn's theorem, replacing the minimal condition by the requirement that there exists a minimal right ideal. [11]) The arguments which Jacobson used are completely in the line of Noether's and could have been given already at Noether's time in the twenties. [12]) We refer to Artin's article [4].

2.3 Representations and modules

The next Chapter III of [20] gives the representation theory of semi-simple algebras. Noether defines a representation of an algebra into a commutative ring L as a homomorphism

$$A \longrightarrow \mathcal{M}_n(L)$$

which sends A into the ring of $n \times n$ matrices with entries in L; the integer n is the *degree* of the representation. If A is a K-algebra then L should also be a K-algebra and the homomorphism should be a K-algebra homomorphism. Thus Noether starts from the classical concept of representation which deals with matrices. She then sets out to explain that any such representation is given by a bi-module M with respect to (A, L) which is a free L-module of rank n. Given any L-basis of M she explains how the matrix represention arises in the way we are used today.

In a footnote she mentiones an idea of van der Waerden who proposes to change the definition of representation such that it will not be necessary to refer to a basis of M, namely:

$$A \longrightarrow \mathrm{End}_L(M)$$

where M is a free L-modul of rank n [13]) and End_L denotes the algebra of L-endomorphisms. [14])

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[11]) In the Zentralblatt review of [15] it is mentioned that the same result had been discovered independently by Chevalley.

[12]) In the years 1934/35 Jacobson, who was 25, studied at the Institute for Advanced Study in Princeton where he attended lectures by E. Noether. The subject of these lectures was representation theory. We infer this from a letter of Noether to Hasse, dated March 6, 1934 where she writes: "*Ich habe mit Darstellungsmoduln, Gruppen mit Operatoren angefangen; Princeton wird diesen Winter zum ersten Mal, aber gleich gründlich, algebraisch behandelt...*" (I have started with representation modules and groups with operators; in this winter Princeton is treated algebraically for the first time but quite thoroughly ...). Jacobson himself in his memoirs described his contact with Emmy Noether as a "memorable experience". In Noether's report to Hasse about we read further: "*Ich merke aber, dass ich vorsichtig sein muss; sie sind doch wesentlich an explizites Rechnen gewöhnt, und einige habe ich schon vertrieben.*" (But I realize that I have to be careful, they are after all essentially used to explicit computation, and I have already driven off some of them.) By "they" she refers to her audience, of which she named Albert and Vandiver.

[13]) Noether speaks of "*Linearformenmodul*" (module of linear forms)

[14]) Noether speaks of "Lineare Transformationen" (linear transformations).

3 Dickson

Let us insert some remarks about Dickson's book and its consequences. Dickson's book is written quite elementary without any prerequisites; it is on a textbook level, addressed to students who wished to learn the subject. It contained the first full presentation of the theory of algebras which included Wedderburn's theorem. In addition, it contained the first steps towards a non-commutative arithmetic; his important idea was to build such theory over *maximal* orders instead of arbitrary ones. Dickson's book had a great influence in German quarters. We said already that E. Noether was inspired by it to develop her new framework for group representations. Another attentive reader was Speiser who developed the first steps towards a non-commutative arithmetic in maximal orders [26]. On Speiser's initiative Dickson's book was translated into German, and Speiser added an appendix where he explained his ideas. This again inspired Artin to give a complete exposition of non-commutative arithmetic in algebras, including the theory of Brandt's gruppoid for the multiplcative structure of two-sided ideals in maximal orders. Artin divided this into three papers which all appeared 1928 in the same volume of *Hamburger Abhandlungen.* [1], [2], [3]. The next paper on the subject was Hasse's [12] who proposed to develop the arithmetic in algebras by means of local methods according to Hensel,

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